# Solutions to Bain and Engelhardt's Introduction to Probability and Mathematical Statistics

**06.01**

## Given: the pdf of x $f_x\left(x\right) = \begin{cases} 4x^3 & , \quad 0 < x < 1 \\ 0 & , \quad o/w \end{cases}$

## Find: PDF of a) $Y = X^4$

**Setup:** Use the CDF technique to get the CDF of Y in terms of a CDF of X
$F_Y(y) = P[Y \leq y] = P[X^4 \leq Y] = P[-y^{\frac{1}{4}} \leq X \leq y^{\frac{1}{4}}] = F_X(y^{\frac{1}{4}}) - F_X(-y^{\frac{1}{4}})$

**Steps: i)** Differentiate with respect to y to find an equation given in terms of the pdf of x:
$f_y(y) = \frac{d}{dy}F_X(y^{\frac{1}{4}}) - \frac{d}{dy}F_X(-y^{\frac{1}{4}}) = f_x(y^{\frac{1}{4}})\frac{d}{dy}y^{\frac{1}{4}} - f_x(-y^{\frac{1}{4}})\frac{d}{dy}-y^{\frac{1}{4}} = f_x(y^{\frac{1}{4}})\frac{y^{-\frac{3}{4}}}{4} - f_x(-y^{\frac{1}{4}})\frac{-y^{-\frac{3}{4}}}{4}$

**ii)** Plug in the original limits and function for the pdf of x, and compute the cdf for y

**Result:** $f_y(y) = \begin{cases} 4y^{\frac{3}{4}}\frac{1}{4y^{\frac{3}{4}}} & , \quad 0 < x < 1 \\ 0 & , \quad o/w \end{cases} = \begin{cases} 1 & , \quad 0 < x < 1 \\ 0 & , \quad o/w \end{cases}$

## Find: PDF of b) $W = e^X$

**Setup:** Use the CDF technique to get the CDF of W in terms of a CDF of X
$F_W(w) = P[W \leq w] = P[e^X \leq W] = P[X \leq lnW] = F_X(lnW)$

**Steps: i)** Differentiate with respect to w to find an equation given in terms of the pdf of x:
$f_w(w) = \frac{d}{dw}F_X(lnW)\frac{d}{dw}(lnw) = f_x(lnw)\frac{1}{w}$

**ii)** Plug in the original limits and function for the pdf of x, and compute the cdf for y

**Result:** $f_W(w) = \begin{cases} \frac{4(lnw)^3}{w} & , \quad 1 < w < e \\ 0 & , \quad o/w \end{cases}$

## Find: PDF of c) $Z = \ln x$

**Setup:** Use the CDF technique to get the CDF of Z in terms of a CDF of X
$F_Z(z) = P[Z \leq z] = P[lnx \leq z] = P[X \leq e^z] = F_X(e^z)$

**Steps: i)** Differentiate with respect to z to find an equation given in terms of the pdf of x:
$f_z(z) = \frac{d}{dz}F_X(e^z) = f_x(e^z)\frac{de^z}{dz}$

**ii)** Plug in the original limits and function for the pdf of x, and compute the cdf for y

**Result:** $f_Z(z) = \begin{cases} 4e^{4z} & , \quad -\infty \leq z < 0 \\ 0 & , \quad o/w \end{cases}$

# Find: PDF of d) $U = (X - 0.5)^2$

**Setup:** Use the CDF technique to get the CDF of U in terms of a CDF of X
$F_U(u) = P[U \leq u] = P[(X - 0.5)^2 \leq u] = P[|X - 0.5| \leq u^{0.5}] = F_X(u^{1/2} + 1/2) - F_X(-u^{1/2} + 1/2)$

**Steps: i)** Differentiate with respect to u to find an equation given in terms of the pdf of x:
$f_U(u) = \frac{d}{du} F_X(u^{1/2} + 1/2) = f_x(u^{1/2} + 1/2)\frac{d}{du}(u^{1/2} + 1/2) - f_x(-u^{1/2} + 1/2)\frac{d}{du}(-u^{1/2} + 1/2)$
$f_x(u^{1/2} + 1/2)1/2u^{-1/2}) - f_x(-u^{1/2} + 1/2)1/2u^{-1/2}$

**ii)** INCOMPLETE

**Result:** $f_Z(z) = \begin{cases} 4e^{4z} & , \quad -\infty \leq z < 0 \\ 0 & , \quad o/w \end{cases}$

**06.02**

# Given: $X \sim Unif(0, 1)$
# Find: a) PDF of $Y = X^{1/4}$

**Setup:** $F_Y(y) = P[Y \leq y] = P[X^{1/4} \leq y] = P[X \leq y^4] = F_X(y^4)$

**Steps: i)** find the pdf of x. Because X is a Uniform distribution with parameters 1 and 0, the pdf, which for Unif(a,b) is 1/b-a where $a < x < b$. Here, Unif(0,1) gives 1/1-0 =1

