Recursive Analogues of Shrewdness and Subtlety, with Applications to Fine Structure

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Abstract

We share both recent and older, well-known results regarding the notions of stable ordinals and shrewd cardinals. We then argue that nonprojectible ordinals may be considered as recursive analogues to subtle cardinals, a highly combinatorial type of cardinal related to Jensen's fine structure, due to the latter possessing a characterisation in terms of shrewdnesss.

1 Introduction

The notion of a shrewd cardinal was introduced by Rathjen in [9]. Shrewd cardinals offer an alternative transfinite extension of indescribability to ξ-indescribability (for $\xi \geq \omega$), since the latter suffers from the following two unfavourable properties:

1. If κ is ξ-indescribable and $\xi' < \xi$, κ need not be ξ'-indescribable.
2. κ cannot be κ-indescribable.

Which imply that the hierarchy is not so nice and stops abruptly, being unable to reach, say, "hyper-indescribability". Shrewdness suffers from neither of these flaws, and therefore can be considered as a linear hierarchy of large cardinal axioms bridging the gap between weak compactness or finite stages of indescribability, and more powerful combinatorial or elementary embedding-based large cardinals in the literature. Let us give their definition:

Definition 1.1. Let $\eta > 0$. A cardinal κ is η-shrewd iff, for all $P \subseteq V_\kappa$ and every formula $\varphi(x)$, if $(V_{\kappa+\eta}, \in, P) \models \varphi(\kappa)$, then there exist $0 < \kappa_0, \eta_0 < \kappa$ so that $(V_{\kappa_0+\eta_0}, \in, P \cap V_{\kappa_0}) \models \varphi(\kappa_0)$.

By slightly modifying the definition, one can obtain various natural strengthenings of these cardinals, as well as delineate the consistency strength hierarchy or shrewdness more finely. In particular, we will consider the definition of two additional parameters: an arbitrary class $\mathcal{A}$ and a set $\mathcal{F}$ of formulae.

The definition of a $\mathcal{A}$-η-$\mathcal{F}$-shrewd cardinal is obtained by the definition of an η-shrewd cardinal by:

1. Restricting the formula φ to be an element of $\mathcal{F}$.
2. Adding an additional predicate for $\mathcal{A}$ in both the hypothesis and the conclusion. Namely, $(V_{\kappa+\eta}, \in, P) \models \varphi(\kappa)$ becomes $(V_{\kappa+\eta}, \in, P, \mathcal{A} \cap V_{\kappa+\eta}) \models \varphi(\kappa)$, and similarly $(V_{\kappa_0+\eta_0}, \in, P \cap V_{\kappa_0}) \models \varphi(\kappa_0)$ becomes $(V_{\kappa_0+\eta_0}, \in, P \cap V_{\kappa_0}, \mathcal{A} \cap V_{\kappa_0+\eta_0}) \models \varphi(\kappa_0)$.

The formula therefore may now contain two additional predicate symbols, one for first-order variables (representing P) and one for the highest-order variables (representing $\mathcal{A}$). Therefore, it makes sense to consider natural $\mathcal{F}$'s which are closed under addition of predicates. The most natural $\mathcal{F}$'s to consider are therefore the classes Π_n and Σ_n of the Lévy hierarchy. By the Tarski-Vaught test, we may eliminate the latter, as $\mathcal{F} = \Sigma_{n+1}$ immediately reduces to $\mathcal{F} = \Pi_n$.

Note that if $\mathcal{A} = V$, one may eliminate the second part and so consider simply η-$\mathcal{F}$-shrewdness. Dually, if $\mathcal{F}$ consists of all formulae, one may eliminate the first part and so consider simply $\mathcal{A}$-η-shrewdness. By eliminating both, one arrives where we started – η-shrewdness.

It is now an immediate observation that κ is η-shrewd iff it is η-Π_n-shrewd for all n. This is since any formula is Π_n for some n, by writing in prenex normal form. This allows us to connect back finite stages of shrewdness to finite stages of indescribability:

Proposition 1.2. *Assume $0 \leq n, m < \omega$ and $n \neq 0$. Then κ is n-Π_m-shrewd iff κ is Π^n_m-indescribable. In particular, κ is n-shrewd iff κ is Π^{n+1}_0-indescribable.*

The proof that, if κ is η-shrewd and $\delta < \eta$, then κ is δ-shrewd (which implies that, say, if κ is η-shrewd and $\eta \geq \omega$, then κ is totally indescribable) can be found in [8].

Subtlety is a combinatorial principle introduced by Jensen relating to his analysis of the combinatorial and fine structural properties of the constructible universe L. While originally formulated in a manner similarly to his $\diamond$ principle, it also has a characterisation in terms of shrewdness. This characterisation is quite reminiscent of Woodin and Vopěnka cardinals. Let us first give the original definition.

Definition 1.3. A cardinal κ is called subtle iff, for any sequence $\langle S_\alpha : \alpha < \kappa\rangle$ satisfying $S_\alpha \subseteq \alpha$ for all $\alpha < \kappa$, and club C in κ, there are $\beta, \delta \in C$ so that $\beta < \delta$ and $S_\delta \cap \beta = S_\beta$.

In essence, any sequence $\langle S_\alpha : \alpha < \kappa\rangle$ satisfying $S_\alpha \subseteq \alpha$ for all $\alpha < \kappa$ can be made to "cohere" club many times. Some properties of subtle cardinals can be found in [4]. For the following characterisation, say κ is $\mathcal{A}$-$<\pi$-shrewd iff it is $\mathcal{A}$-η-shrewd for all $\eta < \pi$.

