Solutions to Bain and Engelhardt's Introduction to Probability and Mathematical Statistics

06.01

Given: the pdf of x $f_x(x) = \begin{cases} 4x^3 & , & 0 < x < 1 \\ 0 & , & o/w \end{cases}$

Find: PDF of a) $Y = X^4$

Setup: Use the CDF technique to get the CDF of Y in terms of a CDF of X

$F_Y(y) = P[Y \le y] = P[X^4 \le Y] = P[-y^{\frac{1}{4}} \le X \le y^{\frac{1}{4}}] = F_X(y^{\frac{1}{4}}) - F_X(-y^{\frac{1}{4}})$

Steps: i) Differentiate with respect to y to find an equation given in terms of the pdf of x:

$f_y(y) = \frac{d}{dy}F_X(y^{\frac{1}{4}}) - \frac{d}{dy}F_X(-y^{\frac{1}{4}}) = f_x(y^{\frac{1}{4}})\frac{d}{dy}y^{\frac{1}{4}} - f_x(-y^{\frac{1}{4}})\frac{d}{dy}-y^{\frac{1}{4}} = f_x(y^{\frac{1}{4}})\frac{y^{-\frac{3}{4}}}{4} - f_x(-y^{\frac{1}{4}})\frac{-y^{-\frac{3}{4}}}{4}$

ii) Plug in the original limits and function for the pdf of x, and compute the cdf for y

Result: $f_y(y) = \begin{cases} 4y^{\frac{3}{4}}\frac{1}{4y^{\frac{3}{4}}} & , & 0 < x < 1 \\ 0 & , & o/w \end{cases} = \begin{cases} 1 & , & 0 < x < 1 \\ 0 & , & o/w \end{cases}$

Find: PDF of b) $W = e^X$

Setup: Use the CDF technique to get the CDF of W in terms of a CDF of X

$F_W(w) = P[W \le w] = P[e^X \le W] = P[X \le lnW] = F_X(lnW)$

Steps: i) Differentiate with respect to w to find an equation given in terms of the pdf of x:

$f_w(w) = \frac{d}{dw}F_X(lnW)\frac{d}{dw}(lnw) = f_x(lnw)\frac{1}{w}$

ii) Plug in the original limits and function for the pdf of x, and compute the cdf for y

Result: $f_W(w) = \begin{cases} \frac{4(lnw)^3}{w} & , & 1 < w < e \\ 0 & , & o/w \end{cases}$

Find: PDF of c) $Z = \ln x$

Setup: Use the CDF technique to get the CDF of Z in terms of a CDF of X

$F_Z(z) = P[Z \le z] = P[lnx \le z] = P[X \le e^z] = F_X(e^z)$

Steps: i) Differentiate with respect to z to find an equation given in terms of the pdf of x:

$f_z(z) = \frac{d}{dz}F_X(e^z) = f_x(e^z)\frac{de^z}{dz}$

ii) Plug in the original limits and function for the pdf of x, and compute the cdf for y

Result: $f_Z(z) = \begin{cases} 4e^{4z} & , \quad -\infty \le z < 0 \\ 0 & , \quad o/w \end{cases}$

Find: PDF of d) $U = (X - 0.5)^2$

Setup: Use the CDF technique to get the CDF of U in terms of a CDF of X

$F_U(u) = P[U \le u] = P[(X - 0.5)^2 \le u] = P[|X - 0.5| \le u^{0.5}] = F_X(u^{1/2} + 1/2) - F_X(-u^{1/2} + 1/2)$

Steps: i) Differentiate with respect to u to find an equation given in terms of the pdf of x:

$f_U(u) = \frac{d}{du}F_X(u^{1/2} + 1/2) = f_x(u^{1/2} + 1/2)\frac{d}{du}(u^{1/2} + 1/2) - f_x(-u^{1/2} + 1/2)\frac{d}{du}(-u^{1/2} + 1/2)$
$f_x(u^{1/2} + 1/2)1/2u^{-1/2}) - f_x(-u^{1/2} + 1/2)1/2u^{-1/2}$

ii) INCOMPLETE

Result: $f_Z(z) = \begin{cases} 4e^{4z} & , \quad -\infty \le z < 0 \\ 0 & , \quad o/w \end{cases}$

06.02

Given: $X \sim Unif(0, 1)$
Find: a) PDF of $Y = X^{1/4}$

Setup: $F_Y(y) = P[Y \le y] = P[X^{1/4} \le y] = P[X \le y^4] = F_X(y^4)$

Steps: i) find the pdf of x. Because X is a Uniform distribution with parameters 1 and 0, the pdf, which for Unif(a,b) is 1/b-a where $a < x < b$. Here, Unif(0,1) gives 1/1-0 =1

ii) Differentiate with respect to y to find an equation given in terms of the pdf of x:
$f_Y(y) = \frac{d}{dy}F_X(y^4) = 4y^3$

Result: $f_Y(y) = \begin{cases} 4y^3 & , \quad 0 < y < 1 \\ 0 & , \quad o/w \end{cases}$

Find: b) PDF of $W = e^{-X}$

Setup: $F_W(z) = P[W \le w] = P[e^{-X} \le W] = P[-X \le lnw] = P[X \ge -lnw] = 1 - F_x(-lnw)$

Steps: i) find the pdf of x. See part a) for an explanation of why it is 1 when $a < x < b$

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ii) Differentiate with respect to w to find an equation given in terms of the pdf of x:
$$f_W(w) = -\frac{d}{dw}F_X\frac{d}{dw}(-lnw) = -f_X(-lnw)\frac{-1}{w} \quad for \quad e^{-1} < w < 1 = -\frac{1}{w}$$

Result: $f_W(w) = \begin{cases} \frac{1}{w} & , \quad e^{-1} < w < 1 \\ 0 & , \quad o/w \end{cases}$

Find: c) PDF of $Z = 1 - e^{-X}$

Setup: $F_Z(z) = P[Z \le z] = P[1 - e^{-X} \le z] = P[-e^{-X} \le z - 1] = P[e^{-X} \ge 1 - z] = P[-X \ge ln(1-z)] = P[X \le -ln(1-z)] = F_x(-ln(1-z))$

Steps: i) find the pdf of x. See part a) for an explanation of why it is 1 when $a < x < b$

ii) Differentiate with respect to w to find an equation given in terms of the pdf of x:
$$f_Z(z) = -ln(1-z) = -\frac{-1}{1-z} = \frac{1}{1-z} \quad for \quad 0 < z < 1 - e^{-1}$$

Result: $f_Z(z) = \begin{cases} \frac{1}{1-z} & , \quad 0 < z < 1 - e^{-1} \\ 0 & , \quad o/w \end{cases}$

Find: d) PDF of $U = X(1 - X)$

Setup: $F_U(u) = P[U \le u] = P[X(1 - x) \le u] = P[-X^2 + X \le u] = P[-(X - 1/2)^2 \le u - 1/4] = P[(X - 1/2)^2 \ge 1/4 - u] == P[|(X - 1/2)| \ge (1/4 - u)^{1/2}] =$

Steps: i) find the pdf of x. See part a) for an explanation of why it is 1 when $a < x < b$

ii) INCOMPLETE:
$$f_Z(z) = -ln(1-z) = -\frac{-1}{1-z} = \frac{1}{1-z} \quad for \quad 0 < z < e^{-1}$$

Result: $f_W(w) = \begin{cases} \frac{1}{1-z} & , \quad 0 < z < e^{-1} \\ 0 & , \quad o/w \end{cases}$

06.03

Given: PDF $f_R(r) = \begin{cases} 6r(1 - r) & , \quad 0 < r < 1 \\ 0 & , \quad o/w \end{cases}$
Find: Distribution of the circumference

Setup: The circumference is $c = 2\pi r$. We have the pdf in terms of x, so this is the transformation:
$$F_C(c) = P[C \le c] = P[2\pi r \le c] = P[r \le c/2\pi] = F_x(c/2\pi)$$

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Steps: i) Differentiate with respect to c to find an equation given in terms of the pdf of x.

$f_C(c) = \frac{d}{dc}F_R(c/2\pi) = f_R(c/2\pi)\frac{d}{dc}(c/2\pi) = f_R(c/2\pi)(1/2\pi)$

ii) Plug the original pdf back into this new form:

$f_C(c) = \frac{6c}{2\pi}(1 - (c/2\pi))(1/2\pi) = \frac{6c(2\pi-c)}{(2\pi)^3} \quad if \quad 0 < c < 2\pi$

Result: $f_C(c) = \begin{cases} \frac{6c(2\pi-c)}{(2\pi)^3} & , \quad 0 < c < 2\pi \\ 0 & , \quad o/w \end{cases}$

Find: Distribution of the area

Setup: The area is $a = \pi r^2$ so the cdf $F_A(a) = P[A \le a] = P[\pi r^2 \le a] = P[r^2 \le a/\pi]$
$= P[|r| \le (a/\pi)^{1/2}] = P[-(a/\pi)^{1/2} \le c \le (a/\pi)^{1/2}] = F_R\left((a/\pi)^{1/2}\right) - F_R\left(-(a/\pi)^{1/2}\right)$

Steps: i) Differentiate with respect to a to find an equation in terms of the pdf of x.

$f_A(a) = \frac{d}{da}F_R\left((a/\pi)^{1/2}\right) - \frac{d}{da}F_R\left(-(a/\pi)^{1/2}\right) = f_R\left((a/\pi)^{1/2}\right)\frac{d}{da}(a/\pi)^{1/2} - f_R\left(-(a/\pi)^{1/2}\right)\frac{d}{da}\left(-(a/\pi)^{1/2}\right)$

Result: $f_A(a) = \begin{cases} \frac{3(\sqrt{\pi}-\sqrt{a})}{\pi^{3/2}} & , \quad 0 < a < \pi \\ 0 & , \quad o/w \end{cases}$

06.04 *Please double check the results of this solution*
For $X \sim WEI(\theta, \beta)$ we have the CDF as $F_X = 1 - e^{\frac{x}{\theta}^\beta}$ and the pdf is $f(x) = \frac{\beta}{\theta^\beta}x^{\beta-1}e^{-\frac{x}{\theta}^\beta}$
a) We make the transformation by the CDF method:

$$\begin{aligned} \Pr(Y \le y) &= \Pr\left(\frac{X^\beta}{\theta} \le y\right) \\ &= \Pr\left(X \le \theta y^{\frac{1}{\beta}}\right) \\ &= F_X\left(\theta y^{\frac{1}{\beta}}\right) \\ &= 1 - e^{-\frac{\theta y^{\frac{1}{\beta}}}{\theta}^\beta} \\ &= 1 - e^{-y}, \text{ where } 0 < y \end{aligned}$$

So we have our CDF. For the pdf we simply take the derivative of the above. So $pdf = e^{-y}$ where $0 < y$

b) $W = \ln X$. Again, the most simply method to get the CDF, and in turn the pdf is the CDF method.

$$\begin{aligned} \Pr(W \le w) &= \Pr(\ln X \le w) & (1) \\ &= \Pr(X \le e^w) & (2) \\ &= F(e^w) & (3) \\ &= 1 - e^{-\frac{e^w}{\theta}^\beta} \text{ where } 0 < w & (4) \\ & & (5) \end{aligned}$$

Again we simply differentiate to get the pdf. which turns out to be $\beta e^{\beta w} \theta^{-\beta} e^{-\frac{e^w \beta}{\theta}}$, $0 < w$

c)

06.08

Let X be a random variable with pdf $f_x(x) = \begin{cases} 4x^3 & , & 0 < x < 1 \\ 0 & , & o/w \end{cases}$ **Use the trans-**

formation method to determine the pdf of each of the following random variables

a) Find the pdf of $Y = X^4$

Setup: solve for $x(y)$ and take the absolute value of it's derivative $|\frac{dx(y)}{dy}|$

$$x(y) = y^{\frac{1}{4}} \qquad\qquad |\frac{dx(y)}{dy}| = y^{\frac{-3}{4}} = \frac{1}{4y^{\frac{3}{4}}}$$

Now plug all of the above information in the below equation

$$f_y(y) = f_x(x(y))|\frac{dx(y)}{dy}|$$

Result: $f_y(y) = \begin{cases} 4y^{\frac{3}{4}} \frac{1}{4y^{\frac{3}{4}}} & , & 0 < y < 1 \\ 0 & , & o/w \end{cases} = \begin{cases} 1 & , & 0 < y < 1 \\ 0 & , & o/w \end{cases}$

b) Find the pdf of $W = e^X$

Setup: solve for $x(w)$ and take the absolute value of it's derivative $|\frac{dx(w)}{dw}|$

$$x(w) = ln(w) \qquad\qquad |\frac{dx(w)}{dw}| = \frac{1}{w}$$

Now plug all of the above information in the below equation

$$f_w(w) = f_w(x(w))|\frac{dx(w)}{dw}|$$

Result: $f_w(w) = \begin{cases} 4ln(w)^3 \frac{1}{w} & \\ 0 & \end{cases} = \begin{cases} \frac{4ln(w)^3}{w} & , & 1 \le w \le e \\ 0 & , & o/w \end{cases}$

c) Find the pdf of $Z = ln(x)$

Setup: solve for $x(z)$ and take the absolute value of it's derivative $|\frac{dx(z)}{dz}|$

$$x(z) = e^z \qquad\qquad |\frac{dx(z)}{dz}| = e^z$$

Now plug all of the above information in the below equation

$$f_z(z) = f_z(x(z)) \left| \frac{dx(z)}{dz} \right|$$

Result: $f_z(z) = \begin{cases} 4e^{3z}e^z \\ 0 \end{cases} = \begin{cases} 4e^{4z} & , & z \leq 0 \\ 0 & , & o/w \end{cases}$

d) Find the pdf of $U = (X - 0.5)^2$

Setup: this is not a one-to-one transformation - did not solve   **06.10** Suppose X has pdf $f_X(x) = \frac{1}{2}e^{-|x|}$ for all real x.

