Solutions to Bain and Engelhardt's Introduction to Probability and Mathematical Statistics

06.01

Given: the pdf of x $f_x(x) = \begin{cases} 4x^3 & , \quad 0 < x < 1 \\ 0 & , \quad o/w \end{cases}$

Find: PDF of a) $Y = X^4$

Setup: Use the CDF technique to get the CDF of Y in terms of a CDF of X
$F_Y(y) = P[Y \le y] = P[X^4 \le Y] = P[-y^{\frac{1}{4}} \le X \le y^{\frac{1}{4}}] = F_X(y^{\frac{1}{4}}) - F_X(-y^{\frac{1}{4}})$

Steps: i) Differentiate with respect to y to find an equation given in terms of the pdf of x:
$f_y(y) = \frac{d}{dy}F_X(y^{\frac{1}{4}}) - \frac{d}{dy}F_X(-y^{\frac{1}{4}}) = f_x(y^{\frac{1}{4}})\frac{d}{dy}y^{\frac{1}{4}} - f_x(-y^{\frac{1}{4}})\frac{d}{dy}-y^{\frac{1}{4}} = f_x(y^{\frac{1}{4}})\frac{y^{-\frac{3}{4}}}{4} - f_x(-y^{\frac{1}{4}})\frac{-y^{-\frac{3}{4}}}{4}$

ii) Plug in the original limits and function for the pdf of x, and compute the cdf for y

Result: $f_y(y) = \begin{cases} 4y^{\frac{3}{4}}\frac{1}{4y^{\frac{3}{4}}} & , \quad 0 < x < 1 \\ 0 & , \quad o/w \end{cases} = \begin{cases} 1 & , \quad 0 < x < 1 \\ 0 & , \quad o/w \end{cases}$

Find: PDF of b) $W = e^X$

Setup: Use the CDF technique to get the CDF of W in terms of a CDF of X
$F_W(w) = P[W \le w] = P[e^X \le W] = P[X \le lnW] = F_X(lnW)$

Steps: i) Differentiate with respect to w to find an equation given in terms of the pdf of x:
$f_w(w) = \frac{d}{dw}F_X(lnW)\frac{d}{dw}(lnw) = f_x(lnw)\frac{1}{w}$

ii) Plug in the original limits and function for the pdf of x, and compute the cdf for y

Result: $f_W(w) = \begin{cases} \frac{4(lnw)^3}{w} & , \quad 1 < w < e \\ 0 & , \quad o/w \end{cases}$

Find: PDF of c) $Z = \ln x$

Setup: Use the CDF technique to get the CDF of Z in terms of a CDF of X
$F_Z(z) = P[Z \le z] = P[lnx \le z] = P[X \le e^z] = F_X(e^z)$

Steps: i) Differentiate with respect to z to find an equation given in terms of the pdf of x:
$f_z(z) = \frac{d}{dz}F_X(e^z) = f_x(e^z)\frac{de^z}{dz}$

ii) Plug in the original limits and function for the pdf of x, and compute the cdf for y

Result: $f_Z(z) = \begin{cases} 4e^{4z} & , \quad -\infty \leq z < 0 \\ 0 & , \quad o/w \end{cases}$

Find: PDF of d) $U = (X - 0.5)^2$

Setup: Use the CDF technique to get the CDF of U in terms of a CDF of X
$F_U(u) = P[U \leq u] = P[(X-0.5)^2 \leq u] = P[|X-0.5| \leq u^{0.5}] = F_X(u^{1/2} + 1/2) - F_X(-u^{1/2} + 1/2)$

Steps: i) Differentiate with respect to u to find an equation given in terms of the pdf of x:
$f_U(u) = \frac{d}{du} F_X(u^{1/2} + 1/2) = f_x(u^{1/2} + 1/2) \frac{d}{du}(u^{1/2} + 1/2) - f_x(-u^{1/2} + 1/2) \frac{d}{du}(-u^{1/2} + 1/2)$
$f_x(u^{1/2} + 1/2)1/2u^{-1/2}) - f_x(-u^{1/2} + 1/2)1/2u^{-1/2}$

ii) INCOMPLETE

Result: $f_Z(z) = \begin{cases} 4e^{4z} & , \quad -\infty \leq z < 0 \\ 0 & , \quad o/w \end{cases}$