Theorem 1.4. *Let π be an inaccessible cardinal. Then TFAE:*

1. *π is subtle.*
2. *For any $\mathcal{A} \subseteq V_\pi$, the set of κ which are $\mathcal{A}$-$<\pi$-shrewd is stationary in π.*

Proof. The forward direction can be found in [8]. For the sake of completeness, we will give it here. So, assume π is subtle. It is known that π is strongly inaccessible, and thus $|V_\pi| = \pi$. Let $F : V_\pi \to \pi$ be a bijection. And since strongly inaccessible cardinals are limits of cardinals $\kappa < \pi$ so that $|V_\kappa| = \kappa$, one may pick F in a way so that the set of $\kappa < \pi$ so that $F''V_\kappa = \kappa$ is club. This club is denoted C_F. Now pick an arbitrary club B in π. We aim to show that B contains a $\mathcal{A}$-$<\pi$-shrewd cardinal. Since π is uncountable regular, the intersection of two clubs in π is also club, so we may assume without loss of generality that $B \subseteq C_F$, replacing B with $B \cap C_F$ if this isn't the case. Therefore, all elements of B can be taken to be cardinals.

Now for a contradiction assume there is no cardinal $\kappa \in B$ so that κ is $\mathcal{A}$-$<\pi$-shrewd. Since B is unbounded, one can use the axiom of choice to pick, for all $\kappa \in B$, a σ_κ witnessing that κ is not $\mathcal{A}$-$<\pi$-shrewd (i.e. for each $\kappa \in B$, κ is not $\mathcal{A}$-σ_κ-shrewd). We may once again assume, without loss of generality, that $\kappa < \sigma_\kappa$ for all $\kappa \in B$, by the fact that if κ is $\mathcal{A}$-η-shrewd and $\delta < \eta$, then κ is $\mathcal{A}$-δ-shrewd (and so, by contrapositive, if κ is not $\mathcal{A}$-η-shrewd for some $\eta \leq \kappa$, then κ is not $\mathcal{A}$-η-shrewd for any $\eta > \kappa$). We can extend σ to a total map with domain κ by simply putting $\sigma_\rho = \rho$ for $\rho \notin B$.

Let E be the set of $\rho \in B$ so that $\sigma_\nu < \rho$ whenever $\nu < \rho$. For example, $\min(B) \in E$, since $\nu < \min(B)$ implies $\nu \notin B$ and whence $\sigma_\nu = \nu$. A simple argument shows that E is club in π as well. It's easy to see that, if $\kappa_0, \kappa_1 \in A$ and $\kappa_0 < \kappa_1$, then $\sigma_{\kappa_0} < \kappa_1$ and so κ_0 is not $\mathcal{A}$-κ_1-shrewd. Therefore, for $\kappa \in E$, let κ^* be the minimal element of E above κ. Note that, by our hypothesis that all elements of B are cardinals, we have $\kappa + \kappa^* = \kappa$. Now since κ is never $\mathcal{A}$-κ^*-shrewd, we may let φ_κ and P_κ witness this, i.e. be so that $(V_{\kappa^*}, \in, P_\kappa, \mathcal{A} \cap V_{\kappa^*}) \models \varphi_\kappa(\kappa)$, but whenever $0 < \nu, \delta < \kappa$, we have $(V_{\nu+\delta}, \in, P_\kappa \cap V_\nu, \mathcal{A} \cap V_{\nu+\delta}) \models \neg\varphi_\kappa(\nu)$.

Now let $\theta_\kappa(u, v)$ be the formula $v \in \mathrm{Ord} \wedge \exists \xi > v(u \subseteq V_\xi \wedge (V_\xi, \in, u, \mathbf{P} \cap V_\xi) \models \varphi_\kappa(u, v))$. Here, $\mathbf{P}$ is the predicate (i.e. definable class) which is interpreted as $\mathcal{A}$. Now, if $\kappa^* < \rho$, then $(V_\rho, \in, \mathcal{A} \cap V_\rho) \models \theta_\kappa(P_\kappa, \kappa)$, witnessed by $\xi = \kappa^*$. Let $\vartheta_\kappa(v)$ be the formula in the language with both additional predicate symbols which states that $\theta_\kappa(u, v)$ holds where u is the subclass of the first-order domain of all objects satisfying the first predicate. Therefore, $\kappa^* < \rho$ implies $(V_\rho, \in, P_\kappa, \mathcal{A} \cap V_\rho) \models \vartheta_\kappa(\kappa)$. Furthermore, for all $0 < \mu < \kappa$, we have $(V_\kappa, \in, P_\kappa \cap V_\mu, \mathcal{A} \cap V_\kappa) \models \neg\vartheta_\kappa(\mu)$ by our assumption that whenever $0 < \nu, \delta < \kappa$, we have $(V_{\nu+\delta}, \in, P_\kappa \cap V_\nu, \mathcal{A} \cap V_{\nu+\delta}) \models \neg\varphi_\kappa(\nu)$.

