

# Stability for anisotropic curvature functionals

Julian Scheuer (Goethe University Frankfurt)  
joint work with Xuwen Zhang (Xiamen/Frankfurt)

**Horní Lysečiny “Integral”**

**Isoperimetric problem:** Determine properties of area minimising surface, given volume constraint.

Round spheres in  $\mathbb{R}^{n+1}$  are unique closed minimisers of

$$\mathcal{R}(\Omega) = \frac{\text{Area}^{\frac{n+1}{n}}(\partial\Omega)}{\text{Volume}(\Omega)}$$

Standard variational methods:

- ▶ Minimisers of  $\mathcal{R}$  have **constant mean curvature**

$$H = \text{tr}(A) = \sum_{i=1}^n \kappa_i$$

$(\kappa_i)$  are eigenvalues of the Weingarten map  $A$ , principal curvatures.

# Alexandrov's theorem

Is a closed embedded constant mean curvature (CMC) hypersurface of  $\mathbb{R}^{n+1}$  necessarily a sphere?

Answer: **YES!** (Alexandrov<sup>1</sup>)

- ▶ Proof: Reflection across moving planes and the maximum principle.
- ▶ We are going to see another elegant proof today.

Relaxed CMC condition: Suppose for some  $\delta > 0$ , on a hypersurface  $M$

$$n - \delta \leq H \leq n + \delta.$$

Can we conclude

$$\text{dist}(M, S) \leq C\epsilon$$

for the unit sphere  $S$ , a constant  $C$  and where

$$\lim_{\delta \rightarrow 0} \epsilon(\delta) = 0 ?$$

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<sup>1</sup>A characteristic property of spheres, Ann. Mat. Pura Appl. **58** (1962), no. 4, 303–315.

# The question of stability

## Theorem (Giulio Ciraolo and Luigi Vezzoni)

*A sharp quantitative version of Alexandrov's theorem via the method of moving planes, J. Eur. Math. Soc. **20** (2018), no. 2, 261–299.*

*Let  $\Omega$  be a smooth domain with connected boundary, then  $\partial\Omega$  **lies within an annulus of thickness  $C \operatorname{osc}(H)$** .  $C$  depends on  $|\partial\Omega|$  and a lower bound for interior and exterior balls.*

Generalization to spaceforms and other curvature functions

$$F = F(\kappa_i)$$

was given by Ciraolo/Roncoroni/Vezzoni.<sup>2</sup>

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<sup>2</sup>*Quantitative stability for hypersurfaces with almost constant curvature in space forms, Ann. Mat. Pura Appl. **200** (2021), no. 5, 2043–2083.*

# The question of stability

## Theorem (Rolando Magnanini and Giorgio Poggesi)

*On the stability for Alexandrov's soap bubble theorem*, J. Anal. Math. **139** (2019), no. 1, 179–205.

*Let  $\Omega$  be a smooth domain with connected boundary, then  $\partial\Omega$  lies within an annulus of thickness at most  $C\|H - H_0\|_{L^1(\partial\Omega)}^{\tau_n}$ , where*

$$H_0 = \frac{n}{n+1} \frac{|\partial\Omega|}{|\Omega|},$$

*$\tau_n$  is a dimensional constant and  $C$  depends on few geometric quantities, such as interior and exterior ball conditions.*

# Ambient anisotropic geometry

- $\mathcal{W}$  (aka Wulff shape) smooth boundary of convex body  $\mathcal{W}_0$  containing the origin.

- ▶ Minkowski norm

$$F^0(x) = \inf\{s > 0: x \in s\mathcal{W}_0\}$$

- ▶ Then  $\mathcal{W} = \{F^0 = 1\}$ .

- $q(x) = \frac{1}{2}(F^0(x))^2$  is smooth, convex and hence

$$\bar{g}(x) := D^2q(x)$$

is a metric on  $\mathbb{R}^{n+1} \setminus \{0\}$ .

- Let  $F(z) = \sup_{x \in \mathcal{W}} \langle x, z \rangle$  be the support function of  $\mathcal{W}$ ,
  - ▶  $\Phi = (D^\#F)|_{\mathbb{S}^n}: \mathbb{S}^n \rightarrow \mathcal{W}$  is an embedding of the Wulff shape.