(a) Find the pdf of $Y = |X|$.
CDF Method
$F_Y(y) = P[Y \leq y] = P[|x| \leq y] = P[-y \leq X \leq y] = F_X(y) - F_X(-y)$
$f_Y(y) = \frac{dF_X(y)}{dy} - \frac{dF_X(-y)}{dy}$
$f_Y(y) = f_X(y)\frac{dy}{dx} - f_X(-y)(\frac{-dy}{dy})$
$f_y = \frac{1}{2}e^{-y} + \frac{1}{2}e^{-y} = e^{-y}$ $y > 0$

(b) Let $W = 0$ if $X \leq 0$ and $W = 1$ if $X > 0$. Find the CDF of W
$F_W(w) = P[W = 0] = \frac{1}{2}$
$F_W(w) = P[W = 1] = \frac{1}{2}$
$F_W(w) =$

$$\begin{cases} 0 & w \leq 0 \\ \frac{1}{2} & 0 \leq w \leq 1 \\ 1 & w > 1 \end{cases}$$

06.13 X has pdf

$$f(x) = \frac{x^2}{24} - 2 < x < 4 \qquad\qquad 0\, \text{otherwise}$$

We want pdf of the CDF $Y = X^2$ with regions: $(-2, 0) \cup [0, 4)$

$$[F_x(\sqrt{y}) - F_x(-\sqrt{y})] = \left[f_x(\sqrt{y})(\frac{1}{2}\sqrt{y}) - f_x(-\sqrt{y})(-\frac{1}{2}\sqrt{y}) \right]$$

06.14

Given: Joint PDF $f(x, y) = \begin{cases} 4e^{-2(x+y)} & , & 0 < x < \infty, 0 < y < \infty \\ 0 & , & o/w \end{cases}$

Find: a) CDF of W=X+Y

Setup: $F_w(w) = P[W \leq w] = P[X + Y \leq w]$

Steps:

i) Express as a sum of probabilities, replace probabilities with binomials

ii) Simplify and Use Combinatorial Identity

Result: $\binom{n+m}{k}$

06.15 This is a simplified version of example 6.4.5.

$X_1, X_2 \sim POI(\lambda)$ so the MGF of both is $e^{\lambda(e^t-1)}$. Thus by theorem 6.4.4

$$M_Y(t) = e^{\lambda(e^t-1)}e^{\lambda(e^t-1)} = e^{2\lambda(e^t-1)} \sim POI(2\lambda)$$

The pdf then of Y is

$$f_Y(y) = \begin{cases} \frac{e^{-2\lambda}(2\lambda)^y}{y!} & y = 0, 1, 2, ... \\ 0 & otherwise. \end{cases}$$

06.16 Note: the pdf of $f_{x_1,x_2} = \frac{1}{x_1^2}\frac{1}{x_2^2}$

a) We need to find $f_{u,v} = f_{x_1,x_2}(x_1(u,v), x_2(u,v))|J|$ where J is our jacobian. First we let $u = x_1 x_2$ and $v = x_1$ thus $x_1 = v$ and $x_2 = \frac{u}{v}$, now we can find J.

$$J = \begin{vmatrix} 0 & 1 \\ \frac{1}{v} & \frac{-u}{v^2} \end{vmatrix} = \frac{1}{v}$$

Finally, our pdf is:

$$
\begin{aligned}
f_{U,V}(u, v) &= f_{x_1,x_2}(v, \frac{u}{v})\left|\frac{1}{v}\right| \\
&= \frac{1}{v^2}\frac{1}{(\frac{u}{v})^2}\left|\frac{1}{v}\right| \\
&= \frac{1}{u^2 v}, 1 < v < u < \infty
\end{aligned}
$$

b) We need to find $f_u(u)$ given $f_{U,V}(u, v) = \frac{1}{u^2 v}, 1 < v < u < \infty$

$$
\begin{aligned}
f_u(u) &= \int_1^u \frac{1}{u^2 v} dv \\
&= \frac{1}{u^2}\ln(v)\Big|_1^u \\
&= \frac{1}{u^2}(\ln(u) - 0) \\
&= \frac{1}{u^2}\ln(u), 1 < u < \infty
\end{aligned}
$$

06.17 Suppose that X_1 and X_2 are a random sample of size 2 from a gamma distribution, $X_i \sim \text{GAM}(2, 1/2)$.

1. Find the pdf of $Y = \sqrt{X_1 + X_2}$;

2. Find the pdf of $W = X_1/X_2$.

[Solution] (a) Since X_1 and X_2 are independent to each other, the joint distribution density function is the product of $f(X_1)$ and $f(X_2)$, which is

$$
f(X_1, X_2) = \begin{cases} \left(\frac{1}{\sqrt{2}\,\Gamma(1/2)}\right)^2 X_1^{-1/2} X_2^{-1/2 - (X_1+X_2)/2} = \frac{1}{2\pi} X_1^{-1/2} X_2^{-1/2 - (X_1+X_2)/2}, & \text{if } X_1, X_2 > 0; \\ 0, & \text{otherwise.} \end{cases}
$$

If $x_1 = w$, then $x_2 = y^2 - w$, which implies that

$$
J = \begin{vmatrix} 1 & 0 \\ -1 & 2y \end{vmatrix} = 2y.
$$

Now,

$$
f(w, y) = \begin{cases} \frac{1}{\pi} y w^{-1/2}(y^2 - w)^{-1/2 - y^2/2}, & \text{if } y^2 > w > 0; \\ 0, & \text{otherwise.} \end{cases}
$$

So, since $u = (y^2 - w)^{1/2}$ implies that $-2u = (y^2 - w)^{-1/2}w$, and $u = y\sin\theta$ implies that $u = y\cos\theta\theta$,

$$
\begin{aligned}
f_Y(y) &= \frac{1}{\pi} \int_0^{y^2} y^{-y^2/2} w^{-1/2}(y^2 - w)^{-1/2} \; w \\
&= \frac{-2}{\pi} \int_y^0 y^{-y^2/2}(y^2 - u^2)^{-1/2} \; u \\
&= \frac{2}{\pi} \int_0^y y^{-y^2/2}(y^2 - u^2)^{-1/2} \; u \\
&= y^{-y^2/2}\frac{2}{\pi} \int_0^y \frac{1}{\sqrt{y^2 - u^2}} \; u \\
&= y^{-y^2/2}\frac{2}{\pi} \int_0^{\pi/2} \frac{y\cos\theta}{\sqrt{y^2 - y^2\sin^2\theta}} \; \theta \\
&= y^{-y^2/2}\frac{2}{\pi} \int_0^{\pi/2} \theta \\
&= y^{-y^2/2}\frac{2}{\pi}\frac{\pi}{2} \\
&= \begin{cases} y^{-y^2/2}, & \text{if } y > 0; \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
$$

(b) If $X_1 = z$, then $X_2 = z/w$, which implies that

$$
J = \begin{vmatrix} 1 & 0 \\ 1/w & z/w^2 \end{vmatrix} = \frac{z}{w^2}.
$$

Now,

$$f(z, w) = \begin{cases} \frac{1}{2\pi} \frac{z}{w^2} \left(\frac{z^2}{w}\right)^{-1/2} e^{-(z+z/w)/2}, & \text{if } z, w > 0; \\ 0, & \text{otherwise.} \end{cases}$$

So, for getting the marginal density function of w, we need to take the integral for the above joint density function with respect to z,

$$\begin{aligned} f_W(w) &= \int_0^\infty \frac{1}{2\pi} w^{-3/2} e^{-z(1+1/w)/2} \, z \\ &= -\frac{1}{2\pi} \frac{2}{1+1/w} e^{-z(1+1/w)/2} w^{-3/2} \Big|_0^\infty \\ &= -\frac{1}{\pi(w+1)\sqrt{w}} e^{-z(1+1/w)/2} \Big|_0^\infty \\ &= \begin{cases} \frac{1}{\pi(w+1)\sqrt{w}}, & \text{if } w > 0; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

6.18 It is given that X and Y have a joint pdf given by

$$f(x, y) = e^{-y} \quad if \quad 0 < x < y < \infty. \tag{6}$$

(a): Find the joint pdf of $S = X + Y$ and $T = X$.

This can be done using the joint transformation method. By rearranging the above formulas we get $X = T$ and $Y = S - T$. Then it is easy to get the jacobian

$$J = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \tag{7}$$

whose determinant is clearly one. Note that the order in which you take partial derivatives is unimportant provided you are consistent - you will get the same determinant either way. Then we substitute in $X = T$ and $Y = S - T$ into the pdf and multiply by the determinant of the jacobian:

$$f_{S,T}(s, t) = f_{X,Y}(x(s,t), y(s,t)) \times 1 = \begin{cases} e^{t-s} & if \quad 0 < t < s/2 \\ 0 & otherwise \end{cases}. \tag{8}$$

The bounds of the function can be found in a few different ways. One way is to consider the bounds of the original function, $0 < x < y < \infty$. We can substitute in the new formulas for X and Y to get

$$0 < t < s - t < \infty. \tag{9}$$

Then it is apparent that

$$0 < 2t < s < \infty, \tag{10}$$

which then yields

$$0 < t < s/2, \tag{11}$$

the bounds of our new function.

(b): Find the marginal pdf of T.

The easiest way to do this is to "integrate out" S from the joint pdf we derived:

$$f_T(t) = \int_{-\infty}^{\infty} f_{S,T}(s,t)ds = \int_{2t}^{\infty} e^{t-s}ds$$

$$= e^t \int_{2t}^{\infty} e^{-s}ds = e^t(-e^{-s}|_{2t}^{\infty}) \tag{12}$$

$$= e^{-t} \quad if \quad t > 0.$$

(c): Find the marginal pdf of S.

This is just like part (b), except this time "integrate out" T:

$$f_S(s) = \int_{-\infty}^{\infty} f_{S,T}(s,t)dt = \int_0^{s/2} e^{t-s}ds$$

$$= e^{-s} \int_0^{s/2} e^t dt = e^{-s}(e^t|_0^{s/2}) \tag{13}$$

$$= e^{-s}(e^{s/2} - 1) \quad if \quad s > 0.$$

6.21 Let X and Y be continuous random variables with a joint density function given by

$$f_{X,Y}(x,y) = 2(x+y) \quad if \quad 0 < x < y < 1 \quad and \quad 0 \quad otherwise. \tag{14}$$

(a) Find the joint density function of $S = X$ and $T = XY$.

We can solve for X and Y in terms of the new variables, to get $X = S$ and $Y = T/S$. Then the jacobian is given by

$$J = \begin{pmatrix} 1 & 0 \\ -T/S^2 & 1/S \end{pmatrix}. \tag{15}$$

Then the new pdf is given by

$$f_{T,S}(s,t) = f_{X,Y}(x(s,t), y(s,t)) \times |1/s| = \begin{cases} 2(s+t/s)\,|1/s|, & 0 < s^2 < t < s < 1 \\ 0 & otherwise. \end{cases} \tag{16}$$

The bounds of this equation can be interpreted in the following way: the old triangular region in the xy plane got transformed to the region in the st plane between the lines $T = S$ and $T = S^2$.