06.02

Given: $X \sim Unif(0, 1)$
Find: a) PDF of $Y = X^{1/4}$

Setup: $F_Y(y) = P[Y \leq y] = P[X^{1/4} \leq y] = P[X \leq y^4] = F_X(y^4)$

Steps: i) find the pdf of x. Because X is a Uniform distribution with parameters 1 and 0, the pdf, which for Unif(a,b) is 1/b-a where $a < x < b$. Here, Unif(0,1) gives 1/1-0 =1

ii) Differentiate with respect to y to find an equation given in terms of the pdf of x:
$f_Y(y) = \frac{d}{dy} F_X(y^4) = 4y^3$

Result: $f_Y(y) = \begin{cases} 4y^3 & , \quad 0 < y < 1 \\ 0 & , \quad o/w \end{cases}$

Find: b) PDF of $W = e^{-X}$

Setup: $F_W(z) = P[W \leq w] = P[e^{-X} \leq W] = P[-X \leq lnw] = P[X \geq -lnw] = 1 - F_x(-lnw)$

Steps: i) find the pdf of x. See part a) for an explanation of why it is 1 when $a < x < b$

ii) Differentiate with respect to w to find an equation given in terms of the pdf of x:
$f_W(w) = -\frac{d}{dw}F_X\frac{d}{dw}(-lnw) = -f_X(-lnw)\frac{-1}{w} \quad for \quad e^{-1} < w < 1 = -\frac{1}{w}$

Result: $f_W(w) = \begin{cases} \frac{1}{w} & , \quad e^{-1} < w < 1 \\ 0 & , \quad o/w \end{cases}$

Find: c) PDF of $Z = 1 - e^{-X}$

Setup: $F_Z(z) = P[Z \leq z] = P[1 - e^{-X} \leq z] = P[-e^{-X} \leq z - 1] = P[e^{-X} \geq 1 - z] = P[-X \geq ln(1-z)] = P[X \leq -ln(1-z)] = F_x(-ln(1-z))$

Steps: i) find the pdf of x. See part a) for an explanation of why it is 1 when $a < x < b$

ii) Differentiate with respect to w to find an equation given in terms of the pdf of x:
$f_Z(z) = -ln(1-z) = -\frac{-1}{1-z} = \frac{1}{1-z} \quad for \quad 0 < z < 1 - e^{-1}$

Result: $f_Z(z) = \begin{cases} \frac{1}{1-z} & , \quad 0 < z < 1 - e^{-1} \\ 0 & , \quad o/w \end{cases}$

Find: d) PDF of $U = X(1 - X)$

Setup: $F_U(u) = P[U \leq u] = P[X(1-x) \leq u] = P[-X^2 + X \leq u] = P[-(X-1/2)^2 \leq u - 1/4] = P[(X-1/2)^2 \geq 1/4 - u] == P[|(X-1/2)| \geq (1/4-u)^{1/2}] =$

Steps: i) find the pdf of x. See part a) for an explanation of why it is 1 when $a < x < b$

ii) INCOMPLETE:
$f_Z(z) = -ln(1-z) = -\frac{-1}{1-z} = \frac{1}{1-z} \quad for \quad 0 < z < e^{-1}$

Result: $f_W(w) = \begin{cases} \frac{1}{1-z} & , \quad 0 < z < e^{-1} \\ 0 & , \quad o/w \end{cases}$

06.03

Given: PDF $f_R(r) = \begin{cases} 6r(1-r) & , \quad 0 < r < 1 \\ 0 & , \quad o/w \end{cases}$

Find: Distribution of the circumference

Setup: The circumference is $c = 2\pi r$. We have the pdf in terms of x, so this is the transformation:
$F_C(c) = P[C \leq c] = P[2\pi r \leq c] = P[r \leq c/2\pi] = F_x(c/2\pi)$