Now let E_∞ be the set of ordinals below π which are limit points of E. Now the upshot is that, whenever $\kappa, \rho \in E_\infty$ and $\kappa < \rho$, then $(V_\rho, \in, P_\kappa, \mathcal{A} \cap V_\rho) \models \vartheta_\kappa(\kappa)$ and, whenever $0 < \mu < \kappa$, $(V_\kappa, \in, P_\kappa \cap V_\mu, \mathcal{A} \cap V_\kappa) \models \neg\vartheta_\kappa(\mu)$. Now we use subtlety to derive a contradiction. Let $\langle \psi_n : n < \omega \rangle$ enumerate all formulae with one free variable in our language with two predicate symbols. Also, for $\kappa \in E_\infty$, let κ^{**} denote the minimal element of E_∞ above κ. Then define P^*_κ as a particular coding of P_κ, namely $P^*_\kappa = (F'' P_\kappa \cap (\kappa \setminus \omega)) \cup \{3n : n \in F'' P_\kappa \cap \omega\} \cup \{3n+1 : (V_{\kappa^{**}}, \in, P_\kappa, \mathcal{A} \cap V_{\kappa^{**}}) \models \psi_n(\kappa)\} \cup \{3n+2 : (V_{\kappa^{**}}, \in, P_\kappa, \mathcal{A} \cap V_{\kappa^{**}}) \models \neg\psi_n(\kappa)\}$. For $\alpha \notin E_\infty$, set $P^*_\alpha = \alpha$.

Since E_∞ is the set of limit points of a particular club and is therefore seen to also be a club, and it is also immediate that $P^*_\alpha \subseteq \alpha$ for all $\alpha < \delta$, we can use subtlety of π to find two $\beta, \gamma \in E_\infty$ so that $\beta < \gamma$ and $P^*_\gamma \cap \beta = P^*_\beta$. By the way we defined P^*_κ for $\kappa \in E_\infty$, the fact that F is injective, and that $E_\infty \subseteq C_F$, it follows that $P_\gamma \cap V_\beta = P_\beta$. By our previous remark that $\kappa, \rho \in E_\infty$ and $\kappa < \rho$ implies $(V_\rho, \in, P_\kappa, \mathcal{A} \cap V_\rho) \models \vartheta_\kappa(\kappa)$, it follows that $(V_{\gamma^{**}}, \in, P_\gamma, \mathcal{A} \cap V_{\gamma^{**}}) \models \vartheta_\gamma(\gamma)$. Combining this with $P^*_\gamma \cap \beta = P^*_\beta$ and the definition of P^*_κ for $\kappa \in E_\infty$, we obtain $(V_{\beta^{**}}, \in, P_\gamma \cap V_\beta, \mathcal{A} \cap V_{\beta^{**}}) \models \vartheta_\gamma(\beta)$. Another application of the precise way P^*_κ is defined for $\kappa \in E_\infty$, together with $P^*_\gamma \cap \beta = P^*_\beta$, we get $(V_\gamma, \in, P_\gamma \cap V_\beta, \mathcal{A} \cap V_\gamma) \models \vartheta_\gamma(\beta)$. But this contradicts our assumption that, whenever $0 < \mu < \kappa$, $(V_\kappa, \in, P_\kappa \cap V_\mu, \mathcal{A} \cap V_\kappa) \models \neg\vartheta_\kappa(\mu)$.

Now, for the backwards direction, we utilize Rathjen's notion of reducibility. Assume that π is inaccessible and, for any $\mathcal{A} \subseteq V_\pi$, the set of κ which are $\mathcal{A}$-$<\pi$-shrewd is stationary in π. Via Appendix 7.8 of [9], we see that any $\mathcal{A}$-$<\pi$-shrewd cardinal below π is $\mathcal{A}$-$<\pi$-reducible. Now let C be a club in π and $\vec{S} = \langle S_\alpha : \alpha < \pi \rangle$ satisfy $S_\alpha \subseteq \alpha$ for all $\alpha < \pi$. It actually suffices for $\vec{S}$ to be partial with domain C, i.e. S_α is only defined for $\alpha \in C$. This will be relevant later, and does not affect the large cardinal axiom as we only consider S_β for $\beta \in C$. Note then that $\vec{S} \subseteq V_\pi$. By stationarity, there is $\kappa \in C$ which is $\vec{S}$-$<\pi$-reducible. We will be interested in the specific case that κ is $\vec{S}$-$\kappa+2$-reducible. Now there is some $0 < \kappa_0 < \eta_0 < \kappa + 2$ so that $\langle V_{\eta_0}, \in, \kappa_0, \vec{S}|\eta_0 \rangle$ is elementarily equivalent to $\langle V_{\kappa+2}, \in, \kappa, \vec{S}|(\kappa+2) \rangle$. We first claim that we must have $\eta_0 = \kappa_0 + 2$: let φ be a formula formalizing "$c+1$ exists and is the largest ordinal", where c is a constant symbol representing κ_0 or κ. Since $\langle V_{\kappa+2}, \in, \kappa \rangle \models \varphi$, we have $\langle V_{\eta_0}, \in, \kappa_0 \rangle \models \varphi$, and so $\kappa_0 + 1$ is the largest ordinal of V_{η_0}. Thus $\eta_0 = \kappa_0 + 2$.

