Curvature energies: From Steiner to Willmore and beyond

Colloquium Universitat de Valencia

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Closed Soap Bubbles

- **A soap bubble wants to be round!**
  - The physical system minimises

$$\text{Energy} = \text{surface tension coefficient} \times \text{surface area}.$$

  - It solves the **isoperimetric problem**: Find the closed surface, that minimises surface area, given fixed enclosed volume.
  - The solution is a round sphere.
- What are properties of minimisers? We first consider this question for domains in the plane.

Solution to the isoperimetric problem in the plane

Theorem (Classical)

If a smooth domain $\Omega \subset \mathbb{R}^2$ minimises the ratio

$$\mathcal{R}(\Omega) = \frac{Length(\partial\Omega)^2}{Area(\Omega)},$$

then $\partial\Omega$ has constant curvature.

Proof.

- From a parametrisation γ_0 of $\partial\Omega$, start **Curve Shortening Flow** (heat equation for curves).
    - $$\begin{aligned} &\gamma\colon [0,T) \times \mathbb{S}^1 \to \mathbb{R}^2, \quad \gamma(0,\cdot) = \gamma_0, \\ &\qquad \partial_t \gamma(t,x) = |\partial_x\gamma|^{-2}(\partial^2_{xx}\gamma)^\perp \equiv \kappa(t,x)\nu(t,x), \end{aligned}$$

      where ν is the inward normal and $\kappa(t,\cdot)$ the curvature of $\gamma(t,\cdot)$.

Solution to the isoperimetric problem in the plane

- Need the variations of A and L.
  - $$\partial_t L(\partial\Omega_t) = \partial_t \int_0^{2\pi} |\partial_x \gamma| = \int_0^{2\pi} \frac{\langle \partial_x(\kappa\nu), \partial_x\gamma\rangle}{|\partial_x\gamma|} = -\int_{\gamma_t} \kappa^2.$$
  - Similarly, Gauss-Bonnet implies $\partial_t A(\Omega_t) = -\int_{\gamma_t}\kappa = -2\pi$.
- Hölder's inequality and Gauss-Bonnet imply

$$\begin{aligned} 0 \le A\partial_t\mathcal{R}(\Omega_t)_{|t=0} = \left(2L\partial_t L - \frac{L^2}{A}\partial_t A\right)_{|t=0} &= -2L\int_{\gamma_0}\kappa^2 + 2\pi\mathcal{R}(\Omega) \\ &\le -2\left(\int_{\gamma_0}\kappa\right)^2 + 8\pi^2 \\ &= 0, \end{aligned}$$

since Ω minimises $\mathcal{R}$, i.e. $\mathcal{R}(\Omega) \le \mathcal{R}(\text{unit ball}) = 4\pi$.

  - The equality case in Hölder's inequality implies $\kappa = \text{const}$.
  - It is an exercise in ODEs to show that it must be a circle.

Surfaces

- Given a local parametrisation of a surface,

$$F \colon U \to \mathbb{R}^3.$$

- The **second fundamental form** encodes how $F(U)$ is bent in $\mathbb{R}^3$:
  - If (e_1, e_2) is a orthonormal basis of the tangent plane at $p \in F(U)$, then it is defined as

$$h(e_i, e_j) = \left\langle D^2 F(e_i, e_j), e_3 \right\rangle$$

    and the **principal curvatures at** p are the eigenvectors $\kappa_i(p)$, where $i = 1, 2$.
  - The functions κ_i are continuous, but NON-SMOOTH in general.
  - NOTE: We need a bilinear form to capture curvature in all directions.