# Induced anisotropic geometry

- Let  $x: M^n \rightarrow \mathbb{R}^{n+1}$  embedding with  $M$  closed manifold,  $\tilde{\nu}: M \rightarrow \mathbb{S}^n$  its normal vector field aka Gauss map.
  - ▶ *Anisotropic normal*  $\nu = \Phi \circ \tilde{\nu} =$  “position  $\nu(x)$  on the Wulff shape, at which the normal of the Wulff shape equals  $\tilde{\nu}(x)$ ”
- Tangent space coincidence:  $T_x M = T_{\tilde{\nu}(x)} \mathbb{S}^n = T_{\nu(x)} \mathcal{W}$ .
  - ▶ Homogeneity of  $F^0$  implies for all tangent vectors  $V \in T_x M$ ,

$$\bar{g}_\nu(\nu, \nu) = 1, \quad g_\nu(\nu, V) = 0$$

- ▶ *Induced anisotropic metric and second fundamental form:*

$$g_{ij}(x) = \bar{g}_{\nu(x)}(\partial_i x, \partial_j x), \quad h_{ij} = \bar{g}_{\nu(x)}(\partial_i \nu, \partial_j x).$$

- ▶ *Anisotropic mean curvature*  $H = g^{ij} h_{ij} \equiv \text{tr}(A)$ .
- *Anisotropic volume element*  $d\mu = F(\tilde{\nu}) d\text{vol}_n$  ( $\neq$  volume element induced from  $g$ ).

# Level sets

- For a function  $f$  on  $\mathbb{R}^{n+1}$ , define the  $F$ -gradient by

$$\nabla_F^\sharp f = F(D^\sharp f)D^\sharp F(D^\sharp f).$$

- ▶ On a regular level set  $M$  of  $f$  this coincides with the definition

$$\bar{g}_{\nu(x)}(\nabla_F^\sharp f, V) = Df(x)V \quad \forall V \in T_x M.$$

- Define the  $F$ -Hessian endomorphism by

$$\nabla_F^2 f = D^2(\tfrac{1}{2}F^2)(D^\sharp f) \circ D^2 f$$

- ▶ On a regular level set  $M$  of  $f$  this coincides with

$$\nabla_F^2 f = \text{tr}(\bar{g}_{\nu(x)}^{-1} \circ D^2 f)$$

- ▶ Operator degenerates where  $Df = 0$ .



# Anisotropic level-set stability

## Theorem (with Xuwen Zhang)

*Stability of the Wulff shape with respect to anisotropic curvature functionals, (2021), arxiv:2308.15999.*

Let  $n \geq 2$ ,  $M \subset \mathbb{R}^{n+1}$  closed hypersurface,  $F$  an elliptic integrand,  $\mu(M) = 1$ . Let  $\mathcal{U}$  be one-sided neighbourhood of  $M$ , given by level sets of  $f \in C^2(\bar{\mathcal{U}})$ ,

$$\bar{\mathcal{U}} = \bigcup_{0 \leq t \leq \max|f|} M_t, \quad M_t = \{|f| = t\},$$

with  $f|_M = 0$  and  $df|_{\bar{\mathcal{U}}} > 0$ . Let  $p > n$  and  $\max_{0 \leq t \leq \max|f|} \|A\|_{p, M_t} \leq C_0$ . Then

$$\text{dist}(M, \mathcal{W}) \leq \frac{C(n, p, F, C_0)}{\min(\max|f|, \min|df|)^{\frac{p}{p+1}}} \left( \int_{\mathcal{U}} |\dot{\nabla}_F^2 f|^p \right)^{\frac{1}{p+1}}$$

for the Wulff shape corresponding to  $F$ , provided the RHS is small.

## Some words about the proof

If  $h$  is the anisotropic  $2^{nd}$  fundamental form of any regular level set of  $f$ , then

$$\nabla^2 f|_M = F(D^\sharp f)h.$$

Hence

$$F^2(D^\sharp f)|\mathring{A}|^2 \leq |\mathring{\nabla}_F^2 f|^2,$$

where  $\mathring{A}$  is the tracefree part of the anisotropic second fundamental form,

$$|\mathring{A}|^2 = c_n \sum_{i < j} (\kappa_i - \kappa_j)^2.$$

## Some words about the proof

Hence  $\mathring{A}$  can be controlled by  $\mathring{\nabla}_F^2 f$ .

Classical result from (isotropic) hypersurface theory, due to Darboux:

$$\mathring{A} = 0 \quad \Rightarrow \quad M = \text{Sphere}.$$

- ▶ A similar result holds in the anisotropic world<sup>3</sup>

Stability versions in the isotropic case are plentyfull and due to De Lellis/Müller, Topping, Grosjean.

- ▶ In the anisotropic world there is a recent one...

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<sup>3</sup>Yijun He and Haizhong Li: *Integral formula of Minkowski type and new characterization of the Wulff shape*, Acta Math. Sin. **24** (2008), no. 4, 697–704.

# Some words about the proof

## Theorem (Antonio De Rosa, Stefano Giofré)

*Absence of bubbling phenomena for non-convex anisotropic nearly umbilical and quasi-Einstein hypersurfaces*, J. Reine Angew. Math. **780** (2021), 1–40.