(b) Find the marginal pdf of T.

To find the marginal of T, s needs to be "integrated out."

$$f_T(t) = \int_{-\infty}^{\infty} f_{S,T}(s,t)ds$$

$$= \int_t^{\sqrt{t}} 2(s+t/s)\,|1/s|\,ds = \int_t^{\sqrt{t}} 2(1+t/s^2)ds$$

$$= 2\int_t^{\sqrt{t}} ds + 2t\int_t^{\sqrt{t}} 1/s^2 ds = 2(\sqrt{t}-t) + 2t(-1/s|_t^{\sqrt{t}})$$
(17)

$$= 2\sqrt{t} - 2t + 2 - 2\sqrt{t} = \begin{cases} 2 - 2t & t \in (0,1) \\ 0 & otherwise. \end{cases}$$

.

06.23 We will use the property that independant identically distributed random variables has the form of 6.4.4, $M_Y(t) = [M_X(t)]^n$ where $Y = X_1 + X_2 + ... + X_n$. then since $X_i \sim GEO(p)$

$$\begin{aligned} Mgf(Y) &= M_{X_1}(t)M_{X_2}(t)...M_{X_k}(t) \\ &= (M_X(t))^k \\ &= (\frac{pe^t}{1-qe^t})^k \sim NegativeBinomial(k,p) \end{aligned}$$

06.25 First note, X_1, X_2, X_3, X_4 are all independant, but they are not IID as only $X_2, X_3, X_4 \sim POI(5)$ with X_1 not being listed. So formula 6.4.5 does not hold. 6.4.4 does though.

A)

$$\begin{aligned} Mgf(Y) &= M_{X_1}(t)M_{X_2+X_3+X_4}(t) \\ &= M_{X_1}(t)(M_{X_i}(t))^3 \end{aligned}$$

Since X_2, X_3, X_4 are iid 6.4.5 holds for moving to this mgf

$$\begin{aligned} &= M_{X_1}(t)(e^{\mu(e^t-1)})^3 \\ &= M_{X_1}(t)e^{3\mu(e^t-1)} \\ &= M_{X_1}(t)e^{15(e^t-1)} \\ e^{25(e^t-1)} &= M_{X_1}(t)e^{15(e^t-1)} \\ \frac{e^{25(e^t-1)}}{e^{15(e^t-1)}} &= M_{X_1}(t) \\ e^{10(e^t-1)} &= M_{X_1}(t) \sim POI(10) \end{aligned}$$

B) For $W = X_1 + X_2$ we have $X_1 \sim POI(10)$ and $X_2 \sim POI(5)$. So $POI(10+5) = POI(15)$

06.29

Given: PDF $f(x) = \begin{cases} \frac{1}{x^2} & , \quad 1 \leq x < \infty, 0 < y < \infty \\ 0 & , \quad o/w \end{cases}$

Find: a) Joint PDF of the order statistics

Setup: $F_w(w) = P[W \leq w] = P[X + Y \leq w]$

Steps: i) Differentiate with respect to a to find an equation in terms of the pdf of x.
$f_A(a) = \frac{d}{da} F_R(a/\pi)^{1/2} - \frac{d}{da} F_R - (a/\pi)^{1/2} = f_R[(a/\pi)^{1/2}]\frac{d}{da}(a/\pi)^{1/2} f_R[-(a/\pi)^{1/2}]\frac{d}{da} - (a/\pi)^{1/2}$

ii) Simplify and Use Combinatorial Identity

Result: $\binom{n+m}{k}$

Find: b) PDF of the smallest order statistic Y_1

Setup:

Steps: i)

Result:

Find: c) PDF of the largest order statistic Y_n

Setup:

Steps: i)

Result:

Find: d) PDF of the sample range $R = Y_n - Y_1, \quad for n = 2$

Setup: The area is $a = \pi r^2$ so the cdf $F_A(a) = P[A \leq a] = P[\pi r^2 \leq a] = P[r^2 \leq a/\pi]$
$= P[|r| \leq (a/\pi)^{1/2}] = P[-(a/\pi)^{1/2} \leq c \leq (a/\pi)^{1/2}] = F_r(a/\pi)^{1/2} - F_r - (a/\pi)^{1/2}$

Steps: i)

Result:

Find: e) PDF of the sample median $R = Y_r - Y_1$, $for\ nodds\ othatr$ $(n+1)/2$

Setup:

Steps: i)

Result: a) The PDF of the smallest order statistic is provided by formula 6.5.4
$g_1(y_1) = n[1 - F(y_1)]^{n-1} f(y_1)$
In this case we have $g_1(y_1) = n(1 - (1 - e^{-y})e^y$ when $y_1 > 0$
b)The PDF of the largest order statistic is provided by formula 6.5.6 $g_n(y_n) = n[F(y_n)]^{n-1} f(y_n)$
$g_n(y_n) = n[1 - e^{-y_n}]^{n-1} e^{y_n}$ Simplifying provides
$ne^{-y_n}(1 - e^{-n})^{n-1}$ when $y_n > 0$ c) Because the exponential distribution has the memoryless property, the difference between the first order statistic and the greatest order statistic won't be conditional on the value of the first order statistic (so we can treat it as zero). The probability that all the other observations $(n-1)$ fall into the range is $P(R < r) \in (0, r)$ So $P(R < r) = [\int_0^r e^{-x} dx)^{n-1} = (1 - e^r)^{n-1}$ This is P(R¡r) which is the CDF, differentiate to get the PDF:
$(n-1)(1 - e^{-r})^{n-2} e^{-r}$
06.35 Suppose X_1, X_2 are independent exponentially distributed random variables $X_i \sim$ EXP(θ), and let $Y = X_1 - X_2$.
(a) Find the MGF of Y.
We can think of $Y = X_1 - X_2$ as $Y = X_1 + (-1)X_2$. Then using Theorem 6.4.1,

$$
\begin{aligned}
M_Y(t) &= (M_{X_1}(t))(M_{-X_2}(t)) \\
M_Y(t) &= (M_{X_1}(t))(M_{X_2}(-t)) \\
M_Y(t) &= \left(\frac{1}{1 - \theta t}\right)\left(\frac{1}{1 - \theta(-t)}\right) \\
M_Y(t) &= \left(\frac{1}{1 - \theta t}\right)\left(\frac{1}{1 + \theta t}\right) \\
M_Y(t) &= \frac{1}{1 - \theta t + \theta t - \theta^2 t^2} \\
M_Y(t) &= \frac{1}{1 - \theta^2 t^2}
\end{aligned}
$$

(b) What is the distribution of Y?
Since $\frac{1}{1-\theta^2 t^2}$ is the MGF of a double exponential, $Y \sim$ DE$(\theta,0)$.

07.01 Consider a random sample of size n from a distribution with $CDF F(x) = 1 - \frac{1}{x} if 1 \leq x \leq \infty$

(a) Derive the CDF of the smallest order statistic, $X_{1:n}$

Solution: $G_1(y_1) = 1 - [1 - F_X(y_1)]^n = 1 - [1 - [1 - \frac{1}{y_1}]]^n = 1 - [\frac{1}{y_1}]^n$

$$
G_1(y_1) = \begin{cases} 1 - \frac{1}{[y_1]^n} & \text{if } 1 \leq y_1 \\ 0 & \text{if } 0 > y_1. \end{cases}
$$

(b) Find the limiting distribution of $X_{1:n}$ Solution:

$$\lim_{n\to\infty} 1 - \frac{1}{y_1{}^n} = \begin{cases} 1 & \text{if } y_1 > 1 \\ 0 & \text{if } y_1 \leq 1 \end{cases}$$

The limiting distribution of $X_{1:n}$ is degenerate at $y = 1$

(c) Find the limiting distribution of $X^n{}_{1:n}$

Solution:

$$F_{X^n{}_{1:n}}(y) = P(X^n{}_{1:n} \leq y) = P(X_{1:n} \leq y^{\frac{1}{n}}) = F_{X_{1:n}}(y^{\frac{1}{n}}) = 1 - \frac{1}{y^{\frac{1}{n}n}} = 1 - \frac{1}{y_n}$$

then, the limiting distribution of $X^n{}_{1:n} = \begin{cases} 1 - \frac{1}{y_n} & \text{if } y > 1 \\ 0 & \text{if } otherwise \end{cases}$

07.02

$$F(x) = \left\{ -\frac{1}{x^2}, \text{all real x} \right.$$

2a. $F_{X_{n:n}}(y) = (\frac{1}{1+e^{-y}})^n$; $\lim_{n\to\infty}(\frac{1}{1+e^{-y}})^n$ has no limiting distribution.

2b. $F_{X_{n:n}-\ln(n)}(y) = P[X_{n:n} - \ln(n) \leq y] = P[X_{n:n} \leq y + \ln(n)]$

$$= F_{X_{n:n}}(y + \ln(n))^n = (\frac{1}{1+e^{-(y+\ln(n))}})^n = (\frac{1}{1+\frac{e^{-y}}{n}})^n;$$

$$\lim_{n\to\infty}(\frac{1}{1+\frac{e^{-y}}{n}})^n = e^{-e^{-y}}$$

07.03 3a. $F(x) = \begin{cases} 1 - \dfrac{1}{x^2}, x > 1 \\ 0, x \leq 0 \end{cases}$

$F_{X_{1:n}}(y) = P[X_{1:n} \leq y] = 1 - P[X_{1:n} \geq y] = 1 - \frac{1}{y^{2n}}, y > 1$

$$\lim_{n\to\infty}(1 - \frac{1}{y^{2n}}) = 1 - 0 = \begin{cases} 1, y > 1 \\ 0, y \leq 0 \end{cases}$$

3b.
$F_{X_{n:n}}(y) = P[X_{n:n} \leq y] = 1 - P[X_{n:n} \geq y] = 1 - (1 - \frac{1}{y^2})^n = \frac{1}{y^{2n}}$; $lim?_{n\to\infty}\frac{1}{y^{2n}} = 0$,
Therefore $F_{X_{n:n}}(y)$ has no limiting distribution.
3c.
$F_{n^{-\frac{1}{2}}X_{n:n}}(y) = P[\frac{1}{\sqrt{n}}X_{n:n} \leq y] = P[X_{n:n} \leq \sqrt{n}y] = F_{X_{n:n}}(\sqrt{n}y) = (1 - (\sqrt{n}y)^{-2})^n, for y > \frac{1}{\sqrt{n}}$

$$\lim_{n\to\infty}(1 - (\sqrt{n}y)^{-2})^n = (1 - \frac{1}{ny^2})^n = \begin{cases} e^{-y^{-2}}, y > 0 \\ 0, y \leq 0 \end{cases}$$

07.07 The WEI $(1, 2)$ distribution has pdf $f(x) = 2xe^{-x^2}$ for $x > 0$, mean $\mu = \Gamma(\frac{3}{2}) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{1}{2}\sqrt{\pi}$ and variance $\sigma^2 = \Gamma(2) - \Gamma(\frac{3}{2})^2 = 1 - \frac{\pi}{4}$
(a)According to the central limit theorem, this holds with $a = \mu - \frac{1.96\sigma}{\sqrt{n}}$ and $b = \mu + \frac{1.96\sigma}{\sqrt{n}}$, so if $n = 35$ we have $a = 0.7328$ and $b = 1.0397$

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(b)For odd n, $X_{\frac{n+1}{2}:n}$ is approximately $N(x_{\frac{1}{2}}, \frac{c^2}{n})$, where $c^2 = \frac{1}{4f(x_{\frac{1}{2}})^2}$. Now $F(x_{\frac{1}{2}}) = \frac{1}{2}$, because $F(x) = 1 - e^{-x^2}$, it implies that $x_{\frac{1}{2}} = \sqrt{ln2}$. Also because $c^2 = \frac{1}{4ln2}$, we have $a = x_{\frac{1}{2}} - \frac{1.96c}{\sqrt{n}}$ and $b = x_{\frac{1}{2}} + \frac{1.96c}{\sqrt{n}}$, so when $n = 35$ we have $a = 0.6336$ and $b = 1.0315$

07.09
Let $X_1, X_2, ..., X_{100}$ be a random sample from an exponential distribution, $X_i \sim EXP(1)$, and let $Y = X_1 + X_2 + ... + X_{100}$.