Mean curvature and Gauss curvature

- We are interested in smooth **curvature invariants**, for example
  - The **mean curvature**
  $$H = \kappa_1 + \kappa_2,$$
  - The **Gauss curvature**
  $$K = \kappa_1 \kappa_2.$$
  - Retrospectively, H and K contain all information on curvature, because when $\kappa_1 \leq \kappa_2$, then
  $$2\kappa_2 = (\kappa_1 + \kappa_2) + (\kappa_2 - \kappa_1) = H + \sqrt{H^2 - 4K}.$$

Alexandrov's theorem

- In $\mathbb{R}^3$, every solution to the isoperimetric problem has constant H.
  - The proof is very similar to the curve case seen earlier, using mean curvature flow instead:
$$\partial_t F = H\nu.$$
  - Contrary to the planar case, it took more than 100 years of partial results, before the following question was finally settled:
- **Is every closed, embedded constant mean curvature hypersurface of $\mathbb{R}^3$ a sphere?** (CMC problem)
- Answer: **YES!** (Alexandrov, 1962)
- Observation: CMC is a local property, but we obtain a global result.
- This means:
  - If a person can determine the curvature of a region of area A,
  - and the earth has area E,
  - then
$$N \geq \frac{E}{A}$$
humans can determine whether the earth is a round sphere.

The question of stability

- This CMC test would not lead anywhere in practice, because the earth is bumpy.
- Instead, what if everybody measures values close to, say, $H_0 \in \mathbb{R}$?
  - Suppose for some $\delta > 0$:

$$H_0 - \delta \leq H \leq H_0 + \delta.$$

  - Can we conclude that

$$\mathrm{dist}(\mathrm{Earth}, S) \leq C\epsilon$$

    for a round sphere S, a constant C and an error ϵ with

$$\lim_{\delta \to 0} \epsilon(\delta) = 0 \ ?$$

- In other words:

The question of stability

Is every closed, embedded "almost CMC" hypersurface of $\mathbb{R}^3$ "close" to a sphere?

- Answer: **Depends on the meaning of "almost" and "close"**.
- Magnanini/Poggesi (2019): "Yes", if
  - "close" $\sim$ Hausdorff-close,
  - "almost" $\sim$ In L^1-sense,
  - C is allowed to depend on an inradius bound.
- A similar result is due to Ciraolo and Vezzoni:

The question of stability

Theorem (Giulio Ciraolo and Luigi Vezzoni)

A sharp quantitative version of Alexandrov's theorem via the method of moving planes, J. Eur. Math. Soc. **20** (2018), no. 2, 261–299.

Let Ω be a smooth domain with connected boundary, then $\partial\Omega$ **lies within an annulus of thickness** *$C\,\mathrm{osc}(H)$. C depends on $|\partial\Omega|$ and a lower bound for interior and exterior balls.*

Integral approach

- Reilly's identity: For C^2-functions f on $\Omega \subset \mathbb{R}^3$, with $f_{|\partial\Omega} = \text{const}$:

$$\int_\Omega (\Delta f)^2 - \int_\Omega |\nabla^2 f|^2 = \int_{\partial\Omega} H(\partial_\nu f)^2.$$

  - Defining the Cauchy-Schwarz-deficit by

$$|\mathring{\nabla}^2 f|^2 = |\nabla^2 f|^2 - \tfrac{1}{3}(\Delta f)^2,$$

    we obtain

$$\int_\Omega |\mathring{\nabla}^2 f|^2 = \frac{2}{3}\int_\Omega (\Delta f)^2 - \int_{\partial\Omega} H(\partial_\nu f)^2.$$

- Solve

$$\begin{aligned} \Delta f &= 1 \quad \text{in } \Omega \\ f &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

Integral approach II

- then

$$\begin{aligned}\int_\Omega |\mathring{\nabla}^2 f|^2 &= \frac{2}{3\operatorname{vol}(\Omega)}\left(\int_\Omega \Delta f\right)^2 - \int_{\partial\Omega} H(\partial_\nu f)^2\\ &\leq \frac{2}{3}\frac{\operatorname{Area}(\partial\Omega)}{\operatorname{vol}(\Omega)}\int_{\partial\Omega}(\partial_\nu f)^2 - \int_{\partial\Omega} H(\partial_\nu f)^2\\ &\equiv \int_{\partial\Omega}(H_0 - H)(\partial_\nu f)^2.\end{aligned}$$

“Almost spherical”-type theorem

Theorem (2021, “Level-Set Stability”)