Utilizing elementary equivalence again, but this time exploiting the predicate, we obtain, for any $\alpha < \kappa_0$, $\langle V_{\eta_0}, \in, \kappa_0, \vec{S}|(\kappa_0+2) \rangle \models \alpha \in S_{\kappa_0}$ iff $\langle V_{\kappa+2}, \in, \kappa, \vec{S}|(\kappa+2) \rangle \models \alpha \in S_\kappa$. Thus, $S_\kappa \cap \kappa_0 = S_{\kappa_0}$, as desired. It remains to show that $\kappa, \kappa_0 \in C$. We already have $\kappa \in C$ by hypothesis, and

$$
\begin{aligned}
\kappa_0 \in C &\iff \kappa_0 \in \mathrm{dom}(\vec{S}) \\
&\iff \kappa_0 \in \mathrm{dom}(\vec{S}|(\kappa_0+2)) \\
&\iff \langle V_{\kappa_0+2}, \in, \kappa_0, \vec{S}|(\kappa_0+2)\rangle \models \text{``}\kappa_0 \in \mathrm{dom}(\vec{S}|(\kappa_0+2)) \\
&\iff \langle V_{\kappa+2}, \in, \kappa, \vec{S}|(\kappa_0+2)\rangle \models \text{``}\kappa \in \mathrm{dom}(\vec{S}|(\kappa+2)) \\
&\iff \kappa \in C
\end{aligned} \tag{1}
$$

which is true. □

In particular, if π is subtle, then for any $\eta < \pi$ there is an $\kappa < \pi$ which is η-shrewd. Of course, the full characterisation we gave is stronger. Note that π itself need not be 1-shrewd: this is since any 1-shrewd cardinal is Π^2_0-indescribable, but subtlety is describable by a Π^1_1 definition and whence the least subtle cardinal isn't even weakly compact.

2 Recursive analogues and stable ordinals

In this brief note, we shall show that nonprojectible ordinals may be considered as so-called "recursive analogues" of subtle cardinals. The notion of a nonprojectible ordinal was a direct byproduct of Jensen's analysis of the fine structure of L, thereby also being perhaps genealogically related to subtlety as well. The notion was first shared in print in the famous manuscript [3], under the name of a strongly admissible ordinal. The term "nonprojectibility" arises from the fact that α is nonprojectible iff, whenever $\eta < \alpha$ and $f : \eta \to \alpha$ is surjective, then f is not Σ_1-definable in L_α with parameters. Equivalently, the ordinals $\eta < \alpha$ so that $L_\eta \prec_{\Sigma_1} L_\alpha$ are unbounded, which is the definition we shall use. The original definition was as ordinals α so that L_α satisfies Kripke-Platek set theory KP with Σ_1-separation, which also is equivalent to ordinals α so that $L_\alpha \cap \mathcal{P}(\omega)$ is a model of Π^1_2-comprehension, where the latter additionally required $\alpha < \omega_1^L$. See [2] and [11] for proofs of these characterisations.

The notion of a recursive analogue is relevant to α-recursion theory, in which one generalizes theorems in classical recursion and computational complexity theory (where primitive objects are hereditarily finite sets, e.g. natural numbers and finite binary sequences) to theorems about L_α where α is an admissible ordinal. For example, for $\alpha > \omega$, hyperarithmetic reals may be treated as primitive objects as they arise in $L_{\omega_1^{\mathrm{CK}}} \cap \mathcal{P}(\omega)$. In essence, a recursive analogue of a large cardinal is a notion describing large countable ordinals, which may behave in a way that mimics the way that large cardinals act. This also makes them useful in proof theory, as the existence of recursive analogues of large cardinals is actually provable in ZFC, unlike the existence of large cardinals, and so one may define ordinal representation systems in ZFC without requiring large cardinal hypotheses. For example, an ordinal analysis of $\mathsf{KP}+$ " every set is contained in a standard transitive model of KP", also denoted KPI, has traditionally required the additional assumption of the existence of a weakly inaccessible cardinal, but this could be eliminated by replacing weakly inaccessible cardinals with their recursive analogue – recursively inaccessible ordinals.

Recursively regular ordinals are typically considered to be admissible ordinals (and whence recursively inaccessible ordinals are precisely the admissible limits of admissible ordinals), because of the similarity between the following two characterisations:

Lemma 2.1. *A cardinal κ is regular iff, for any function $f : \kappa \to \kappa$, there is an $\alpha < \kappa$ so that $f''\alpha \subseteq \alpha$.*

An ordinal κ is admissible iff the above holds when f is restricted to be Δ_1-definable in L_κ with parameters.

Unfortunately, this substitution means that slightly more heavy lifting is required, as one needs to verify that the desired projection functions are sufficiently definable. The notions of Δ_1- and Σ_1-definability in L_α with parameters arises often, so we adopt the following abbreviative convention from generalized recursion theory: $A \subseteq L_\alpha$ is α-recursive (resp. α-recursively enumerable) iff it is Δ_1-definable (resp. Σ_1-definable) in L_α with parameters. Back to the topic at hand, let us define the notion of a ξ-Π_n-reflecting ordinal.

Definition 2.2. Let A be a class of ordinals. An ordinal α is called ξ-Π_n-reflecting onto A iff, for every Π_n-formula $\varphi(x)$, and for all $b \in L_\alpha$, if $L_{\alpha+\xi} \models \varphi(b)$, then there exist $\alpha_0, \xi_0 < \alpha$ so that $\alpha_0 \in A$, $b \in L_{\alpha_0}$ and $L_{\alpha_0+\xi_0} \models \varphi(b)$.

α is called ξ-Π_n-reflecting iff it is ξ-Π_n-reflecting onto Ord, and Π_n-reflecting iff it is 0-Π_n-reflecting.