**ii)** Differentiate with respect to y to find an equation given in terms of the pdf of x:
$f_Y(y) = \frac{d}{dy} F_X(y^4) = 4y^3$

**Result:** $f_Y(y) = \begin{cases} 4y^3 & , \quad 0 < y < 1 \\ 0 & , \quad o/w \end{cases}$

# Find: b) PDF of $W = e^{-X}$

**Setup:** $F_W(z) = P[W \leq w] = P[e^{-X} \leq W] = P[-X \leq lnw] = P[X \geq -lnw] = 1 - F_x(-lnw)$

**Steps: i)** find the pdf of x. See part a) for an explanation of why it is 1 when $a < x < b$

**ii)** Differentiate with respect to w to find an equation given in terms of the pdf of x:
$f_W(w) = -\frac{d}{dw}F_X\frac{d}{dw}(-lnw) = -f_X(-lnw)\frac{-1}{w} \quad for \quad e^{-1} < w < 1 = -\frac{1}{w}$

**Result:** $f_W(w) = \begin{cases} \frac{1}{w} & , \quad e^{-1} < w < 1 \\ 0 & , \quad o/w \end{cases}$

# Find: c) PDF of $Z = 1 - e^{-X}$

**Setup:** $F_Z(z) = P[Z \leq z] = P[1 - e^{-X} \leq z] = P[-e^{-X} \leq z - 1] = P[e^{-X} \geq 1 - z] = P[-X \geq ln(1-z)] = P[X \leq -ln(1-z)] = F_x(-ln(1-z))$

**Steps: i)** find the pdf of x. See part a) for an explanation of why it is 1 when $a < x < b$

**ii)** Differentiate with respect to w to find an equation given in terms of the pdf of x:
$f_Z(z) = -ln(1-z) = -\frac{-1}{1-z} = \frac{1}{1-z} \quad for \quad 0 < z < 1 - e^{-1}$

**Result:** $f_Z(z) = \begin{cases} \frac{1}{1-z} & , \quad 0 < z < 1 - e^{-1} \\ 0 & , \quad o/w \end{cases}$

# Find: d) PDF of $U = X(1 - X)$

**Setup:** $F_U(u) = P[U \leq u] = P[X(1-x) \leq u] = P[-X^2 + X \leq u] = P[-(X-1/2)^2 \leq u - 1/4] = P[(X-1/2)^2 \geq 1/4 - u] == P[|(X-1/2)| \geq (1/4-u)^{1/2}] =$

**Steps: i)** find the pdf of x. See part a) for an explanation of why it is 1 when $a < x < b$

**ii)** INCOMPLETE:
$f_Z(z) = -ln(1-z) = -\frac{-1}{1-z} = \frac{1}{1-z} \quad for \quad 0 < z < e^{-1}$

**Result:** $f_W(w) = \begin{cases} \frac{1}{1-z} & , \quad 0 < z < e^{-1} \\ 0 & , \quad o/w \end{cases}$

**06.03**

# Given: PDF $f_R(r) = \begin{cases} 6r(1-r) & , \quad 0 < r < 1 \\ 0 & , \quad o/w \end{cases}$

# Find: Distribution of the circumference

**Setup:** The circumference is $c = 2\pi r$. We have the pdf in terms of x, so this is the transformation:
$F_C(c) = P[C \leq c] = P[2\pi r \leq c] = P[r \leq c/2\pi] = F_x(c/2\pi)$

**Steps: i)** Differentiate with respect to c to find an equation given in terms of the pdf of x.
$f_C(c) = \frac{d}{dc}F_R(c/2\pi) = f_R(c/2\pi)\frac{d}{dc}(c/2\pi) = f_R(c/2\pi)(1/2\pi)$

**ii)** Plug the original pdf back into this new form:
$f_C(c) = \frac{6c}{2\pi}(1-(c/2\pi))(1/2\pi) = \frac{6c(2\pi-c)}{(2\pi)^3} \quad if \quad 0 < c < 2\pi$

**Result:** $f_C(c) = \begin{cases} \frac{6c(2\pi-c)}{(2\pi)^3} & , \quad 0 < c < 2\pi \\ 0 & , \quad o/w \end{cases}$

# Find: Distribution of the area

**Setup:** The area is $a = \pi r^2$ so the cdf $F_A(a) = P[A \le a] = P[\pi r^2 \le a] = P[r^2 \le a/\pi]$
$= P[|r| \le (a/\pi)^{1/2}] = P[-(a/\pi)^{1/2} \le c \le (a/\pi)^{1/2}] = F_R\left((a/\pi)^{1/2}\right) - F_R\left(-(a/\pi)^{1/2}\right)$