Steps: i) Differentiate with respect to c to find an equation given in terms of the pdf of x.
$f_C(c) = \frac{d}{dc}F_R(c/2\pi) = f_R(c/2\pi)\frac{d}{dc}(c/2\pi) = f_R(c/2\pi)(1/2\pi)$

ii) Plug the original pdf back into this new form:
$f_C(c) = \frac{6c}{2\pi}(1-(c/2\pi))(1/2\pi) = \frac{6c(2\pi-c)}{(2\pi)^3} \quad if \quad 0 < c < 2\pi$

Result: $f_C(c) = \begin{cases} \frac{6c(2\pi-c)}{(2\pi)^3} & , \quad 0 < c < 2\pi \\ 0 & , \quad o/w \end{cases}$

Find: Distribution of the area

Setup: The area is $a = \pi r^2$ so the cdf $F_A(a) = P[A \leq a] = P[\pi r^2 \leq a] = P[r^2 \leq a/\pi]$
$= P[|r| \leq (a/\pi)^{1/2}] = P[-(a/\pi)^{1/2} \leq c \leq (a/\pi)^{1/2}] = F_R\left((a/\pi)^{1/2}\right) - F_R\left(-(a/\pi)^{1/2}\right)$

Steps: i) Differentiate with respect to a to find an equation in terms of the pdf of x.
$f_A(a) = \frac{d}{da}F_R\left((a/\pi)^{1/2}\right) - \frac{d}{da}F_R\left(-(a/\pi)^{1/2}\right) = f_R\left((a/\pi)^{1/2}\right)\frac{d}{da}(a/\pi)^{1/2} - f_R\left(-(a/\pi)^{1/2}\right)\frac{d}{da}\left(-(a/\pi)^{1/2}\right)$

Result: $f_A(a) = \begin{cases} \frac{3(\sqrt{\pi}-\sqrt{a})}{\pi^{3/2}} & , \quad 0 < a < \pi \\ 0 & , \quad o/w \end{cases}$

06.04 *Please double check the results of this solution*
For $X \sim WEI(\theta, \beta)$ we have the CDF as $F_X = 1 - e^{\frac{x}{\theta}^{\beta}}$ and the pdf is $f(x) = \frac{\beta}{\theta^\beta}x^{\beta-1}e^{-\frac{x}{\theta}^{\beta}}$
a) We make the transformation by the CDF method:

$$\begin{aligned}
\Pr(Y \leq y) &= \Pr\left(\frac{X}{\theta}^{\beta} \leq y\right) \\
&= \Pr\left(X \leq \theta y^{\frac{1}{\beta}}\right) \\
&= F_X\left(\theta y^{\frac{1}{\beta}}\right) \\
&= 1 - e^{-\frac{\theta y^{\frac{1}{\beta}}}{\theta}^{\beta}} \\
&= 1 - e^{-y}, \text{ where } 0 < y
\end{aligned}$$

So we have our CDF. For the pdf we simply take the derivative of the above. So $pdf = e^{-y}$ where $0 < y$

b) $W = \ln X$. Again, the most simply method to get the CDF, and in turn the pdf is the CDF method.

$$\begin{aligned}
\Pr(W \leq w) &= \Pr(\ln X \leq w) && (1) \\
&= \Pr(X \leq e^w) && (2) \\
&= F(e^w) && (3) \\
&= 1 - e^{-\frac{e^w}{\theta}^{\beta}} \text{ where } 0 < w && (4) \\
& && (5)
\end{aligned}$$

Again we simply differentiate to get the pdf. which turns out to be $\beta e^{\beta w}\theta^{-\beta}e^{-\frac{e^{w\beta}}{\theta}}$, $0 < w$
c)
06.10 Suppose X has pdf $f_X(x) = \frac{1}{2}e^{-|x|}$ for all real x.