Stability from rigidity via umbilicity, Adv. Calc. Var. (2024), doi:10.1515/acv-2023-0119

Let $n \geq 2$ and $\Omega \Subset \mathbb{R}^{n+1}$ be a domain with connected C^2-boundary and let $f \in C^2(\bar{\Omega})$ satisfy

$$f_{|\partial\Omega} = 0, \quad \nabla f_{|\partial\Omega} \neq 0.$$

Then there exist constants $\alpha = \alpha(n)$ and

$$C = C\left(n, \frac{\mathrm{Area}(\partial\Omega)^{\frac{1}{n}} \|\nabla^2 f\|_{\infty,\Omega}}{\min_{\partial\Omega}|\nabla f|}\right),$$

such that

$$\mathrm{dist}(\partial\Omega, S) \leq C\,\mathrm{Area}(\partial\Omega)^{\frac{1}{n}} \left(\frac{1}{\min_{\partial\Omega}|\nabla f|^{n+1}} \int_\Omega |\mathring{\nabla}^2 f|^{n+1}\right)^{\alpha}.$$

Some words about the proof

- If h is the 2^{nd} fundamental form of $\partial\Omega$, then

$$\nabla^2 f_{|T\partial\Omega} = -|\nabla f| h.$$

- A further calculation shows

$$|\nabla f|^2 |A^o|^2 = |\mathring{\nabla}^2 f_{|T\partial\Omega}|^2 - \tfrac{1}{n+1}(\mathring{\nabla}^2 f(\nu,\nu))^2.$$

  - A^o is the tracefree part of the Weingarten operator A,

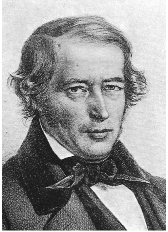
$$|A^o|^2 = c_n \sum_{i<j} (\kappa_i - \kappa_j)^2.$$

- Co-area formula: $\|\mathring{\nabla}^2 f\|_{L^{n+1}(\Omega)}$ controls $\|A^o\|_{L^{n+1}(\partial\Omega)}$.
- We will see later, how the proof is completed.

Geometric meaning of H and K

- **Steiner formula**: For convex $\Omega \subset \mathbb{R}^3$, the unit ball B and all $\epsilon \geq 0$:

$$\mathrm{vol}(\bar{\Omega} + \epsilon B) = W_0(\Omega) + 3W_1(\Omega)\epsilon + 3W_2(\Omega)\epsilon^2 + W_3(\Omega)\epsilon^3.$$



Jakob Steiner, Swiss Mathematician, 1796-1863

Geometric meaning of H and K

- The $W_k(\Omega)$ are called "quermassintegrals of Ω" and if $\partial\Omega$ is twice differentiable, they are

$$W_0(\Omega) = \mathrm{vol}(\Omega), \quad W_1(\Omega) = \tfrac{1}{3}\,\mathrm{Area}(\partial\Omega),$$

$$W_2(\Omega) = \frac{1}{6}\int_{\partial\Omega} H = \frac{1}{3}\int_{\partial\Omega} uK = \frac{1}{3}\int_{\mathbb{S}^2} u = \frac{1}{6}\int_{\mathbb{S}^2} (u(z) - u(-z))\, dz$$

$$W_3(\Omega) = \frac{1}{3}\int_{\partial\Omega} K = \frac{1}{3}\int_{\mathbb{S}^2} 1\, dz = \frac{4\pi}{3}.$$

- So up to constants, W_2 is the mean width of Ω and W_3 is the surface area of the sphere.