**Steps: i)** Differentiate with respect to a to find an equation in terms of the pdf of x.
$f_A(a) = \frac{d}{da}F_R\left((a/\pi)^{1/2}\right) - \frac{d}{da}F_R\left(-(a/\pi)^{1/2}\right) = f_R\left((a/\pi)^{1/2}\right)\frac{d}{da}(a/\pi)^{1/2} - f_R\left(-(a/\pi)^{1/2}\right)\frac{d}{da}\left(-(a/\pi)^{1/2}\right)$

**Result:** $f_A(a) = \begin{cases} \frac{3(\sqrt{\pi}-\sqrt{a})}{\pi^{3/2}} & , \quad 0 < a < \pi \\ 0 & , \quad o/w \end{cases}$

**06.04** *Please double check the results of this solution*
For $X \sim WEI(\theta, \beta)$ we have the CDF as $F_X = 1 - e^{\frac{x}{\theta}^{\beta}}$ and the pdf is $f(x) = \frac{\beta}{\theta^\beta}x^{\beta-1}e^{-\frac{x}{\theta}^{\beta}}$
a) We make the transformation by the CDF method:

$$\begin{aligned} \Pr(Y \le y) &= \Pr\left(\frac{X}{\theta}^{\beta} \le y\right) \\ &= \Pr\left(X \le \theta y^{\frac{1}{\beta}}\right) \\ &= F_X\left(\theta y^{\frac{1}{\beta}}\right) \\ &= 1 - e^{-\frac{\theta y^{\frac{1}{\beta}}}{\theta}^{\beta}} \\ &= 1 - e^{-y}, \text{ where } 0 < y \end{aligned}$$

So we have our CDF. For the pdf we simply take the derivative of the above. So $pdf = e^{-y}$ where $0 < y$

b) $W = \ln X$. Again, the most simply method to get the CDF, and in turn the pdf is the CDF method.

$$\begin{aligned} \Pr(W \le w) &= \Pr(\ln X \le w) && (1) \\ &= \Pr(X \le e^w) && (2) \\ &= F(e^w) && (3) \\ &= 1 - e^{-\frac{e^w}{\theta}^{\beta}} \text{ where } 0 < w && (4) \\ & && (5) \end{aligned}$$

Again we simply differentiate to get the pdf. which turns out to be $\beta e^{\beta w}\theta^{-\beta}e^{-\frac{e^{w\beta}}{\theta}}$ , $0 < w$
c)

## 06.08

**Let X be a random variable with pdf** $f_x(x) = \begin{cases} 4x^3 & , \quad 0 < x < 1 \\ 0 & , \quad o/w \end{cases}$ **Use the transformation method to determine the pdf of each of the following random variables**

**a) Find the pdf of** $Y = X^4$

**Setup:** solve for $x(y)$ and take the absolute value of it's derivative $|\frac{dx(y)}{dy}|$

$$x(y) = y^{\frac{1}{4}} \qquad |\frac{dx(y)}{dy}| = y^{\frac{-3}{4}} = \frac{1}{4y^{\frac{3}{4}}}$$

Now plug all of the above information in the below equation

$$f_y(y) = f_x(x(y))|\frac{dx(y)}{dy}|$$

**Result:** $f_y(y) = \begin{cases} 4y^{\frac{3}{4}}\frac{1}{4y^{\frac{3}{4}}} & , \quad 0 < y < 1 \\ 0 & , \quad o/w \end{cases} = \begin{cases} 1 & , \quad 0 < y < 1 \\ 0 & , \quad o/w \end{cases}$

**b) Find the pdf of** $W = e^X$

**Setup:** solve for $x(w)$ and take the absolute value of it's derivative $|\frac{dx(w)}{dw}|$

$$x(w) = ln(w) \qquad |\frac{dx(w)}{dw}| = \frac{1}{w}$$

Now plug all of the above information in the below equation

$$f_w(w) = f_w(x(w))|\frac{dx(w)}{dw}|$$

**Result:** $f_w(w) = \begin{cases} 4ln(w)^3\frac{1}{w} \\ 0 \end{cases} = \begin{cases} \frac{4ln(w)^3}{w} & , \quad 1 \le w \le e \\ 0 & , \quad o/w \end{cases}$

**c) Find the pdf of** $Z = ln(x)$

**Setup:** solve for $x(z)$ and take the absolute value of it's derivative $|\frac{dx(z)}{dz}|$

$$x(z) = e^z \qquad |\frac{dx(z)}{dz}| = e^z$$

Now plug all of the above information in the below equation

$$f_z(z) = f_z(x(z))|\frac{dx(z)}{dz}|$$

**Result:** $f_z(z) = \begin{cases} 4e^{3z}e^{z} \\ 0 \end{cases} = \begin{cases} 4e^{4z} & , \quad z \leq 0 \\ 0 & , \quad o/w \end{cases}$

**d) Find the pdf of** $U = (X - 0.5)^2$

**Setup:** this is not a one-to-one transformation - did not solve **06.10** Suppose $X$ has pdf $f_X(x) = \frac{1}{2}e^{-|x|}$ for all real $x$.

