SODOM-2 Program

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1 Introduction

SODOM-2 is an algorithm formulated by K. Yokoya [1] to calculate the invariant spin field (ISF) in an accelerator by decomposing the spin and orbit motion into their Fourier components. The derivation shown here is taken nearly exactly from [2]. The ISF is a special 2π-periodic spin field that solves the Thomas-BMT equation.

$$\boldsymbol{n}(\boldsymbol{z},\theta) = \underline{R}(\boldsymbol{z}_0,\theta_0;\theta)\boldsymbol{n}(\boldsymbol{z}_0,\theta_0), \qquad \boldsymbol{n}(\boldsymbol{z},\theta_0+2\pi) = \boldsymbol{n}(\boldsymbol{z},\theta_0) \tag{1}$$

For 3D linear orbit motion, a particle lies on the invariant torus defined by $\boldsymbol{J} = (J_I, J_{II}, J_{III})$, where each J_i is the action in the i-th oscillation mode. For example, in an uncoupled ring, $J_I = J_x$ and a particle's x-coordinate is $x(s) = \sqrt{2J_x\beta_x(s)}\cos\phi_x(s)$. See [3] for more details. In the following expressions the actions $\boldsymbol{J}$ are omitted because they are constants. The ISF can be expressed as a spinor $\Psi(\boldsymbol{\phi},\theta)$ where $\boldsymbol{n}(\boldsymbol{\phi},\boldsymbol{\theta}) = \Psi^\dagger\boldsymbol{\sigma}\Psi$ and $\boldsymbol{\sigma}$ are the Pauli matrices. Omitting the azimuth position θ, starting at $\boldsymbol{\phi}$ after one turn the invariant spin direction at the angle coordinates $\boldsymbol{\phi}$ agrees with the invariant spin direction at $\boldsymbol{\phi}+2\pi\boldsymbol{Q}$, where $\boldsymbol{Q}$ are the orbital tunes in each mode, up to some arbitrary phase factor $\tilde{\nu}_{\boldsymbol{J}}(\boldsymbol{\phi})$:

$$\underline{A}(\boldsymbol{\phi})\Psi(\boldsymbol{\phi}) = e^{-\mathrm{i}\pi\tilde{\nu}_{\boldsymbol{J}}(\boldsymbol{\phi})}\Psi(\boldsymbol{\phi}+2\pi\boldsymbol{Q})\ , \tag{2}$$

where $\underline{A}(\boldsymbol{\phi})$ is the 1-turn spin transport quaternion at initial angle $\boldsymbol{\phi}$. A phase function $\varphi_{\boldsymbol{J}}(\boldsymbol{\phi})$ is used such that the new spinor $\Psi_n(\boldsymbol{\phi}) = e^{\mathrm{i}\frac{1}{2}\varphi_{\boldsymbol{J}}(\boldsymbol{\phi})}\Psi(\boldsymbol{\phi})$ has the periodicity condition

$$\underline{A}(\boldsymbol{\phi})\Psi_n(\boldsymbol{\phi}) = e^{-\mathrm{i}\pi\nu(\boldsymbol{J})}\Psi_n(\boldsymbol{\phi}) \ , \tag{3}$$

where the phase factor $\nu(\boldsymbol{J}) = 2\pi\tilde{\nu}_{\boldsymbol{J}}(\boldsymbol{\phi}) - \varphi_{\boldsymbol{J}}(\boldsymbol{\phi}) + \varphi_{\boldsymbol{J}}(\boldsymbol{\phi} + 2\pi\boldsymbol{Q})$ is independent of the of the angle coordinates $\boldsymbol{\phi}$. This is the amplitude-dependent spin tune $\nu(\boldsymbol{J})$. The 1-turn quaternion $\underline{A}(\boldsymbol{\phi})$ and the ISF $\Psi_n(\boldsymbol{\phi})$ are 2π-periodic functions of $\boldsymbol{\phi}$ and can therefore be expressed as a Fourier series.

$$\underline{A}(\boldsymbol{\phi}) = \sum_{\boldsymbol{j}} \underline{A}_{\boldsymbol{j}} e^{\mathrm{i}\boldsymbol{j}\cdot\boldsymbol{\phi}} \ , \quad \Psi_n(\boldsymbol{\phi}) = \sum_{\boldsymbol{j}} \Psi_{n,\boldsymbol{j}} e^{\mathrm{i}\boldsymbol{j}\cdot\boldsymbol{\phi}} \tag{4}$$

Equation (3) can then be expressed as

$$e^{-\mathrm{i}2\pi\boldsymbol{j}\cdot\boldsymbol{Q}} \sum_{\boldsymbol{k}} \underline{A}_{\boldsymbol{j}-\boldsymbol{k}} \Psi_{n,\boldsymbol{k}} = e^{-\mathrm{i}\pi\nu} \Psi_{n,\boldsymbol{j}} \quad . \tag{5}$$