Willmore Energy

- **The Willmore energy** does not arise from this construction, but has a special role due to its conformal invariance and its application to mathematical biology.

$$\mathcal{W} = \frac{1}{4}\int_{\partial\Omega} H^2.$$

- From the Gauss-Bonnet theorem,

$$\begin{aligned} 2\pi\chi(\partial\Omega) = \int_{\partial\Omega} K &= \frac{1}{2}\int_{\partial\Omega}\kappa_1\kappa_2 + \frac{1}{2}\int_{\partial\Omega}\kappa_1\kappa_2 \\ &\leq \frac{1}{4}\int_{\partial\Omega}(\kappa_1^2+\kappa_2^2) + \frac{1}{2}\int_{\partial\Omega}\kappa_1\kappa_2 \\ &= \frac{1}{4}\int_{\partial\Omega} H^2. \end{aligned}$$

  - If $\partial\Omega$ is a topological sphere, then, with equality iff $\partial\Omega$ is round,

$$\mathcal{W} = \frac{1}{4}\int_{\partial\Omega} H^2 \geq 4\pi.$$

Willmore conjecture

Willmore conjecture (1965): If Σ is a surface of genus 1, then

$$\mathcal{W}(\Sigma) \geq 2\pi^2$$

with equality iff Σ is the embedded torus

$$(u, v) \mapsto ((\sqrt{2} + \cos u)\cos v, (\sqrt{2} + \cos u)\sin v, \sin u).$$

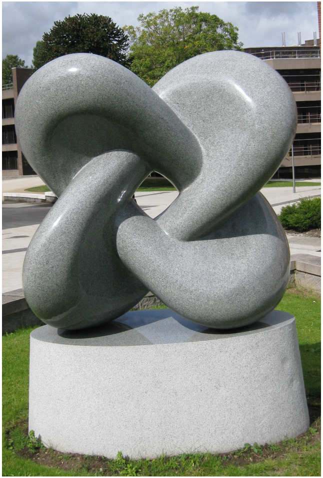
Thomas Willmore, English Geometer, 1919-2005

Theorem (Fernando C. Marques and André Neves)

Min-Max theory and the Willmore conjecture, Ann. Math. **179** (2014), no. 2, 683–782.

The Willmore conjecture holds true!

A Willmore torus



$\Delta H + (\frac{1}{4}H^2 - K)H = 0$, Durham University

Quermassintegral inequalities

- Recall **Steiner's formula**: For convex $\Omega \subset \mathbb{R}^3$ and the unit ball B:

$$\mathrm{vol}(\bar{\Omega} + \epsilon B) = W_0(\Omega) + 3W_1(\Omega)\epsilon + 3W_2(\Omega)\epsilon^2 + W_3(\Omega)\epsilon^3,$$

where $W_0(\Omega) = \mathrm{vol}(\Omega)$, $W_1(\Omega) = \frac{1}{3}\,\mathrm{Area}(\partial\Omega)$,

$$W_2(\Omega) = \frac{1}{6}\int_{\partial\Omega} H, \quad W_3(\Omega) = \frac{4\pi}{3}.$$

- **Classical quermassintegral inequalities:**

$$\frac{W_{k+1}(\Omega)}{W_{k+1}(B)} \geq \left(\frac{W_k(\Omega)}{W_k(B)}\right)^{\frac{2-k}{3-k}}, \quad 0 \leq k \leq 2,$$

with equality iff Ω is a ball (rigidity). For $k = 0$ this is the isoperimetric inequality.

- We also recall, with equality iff Ω is a ball,

$$\int_{\partial\Omega} H^2 \geq 16\pi$$

Quermassintegral inequalities

- Due to the characterisation of the equality case, we can ask what happens in the **almost equality case**.
  - Must $\partial\Omega$ be **close** to a sphere?
- For example, suppose
$$\int_{\partial\Omega} H^2 \leq 16\pi + \delta.$$
- Then
$$16\pi + \delta \geq \int_{\partial\Omega} (\kappa_1^2 + \kappa_2^2 - 2\kappa_1\kappa_2) + 4\int_{\partial\Omega} K = \int_{\partial\Omega} (\kappa_1 - \kappa_2)^2 + 16\pi.$$
- The quantity
$$\|A^0\|^2_{L^2(\partial\Omega)} := \int_{\partial\Omega} (\kappa_1 - \kappa_2)^2$$
is at the heart of all of our stability investigations.
  - In the above example, we have calculated that
$$\|A^0\|^2_{L^2(\partial\Omega)} < \delta.$$

The umbilicity theorem

- What does A^0 have to do with proximity to a sphere?
- A very classical result from hypersurface theory, due to

Jean Gaston Darboux, 1842-1917

says that

$$\kappa_1 = \kappa_2 \text{ throughout } \partial\Omega \quad \Rightarrow \quad \partial\Omega = \text{Sphere.}$$

- Key: For *this* rigidity result, stability versions are available, for example due Camillo De Lellis & Stefan Müller, Peter Topping and ...