(a) Find the pdf of $Y = |X|$.
CDF Method
$F_Y(y) = P[Y \leq y] = P[|x| \leq y] = P[-y \leq X \leq y] = F_X(y) - F_X(-y)$
$f_Y(y) = \frac{dF_X(y)}{dy} - \frac{dF_X(-y)}{dy}$
$f_Y(y) = f_X(y)\frac{dy}{dx} - f_X(-y)(\frac{-dy}{dy})$
$f_y = \frac{1}{2}e^{-y} + \frac{1}{2}e^{-y} = e^{-y}\ y > 0$

(b) Let $W = 0$ if $X \leq 0$ and $W = 1$ if $X > 0$. Find the CDF of W
$F_W(w) = P[W = 0] = \frac{1}{2}$
$F_W(w) = P[W = 1] = \frac{1}{2}$
$F_W(w) =$

$$\begin{cases} 0 & w \leq 0 \\ \frac{1}{2} & 0 \leq w \leq 1 \\ 1 & w > 1 \end{cases}$$

06.13 X has pdf

$$f(x) = \frac{x^2}{24} - 2 < x < 4 \qquad\qquad 0\text{otherwise}$$

We want pdf of the CDF $Y = X^2$ with regions: $(-2, 0) \cup [0, 4)$

$$[F_x(\sqrt{y}) - F_x(-\sqrt{y})] = \left[f_x(\sqrt{y})(\frac{1}{2}\sqrt{y}) - f_x(-\sqrt{y})(-\frac{1}{2}\sqrt{y})\right]$$

06.14

Given: Joint PDF $f(x, y) = \begin{cases} 4e^{-2(x+y)} & , \quad 0 < x < \infty, 0 < y < \infty \\ 0 \quad , & o/w \end{cases}$

Find: a) CDF of W=X+Y

Setup: $F_w(w) = P[W \leq w] = P[X + Y \leq w]$

Steps:

i) Express as a sum of probabilities, replace probabilities with binomials

ii) Simplify and Use Combinatorial Identity

Result: $\binom{n+m}{k}$

06.15 This is a simplified version of example 6.4.5.

$X_1, X_2 \sim POI(\lambda)$ so the MGF of both is $e^{\lambda(e^t-1)}$. Thus by theorem 6.4.4

$$M_Y(t) = e^{\lambda(e^t-1)}e^{\lambda(e^t-1)} = e^{2\lambda(e^t-1)} \sim POI(2\lambda)$$

The pdf then of Y is

$$f_Y(y) = \begin{cases} \frac{e^{-2\lambda}(2\lambda)^y}{y!} & y = 0, 1, 2, ... \\ 0 & otherwise. \end{cases}$$

06.16 Note: the pdf of $f_{x_1,x_2} = \frac{1}{x_1^2}\frac{1}{x_2^2}$
a) We need to find $f_{u,v} = f_{x_1,x_2}(x_1(u,v), x_2(u,v))|J|$ where J is our jacobian. First we let $u = x_1x_2$ and $v = x_1$ thus $x_1 = v$ and $x_2 = \frac{u}{v}$, now we can find J.

$$J = \begin{vmatrix} 0 & 1 \\ \frac{1}{v} & \frac{-u}{v^2} \end{vmatrix} = \frac{1}{v}$$

Finally, our pdf is:

$$\begin{aligned} f_{U,V}(u,v) &= f_{x_1,x_2}(v, \frac{u}{v})\left|\frac{1}{v}\right| \\ &= \frac{1}{v^2}\frac{1}{(\frac{u}{v})^2}\left|\frac{1}{v}\right| \\ &= \frac{1}{u^2v}, 1 < v < u < \infty \end{aligned}$$

b) We need to find $f_u(u)$ given $f_{U,V}(u,v) = \frac{1}{u^2v}, 1 < v < u < \infty$

$$\begin{aligned} f_u(u) &= \int_1^u \frac{1}{u^2v}dv \\ &= \frac{1}{u^2}\ln(v)\Big|_1^u \\ &= \frac{1}{u^2}(\ln(u) - 0) \\ &= \frac{1}{u^2}\ln(u), 1 < u < \infty \end{aligned}$$

06.17 Suppose that X_1 and X_2 are a random sample of size 2 from a gamma distribution, $X_i \sim \mathrm{GAM}(2, 1/2)$.

1. **Find the pdf of** $Y = \sqrt{X_1 + X_2}$;
2. **Find the pdf of** $W = X_1/X_2$**.**