(a) Give an approximation for $P[Y > 110]$.
$100 \cdot E[X] = 100 \cdot 1 = 100$
$100 \cdot Var(X) = 100 \cdot 1^2 = 100$
$P[Y > 110] = P[\frac{Y-100}{\sqrt{100}} > \frac{110-100}{\sqrt{100}}]$
$= 1 - \Phi(\frac{110-100}{\sqrt{100}}) = 1 - \Phi(1)$
$= 1 - 0.8413 = 0.1587$

(b) If $\bar{X}$ is the sample mean, then approximate $P[1.1 < \bar{X} < 1.2]$.
$P[1.1 < \bar{X} < 1.2] = P[110 < Y < 120] = \Phi(2) - \Phi(1) = 0.9772 - 0.8413 = 0.1359$

07.11 a) First we need to know the μ and the σ. For a Uniform variable with $a = 0, b = 1$ we have $\mu = 1/2$ and $\sigma = 1/\sqrt{12}$ (Note: it is not σ^2). We also need to know that $n = 20$ from there we can use the CLT:

$$\Pr\left(\sum_{}^{20} X_i < 12\right) = \Pr\left(\frac{\sum X_i - 10}{\sqrt{20}\frac{1}{\sqrt{12}}} < \frac{12 - 10}{\sqrt{20}\frac{1}{\sqrt{12}}}\right)$$

$$= \Phi\left(\frac{12 - 10}{\sqrt{20}\frac{1}{\sqrt{12}}}\right)$$

$$\approx .9394$$

b) We let $Y = \sum^{20} X_i$, let Y' be our 90th percentile that we want to find. So we setup our probability as $\Pr(Y \le Y') = .9$, .9 as we are interested in the 90th percentile. Using μ, σ, and n from part (a) we solve with CLT:

$$\Pr(Y \le Y') = \Pr\left(\frac{Y - \mu n}{\sigma\sqrt{n}} \le \frac{Y' - \mu n}{\sigma\sqrt{n}}\right)$$

$$= \Pr\left(Z \le \frac{Y' - 10}{\sqrt{20}\frac{1}{\sqrt{12}}}\right) \text{ Note: Z is standard normal due to CLT}$$

$$.9 = \Phi\left(\frac{Y' - 10}{\sqrt{20}\frac{1}{\sqrt{12}}}\right)$$

We now solve for Y'. We know (from a chart or list) that .9 from Φ is $z \approx 1.285$. So we set our final equation for finding out Y' with that in mind.

$$\frac{Y' - 10}{1.291} = 1.285$$

$$Y' \approx 11.658$$

07.12   a) First, an understanding that the wording here implies that X is actually "failures" of weapons. So the given p would normally be q in other contexts. So using the binomial theorem we would have $p = .05$ and $q = .95$. Knowing that we can use the Binomial theorem easily:

$$\Pr(X \geq 1) = 1 - \Pr(X < 1)$$
$$= 1 - \binom{n}{0}(.05)^0(.95)^n$$

We now solve for n from the above equation knowing that the desired probability is .99

$$.99 = 1 - (.95)^n$$
$$\ln .95^n = \ln .01$$
$$n = \frac{\ln .01}{\ln .95}$$

So n, since it must be an integer, is rounded to 90.
b)
07.13   From the hint we know that $Y_n = \sum^n X_i$ where $X_i \sim Geo(p)$. So for $\sum^n X_i$ the $\mu = \frac{n}{p}$ and $\sigma^2 = \frac{nq}{p^2}$. Then by the CLT:

$$\Pr(Y_n \leq y) = \Pr\left(\sum_{}^{n} X_i \leq y\right)$$
$$= \Pr\left(\frac{\sum^n X_i - \frac{n}{p}n}{\sqrt{n}\sqrt{\frac{nq}{p^2}}} \leq \frac{y - \frac{n}{p}n}{\sqrt{n}\sqrt{\frac{nq}{p^2}}}\right)$$
$$= \Phi\left(\frac{y - \frac{n}{p}n}{\sqrt{n}\sqrt{\frac{nq}{p^2}}}\right)$$

07.16   a) We need two things for this proof. First, we need to know μ and σ^2 of $\overline{X}$. We know this is $\mu = \mu$ and $\sigma^2 = \frac{\mu^2}{n}$ from facts of the sample mean distribution of $POI(\mu)$. Next the theorems from section 7.6, namely 7.6.2 and from 7.7, 7.7.2. These will let us prove the following:

$$\Pr\left[|\overline{X_n} - \mu| < \epsilon\right] \geq 1 - \frac{\mu^2}{\epsilon^2 n}$$
$$\lim_{n \to \infty} \Pr\left[|\overline{X_n} - \mu| < \epsilon\right] = 1$$

From this we now know that $\overline{X} \xrightarrow{P} \mu$ from 7.6.3. For our goal, $e^{\overline{X_n}}$ we simply need to know 7.7.2. Since $\overline{X} \xrightarrow{P} \mu$ then $e^{\overline{X}} \xrightarrow{P} e^{\mu}$

b) It has been shown elsewhere in the text that any $\overline{X_n}$ will converge to $N(0,1)$ if standardized. The theorem we need to use then, is 7.7.6 which states that a function of an already convergent series also converges to an asymptotic normal distribution. (For an almost direct example see Example 7.7.3)

Our $g(y)$ here is $e^{-\overline{X_n}}$ where $g(y) = e^y$. So then $g'(y) = -e^{-y}$ and using 7.7.6 we can find our distribution if $\frac{d}{d\mu} e^{-\mu} = -e^{-\mu}$ then $N(e^{\mu}, \frac{-e^{-2\mu}\mu^2}{n})$

c) From parts (a) we know that $\overline{X_n} \xrightarrow{P} \mu$ and $e^{-\overline{X}} \xrightarrow{P} e^{-\mu}$. So we can use theorem 7.7.3 via section (2), which states that $X_n Y_n \xrightarrow{P} cd$. In our case we have the prior two found distributions. So then by the theorem $\overline{X_n} e^{\overline{X_n}} \xrightarrow{P} \mu e^{-\mu}$

08.01 $X_i \sim N(101, 4)$ so then $\mu = 101$ and $\sigma^2 = 4$ or $\sigma = 2$. We just use the CLT to solve this:

$$\Pr\left(20 \text{ bags will weigh at least 1 ton}\right) = \Pr\left(\sum^{20} X_i \geq 2000\right)$$

$$= \Pr\left(\frac{\sum^{20} X_i - 2020}{2\sqrt{20}} \geq \frac{2000 - 2020}{2\sqrt{20}}\right)$$

$$= 1 - \Phi\left(-2.23\right)$$

$$\approx 0.987$$

08.02 a) Since both S and B are normal variables we may transform them, via theorem 8.3.1 into a new normal variable. The values we need are μ and σ of both S and B. For S we have $\mu = 1$, $\sigma^2 = .0004$, for B $\mu = 1.01$ and $\sigma^2 = .0009$. For the question, we want the probability that $S > B$ so in other words $\Pr(S - B > 0)$. This means $S - B = Y$ is a new normal variable (by theorem), with values $\mu = -.01$ and $\sigma^2 = .0013$. We will use the CLT to solve the probability, so we need $\sigma = .036$. Using that we solve:

$$\Pr\left(S - B > 0\right) = \Pr\left(Y > 0\right)$$

$$= \Pr\left(\frac{Y - (-.01)}{.036} > \frac{0 - (-.01)}{.036}\right)$$

$$= \Phi\left(\frac{.01}{.036}\right)$$

$$\approx 0.39$$

b) We now assume that for S and B that σ^2 are identical for each, but unknown. We do know our desired probability, .95 so we will solve for that instead. Very similar in approach to part (a),

we just solve for σ now. Important fact is that the $N(-.01, \sigma^2 + \sigma^2) = N(-.01, 2\sigma^2)$ so $\sigma = \sigma\sqrt{2}$

$$\Phi\left(\frac{.01}{\sigma\sqrt{2}}\right) = .95$$

So we find the value in our table, 1.65 and solve for σ

$$\frac{.01}{\sigma\sqrt{2}} = 1.65$$
$$\approx .00428$$

Problem 8.3 Let $X_1, X_2, ..., X_n$ be a random sample of size n from a normal distribution, X_i is appoximately $N(\mu, \sigma^2)$, and define $U = \sum\limits_{i=1}^{n} X_i$ and $W = \sum\limits_{i=1}^{n} X_i^2$

a) Find a statistic that is a function of U and W and unbiased for the parameter $\theta = 2\mu - 5\sigma^2$:
First we find for μ

$$\mu = E(\bar{x})$$
$$= E(\frac{\sum\limits_{i=1}^{n} X_i}{n})$$
$$= \frac{1}{n}E(U)$$

Then find for σ^2

$$\sigma^2 = E(S^2)$$
$$= E(\frac{\sum\limits_{i=1}^{n} X_i^2 - n\bar{x}^2)}{n-1})$$
$$= \frac{1}{n-1}E(W - n(\frac{(\sum\limits_{i=1}^{n} X_i)^2}{n^2}))$$
$$= \frac{1}{n-1}E(W - \frac{U^2}{n})$$
$$= E(\frac{1}{n-1}(W - \frac{U^2}{n}))$$

thus

$$\theta = 2\mu - 5\sigma^2$$

$$= 2(\frac{1}{n}E(U)) - 5E(\frac{1}{n-1}(W - \frac{U^2}{n}))$$

$$= E(\frac{2}{n}(U) - \frac{5}{n-1}(W - \frac{U^2}{n}))$$

$$= \frac{2U}{n} - \frac{5}{n-1}(W - \frac{U^2}{n})$$

which is an unbiased estimator for θ

b) Find a statistic that is unbiased for $\sigma^2 + \mu^2$

$$\mu = E(\frac{\sum\limits_{i=1}^{n} X_i}{n})$$

$$\mu^2 = (E(\frac{\sum\limits_{i=1}^{n} X_i}{n}))^2$$

$$= E((\frac{\sum\limits_{i=1}^{n} X_i}{n})^2) - Var(\frac{\sum\limits_{i=1}^{n} X_i}{n})$$

$$= E(\frac{1}{n^2}(\sum_{i=1}^{n} X_i)^2) - \frac{1}{n^2}Var(\sum_{i=1}^{n} X_i)$$

$$= E(\frac{U^2}{n^2}) - \frac{1}{n^2}n\sigma^2$$

$$= E(\frac{U^2}{n^2}) - \frac{\sigma^2}{n}$$

From this then, $\sigma^2 = \frac{1}{n-1}E(W - \frac{U^2}{n})$. then finally the following:

$$\sigma^2 + \mu^2 = \sigma^2 + E(\frac{U^2}{n^2}) - \frac{\sigma^2}{n}$$

$$= \frac{n-1}{n}\sigma^2 + E(\frac{U^2}{n^2})$$

$$= \frac{n-1}{n}\frac{1}{n-1}E(W - \frac{U^2}{n}) + E(\frac{U^2}{n^2})$$

$$= E(\frac{W}{n} - \frac{U^2}{n}) + E(\frac{U^2}{n^2})$$

$$= E(\frac{W}{n})$$

$$= \frac{W}{n}$$

which is an unbiased estimator for $\sigma^2 + \mu^2$

c) Let c be a constant, and define $Y_i = 1$ if $X_i \leq c$ and zero otherwise. Find a statistic that is a function of $Y_1, Y_2, ...Y_n$ and also unbiased for $F_x(c) = \phi(\frac{c-\mu}{\sigma})$:

$$P(Y_i = 1) = P(X_i \leq c)$$

$$= P(\frac{X_i - \mu}{\sigma} \leq \frac{c - \mu}{\sigma})$$

$$= P(N(0,1) \leq \frac{c - \mu}{\sigma})$$

$$= \phi(\frac{c - \mu}{\sigma})$$

$$= F_x(c)$$

$$E(Y_i) = 1(P(Y_i = 1)) + 0(P(Y_i = 0))$$

$$= P(Y_i = 1)$$

$$= \phi(\frac{c - \mu}{\sigma})$$

$$E(\frac{1}{n}\sum_{i=1}^{n}Y_i) = \frac{1}{n}\sum_{i=1}^{n}E(Y_i)$$

$$= \frac{1}{n}n\phi(\frac{c - \mu}{\sigma})$$

$$= \phi(\frac{c - \mu}{\sigma})$$

Then $\frac{1}{n} \sum\limits_{i=1}^{n} Y_i = \bar{Y}$ which is unbiased for $F_x(c) = \phi(\frac{c-\mu}{\sigma})$

08.04

Using a change of variables: $\quad X_1 = \frac{Y_1+Y_2}{2}, \quad X_2 = \frac{Y_1-Y_2}{2}$