(a) Find the pdf of $Y = |X|$.
CDF Method
$F_Y(y) = P[Y \leq y] = P[|x| \leq y] = P[-y \leq X \leq y] = F_X(y) - F_X(-y)$
$f_Y(y) = \frac{dF_X(y)}{dy} - \frac{dF_X(-y)}{dy}$
$f_Y(y) = f_X(y)\frac{dy}{dx} - f_X(-y)(\frac{-dy}{dy})$
$f_y = \frac{1}{2}e^{-y} + \frac{1}{2}e^{-y} = e^{-y}\ y > 0$

(b) Let $W = 0$ if $X \leq 0$ and $W = 1$ if $X > 0$. Find the CDF of $W$
$F_W(w) = P[W = 0] = \frac{1}{2}$
$F_W(w) = P[W = 1] = \frac{1}{2}$
$F_W(w) =$

$$\begin{cases} 0 & w \leq 0 \\ \frac{1}{2} & 0 \leq w \leq 1 \\ 1 & w > 1 \end{cases}$$

**06.13** X has pdf

$$f(x) = \frac{x^2}{24} - 2 < x < 4 \qquad\qquad 0\text{otherwise}$$

We want pdf of the CDF $Y = X^2$ with regions: $(-2, 0) \cup [0, 4)$

$$[F_x(\sqrt{y}) - F_x(-\sqrt{y})] = \left[ f_x(\sqrt{y})(\frac{1}{2}\sqrt{y}) - f_x(-\sqrt{y})(-\frac{1}{2}\sqrt{y}) \right]$$

**06.14**

# Given: Joint PDF $f(x, y) = \begin{cases} 4e^{-2(x+y)} & , \quad 0 < x < \infty, 0 < y < \infty \\ 0 & , \quad o/w \end{cases}$

# Find: a) CDF of W=X+Y

**Setup:** $F_w(w) = P[W \leq w] = P[X + Y \leq w]$

**Steps:**

**i)** Express as a sum of probabilities, replace probabilities with binomials

**ii)** Simplify and Use Combinatorial Identity

**Result:** $\binom{n+m}{k}$

**06.15** This is a simplified version of example 6.4.5.

$X_1, X_2 \sim POI(\lambda)$ so the MGF of both is $e^{\lambda(e^t-1)}$. Thus by theorem 6.4.4

$$M_Y(t) = e^{\lambda(e^t-1)}e^{\lambda(e^t-1)} = e^{2\lambda(e^t-1)} \sim POI(2\lambda)$$

The pdf then of Y is

$$f_Y(y) = \begin{cases} \frac{e^{-2\lambda}(2\lambda)^y}{y!} & y = 0, 1, 2, ... \\ 0 & otherwise. \end{cases}$$

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**06.16** Note: the pdf of $f_{x_1,x_2} = \frac{1}{x_1^2}\frac{1}{x_2^2}$

a) We need to find $f_{u,v} = f_{x_1,x_2}(x_1(u,v), x_2(u,v))|J|$ where J is our jacobian. First we let $u = x_1x_2$ and $v = x_1$ thus $x_1 = v$ and $x_2 = \frac{u}{v}$, now we can find J.

$$J = \begin{vmatrix} 0 & 1 \\ \frac{1}{v} & \frac{-u}{v^2} \end{vmatrix} = \frac{1}{v}$$

Finally, our pdf is:

$$\begin{aligned} f_{U,V}(u,v) &= f_{x_1,x_2}(v, \frac{u}{v})\left|\frac{1}{v}\right| \\ &= \frac{1}{v^2}\frac{1}{(\frac{u}{v})^2}\left|\frac{1}{v}\right| \\ &= \frac{1}{u^2v}, 1 < v < u < \infty \end{aligned}$$

b) We need to find $f_u(u)$ given $f_{U,V}(u,v) = \frac{1}{u^2v}, 1 < v < u < \infty$

$$\begin{aligned} f_u(u) &= \int_1^u \frac{1}{u^2v}dv \\ &= \frac{1}{u^2}\ln(v)\Big|_1^u \\ &= \frac{1}{u^2}(\ln(u) - 0) \\ &= \frac{1}{u^2}\ln(u), 1 < u < \infty \end{aligned}$$