[Solution] (a) Since X_1 and X_2 are independent to each other, the joint distribution density function is the product of $f(X_1)$ and $f(X_2)$, which is

$$f(X_1, X_2) = \begin{cases} \left(\frac{1}{\sqrt{2}\,\Gamma(1/2)}\right)^2 X_1^{-1/2} X_2^{-1/2-(X_1+X_2)/2} = \frac{1}{2\pi} X_1^{-1/2} X_2^{-1/2-(X_1+X_2)/2}, & \text{if } X_1, X_2 > 0; \\ 0, & \text{otherwise.} \end{cases}$$

If $x_1 = w$, then $x_2 = y^2 - w$, which implies that

$$J = \begin{vmatrix} 1 & 0 \\ -1 & 2y \end{vmatrix} = 2y.$$

Now,

$$f(w, y) = \begin{cases} \frac{1}{\pi} y w^{-1/2} (y^2 - w)^{-1/2-y^2/2}, & \text{if } y^2 > w > 0; \\ 0, & \text{otherwise.} \end{cases}$$

So, since $u = (y^2 - w)^{1/2}$ implies that $-2\,u = (y^2 - w)^{-1/2} w$, and $u = y \sin\theta$ implies that $u = y \cos\theta\theta$,

$$\begin{aligned} f_Y(y) &= \frac{1}{\pi} \int_0^{y^2} y^{-y^2/2} w^{-1/2} (y^2 - w)^{-1/2}\ w \\ &= \frac{-2}{\pi} \int_y^0 y^{-y^2/2} (y^2 - u^2)^{-1/2}\ u \\ &= \frac{2}{\pi} \int_0^y y^{-y^2/2} (y^2 - u^2)^{-1/2}\ u \\ &= y^{-y^2/2} \frac{2}{\pi} \int_0^y \frac{1}{\sqrt{y^2 - u^2}}\ u \\ &= y^{-y^2/2} \frac{2}{\pi} \int_0^{\pi/2} \frac{y \cos\theta}{\sqrt{y^2 - y^2 \sin^2\theta}}\ \theta \\ &= y^{-y^2/2} \frac{2}{\pi} \int_0^{\pi/2} \theta \\ &= y^{-y^2/2} \frac{2}{\pi} \frac{\pi}{2} \\ &= \begin{cases} y^{-y^2/2}, & \text{if } y > 0; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

(b) If $X_1 = z$,then $X_2 = z/w$, which implies that

$$J = \begin{vmatrix} 1 & 0 \\ 1/w & z/w^2 \end{vmatrix} = \frac{z}{w^2}.$$

Now,

$$f(z, w) = \begin{cases} \frac{1}{2\pi} \frac{z}{w^2} \left(\frac{z^2}{w}\right)^{-1/2\ -(z+z/w)/2}, & \text{if } z, w > 0; \\ 0, & \text{otherwise.} \end{cases}$$

So, for getting the marginal density function of w, we need to take the integral for the above joint density function with respect to z,

$$
\begin{aligned}
f_W(w) &= \int_0^\infty \frac{1}{2\pi} w^{-3/2\ -z(1+1/w)/2}\ z \\
&= -\frac{1}{2\pi}\frac{2}{1+1/w}{}^{-z(1+1/w)/2} w^{-3/2}\Big|_0^\infty \\
&= -\frac{1}{\pi(w+1)\sqrt{w}}{}^{-z(1+1/w)/2}\Big|_0^\infty \\
&= \begin{cases} \frac{1}{\pi(w+1)\sqrt{w}}, & \text{if } w > 0; \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
$$

6.18 It is given that X and Y have a joint pdf given by

$$f(x, y) = e^{-y} \quad if \quad 0 < x < y < \infty. \tag{6}$$

(a): Find the joint pdf of $S = X + Y$ and $T = X$.