This is simply an eigenproblem for the matrix $e^{-\mathrm{i}2\pi\boldsymbol{j}\cdot\boldsymbol{Q}}\underline{A}_{\boldsymbol{j}-\boldsymbol{k}}$. The eigenvalues give the amplitude-dependent spin tune, and an eigenvector gives the Fourier coefficients $\Psi_{n,\boldsymbol{j}}$ which can then be used to construct the ISF as a function of the angle coordinates $\boldsymbol{\phi}$ per Eq. (4). It can be checked that the eigenvector with components $\Psi'_{n,\boldsymbol{j}} = \Psi_{n,\boldsymbol{j}-\boldsymbol{l}}$ for some vector of integers $\boldsymbol{l}$ is also an eigenvector with eigenvalue $e^{-\mathrm{i}\pi(\nu - 2\boldsymbol{l}\cdot\boldsymbol{Q})}$, and so the spin tune obtained from the eigenvalue may be any $2\times$integer multiple of the orbital tunes. The best choice of eigenvector/eigenvalue pair is chosen to be the one with a maximum $|\Psi_{n,(0,0,0)}|$.

2 Running the SODOM2 Program

The **sodom2** program comes with the "Bmad Distribution" which is a package that contains the Bmad toolkit library along with a number of Bmad based programs. See the Bmad website for more details. The syntax for invoking the program is:

```
sodom2 {<master_input_file_name>}
```

Example:

```
sodom2 my_input_file.init
```

The <**master_input_file_name**> optional argument is used to set the master input file name. The default value is "**sodom2.init**". The syntax of the master input file is explained in §3.

Example input files are in the directory (relative to the root of a Distribution):

```
bsim/sodom2/example
```

3 Master Input File

The **master input file** holds the parameters needed for running the **sodom2** program. The master input file must contain a single namelist named **params**. Example:

```
&params
  sodom2%lat_file = 'esr-18GeV.bmad'
  sodom2%ele_eval = '107'
  sodom2%J = 0, 100e-9, 0
  sodom2%n_samples = 35, 35, 1
  sodom2%n_axis_output_file = 'n_axis.out'
  sodom2%particle_output_file = 'sodom2.out'
  sodom2%write_as_beam_init = T
  sodom2%add_closed_orbit_to_particle_output = F
  sodom2%print_n_mat = T
  sodom2%linear_tracking = T
/
```

Parameters in the master input file that affect the program are:

sodom2%add_closed_orbit_to_particle_output
: If set **False** (the default), the particle_output_file includes the particle positions with respect to the closed orbit. If set **True**, the output positions are with respect to the zero orbit.

sodom2%ele_eval
: Name or element index of the element to evaluate the n-axis at. Examples:

```
ele_eval = "Q3##2"   ! 2nd element named Q3 in the lattice.
ele_eval = 37        ! 37th element in the lattice.
```

The default is to start at the beginning of the lattice. Note that the evaluation is performed at the downstream end of the element, so the n-axis is evlauated at the start of the element after ele_eval.

sodom2%J
: Array of the particle actions in each oscillation mode (J_I, J_{II}, J_{III}). Atleast one J_i must be specified.

sodom2%lat_file
: Name of the Bmad lattice file to use. This name is required.

sodom2%linear_tracking
: If set **True** (the default), **sodom2** will set the orbital tracking method for every element in the lattice to linear before computing the 1-turn quaternions for each sample particle. SODOM-2 assumes the linear actions $\boldsymbol{J}$ are constants, and therefore this flag should generally be set to **True**. If set **False**, **sodom2** will use the tracking methods specified in the lattice file.

sodom2%n_axis_output_file
: Name of the output file to write the spinor Fourier components of the ISF to. Default is 'n_axis.out'.

sodom2%n_samples
Array of the number of Fourier coefficients to compute in each oscillation mode. In order to center the harmonics around 0, **sodom2** will automatically set these quantities to be the nearest larger odd number if even numbers are inputted.

sodom2%particle_output_file
Name of the output file to write the phase space coordinates and $\boldsymbol{n}$ axis of each of the sample particles used to calculate the ISF. Default is 'sodom2.out'

sodom2%print_n_mat
If set **True**, the conversion matrix (**N** matrix) from action-angle coordinates to phase space coordinates (x, p_x, y, p_y, z, p_z) as described by Wolski [3] is printed to the terminal. Default is **False**.

sodom2%write_as_beam_init
If set **True**, the particle_output_file is printed in a Bmad beam_init format. The default is **False**.

4 References

[1] K. Yokoya, "An algorithm for calculating the spin tune in circular accelerators", DESY–99–006 (1999).

[2] G. H. Hoffstaetter, *High-Energy Polarized Proton Beams, A Modern View*, Springer. Springer Tracks in Modern Physics Vol 218, (2006).

[3] A. Wolski, "Alternate approach to general coupled linear optics", Phys. Rev. Special Topics, Accel & Beams, **9**, 024001 (2006).