It is known that α is Π_2-reflecting iff $\alpha > \omega$ and it is admissible. See [1] for a proof of this. Therefore, it may be argued that Π_2-reflecting ordinals serve as a countable analogue of uncountable regular ordinals, as already mentioned. It is known that Π_3-reflecting ordinals are Π_2-reflecting onto the class of Π_2-reflecting ordinals, and much more. Whence, it was argued by Richter and Aczel in [10] that Π_{n+2}-reflecting ordinals should serve as recursive analogues to Π^1_n-describable cardinals for $n > 0$. In general, for $\xi > 0$, ξ-Π_n-reflecting ordinals should serve as recursive analogues to $1+\xi$-Π_n-shrewd ordinals. A large focus has become stability, which links back to our previous mention of nonprojectibility:

Definition 2.3. Say α is ξ-stable iff $L_\alpha \prec_{\Sigma_1} L_\xi$, where $\prec_{\Sigma_1}$ denotes the relation of being a Σ_1-elementary substructure.

It is easy to see that if α is $\alpha+1$-stable, then it is Π_n-reflecting for all $n < \omega$. Namely, let $\varphi(x)$ be an arbitrary formula, and $b \in L_\alpha$. Assume $L_\alpha \models \varphi(b)$. Then $L_{\alpha+1}$ satisfies "there is an ordinal β so that $b \in L_\beta$ and $L_\beta \models \varphi(b)$", which can be written in Σ_1 form. Thus, by $L_\alpha \prec_{\Sigma_1} L_{\alpha+1}$, L_α satisfies the same thing, and so the β witnessing this satisfies $b \in L_\beta$ and $L_\beta \models \varphi(b)$. It turns out that the converse is also true, and a more general result holds for ξ-Π_n-reflection.

Lemma 2.4. *For an ordinal α, α is $\alpha+\xi+1$-stable iff it is ξ-Π_n-reflecting for all n.*

Using Theorem 1.4 and Lemma 2.4, this motivates the definition of a recursively subtle ordinal, as well as a slight strengthening.

Definition 2.5. Let $\mathcal{A}$ be an arbitrary class. Say that α is $\mathcal{A}$-ξ-stable iff $\langle L_\alpha, \in, \mathcal{A} \cap L_\alpha\rangle \prec_{\Sigma_1} \langle L_\xi, \in, \mathcal{A} \cap L_\xi\rangle$. Now say an ordinal ρ is recursively subtle iff, for any ρ-recursively enumerable $\mathcal{A} \subseteq L_\rho$, ρ is Π_2-reflecting onto the set of $\kappa < \rho$ which are $\mathcal{A}$-ρ-stable.

Note that it is a known result that if κ is $\mathcal{A}$-$<\rho$-stable, it is already $\mathcal{A}$-ρ-stable, and so we did not deviate from the "spirit" of Theorem 1.4 too much. Actually, let us briefly state some well-known properties of stability:

Lemma 2.6.
1. *If $\alpha < \beta < \gamma$ and α is γ-stable, then α is β-stable.*
2. *If α is β-stable and β is γ-stable, then α is γ-stable.*
3. *If α is β-stable for all $\alpha \in A$, then either $\sup A = \beta$ or $\sup A$ is β-stable.*

4. *Dually, if α is β-stable for all $\beta \in A$, then α is $\sup B$-stable.*
5. *If α is $\alpha+2$-stable, then it is Π_n-reflecting on the class of ordinals ξ which are $\xi+1$-stable, for all $n<\omega$.*
6. *The least ordinal α that is a limit of ordinals ξ which are $\xi+1$-stable is not itself even admissible.*

Proof. (1) This follows from an easy upwards absoluteness argument.

(2) This is trivial.

(3) Assume that, for all $\alpha \in A$, we have $L_\alpha \prec_{\Sigma_1} L_\beta$. We aim to show that $L_{\sup A} \prec_{\Sigma_1} L_\beta$, assuming $\sup A < \beta$. In the case when $\sup A = \max A$, it is trivial, so we may assume, without loss of generality, that $\sup A$ is strictly greater than all elements of A. Let $x \in L_{\sup A}$ and φ be an arbitrary Σ_1 formula. We aim to show that $L_{\sup A} \models \varphi(x)$ iff $L_\beta \models \varphi(x)$. The forwards direction follows by upwards absoluteness. For the converse direction, let $\varphi(x)$ be of the form $\exists y \psi(x,y)$ where ψ is Δ_0. Assume $L_\beta \models \varphi(x)$. Since $\sup A$ is a limit ordinal and A is cofinal in $\sup A$, there is some $\alpha \in A$ so that $x \in L_\alpha$. By $L_\alpha \prec_{\Sigma_1} L_\beta$, we have $L_\alpha \models \varphi(x)$. Then $L_{\sup A} \models \varphi(x)$, once again by upwards absoluteness.

(4) This follows by a similar argument to (3).

(5) Let φ be a Π_n formula so that $L_\alpha \models \varphi$, and $x \in L_\alpha$. Then $L_{\alpha+2} \models \exists\xi(L_\xi \prec_{\Sigma_1} L_{\xi+1} \wedge x \in L_\xi \wedge L_\xi \models \varphi(x))$, with witness $\xi = \alpha$, which can be rendered in Σ_1 form. Thus $L_\alpha \models \exists\xi(L_\xi \prec_{\Sigma_1} L_{\xi+1} \wedge x \in L_\xi \wedge L_\xi \models \varphi(x))$, and so there is some $\xi < \alpha$ so that ξ is $\xi+1$-stable, $x \in L_\xi$ and $L_\xi \models \varphi$. Thus $\xi \in A \cap \alpha$ where A is the class of ξ which are $\xi+1$-stable, and so ξ witnesses Π_n-reflection onto A in this instance.