Proof of the umbilicity theorem

- By assumption we have for some function f

$$h^i_j = f(x)\delta^i_j.$$

- Tracing this, we deduce $f = H/2$.
- First prove the mean curvature is constant. Differentiate:

$$\partial_i H = \partial_i h^k_k = \nabla_k h^k_i = \tfrac{1}{2}\partial_k H \delta^k_i = \tfrac{1}{2}\partial_i H,$$

  which implies $\nabla H = 0$. Hence $\lambda := f$ is a constant.
- Now let F be the embedding vector of the surface. Then

$$\partial_i(\nu - \lambda F) = h^k_i \partial_k F - \lambda \delta^k_i \partial_k F = 0.$$

- So $\nu - \lambda F =: -\lambda p$ is a constant vector and we finally deduce

$$|F - p| = 1/\lambda,$$

  and hence F has constant distance to the point p, and must be a piece of a sphere.

An almost-umbilicity theorem

For *this* rigidity result, stability versions are available, for example due Camillo De Lellis & Stefan Müller, Peter Topping and ...

Theorem (Antonio De Rosa, Stefano Gioffré)

Absence of bubbling phenomena for non-convex anisotropic nearly umbilical and quasi-Einstein hypersurfaces, J. Reine Angew. Math. **780** (2021), 1–40.

Let $M \subset \mathbb{R}^{n+1}$ be a closed hypersurface, $p > n$ and

$$|M| = |\mathbb{S}^n|, \quad \|A\|_{L^p(M)} \leq c_0.$$

Then there exist $C = C(n, p, c_0) > 0$, such that: if $\|A^0\|_{L^p(M)} \leq 1/C$, then there exists $c \in \mathbb{R}^{n+1}$, such that $M - c$ is a graph over the sphere,

$$\psi \colon \mathbb{S}^n \to M, \quad \psi(x) = e^{f(x)}x + c,$$

$$\|f\|_{W^{2,p}(\mathbb{S}^n)} \leq C\|A^0\|_{L^p(M)}.$$

Application of Almost-Umbilicity - Quermassintegral inequalities

Theorem (Stability in the non-convex Minkowski inequality)

Stability from rigidity via umbilicity, Adv. Calc. Var. (2024), doi:10.1515/acv-2023-0119

Let $M = \partial\Omega \subset \mathbb{R}^3$ be a closed and starshaped hypersurface with $H > 0$. Then there exists a constant $C > 0$, such that

$$\operatorname{dist}(M, \partial B_R) \leq CR\left(\left(\frac{\int_{\partial\Omega} H}{8\pi}\right)^2 - \frac{|\partial\Omega|}{4\pi}\right)^\alpha$$

for a suitable ball B_R.

Application of Almost-Umbilicity - Quermassintegral inequalities

- **Proof Ansatz** (for simplicity focus on $k = 1$): We use the rescaled **inverse mean curvature flow**

$$\partial_t x = \left(\frac{2}{H} - \langle x, \nu \rangle\right) \nu,$$

  where $x \colon [0, \infty) \times \mathbb{S}^2 \to \mathbb{R}^3$.
  - Smooth convergence to the unit sphere settled by Claus Gerhardt and John Urbas in the 90's.
  - Flow preserves surface area ($= W_1$) and decreases $\int_{\partial\Omega} H$ ($= W_2$).
- Along this flow, it is possible to directly relate $\|A^0\|_{L^2(M)}$ to $W_2^2 - W_1$.
- Then use Almost-Umbilicity.

¡Muchas Gracias!