$J = \det = -\frac{1}{2}$

$f_y(y_1, y_2) = f_x\left(\frac{Y_1+Y_2}{2}, \frac{Y_1-Y_2}{2}\right)$

$= \frac{1}{2\pi\sigma^2} exp\frac{1}{2\sigma^2}\left[\left(\frac{Y_1+Y_2}{2}\right)^2 + \left(\frac{Y_1-Y_2}{2}\right)^2\right]$

$= \frac{1}{2\pi\sigma^2} exp\frac{1}{2\sigma^2}\left(\frac{Y_1^2}{4} + \frac{Y_2^2}{4} + \frac{Y_1^2}{4} + \frac{Y_2^2}{4}\right)$

$= \frac{1}{2\pi\sigma^2} exp\frac{1}{2\sigma^2}\left(\frac{Y_1^2}{4} + \frac{Y_1^2}{4}\right) + \left(\frac{Y_2^2}{4} + \frac{Y_2^2}{4}\right)$

$= \frac{1}{2\pi\sigma^2} exp\frac{1}{2\sigma^2}\left[\left(\frac{Y_1}{2}\right)^2 + \left(\frac{Y_2}{2}\right)^2\right]$

Hence, $f_y(y_1, y_2) = f_x\left(\frac{Y_1+Y_2}{2}, \frac{Y_1-Y_2}{2}\right) = f_{x1}(Y_1), f_{x2}(Y_2)$

Therefore Y_1 and Y_2 are independent and normally distributed.
08.05 a) We can find the distribution by use of moment generating functions. Since $\sum^{10} T$
then we have $M_{\sum^{10} T}(t)$. Evaluating this out will lead as follows:

$$M_{\sum^{10} T}(t) = E\left(e^{t\sum^{10} T}\right)$$
$$= \prod_{}^{10} E\left(e^{tT_i}\right)$$
$$= \left(\frac{1}{1-100t}\right)^{10}$$

Which is known to be a gamma random variable with parameters $GAM(100, 10)$

b) Let $X \sim GAM(100, 10)$. We need to find $\Pr(X \geq 548)$ as the days in 1.5 years is 548. By using our hint we transform this probability into one in which we are able to find readily (by use of a table).

$$\Pr(X \geq 548) = \Pr\left(\frac{2X}{100} \geq \frac{2*548}{100}\right)$$
$$= \Pr\left(\chi^2 \geq \frac{548}{50}\right)$$

Since our $\nu = 20$ we find that our probability is $1 - .05 = .95$

c) For two years, ie 730 days, we need to find $\Pr(X \geq 730) = .95$. All of our solutions rely on finding a needed quantity in a table. We are solving for the κ of our $GAM(100, \kappa)$ variable. So we get our percentile as $\frac{730}{50} = 14.6$ then using either table 4 or table 5, we find that for the .05 percent we need a $\nu \approx 24$. Actually a little less, but due to the fact we are parts, 24 is needed. Then our $\kappa = \frac{\nu}{2} = 12$

8.8 Suppose that $X \sim \chi^2(m)$ and $Y \sim \chi^2(n)$, and X and Y are independent. Is $X - Y \sim \chi^2$ if $n > m$? No, $X - Y$ can take on negative values and a random variable with a χ^2 distribution cannot.

8.9 Suppose that X is approximately $\chi^2(m)$ and $S = X + Y$ is approximately $\chi^2(m+n)$ and X and Y are independent. Use MGFs to show that $S - X$ is approximately $\chi^2(n)$:

First we have $M_x(t) = (1 - 2t)^{\frac{-m}{2}}$ and $M_s(t) = (1 - 2t)^{\frac{-(m+n)}{2}}$. With these then we can get our MGF of interest, so $M_s(t) = M_x(t)M_y(t)$, finally then:

$$
\begin{aligned}
M_y(t) &= \frac{M_s(t)}{M_x(t)} \\
&= \frac{(1 - 2t)^{\frac{-(m+n)}{2}}}{(1 - 2t)^{\frac{-m}{2}}} \\
&= (1 - 2t)^{\frac{-n}{2}}
\end{aligned}
$$

which is the MGF of $\chi^2(n)$ Thus $Y = S - X$ is approximately $\chi^2(n)$.

08.10 We have $n = 15$. With $\overline{X} = \frac{\sum_{i=1}^{15} Y_i}{15}$ where $Y \sim EXP(\theta)$ we have in fact $\frac{Z}{15}$ where $Z \sim GAM(\theta, 15)$. So to solve our probability, we proceed as follows:

$$
\begin{aligned}
\Pr\left(\frac{cZ}{15} \leq \theta\right) &= \Pr\left(15cZ \leq 15\theta\right) \\
&= \Pr\left(\frac{2cZ}{\theta} \leq \frac{15\theta 2}{\theta}\right) \\
&= \Pr\left(\frac{2Z}{\theta} \leq \frac{30}{c}\right)
\end{aligned}
$$

We now transform $\frac{2Z}{\theta} \sim \chi^2(30)$. For the probability of .95 we look in our table and find that for $\kappa = 30$ we need a $\gamma = 43.77$ so then $\frac{30}{c} = \gamma = 43.77$ and then $c = .685$

08.12 Since X_1 and X_2 are $\sim N(0, 25)$ we know that being squared they are "close" to the χ^2 distribution. So our goal in finding the probability will be to rework D into something involving χ^2. First, since $D = \sqrt{X_1^2 + X_2^2}$ we will get rid of the square root.

$$\Pr\left(\sqrt{X_1^2 + X_2^2} \le 12.25\right) = \Pr\left(X_1^2 + X_2^2 \le (12.25)^2\right)$$

Since $X^2 = (X - 0)^2$ and since $\mu = 0$ we can rework the remaining distribution into a χ^2 as follows

$$
\begin{aligned}
\Pr\left(X_1^2 + X_2^2 \le (12.25)^2\right) &= \Pr\left((X - \mu)^2 + (X^2 - \mu)^2 \le (12.25)^2\right) \\
&= \Pr\left(\frac{(X - \mu)^2 + (X - \mu)^2}{\sigma^2} \le \frac{(12.25)^2}{\sigma^2}\right) \\
&= \Pr\left(\sum^2 \frac{(X - \mu)^2}{\sigma^2} \le \frac{(12.25)^2}{\sigma^2}\right) \\
&= \Pr\left(\sum^2 \frac{(X - 0)^2}{25} \le \frac{(12.25)^2}{25}\right) \\
&= \Pr\left(\chi^2(2) \le \frac{(12.25)^2}{25}\right) \\
&= \Pr\left(\chi^2(2) \le 6\right)
\end{aligned}
$$

Using a table we find this probability to be $\approx .95$

08.13 a) We just convert the $\overline{X}$ to standard normal, and evaluate. $\Pr\left(\frac{\overline{X} - 0}{\frac{1}{4}} \le \frac{\frac{1}{2} - 0}{\frac{1}{4}}\right)$ which we see is then $\Phi(2)$

b) Let $Y = Z_1 - Z_2$ then $Y \sim N(0, 2)$. Thus $\Pr\left(\frac{Y}{\sqrt{2}} \le \frac{2}{\sqrt{2}}\right)$ which is $\Phi(1.41) \approx .921$

c) By theorem, this is the same as part (b) as our $Y = X_1 + X_2$ is still $Y \sim N(0, 2)$

d) By theorem corollary 8.3.2, $\sum^{16} Z^2 \sim \chi^2(16)$ as our $\mu = 0$ and $\sigma^2 = 1$. So then $\Pr\left(\chi^2(16) \le 32\right) \approx .99$

e) We want to use theorem 8.3.6 part 3. First we divide both by $n - 1$ to get S^2. Next we will multiply both sides by $n - 1$ and divide by σ^2 which will give us a distribution of $\chi^2(15)$.

$$
\begin{aligned}
\Pr\left(\sum^{16}(X - \overline{X})^2 \le 25\right) &= \Pr\left(\frac{15 \sum^{16}(X - \overline{X})^2}{(15)(1)} \le \frac{15 * 25}{15 * 1}\right) \\
&= \Pr\left(\chi^2(15) \le 25\right)
\end{aligned}
$$

Which we find to be $\approx .95$

08.16 a) We make $\overline{X}$ into standard normal then calculate.

$$\Pr\left(3 < \overline{X} < 7\right) = \Pr\left(\frac{3-6}{\frac{5}{3}} < \frac{\overline{X}-6}{\frac{5}{3}} < \frac{7-6}{\frac{5}{3}}\right)$$
$$= \Pr\left(-1.8 < \Phi < .6\right)$$
$$= \Phi(.6) - \Phi(-1.8)$$
$$\approx .6898$$

b) By theorem 8.4.3 we have $t(n-1) = t(8)$. So $\Pr(t(8) > 1.86) = 1 - \Pr(t(8) < 1.86) \approx .05$

c) By theorem 8.3.6 we multiply by $n-1$ and divide by σ^2 to give us a $\chi^2(8)$. Then $\Pr(\chi^2(8) \le 10.22) = .75$

08.17 a) Simply use the table to find $Y(22.31) - Y(7.26)$

b) Using tables, this is found to be 27.14

c) We use some simply manipulations to get Y isolated.

$$\Pr\left(\frac{Y}{1+Y} \ge \frac{11}{16}\right) = \Pr\left(\frac{1+Y}{Y} \le \frac{16}{11}\right)$$
$$= \Pr\left(\frac{1}{Y} \le \frac{16}{11} - 1\right)$$
$$= \Pr\left(Y \ge \frac{11}{5}\right)$$
$$= 1 - .1 = .90$$

d) Use of tables for $T(2.65) - T(.87)$

e) Again, this is found to be .265 using a table

f) The absolute value will put this as a $1 - T(c) + T(-c) = .02$ or $T(c) = .98 + T(-c)$. So we need $\Pr(T \le c) = .99$ which found gives us a $c = 2.5$.

g) Use of tables to find the probability.

h) The probability is simplified to $\Pr(X \le \frac{1}{.25})$ which we find to be .975

08.18 a) By theorem 8.3.4 we know that $V_1 + V_2 = \chi^2(14)$ Thus our result, via table, is $\approx .144$

b) By definition of the t distribution $\frac{Z}{\sqrt{\frac{V_1}{5}}} \sim t(5)$. By table the probability is then .95

c) We need to make the distribution a t distribution. We need to divide by our V_2 and multiply by $\sqrt{9}$ to get the form we need for the t. Thus:

$$\Pr\left(Z \ge .611\sqrt{V_2}\right) = \Pr\left(\frac{Z}{\sqrt{\frac{V_2}{9}}} \ge 3 * .611\right)$$
$$= \Pr\left(t(9) \ge 1.83\right)$$
$$= 1 - .95 = .05$$

d) If we multiply both sides by a $\frac{9}{5}$ we will get an F distribution which we can use a table to calculate the result. Thus, $\Pr(F(5,9) \le 2.61) = .9$

e) We can convert this to an F distribution with a little work:

$$\Pr\left(\frac{V_1}{V_1 + V_2} \le b\right) = \Pr\left(\frac{V_1 + V_2}{V_1} \ge \frac{1}{b}\right)$$
$$= \Pr\left(1 + \frac{V_2}{V_1} \ge \frac{1}{b}\right)$$
$$= \Pr\left(\frac{V_2}{V_1} \ge \frac{1}{b} - 1\right)$$
$$= \Pr\left(\frac{5}{9}\frac{V_2}{V_1} \ge \frac{5}{9}\left(\frac{1}{b} - 1\right)\right)$$
$$= \Pr\left(F(9,5) \ge \frac{5}{9}\left(\frac{1}{b} - 1\right)\right)$$
$$= \Pr\left(F(9,5) \ge \frac{5}{9}\left(\frac{1}{b} - 1\right)\right)$$

We now find our value for an $F(9,5) = .90$ which turns out to be 2.61. We must take the inverse of this since we are $\ge$ and our table is in a format for $\le$. We now set that equal to our percentile and solve for b.

$$\frac{1}{2.61} = \left(\frac{5}{9}\left(\frac{1}{b} - 1\right)\right)$$
$$\frac{9}{5}\frac{1}{2.61} + 1 = \frac{1}{b}$$
$$b \approx 5.92$$

9.01 Find the MMEs of θ based on a random sample $X_1, ..., X_n$ from each of the following pdf's:

a) $f(x;\theta) = \theta x^{\theta-1}; 0 < x < 1, 0 < \theta$

First set the sample moment to the population moment.