(6) This proof uses Richter and Aczel's notion of Σ_1-collection; note that this is different to many other notions of $\mathcal{F}$-collection. Now for contradiction, assume that α is admissible. Let $\varphi(n,\gamma)$ be a formula that asserts that γ is the n'th, starting from $n=0$, ordinal so that γ is $\gamma+1$-stable. This formula can be defined uniformly in n. Then apply Σ_1-collection to the formula $L_\alpha \models \forall n < \omega \exists \nu \varphi(n,\nu)$ to obtain "for some $b \in L_\alpha$, we have $L_\alpha \models \forall n < \omega \exists \nu (b \models \varphi(n,\nu))$", i.e. some set in L_α contains ω many ordinals γ so that γ is $\gamma+1$-stable. But this contradicts minimality of α. □

The reason why we used Π_2-reflection in our formulation of recursive subtlety is because, recalling the statement that Π_{n+2}-reflection serves as a recursive analogue of Π^1_n-indescribability, it is known that κ is Π^1_0-indescribable onto A iff it is strongly inaccessible and A is stationary in κ, thus if ρ is recursively inaccessible (which we shall show all recursively subtle ordinals are) it yields another adequate generalizaton. For example, an ordinal is considered to be recursively Mahlo iff it is Π_2-reflecting onto the set of admissible ordinals below, analogously to how a cardinal is Mahlo iff it is Π^1_0-indescribable onto the set of regular cardinals below.

As mentioned, our main theorem that ρ is recursively subtle iff it is nonprojectible. First, we shall state some results regarding stability and nonprojectibility, including some direct implications and size comparisons. Recall that ρ is nonprojectible iff, for any $\eta < \rho$, there is some $\kappa < \rho$ so that $\eta < \kappa$ and κ is ρ-stable. It follows that ρ's nonprojectibility is a Π_3 property of L_ρ. Therefore, there is already some behaviour-wise similarity between nonprojectibility and subtlety: the least subtle cardinal is much greater than the least 1-shrewd cardinal, and is itself Mahlo but not weakly compact. Similarly, the least nonprojectible ordinal is much greater than the least α which is $\alpha+1$-stable, and is (as we shall prove in a moment) itself recursively Mahlo but not Π_3-reflecting.

That nonprojectible ordinals are recursively Mahlo follows from the following theorem and the fact that if η is ρ-stable, it is very obviously admissible:

Theorem 2.7. *If ρ is nonprojectible, then ρ is Π_2-reflecting onto the set of $\kappa < \rho$ so that κ is ρ-stable.*

This theorem came as a surprise, since previously Π_2-reflection is generally considered stronger than even iterated limit point taking. However, there are already other contexts in which Π_2-reflection may fail to imply iterated limit points (e.g. $\aleph_\omega$ is Π_2-reflecting onto the set of cardinals, despite not being a limit of limit cardinals), and whence this is not too surprising.

Proof. Assume that ρ is nonprojectible, $L_\rho \models \forall x \exists y \varphi(x, y, b)$, where $\varphi(x, y, p)$ is a Δ_0-formula, and $b \in L_\rho$. Let β be so that β is ρ-stable and $b \in L_\beta$. Now let $x \in L_\beta$. Since β is ρ-stable, there is some $y \in L_\beta$ so that $\varphi^{L_\beta}(x, y, b)$ since $L_\rho \models \exists y \varphi(x, y, b)$ and $L_\beta \prec_{\Sigma_1} L_\rho$. Now let $y \in L_\beta$ be so that $L_\beta \models \varphi(x, y, b)$. Since φ is Δ_0 and such formulae are absolute for transitive sets, it follows that $\varphi^{L_\rho}(x, y, b)$ iff $\varphi^{L_\beta}(x, y, b)$, thus $L_\beta \models \forall x \exists y \varphi(x, y, b)$. The desired result follows. □

Also let us briefly state some results regarding nonprojectibility.

Theorem 2.8. *Say α is ω-fold stable iff there is a map $f : \omega \to \mathrm{Ord}$ so that $f(0) = \alpha$ and, for all $i < \omega$, $f(i)$ is $f(i+1)$-stable.*

1. *Assume α is ω-fold stable, witnessed by f. Then $\sup\{f(i) : i < \omega\}$ is nonprojectible.*
2. *As a sort of converse, assume ρ is nonprojectible, $\mathrm{cof}(\rho) = \omega$ and α is ρ-stable. Then α is ω-fold stable.*

Proof. (1) Let $\alpha_i = f(i)$, where f witnesses α's ω-fold stability, and $\rho = \sup\{\alpha_i : i < \omega\}$. Then by definition we have $L_{\alpha_0} \prec_{\Sigma_1} L_{\alpha_1} \prec_{\Sigma_1} L_{\alpha_2} \prec_{\Sigma_1} \cdots$. By Lemma 2.6.2 and 2.6.4, we see that α_i is α_j-stable whenever $i < j$, and therefore α_i is ρ-stable for all i. Now let $\tau < \rho$. Then there is some $i < \omega$ so that $\tau < \alpha_i$, and so α_i is ρ-stable, witnessing ρ's nonprojectibility in this case.