This can be done using the joint transformation method. By rearranging the above formulas we get $X = T$ and $Y = S - T$. Then it is easy to get the jacobian

$$J = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \tag{7}$$

whose determinant is clearly one. Note that the order in which you take partial derivatives is unimportant provided you are consistent - you will get the same determinant either way. Then we substitute in $X = T$ and $Y = S - T$ into the pdf and multiply by the determinant of the jacobian:

$$f_{S,T}(s, t) = f_{X,Y}(x(s, t), y(s, t)) \times 1 = \begin{cases} e^{t-s} & if \quad 0 < t < s/2 \\ 0 & \qquad otherwise \end{cases} . \tag{8}$$

The bounds of the function can be found in a few different ways. One way is to consider the bounds of the original function, $0 < x < y < \infty$. We can substitute in the new formulas for X and Y to get

$$0 < t < s - t < \infty. \tag{9}$$

Then it is apparent that

$$0 < 2t < s < \infty, \tag{10}$$

which then yields

$$0 < t < s/2, \tag{11}$$

the bounds of our new function.

(b): Find the marginal pdf of T.

The easiest way to do this is to "integrate out" S from the joint pdf we derived:

$$\begin{aligned} f_T(t) &= \int_{-\infty}^{\infty} f_{S,T}(s,t)ds = \int_{2t}^{\infty} e^{t-s}ds \\ &= e^t \int_{2t}^{\infty} e^{-s}ds = e^t(-e^{-s}|_{2t}^{\infty}) \\ &= e^{-t} \quad if \quad t > 0. \end{aligned} \tag{12}$$

(c): Find the marginal pdf of S.

This is just like part (b), except this time "integrate out" T:

$$\begin{aligned} f_S(s) &= \int_{-\infty}^{\infty} f_{S,T}(s,t)dt = \int_{0}^{s/2} e^{t-s}ds \\ &= e^{-s} \int_{0}^{s/2} e^{t}dt = e^{-s}(e^t|_0^{s/2}) \\ &= e^{-s}(e^{s/2} - 1) \quad if \quad s > 0. \end{aligned} \tag{13}$$

6.21 Let X and Y be continuous random variables with a joint density function given by

$$f_{X,Y}(x,y) = 2(x+y) \quad if \quad 0 < x < y < 1 \quad and \quad 0 \quad otherwise. \tag{14}$$

(a) Find the joint density function of $S = X$ and $T = XY$.

We can solve for X and Y in terms of the new variables, to get $X = S$ and $Y = T/S$. Then the jacobian is given by

$$J = \begin{pmatrix} 1 & 0 \\ -T/S^2 & 1/S \end{pmatrix}. \tag{15}$$

Then the new pdf is given by

$$f_{T,S}(s,t) = f_{X,Y}(x(s,t), y(s,t)) \times |1/s| = \begin{cases} 2(s + t/s)\,|1/s|\,, & 0 < s^2 < t < s < 1 \\ 0 & otherwise. \end{cases} \tag{16}$$

The bounds of this equation can be interpreted in the following way: the old triangular region in the xy plane got transformed to the region in the st plane between the lines $T = S$ and $T = S^2$.

(b) Find the marginal pdf of T.

To find the marginal of T, s needs to be "integrated out."

$$\begin{aligned} f_T(t) &= \int_{-\infty}^{\infty} f_{S,T}(s,t)ds \\ &= \int_t^{\sqrt{t}} 2(s+t/s)\,|1/s|\,ds = \int_t^{\sqrt{t}} 2(1+t/s^2)ds \\ &= 2\int_t^{\sqrt{t}} ds + 2t\int_t^{\sqrt{t}} 1/s^2 ds = 2(\sqrt{t}-t) + 2t(-1/s|_t^{\sqrt{t}} \\ &= 2\sqrt{t} - 2t + 2 - 2\sqrt{t} = \begin{cases} 2-2t & t \in (0,1) \\ 0 & otherwise. \end{cases} \end{aligned} \tag{17}$$

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06.23 We will use the property that independant identically distributed random variables has the form of 6.4.4, $M_Y(t) = [M_X(t)]^n$ where $Y = X_1 + X_2 + ... + X_n$. then since $X_i \sim GEO(p)$

$$\begin{aligned} Mgf(Y) &= M_{X_1}(t)M_{X_2}(t)...M_{X_k}(t) \\ &= (M_X(t))^k \\ &= (\frac{pe^t}{1-qe^t})^k \sim NegativeBinomial(k,p) \end{aligned}$$