$$\overline{X} = E[X]$$

$$= \int_0^1 x\theta x^{\theta-1}dx$$

$$= \int_0^1 \theta x^\theta dx$$

$$= \frac{\theta x^{\theta+1}}{\theta+1}\Big|_0^1$$

$$= \frac{\theta}{\theta+1}$$

Now solve for θ in terms of $\overline{X}$. $\frac{\theta}{\theta+1} = \overline{X} \longrightarrow \hat{\theta} = \frac{\overline{X}}{1-\overline{X}}$

b) $f(x;\theta) = (\theta+1)x^{-\theta-2}; 1 < x, 0 < \theta$

First set the sample moment to the population moment.

$$\overline{X} = E[X] = \int_1^\infty x(\theta+1)x^{-\theta-2}dx$$

$$= \int_1^\infty (\theta+1)x^{-\theta-1}dx$$

$$= \frac{\theta+1}{-\theta}x^{-\theta}\Big|_1^\infty$$

$$= \frac{\theta+1}{\theta}$$

Now solve for θ in terms of $\overline{X}$. $\frac{\theta+1}{\theta} = \overline{X} \longrightarrow \hat{\theta} = \frac{1}{\overline{X}-1}$

c) $f(x;\theta) = \theta^2 x e^{-\theta x}; 0 < x, 0 < \theta$

First set the sample moment to the population moment.

$$\overline{X} = E[X]$$

$$= \int_0^\infty \theta^2 x^2 e^{-\theta x}dx$$

$$= -\theta e^{-\theta x}x^2 - 2xe^{-\theta x} - \frac{2}{\theta}e^{-\theta x}\Big|_0^\infty$$

$$= \frac{2}{\theta}$$

Now solve for θ in terms of $\overline{X}$. $\frac{2}{\theta} = \overline{X} \longrightarrow \hat{\theta} = \frac{2}{\overline{X}}$

9.2 Find Methods of Moments Estimators based on a random sample of size n.

a) $X_i \sim NB(3,p)$
$\mu_1' = \frac{r}{p} = \frac{3}{p} = \overline{X}$
$\hat{p} = \frac{3}{\overline{X}}$

b) $X_i \sim Gam(2, k)$
$\mu_1' = k\theta = 2k = \overline{X}$
$\hat{k} = \frac{\overline{X}}{2}$

c) $X_i \sim Wei(\theta, 1/2)$
$\mu_1' = \theta\Gamma(1 + \frac{1}{\beta}) = \theta\Gamma(1 + \frac{1}{1/2}) = \theta\Gamma(3) = \theta(3-1)! = 2\theta = \overline{X}$
$\hat{\theta} = \frac{\overline{X}}{2}$

d) $X_i \sim DE(\theta, \eta)$
$\mu_1' = \eta = \overline{X} = \hat{\nu}$
$\mu_2' = \sigma^2 + \mu^2 = 2\theta^2 + \eta^2$
$\hat{\theta} = \sqrt{\frac{\frac{\sum X^2}{n} - \overline{X}^2}{2}}$

e) $X_i \sim EV(\theta, \eta)$
$\mu_1' = \eta - \gamma\theta = \overline{X}$
$\mu_2' = \sigma^2 + \mu^2 = \frac{\pi^2\theta^2}{6} + (\eta - \gamma\theta)^2$
$\hat{\theta} = \frac{\sqrt{6}}{\pi}\sqrt{\frac{\sum X^2}{n} - \overline{X}^2}$
$\hat{\eta} = \overline{X} + \gamma(\frac{\sqrt{6}}{\pi})\sqrt{\frac{\sum X^2}{n} - \overline{X}^2}$

f) $X_i \sim Par(\theta, k)$
$\mu_1' = \frac{\theta}{(k-1)} = \overline{X}$
$\mu_2' = \sigma^2 + \mu^2 = \frac{\theta^2 k}{[(k-2)(k-1)^2]} + \frac{\theta^2}{(k-1)^2}$
$\frac{\hat{k}}{\hat{k}-2} = \frac{\hat{\sigma}^2}{\overline{X}^2} \rightarrow \frac{\frac{2\hat{\sigma}^2}{\overline{X}^2}}{\frac{\hat{\sigma}^2}{\overline{X}^2} - 1}$
$\hat{\theta} = \overline{X}(\hat{k} - 1)$

9.3 Find MLEs for the following pdfs, based on a sample of $X_1, ..., X_n$ random variables. (a),(b), and (c) are all found in a similar manner. We will go over the steps in part (a) and the remaining will just show the work without comment.

a) $f(x; \theta) = \begin{cases} \theta x^{\theta-1} & 0 < x < 1 \\ 0 & otherwise \end{cases}$

$$L = \theta^n (x_1...x_n)^{\theta-1}$$
Set the likelyhood function equal to our joint pdf

$$\ln L = n \ln \theta + (\theta - 1)(\ln x_1 ... \ln x_n)$$
Now we take the natural log of this to get it in a workable state

$$\frac{d}{d\theta} \ln L = \frac{n}{\theta} + (\ln x_1 ... \ln x_n) = 0$$
We now set our equation equal to zero to find the estimator

Solving the last equation for θ will yield our estimator, which in this case is $\hat{\theta} = \frac{-n}{(\ln x_1 ... \ln x_n)}$.

b) $f(x; \theta) = \begin{cases} (\theta+1)x^{-\theta-2} & 1 < x \\ 0 & otherwise \end{cases}$

$$L = (\theta+1)^n (x_1...x_n)^{-\theta-2}$$
$$\ln L = n\ln(\theta+1) - (\theta+2)(\ln x_1 ... \ln x_n)$$
$$\frac{d}{d\theta}\ln L = \frac{n}{(\theta+1)} - (\ln x_1 ... \ln x_n) = 0$$
$$\hat{\theta} = \frac{n}{(\ln x_1 ... \ln x_n)} - 1$$

c) $f(x; \theta) = \begin{cases} \theta^2 x e^{-\theta x} & 0 < x < 1 \\ 0 & otherwise \end{cases}$

$$L = \theta^{2n}(x_1...x_n)e^{-\theta(x_1...x_n)}$$
$$\ln L = 2n\ln\theta + \ln x_1 + ... + \ln x_n - \theta(x_1 + ... + x_n)$$
$$\frac{d}{d\theta}\ln L = \frac{2n}{\theta} - (x_1...x_n) = 0$$
$$\hat{\theta} = \frac{2}{\overline{X}}$$

09.05 $f(x; \theta) = \frac{2\theta^2}{x^3}$, $x \geq \theta, \theta > 0$. $L(\theta) = \frac{2^n\theta^{2n}}{(x_1...x_n)^3}$ if $x_1 \geq \theta$, that is, if $x_{1:n} \geq \theta, ..., x_n \geq \theta$. Then $L(\theta)$ is maximized at $\hat{\theta} = X_{1:n}$.

9.6 Find MLEs based on a random sample $X_1...X_n$ for the following pdfs.

a) $f(x; \theta_1, \theta_2) = \begin{cases} \frac{1}{\theta_2-\theta_1} & \theta_1 < x < \theta_2 \\ 0 & otherwise \end{cases}$

Since the pdf doesn't depend on X, we can write the likelihood function as:

$$L(\theta_1, \theta_2) = \left(\frac{1}{\theta_2 - \theta_1}\right)^n$$

We want this function to be maximized. So we need the denominator to be as small as possible which would give us the biggest fraction, namely we want to minimize $(\theta_2 - \theta_1)$. This means that we want the smallest value of θ_2 and the largest value of θ_1. We know that $\theta_1 \leq x$, or in other words $\theta_1 \leq X_{1:n}$. This means that the largest value of θ_1 will be $X_{1:n}$. Similarily, to obtain the smallest value of θ_2 we notice that $\theta_2 \geq X_{n:n}$.
Therefore our MLEs for this pdf are $\hat{\theta}_1 = X_{1:n}$ and $\hat{\theta}_2 = X_{n:n}$.

b) $f(x; \theta, \eta) = \begin{cases} \theta\eta^\theta x^{-\theta-1} & \eta \leq x, 0 < \theta, 0 < \eta < \infty \\ 0 & otherwise \end{cases}$

The likelihood function for this pdf will be:

$$
\begin{aligned}
L(\theta, \eta) &= \prod \theta \eta^n x_i^{-\theta-1} \\
ln(L) &= \sum (ln(\theta) + \theta ln(\eta) + (-\theta - 1)ln(x_i)) \\
ln(L) &= nln(\theta) + n\theta ln(\eta) - \theta \sum ln(x_i) - \sum ln(x_i) \\
\frac{d}{d\theta} ln(L) &= \frac{n}{\theta} + nln(\eta) - \sum ln(x_i) \\
0 &= \frac{n}{\theta} + nln(\eta) - \sum ln(x_i) \\
\hat{\theta} &= \frac{n}{\sum ln(x_i) - nln(\eta)}
\end{aligned}
$$

If we look at the fact that $\eta \leq x$ (meaning that $\eta \leq X_{1:n}$) and notice that the term η^θ in the liklihood function will be maximized when η is maximized, we can say right off the bat that $\hat{\eta} = X_{1:n}$.

This gives the MLEs of $\hat{\eta} = X_{1:n}$ and $\hat{\theta} = \frac{n}{\sum ln(x_i) - nln(X_{1:n})}$

9.7 Let $x_1...x_n$ be a random sample from $X_i \sim Geo(p)$. Find MLEs for the following. We must first find the MLE of $GEO(p)$:

$$
\begin{aligned}
L(p) &= \prod p(1-p)^{x-1} \\
\ell &= \sum (\ln p - (x-1)\ln(1-p)) \\
\frac{d\ell}{dp} &= \frac{n}{p} - \frac{\sum x - 1}{1 - p}
\end{aligned}
$$

Setting equal to zero then we get:

$$
\hat{p} = \frac{1}{\overline{X}}
$$

We now use $\hat{p}$ to solve for the following with the invariance property.

a) For $\tau(p) = E(X) = \frac{1}{p}$ we see that $\tau(\hat{p}) = \frac{1}{\frac{1}{\overline{X}}} = \overline{X}$

b) We let $\tau(p) = Var(X) = \frac{(1-p)}{p^2}$ then $\tau(\hat{p}) = \frac{1 - \frac{1}{\overline{X}}}{\frac{1}{\overline{X}^2}} = \overline{X}(\overline{X} - 1)$

c) Finally, $P[X > k] = (1-p)^k$ then we get $(1 - \frac{1}{\overline{X}})^k$

9.8

8a

$P[X > c]$ for some c is $1 - \phi\left(\frac{c-\mu}{\theta}\right)$

Therefore, the MLE is $1 - \phi\left(\frac{c - \bar{X}}{\sqrt{n^{-1}\sum (X_j - \bar{X})^2}}\right)$

8b

To find the 95th percentile of X we need to find

$$0.95 \;=\; P[X > x] \;=\; \phi\!\left(\tfrac{x-\mu}{\theta}\right)$$

Hence, $\tfrac{x-\mu}{\theta} = 1.645 \;\Rightarrow\; x = 1.645\theta + \mu$

By applying the MLE from Part(a) it is shown that

$$\hat{x} \;=\; 1.645\sqrt{n^{-1}\sum(X_j - \bar{X})^2} + \bar{X}$$

9.9 Suppose that $x_{1:n}$ and $x_{n:n}$ are the smallest and largest observed values of a random sample of size n from a distribution with pdf $f(x;\theta); 0 < \theta$

a) if $f(x;\theta) = 1$ for $\theta - 0.5 \le x \le \theta + 0.5$, zero otherwise, show that any value $\hat{(\theta)}$ such that $x_{n:n} - 0.5 \le \hat{(\theta)} \le x_{1:n} + 0.5$ is an ML estimate of θ.
solutions:$L(\theta) = 1$ for $\theta - 0.5 \le x_{1:n}$ and $x_{n:n} \ge \theta + 0.5$, and $l(\theta) = 0$ otherwise. Therefore, as long as $x_{n:n} - 0.5 \le \hat{\theta} \le x_{1:n} + 0.5$, we have an MLE.
b) if $f(x;\theta) = \tfrac{1}{\theta}$ for $\theta \le x \le 2\theta$, zero otherwise, show that any value $\hat{(\theta)} = 0.5x_{n:n}$ is an ML estimate of θ.
solutions:$L(\theta) = \tfrac{1}{\theta^n}$ if $(\theta) \le x_{1:n}$ and $x_{n:n} \le 2(\theta)$ or if$\tfrac{x_{n:n}}{2} \le (\theta) \le x_{1:n}$.$L(\theta)$ is maximized in this interval when(θ) is minimized, so we must have $\hat{\theta} = \tfrac{x_{n:n}}{2}$ as the MLE.