(2) Let $\{\rho_i : i < \omega\}$ be a cofinal subset of ρ, whose existence is guaranteed by our hypothesis $\mathrm{cof}(\rho) = \omega$. For each $i < \omega$, pick a δ_i so that $\rho_i < \delta_i$ and δ_i is ρ-stable, whose existence is guaranteed by our hypothesis that ρ is nonprojectible. By Lemma 2.6.1, δ_i is δ_j-stable whenever $i < j$. Let t be the least natural number so that $\alpha < \rho_t$. Then define $f : \omega \to \mathrm{Ord}$ by $f(0) = \alpha$ and $f(i+1) = \delta_{i+t}$. Another application of Lemma 2.6.1 shows that α is δ_i-stable for all $i \geq t$, and so f witnesses α's ω-fold stability. □

Definition 2.9. Let κ, ρ be ordinals with $\kappa < \rho$. Then $\mathcal{A} \subseteq L_\kappa$ is called (κ, ρ)-interpretable iff $\mathcal{A}$ is κ-recursively enumerable and, for any Σ_1-formula φ and parameters $z_1, z_2, \cdots, z_n$ witnessing this (i.e. $\mathcal{A} = \{x \in L_\kappa : L_\kappa \models \varphi(x, z_1, z_2, \cdots, z_n)\}$), if one sets $\mathcal{A}^{L_\rho} = \{x \in L_\rho : L_\rho \models \varphi(x, z_1, z_2, \cdots, z_n)\}$, then $\mathcal{A} = \mathcal{A}^{L_\rho} \cap L_\kappa$.

In other words, $\mathcal{A}$ is (κ, ρ)-interpretable iff it is κ-recursively enumerable and any such Σ_1-definition gives a way of extending $\mathcal{A}$ to a subset of L_ρ without adding new elements of L_κ.

Lemma 2.10. *Let κ, ρ be ordinals with $\kappa < \rho$. Then the following are equivalent:*

1. *κ is ρ-stable.*
2. *Every κ-recursively enumerable subset of L_κ is (κ, ρ)-interpretable.*

Proof. For the forward direction, assume κ is ρ-stable, $\mathcal{A} \subseteq L_\kappa$ is κ-recursively enumerable, and let $\varphi, \vec{z}$ be so that $\mathcal{A} = \{x \in L_\kappa : L_\kappa \models \varphi(x, \vec{z})\}$. By $L_\kappa \prec_{\Sigma_1} L_\rho$, we have $L_\kappa \models \varphi(x, \vec{z})$ iff $L_\rho \models \varphi(x, \vec{z})$ for $x \in L_\kappa$, i.e. $x \in \mathcal{A}$ iff $x \in \mathcal{A}^{L_\rho}$ for all $x \in L_\kappa$. Therefore $\mathcal{A} \cap L_\kappa = \mathcal{A}^{L_\rho} \cap L_\kappa$ and, since $\mathcal{A} \subseteq L_\kappa$, $\mathcal{A} \cap L_\kappa = \mathcal{A}$. Thus, $\mathcal{A} = \mathcal{A}^{L_\rho} \cap L_\kappa$ and, since $\mathcal{A}, \varphi, \vec{z}$ were arbitrary, any κ-recursively enumerable subset of L_κ is (κ, ρ)-interpretable.

For the converse direction, let $\vec{z} \in L_\kappa$, and φ be a Σ_1 formula. We aim to show that $L_\kappa \models \varphi(\vec{z})$ iff $L_\rho \models \varphi(\vec{z})$. Let $\mathcal{A} = \{x \in L_\kappa : L_\kappa \models \varphi(\vec{z})\}$. Then, by hypothesis, $\mathcal{A} = \mathcal{A}^{L_\rho} \cap L_\kappa$. Note that, since $\mathcal{A} \subseteq L_\kappa$, we have $\mathcal{A} \cap L_\kappa = \mathcal{A}$. It follows that, for all $x \in L_\kappa$, $x \in \mathcal{A}$ iff $x \in \mathcal{A}^{L_\rho}$, i.e. $L_\kappa \models \varphi(\vec{z})$ iff $L_\rho \models \varphi(\vec{z})$. Since $\varphi(\vec{z})$ is independent of x, this gives us the desired result. □

We give our main theorem now.

Theorem 2.11. *Let $\kappa < \rho$ be ordinals so that κ is ρ-stable and $\mathcal{A} \subseteq L_\rho$ is $\Sigma_1^{L_\rho}(L_\kappa)$. Then κ is $\mathcal{A}$-ρ-stable.*

If one is to require that $\mathcal{A} \subseteq L_\kappa$ instead, and that $\mathcal{A}$ is $\Sigma_1(L_\kappa)$, the theorem is significantly easier to prove. But unfortunately $\mathcal{A}$-ρ-stability is trivial if $\mathcal{A} \subseteq L_\kappa$. On the other side of things, one can not strengthen this to when $\mathcal{A}$ is $\Sigma_1(L_\rho)$. This is because $\{\kappa\}$ is $\Sigma_1(L_\rho)$ and κ is never $\{\kappa\}$-ρ-stable, since $\{\kappa\} \cap L_\kappa = \emptyset$ and $\{\kappa\} \cap L_\rho \neq \emptyset$, thus L_κ and L_ρ disagree about the Σ_1-sentence $\exists x(\mathbf{P}(x))$. This has no bearing on the truth of our main theorem, since we quantify $\mathcal{A}$ before considering the $\mathcal{A}$-ρ-stable ordinals. For example, if $\alpha < \kappa$ and κ is ρ-stable, then κ is $\{\alpha\}$-ρ-stable. This should be obvious, although it follows as an immediate corollary of this theorem which we shall finally prove after this brief discussion:

Proof. So κ is ρ-stable and $\mathcal{A}$ is $\Sigma_1^{L_\rho}(L_\kappa)$, witnessed by a Σ_1 formula φ and parameters $\vec{z} \in L_\kappa$. Then $L_\kappa \models \varphi(x, \vec{z})$ iff $x \in \mathcal{A}$, for $x \in L_\kappa$, by stability, so consequently $(L_\kappa, \in, \mathcal{A} \cap L_\kappa) \models \psi(\vec{a})$, where ψ can include the predicate symbol $\mathbf{P}$, iff $(L_\kappa, \in) \models \psi^*(\vec{a})$. Here, ψ^* is obtained from ψ by replacing all instances of the predicate symbol $\mathbf{P}(v)$ with $\varphi(v, \vec{z})$. It is easy to see that this doesn't increase the complexity from Σ_1 to some higher Lévy rank.