06.25 First note, X_1, X_2, X_3, X_4 are all independant, but they are not IID as only $X_2, X_3, X_4 \sim POI(5)$ with X_1 not being listed. So formula 6.4.5 does not hold. 6.4.4 does though.
A)

$$\begin{aligned} Mgf(Y) &= M_{X_1}(t)M_{X_2+X_3+X_4}(t) \\ &= M_{X_1}(t)(M_{X_i}(t))^3 \end{aligned}$$

Since X_2, X_3, X_4 are iid 6.4.5 holds for moving to this mgf

$$\begin{aligned} &= M_{X_1}(t)(e^{\mu(e^t-1)})^3 \\ &= M_{X_1}(t)e^{3\mu(e^t-1)} \\ &= M_{X_1}(t)e^{15(e^t-1)} \\ e^{25(e^t-1)} &= M_{X_1}(t)e^{15(e^t-1)} \\ \frac{e^{25(e^t-1)}}{e^{15(e^t-1)}} &= M_{X_1}(t) \\ e^{10(e^t-1)} &= M_{X_1}(t) \sim POI(10) \end{aligned}$$

B) For $W = X_1 + X_2$ we have $X_1 \sim POI(10)$ and $X_2 \sim POI(5)$. So $POI(10+5) = POI(15)$
06.29

Given: PDF $f(x) = \begin{cases} \frac{1}{x^2} & , \quad 1 \le x < \infty, 0 < y < \infty \\ 0 & , \quad o/w \end{cases}$

Find: a) Joint PDF of the order statistics

Setup: $F_w(w) = P[W \le w] = P[X + Y \le w]$

Steps: i) Differentiate with respect to a to find an equation in terms of the pdf of x.
$f_A(a) = \frac{d}{da} F_R(a/\pi)^{1/2} - \frac{d}{da} F_R - (a/\pi)^{1/2} = f_R[(a/\pi)^{1/2}] \frac{d}{da} (a/\pi)^{1/2} f_R[-(a/\pi)^{1/2}] \frac{d}{da} - (a/\pi)^{1/2}$

ii) Simplify and Use Combinatorial Identity

Result: $\binom{n+m}{k}$

Find: b) PDF of the smallest order statisticY_1

Setup:

Steps: i)

Result:

Find: c) PDF of the largest order statistic Y_n

Setup:

Steps: i)

Result:

Find: d) PDF of the sample range $R = Y_n - Y_1, \quad forn = 2$

Setup: The area is $a = \pi r^2$ so the cdf $F_A(a) = P[A \le a] = P[\pi r^2 \le a] = P[r^2 \le a/\pi]$
=P$[|r| \le (a/\pi)^{1/2}] = P[-(a/\pi)^{1/2} \le c \le (a/\pi)^{1/2}] = F_r(a/\pi)^{1/2} - F_r - (a/\pi)^{1/2}$

Steps: i)

Result:

Find: e) PDF of the sample median
$R = Y_r - Y_1, \quad fornoddsothatr = (n+1)/2$

Setup:

Steps: i)

Result: [11pt]article
a) The PDF of the smallest order statistic is provided by formula 6.5.4
$g_1(y_1) = n[1 - F(y_1)]^{n-1} f(y_1)$
In this case we have $g_1(y_1) = n(1 - (1 - e^{-y})e^y$ when $y_1 > 0$
b)The PDF of the largest order statistic is provided by formula 6.5.6
$g_n(y_n) = n[F(y_n)]^{n-1} f(y_n)$
$g_n(y_n) = n[1 - e^{-y_n}]^{n-1} e^{y_n}$ Simplifying provides
$ne^{-y_n}(1 - e^{-_n})^{n-1}$ when $y_n > 0$ c) Because the exponential distribution has the memoryless property, the difference between the first order statistic and the greatest order statistic won't be conditional on the value of the first order statistic (so we can treat it as zero). The probability that all the other observations $(n-1)$ fall into the range is $P(R < r) \in (0, r)$ So $P(R < r) = [\int_0^r e^{-x} dx)^{n-1} = (1 - e^r)^{n-1}$ This is P(R¡r) which is the CDF, differentiate to get the PDF: $(n-1)(1 - e^{-r})^{n-2}\ e^{-r}$