09.10 *Please check this solution for accuracy*
Before we start, the pdf of $X \sim DE(\eta, \theta)$ is given by $f(\eta,\theta) = \tfrac{1}{2\theta}e^{-\tfrac{|x-\eta|}{\theta}}$. With $E(X) = \eta$ and $Var(X) = 2\theta^2$. Of interest is the sum of the samples at a point a given by $\sum|x-a|$. We can take the derivative of this at all points save for $x = a$. This yields something along the lines of $\tfrac{\sum(x-a)}{|x-a|}$ which will be a sum of $+1$ and -1's. We should also be able to say that when a is the sample median that this should be 0 as well since the most cancelation occurs when this is true.

a) We want to maximize $f(\eta, 1) = \tfrac{1}{2}e^{-\tfrac{|x-\eta|}{1}}$. We will use the above fact to proceed.

$$L(\theta) = \prod \frac{1}{2}e^{-|x-\eta|}$$
$$\ell = \ln\left(\prod \frac{1}{2}e^{-|x-\eta|}\right)$$
$$= \sum\left(\ln 1 - \ln 2 - |x - \eta|\right)$$
$$\frac{d\ell}{d\eta} = \frac{\sum(x-\eta)}{|x-\eta|}$$

From above, we know then that we have a maximum when $\hat{\eta}$ is our sample median.

b) With θ not set, we have instead two variables to maximize:

$$L(0) = \prod \frac{1}{2\theta} e^{-\frac{|x-\eta|}{\theta}}$$

$$\ell = \ln\left(\prod \frac{1}{2\theta} e^{-\frac{|x-\eta|}{\theta}}\right)$$

$$= \sum \left(\ln 1 - \ln 2 - \ln \theta - \frac{|x-\eta|}{\theta}\right)$$

We will solve for θ first

$$\frac{d\ell}{d\theta} = \frac{-n}{\theta} + \frac{\sum |x-\eta|}{\theta^2} = 0$$

$$\hat{\theta} = \frac{\sum |x-\eta|}{n}$$

Now for $\hat{\eta}$ we follow the same logic as in part (a) and will find it again as to be the sample median.

09.15 a) We will replace the $\hat{p}$ with our $\frac{X}{n}$ then solve for c:

$$p(1-p) = E\left(c\hat{p}(1-\hat{p})\right)$$

$$= E\left(c\frac{X}{n}\left(1 - \frac{X}{n}\right)\right)$$

$$= E\left(c\left(\frac{X}{n} - \left(\frac{X}{n}\right)^2\right)\right)$$

$$= \frac{c}{n}\left(E(X) - E\left(\frac{X^2}{n}\right)\right)$$

$$= \frac{c}{n}\left(np - \left(\frac{npq + (np)^2}{n}\right)\right)$$

After a little work we get the following

$$= \frac{c}{n}\left(p(n-1)(1-p)\right)$$

Solving for c against $p(1-p)$ then we get

$$c = \frac{n}{n-1}$$

b) From part (a) we can see that $\frac{n}{n-1}\hat{p}(1-\hat{p})$ is pq. For $Var(X)$ we simply need npq so our solution, due to invariance, is $\frac{n^2}{n-1}\hat{p}(1-\hat{p})$

c) Since we have found estimator's for the mean and variance, we can use invariance to get estimators for the sample mean and sample variance. For p we see that it is $\frac{\sum^N \hat{p}}{N}$ while $Var(X) = \sum^N \frac{n^2}{n-1} \frac{\hat{p}(1-\hat{p})}{N}$

9.17 Let $X_1...X_n$ be a random sample from a uniform distribution $X_i \sim Unif(\theta - 1, \theta + 1$

a) Show that the sample mean $(\overline{X})$ is an unbiased estimator of θ

$E[\overline{X}] = \frac{[(\theta-1)+(\theta+1)]}{2} = \theta$

b) Show that the midrange $\frac{(X_{1:n}+X_{n:n})}{2}$ is an unbiased estimator of θ

$F_{n:n}(t) = \frac{(t-(\theta-1))^n}{2^n}$, so $\frac{d}{dt}F_{n:n}(t) = \frac{d}{dt}\frac{(t-(\theta-1))^n}{2^n} = \frac{n(t-\theta+1)^{n-1}}{2^n}$

$E[X_{n:n}] = \int_{\theta-1}^{\theta+1} \frac{t[n(t-\theta+1)^{n-1}]}{2^n}dt$,

Use the Convolution formula: $= \int_{\theta-1}^{\theta+1} \frac{n(t-\theta+1)^n}{2^n}dt + (\theta - 1)\int_{\theta-1}^{\theta+1} \frac{n(t-\theta+1)^{n-1}}{2^n}dt = \theta + \frac{2n}{n+1} - 1$

$E[X_{1:n}] = \theta - \frac{2n}{n+1} + 1, \quad E[\frac{X_{1:n}+X_{1:n}}{2}]$

09.19

9.21 Consider a random sample of size n from a Bernoulli distribution X_i is approximately $BIN(1,p)$

a) find the CRLB for the variances of unbiased estimators of p.

$$f(x;p) = p^x(1-p)^{1-x}$$

$$ln(f(x;p)) = xln(p) + (1-x)ln(1-p)$$

$$(ln(f(x;p)))' = \frac{x}{p} - \frac{1-x}{1-p} = \frac{x-p}{p(1-p)}$$

$$E(((ln(f(x;p)))')^2) = E((\frac{x-p}{p(1-p)})^2) = \frac{1}{(p(1-p))^2}E((x-p)^2)$$

$$= \frac{E(x^2 - 2px + p^2)}{(p(1-p))^2} = \frac{p(1-p) + p^2 - 2p^2 + p^2}{(p(1-p))^2} =$$

$$\frac{1}{p(1-p)}$$

$$CLRB = \frac{(p')^2}{n(\frac{1}{p(1-p)})} = \frac{p(1-p)}{n}$$

b) Find the CRLB for the variances of unbiased estimators of p(1-p).

$$(p(1-p))' = 1 - 2p$$

$$E(((ln(f(x;p)))')^2) = \frac{1}{p(1-p)}$$

$$CLRB = \frac{(1-2p)^2}{n(\frac{1}{p(1-p)})} = \frac{p(1-p)(1-2p)^2}{n}$$

c) Find a UMVUE of p.

$$Var(\bar{x}) = \frac{p(1-p)}{n}$$

This is the same as the CRLB for p, thus, $\bar{x}$ is a UMVUE of p

09.22 a) First we find $\tau(\theta)$, in our case it is μ so we compute $(\tau'(\theta))^2 = 1$. Next we turn our attention to the denominator of our CRLB:

$$f(x;\mu,9) = \frac{1}{\sqrt{2\pi 9}}e^{-\frac{(x-\mu)^2}{(2)(9)}}$$

$$\ln f = \ln\left(\frac{1}{\sqrt{2\pi 9}}\right) + -\frac{(x-\mu)^2}{(2)(9)}$$

$$\frac{d\ln f}{d\mu} = \frac{(x-\mu)}{(9)}$$

$$\frac{d^2\ln f}{d\mu^2} = -\frac{1}{9}$$

We would normally need to take the expectation here but a constant will yield a constant so we may simply continue and compute the rest of our CRLB.

$$Var(T) \geq \frac{1}{-n\left(\frac{-1}{9}\right)}$$

$$\geq \frac{9}{n}$$

Which is our CRLB.

b) Since $Var(\overline{X}) = \frac{\sigma^2}{n}$ we know that this MLE is a UMVUE since it is exactly our CRLB.

c) We have to find the 95th percentile first, $\Pr(X \leq x_{.95})$. Since we are normal, we just transform to a standard normal, then find the actual value using a table. $\Phi\left(\frac{x_{.95}-\mu}{3}\right)$ and then $\frac{x_{.95}-\mu}{3} = 1.645$ so $x_{.95} = 4.935 + \mu$. Due to invariance we can set our $\tau(\hat{\theta}) = 4.935 + \hat{\mu}$ where

$\hat{\mu} = \bar{X}$, our MLE of μ. The variance of $\hat{\theta}$ is clearly a UMVUE since it is the CRLB as well, this being due to the rules of variances having any constant zeroed out, leaving us with our normal μ.

09.23   09.26 Consider a random sample of size n from a distribution with pdf $f(x; \theta) = \frac{1}{\theta}$ if $0 < x \leq \theta$, and zero otherwise; $0 < \theta$.
(a) Find the MLE $\hat{\theta}$.
$$\hat{\theta} = X_{n:n}$$
(b) Find the MME $\tilde{\theta}$.
$$E[X] = \int_0^\theta \frac{1}{\theta} x \, dx = \frac{1}{\theta}\left(\frac{x^2}{2}\big|_0^\theta\right)$$
$$= \frac{\theta^2}{2\theta} = \frac{\theta}{2} = \bar{X}$$
$$\Rightarrow \hat{\theta} = 2\bar{X}$$
(c) Is the $\hat{\theta}$ unbiased?
No, because $E[\hat{\theta}] = \frac{n}{n+1}\theta$
(d) Is the $\tilde{\theta}$ unbiased?
Yes, because $E[\bar{\theta}] = \theta$
(e) Compare the MSEs of the $\hat{\theta}$ and $\tilde{\theta}$.
$$MSE(\hat{\theta}) = Var(\hat{\theta}) + \left(\frac{\theta n}{n+1} - \theta\right)^2$$
$$= \left[\frac{n}{n+2} - \left(\frac{n}{n+1}\right)^2 + \frac{1}{(n+1)^2}\right]\theta^2 = \frac{2\theta^2}{(n+1)(n+2)}$$
$$MSE(\tilde{\theta}) = Var(2\bar{X}) = 4Var(\bar{X}) = \frac{4}{n}Var(X_1) = \frac{4}{n}\frac{\theta^2}{12} = \frac{\theta^2}{3n}$$
So, $MSE(\hat{\theta})$ is smaller unless n=1.

9.28
28a

$$Var(\hat{\theta}_1) = Var(\bar{X}) = \frac{Var(X_1)}{n} = \frac{\theta^2}{n}$$

$$Var(\hat{\theta}_2) = \left(\frac{n}{n+1}\right)^2 Var(\bar{X}) = \left(\frac{n}{n+1}\right)^2 \frac{Var(X_1)}{n} = \frac{n\theta^2}{(n+1)^2}$$

28b

$$E(\hat{\theta}_1) = E(\bar{X}) = \theta$$

$$E(\hat{\theta}_2) = \frac{n\theta}{n+1}$$

Therefore it is shown that $\hat{\theta}_1$ is unbiased

$$bias(\hat{\theta}_2) = \frac{n\theta}{n+1} - \theta = -\frac{\theta}{n+1}$$

$$MSE(\hat{\theta}_1) = \frac{\hat{\theta}_{2)}}{n}, \text{ and } MSE(\hat{\theta}_2) = \frac{n\theta^2}{(n+1)^2} + \frac{\theta^2}{(n+1)^2} = \frac{\theta^2}{n+1}$$

28c

Since $\frac{n}{n+1} < 1$, then $Var(\hat{\theta}_1)$ will always be greater than $Var(\hat{\theta}_2)$

Hence there is no real need to compare the variances

28d

From Part(c), we know that $Var(\hat{\theta}_2)$ is always better than $Var(\hat{\theta}_1)$

Hence there is no real need to compare the MSEs

09.31   09.33   Solutions 9.34

34a

To find the MLE of p we need to calculate the likelihood function

$$L(p) = p^n (1-p)^{n\bar{X}}$$

$$lnL(p) = nlog(p) + n\bar{X}log(1-p)$$

$$\frac{\partial}{\partial p}lnL(p) = \frac{n}{p} - \frac{n\bar{X}}{1-p} \;=\; 0$$

$$\frac{1}{\hat{p}} = \frac{\bar{X}}{1-\hat{p}} \;\Rightarrow\; \hat{p} = \frac{1}{1+\bar{X}}$$

34b

To find the MLE of $\theta = \frac{1-\hat{p}}{\hat{p}}$ we use the principle of MLEs to say,

$$\hat{\theta} = \frac{1-\hat{p}}{\hat{p}} = \frac{1-(1+\bar{X})^{-1}}{(1+\bar{X})^{-1}} = \bar{X}$$