Now, by regular stability again, $\mathcal{A}$ is (κ, ρ)-interpretable and so $(L_\kappa, \in) \models \psi^*(\vec{a})$ iff $(L_\rho, \in) \models \psi^*(\vec{a})$ whenever $\vec{a} \in L_\kappa$. And then the translation works backwards to show that this happens iff $(L_\kappa, \in, \mathcal{A}) \models \psi(\vec{a})$. □

Corollary 2.12. *Assume ρ is nonprojectible. Then ρ is recursively subtle.*

The proof is quite intuitive – one shows that for sufficiently large $\kappa < \rho$, the additional predicate for $\mathcal{A}$ can be eliminated, by letting κ be greater than the ranks of all the parameters in the definition of $\mathcal{A}$.

Proof. By Theorem 2.7 it suffices to show that, for a tail of $\kappa < \rho$, the predicate in the notion of recursive subtlety may be eliminated. Formally restated, we aim to show that, for any ρ-recursive $\mathcal{A}$, there is some $\tau < \rho$ so that, for all $\kappa < \rho$ so that κ is ρ-stable and $\tau < \kappa$, κ is $\mathcal{A}$-stable. Then the desired result follows from the fairly obvious fact that if α is Π_n-reflecting onto A and $\xi < \alpha$, then α is Π_n-reflecting onto $A \cap [\xi, \alpha)$.

Now let $\mathcal{A}$ be ρ-recursively enumerable. By hypothesis, there is some Σ_1-formula φ and parameters $\vec{z} \in L_\rho$ so that $\mathcal{A} = \{x \in L_\rho : L_\rho \models \varphi(x, \vec{z})\}$. For each $0 < i \leq \text{len}(\vec{z})$, let α_i denote the rank in the constructible hierarchy of z_i, i.e. α_i is the least ordinal so that $z_i \in L_{\alpha_i+1}$. Set $\tau = \max\{\alpha_i : 0 < i \leq \text{len}(\vec{z})\}$, and assume that $\kappa < \rho$ is so that κ is ρ-stable and $\tau < \kappa$. Then, since

$\vec{z} \in L_{\tau+1} \subseteq L_\kappa$, it follows that $\mathcal{A}$ is $\Sigma_1^{L_\rho}(L_\kappa)$. By the previous theorem, κ is $\mathcal{A}$-ρ-stable. This gives the desired result. $\square$

We would like to note that it is still an open question what recursive analogues of higher large cardinal axioms could be. It is a relatively vague question, and not of such high priority as recursive analogues are typically studied for their applications to recursion theory and proof theory. However, as a closing remark, we will explain how it is possible that one could consider the recursive analogue of measurability to be Σ_2-extendibility:

Definition 2.13. An ordinal α is called Σ_2-extendible iff there is $\beta > \alpha$ so that $L_\alpha \prec_{\Sigma_2} L_\beta$. Let ζ denote the least Σ_2-extendible ordinal.

ζ does indeed have applications to recursion theory – for example, the $\Sigma_2(L_\zeta)$ subsets of ω are precisely the arithmetically quasi-inductive subsets of ω, analogously to how the ω_1^{CK}-recursive subsets of ω are precisely the arithmetically inductive subsets of ω. ζ itself also has a characterisation via generalized computability, namely ζ is the supremum of the eventually writable ordinals with respect to an infinite time Turing machine. There may be an analogue of hyperarithmetical theory, namely hyperinductive theory, at this stage – we direct the reader to [7].

Say that a sequence $\vec{X} = \langle X_\beta : \beta < \lambda \rangle$ of λ many subsets of an admissible ordinal $\kappa > \lambda$ is recursive iff $\{(\alpha, \beta) : \alpha \in X_\beta\}$, the subset of $\kappa \times \lambda$ coding $\vec{X}$, is κ-recursive. It is known that, if $\vec{X}$ is recursive, then $\bigcap_{\beta<\lambda} X_\beta$ is κ-recursive. Say an ultrafilter $\mathcal{U}$ on κ is L_κ-complete iff, whenever $\vec{X}$ is recursive and $X_\beta \in \mathcal{U}$ for all $\beta < \lambda$, then $\bigcap_{\beta<\lambda} X_\beta \in \mathcal{U}$. Then say κ is recursively measurable iff there is an L_κ-complete nonprincipal ultrafilter on the Boolean algebra of κ-recursive subsets of κ.

Then κ is recursively measurable iff it is Σ_2-extendible. This is "consistent" in a sense with relations to other large cardinal axioms – for example, any measurable cardinal is weakly compact and subtle, and:

Proposition 2.14. *If α is Σ_2-extendible, then α is Π_3-reflecting, nonprojectible, and in fact a limit of smaller Π_3-reflecting nonprojectible ordinals.*

The proof is an easy reflection argument, utilizing the truth predicate from [6]. For more information on the notion of recursive measurability in particular, see [5]. In an upcoming paper, we attempt to tackle recursive analogues in full generality.

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