34c

Note: $\theta = \tau(p)$, in which $\tau(p) = \frac{1-p}{p}$, then $\tau'(p) = -\frac{1}{p^2}$

$$Var\left(\frac{\partial}{\partial p}lnL(p)\right) = \left(\frac{n}{1-p}\right)^2 Var(\bar{X}) = \frac{nVar(X_1)}{(1-p)^2} = \frac{n}{p^2(1-p)}$$

and that the variance of GEO(p) is $\frac{1-p)}{p^2}$ then the CRLB is

$$\frac{[\tau'(p)]^2}{n/(p^2(1-p))} = \frac{p^{-4}}{n/(p^2(1-p))} = \frac{1-p}{np^2}$$

34d

No, the MLE is not a UMVUE because the MLE is biased since

$$E(\hat{\theta}) = E(\bar{X}) = E(X_1) = \frac{1}{p} \neq \frac{1-p}{p} \;\forall\; 0 < p < 1$$

34e

No it is not consistent because the bias doesn't converge to zero as $n \to$ inf

In fact the bias$(\hat{\theta}) = \frac{1}{p} - \frac{1-p}{p} = 1$, always

34f

Since $E(\bar{X}) = \frac{1}{p}$ and $Var(\bar{X}) = \frac{Var(X_1)}{n} = \frac{1-p}{np^2}$ Then by CLT

$$\frac{\bar{X}-(1/p)}{\sqrt{(1-p)/(np^2)}} \to N(0,1)$$

Hence, $\sqrt{n}\left(\bar{X} - \frac{1}{p}\right) \to N\left(0, \frac{1-p}{p^2}\right)$

Problem 10.2 Consider a random sample of size n from a geometric distribution, $X_i \sim GEO(p)$. Show that $S = \sum_{i=1}^{n} X_i$ is sufficient for p.

$$S = \sum X_i$$

$$f(x) = pq^{x-1}$$

$$f(\underline{x};\theta) = \prod_{i=1}^{n} \theta(1-\theta)^{x_i-1} = \theta^n(1-\theta)^{\sum(x_i)-n}[1(x = 1, 2, ...)]$$

Therefore, $\sum X_i$ is sufficient.

Problem 10.3

Suppose that $X_1, ..., X_n$ is a random sample from a normal distribution, $X_i \sim N(0, \theta)$. Show that it does not depend on θ if $S = \sum X_i^2$.

$$S = \sum X_i^2$$

$$f(x) = \frac{1}{\sqrt{2\pi\theta^2}} e^{\frac{-1}{2\theta^2}(x_i)^2}$$

$$f(\underline{x}; 0, \theta) = \frac{1}{(2\pi\theta^2)^{\frac{n}{2}}} e^{\frac{-1}{2\theta^2}(\sum x_i)^2}[1(X \in \Re)]$$

Therefore, $\sum X_i^2$ is sufficient.

10.4

$S_1, ..., S_n$ is jointly sufficient if equation 10.2.1 is not dependent on θ

Equation 10.2.1 $\quad f_{X|S}(x_1, ..., x_n) = \begin{cases} \frac{f(x_1,...,x_n);\theta}{f_s(s;\theta)} & if \quad s(x_1, ..., x_n) = s, \\ 0 & , \quad o/w \end{cases}$

For a two parameter exponential with $\theta=1$ and $\eta = \eta$ then

$$f(t; \eta) = \begin{cases} e^{-(t-\eta)} & , \quad x > \eta \\ 0 & , \quad o/w \end{cases}$$

Hence,

Therefore $f(\mathbf{x}; \eta) = e^{-\sum(x_i-\eta)}I[X_{1:n} > \eta]$

Notice the largest η can ever be is $X_{1:n}$, hence $S = X_{1:n}$

The pdf of the $X_{1:n}$ is needed to compute equation 10.2.1

$$P(S \leq s) = 1 - (P(X_{1:n} > s))^n \quad \forall S$$

$$F(s) = \int_\eta^s e^{-(x-\eta)} dx \Rightarrow -e^{-(x-\eta)} \Big]_\eta^s \Rightarrow \begin{cases} 1 - e^{-(s-\eta)} & , \ s > \eta \\ 0 & , \ o/w \end{cases}$$

$$f_S(s) = \frac{d}{ds}(1 - (1 - F(s))^n) = n(1 - F(s))^{n-1} f_S(s) = ne^{-(s-\eta)} I[s > \eta]$$

By 10.2.1, $\quad f_{X|S}(x_1, ..., x_n) = \frac{f(x_1,...,x_n);\eta}{f_s(s;\eta)} I[X_{1:n} > \eta]$

$$= \frac{e^{-\sum(x_i-\eta)} I[X_{1:n} > \eta]}{ne^{-(s-\eta)} I[s > \eta]} I[X_{1:n} > \eta] = \frac{e^{-\sum(x_i)}}{ne^{-ns}} I[X_{1:n}] = \frac{e^{-\sum(x_i-s)}}{n} I[X_{1:n}]$$

Which is not dependent on η, therefore $S = X_{1:n}$ is sufficient.

Solutions 10.6

The joint pmf for $BIN(m_i, p)$

$$= f(x_1, ..., x_n; \theta) = \prod \binom{m_i}{x_i} p^{x_i} q^{m_i - x_i} I[x_i = 0, ..., m_i \ \forall \ i = 1, ...n]$$

$$= \prod \binom{m_i}{x_i} p^{\sum x_i} q^{\sum(m_i - x_i)} I[x_i = 0, ..., m_i \ \forall \ i = 1, ...n]$$

$$= \prod \binom{m_i}{x_i} p^{\sum x_i} q^{\sum m_i} \frac{1}{q^{\sum x_i}} I[x_i = 0, ..., m_i \ \forall \ i = 1, ...n]$$

$$= \prod \binom{m_i}{x_i} \frac{p^{\sum x_i}}{q^{\sum x_i}} q^{\sum m_i} I[x_i = 0, ..., m_i \ \forall \ i = 1, ...n]$$

$$= \prod \binom{m_i}{x_i} \left(\frac{p}{q}\right)^{\sum x_i} q^{\sum m_i} I[x_i = 0, ..., m_i \ \forall \ i = 1, ...n]$$

Let C$= \prod \binom{m_i}{x_i}$

$$= C \left(\frac{p}{q}\right)^{\sum x_i} q^{\sum m_i} I[x_i = 0, ..., m_i \ \forall \ i = 1, ...n]$$

Now let $C' = \prod \binom{m_i}{x_i} q^{\sum m_i}$

$$= C' \left(\frac{p}{q}\right)^{\sum x_i} I[x_i = 0, ..., m_i \ \forall \ i = 1, ...n]$$

Now let $S = \sum X_i$

$$= C' \left(\frac{p}{q}\right)^S I[x_i = 0, ..., m_i \ \forall \ i = 1, ...n]$$

By prime factorization, it is shown that $S = \sum X_i$ is sufficient.

10.7

The joint pmf for NB(r_i,p) is

$$f(x_1, ..., x_n; \theta) = \prod \binom{x_i - 1}{r_i - 1} p^{r_i} q^{x_i - r_i} I[x_i = r_1, r_1 + 1, ... \ \forall \ i \leq n]$$

$$= \prod \binom{x_i - 1}{r_i - 1} p^{\sum r_i} q^{\sum x_i} \frac{1}{q^{\sum r_i}} I[x_i = r_1, r_1 + 1, ... \ \forall \ i \leq n]$$

$$= \prod \binom{x_i - 1}{r_i - 1} \frac{p^{\sum r_i}}{q^{\sum r_i}} q^{\sum x_i} I[x_i = r_1, r_1 + 1, ... \ \forall \ i \leq n]$$

$$= \prod \binom{x_i - 1}{r_i - 1} \left(\frac{p}{q}\right)^{\sum r_i} q^{\sum x_i} I[x_i = r_1, r_1 + 1, ... \ \forall \ i \leq n]$$

Let C=$\prod \binom{x_i - 1}{r_i - 1} \left(\frac{p}{q}\right)^{\sum r_i}$

$$= C q^{\sum x_i} I[x_i = r_1, r_1 + 1, ... \ \forall \ i \leq n]$$

Therefore, by factorization criterion it is shown that S=$\sum X_i$ is sufficient

10.08
Consider a random sample of size n from a two-parameter exponential distribution, $X_i \ EXP(1, \eta)$.
Show that $S = X_{1:n}$ is sufficient for η by using the factorization criterion.
$f(\underline{x}; \theta) = g(s, \theta) h(\underline{x})$
$= e^{-\Sigma(x_i - n)} 1[x > \eta]$
$= e^{-\Sigma x_i} e^{n\eta} 1[X_{1:n} > \eta]$
Where $g(s; \theta) = e^{n\eta} 1[X_{1:n} > \eta]$
and $h(\underline{x}) = e^{-\Sigma x_i}$
Therefore, $S = X_{1:n}$ is sufficient.

10.17

17(a)

With a two parameter exponential EXP(1,η) the pdf is

$$f(x) = e^{-(x-\eta)} I[x > \eta]$$

$$= e^{-x} e^{\eta} I[x > \eta]$$

This is a range-dependent exponential. Following the range-dependent theorem, stating if $q_1(\theta)$ is increasing and $q_2(\theta)$ is decreasing then $T_1 = min[q_1^{-1}, q_2^{-1}]$ is a sufficient statistic. Therefore S=$X_{1:n}$ is complete and sufficient for η

17(b)

In order to prove $X_{1:n} - \frac{1}{n}$ is a UMVUE of η we need to show it is unbiased.
Note: If $X_i \sim$ iid f(x;θ), S is complete and sufficient, and T=t(s) is unbiased, then T is a UMVUE.

bias is given by $b(T) = E(T) - \tau\theta$

$E(X_{1:n}) = \eta + \frac{1}{n}$

$b(\eta) = \eta + \frac{1}{n} - \eta = \frac{1}{n}$

Therefore $X_{1:n} - \frac{1}{n}$ is an unbiased estimator and thus a UMVUE for η

17(c)

To find the pth percentile (x_p) in terms of η we need to solve

$$p = P(X_1 < x_p) = \int_\eta^{x_p} e^{-(x-\eta)}dx = \begin{cases} 1 - e^{-(x_p-\eta)} & if \ x_p > \eta \\ 0, & o/w \end{cases}$$

Therefore, $p = 1 - e^{-(x_p-\eta)} \ \Rightarrow \ e^{-(x_p-\eta)} = 1 - p$

Hence, $log(e^{-(x_p-\eta)}) = log(1-p) \ \Rightarrow \ -x_p + \eta = log(1-p)$

Therefore, $x_p = \eta - log(1-p)$

For the UMVUE of x_p, find a function of S that is unbiased.

It follows that $E(\hat{x}_p) = \eta - log(1-p)$

Using Part(b) we get $\hat{x}_p = X_{1:n} - \frac{1}{n} - log(1-p)$

10.18
10.21

21(a)

The pdf for BIN(1,p) is

$f(x,p) = p^x(1-p)^{1-x}I[x = 0,1]$

$\qquad = (1-p)\left(\frac{p}{1-p}\right)^x I[x = 0,1]$

$\qquad = (1-p)e^{log(\frac{p}{1-p})x}I[x = 0,1]$

This is the Regular Exponential Class

Which implies $\sum X_i$ is sufficient and complete for p.

It is known that if $S = \frac{1}{n-1}\sum(X_i - \bar{X})^2 = \frac{1}{n-1}\sum X_i^2 - \frac{n}{n-1}\sum \bar{X}^2$

then $E(S) = p(1-p)$. Since X can only be 0 or 1, then with $\sum X_i^2$, we have

$$S = \frac{1}{n-1}\sum X_i - \frac{n}{n-1}\bar{X}^2 \quad = \quad \frac{n}{n-1}\bar{X}(1-\bar{X})$$

Which is both unbiased and a complete and sufficient statistic.

Therefore S is the UMVUE for p(1-p).

21(b)

Using S from Part(a), we know that $E(S)=p(1-p) = p - p^2 = E(\bar{X}) - p^2$

Let $T= \bar{X}$ - S, then ET - p-p(1-p) = p^2.

Notice T is a function of $\sum X_i$ is a UMVUE for p^2

T can be simplified further,

$$T = \bar{X} - \frac{n}{n-1}\bar{X}(1-\bar{X}) = \bar{X}\left[1 - \frac{1-\bar{X}}{n-1}\right] = \bar{X}\left(\frac{n+\bar{X}}{n-1}\right)$$

10.24