Let  $M \subset \mathbb{R}^{n+1}$  be a closed hypersurface,  $\mathcal{W}$  a Wulff shape,  $p > n$  and

$$|M| = |\mathcal{W}|, \quad \|A\|_{L^p(M)} \leq c_0.$$

Then there exist  $C = C(n, p, F, c_0) > 0$ , such that: if

$$\|\mathring{A}\|_{L^p(M)} \leq \frac{1}{C},$$

then there exists  $c \in \mathbb{R}^{n+1}$  and a parametrization  $\psi: \mathcal{W} \rightarrow M$ , such that

$$\|\psi - \text{id} - c\|_{W^{2,p}(\mathcal{W})} \leq C \|\mathring{A}\|_{L^p(M)}.$$

# Application I: Anisotropic Heintze-Karcher

- **Stability of the domain in the Heintze-Karcher inequality.** In the Euclidean space  $\mathbb{R}^{n+1}$ , for every domain  $\Omega$  with mean-convex  $\partial\Omega$ :

$$\int_{\partial\Omega} \frac{n}{H} \geq (n+1) \operatorname{vol}(\Omega)$$

with equality precisely when  $\Omega$  is a ball.

- ▶ The same holds in the anisotropic setting, if we integrate w.r.t. the anisotropic area measure.

# Application I: Anisotropic Heintze-Karcher

## Theorem (with Xuwen Zhang, Stability in the anisotropic Heintze-Karcher)

*Let  $n \geq 2$ ,  $\alpha > 0$  and  $F$  be an elliptic integrand on  $\mathbb{R}^{n+1}$ . Let  $\Omega \subset \mathbb{R}^{n+1}$  be a bounded domain with connected  $F$ -mean convex  $C^{2,\alpha}$ -boundary that satisfies a uniform interior Wulff sphere condition with radius  $r$ . Then there exists a positive constant  $C$  depending only on  $n, \alpha, F, r, \mu(\partial\Omega)$  and  $|\partial\Omega|_{2,\alpha}$ , such that*

$$\text{dist}(\partial\Omega, \mathcal{W}) \leq C \left( \int_{\partial\Omega} \frac{1}{H} d\mu - \frac{n+1}{n} |\Omega| \right)^{\frac{1}{n+2}}$$

*for some Wulff sphere  $\mathcal{W} \subset \mathbb{R}^{n+1}$ , provided the RHS is sufficiently small.*

# Proof of Heintze-Karcher stability

- The key for stability is the following estimate

$$\int_{\Omega} |\dot{\nabla}_F^2 f|^2 dx \leq \left( \frac{n}{n+1} \right)^2 \int_{\partial\Omega} \frac{1}{H} d\mu - \frac{n+1}{n} |\Omega|,$$

where

$$\begin{aligned} \operatorname{div}(D^{\sharp}(\tfrac{1}{2}F^2)(D^{\sharp}f)) &=: \Delta_F f = 1 \quad \text{in } \Omega \\ f &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

- ▶ Follows from divergence theorem type argument and Hölder's inequality.
- ▶  $f$  shall serve as the foliation function in a neighbourhood of  $\partial\Omega$ .
- ▶ For this we need a lower gradient bound of  $f$  on  $\partial\Omega$ .

# Proof of Heintze-Karcher stability

## Lemma (Gradient bound on $\partial\Omega$ )

Let  $\Omega \subset \mathbb{R}^{n+1}$  be a bounded domain with  $C^2$ -boundary that satisfies the uniform interior Wulff sphere condition with radius  $r$  and let  $f \in C^{1,\beta}(\overline{\Omega}) \cap W^{2,2}(\Omega)$  for some  $\beta \in (0,1)$  be a solution of

$$\begin{aligned} \operatorname{div}(D^\sharp(\tfrac{1}{2}F^2)(D^\sharp f)) &= 1 && \text{in } \Omega \\ f &= 0 && \text{on } \partial\Omega. \end{aligned}$$

then

$$|D^\sharp f| \geq C(n, F)r \quad \text{on } \partial\Omega.$$

- From here, higher regularity in a controlled neighbourhood of  $\partial\Omega$  follows from Schauder theory.
- The level set stability theorem completes the proof of the anisotropic Heintze-Karcher stability.



## Application II: Stability in the anisotropic soap bubble theorem

Theorem (with Xuwen Zhang, Stability in the anisotropic soap bubble theorem)

Let  $n \geq 2$ ,  $\alpha > 0$  and  $F$  be an elliptic integrand on  $\mathbb{R}^{n+1}$ . Let  $\Omega \subset \mathbb{R}^{n+1}$  be a bounded domain with connected boundary  $\partial\Omega \in C^{2,\alpha}$  that satisfies a uniform interior Wulff sphere condition with radius  $r$ . Then there exists a positive constant  $C$  depending only on  $n, \alpha, F, r, \mu(\partial\Omega)$ , such that

$$\text{dist}(\partial\Omega, \mathcal{W}) \leq C \left\| H - \frac{n}{n+1} \frac{\mu(\partial\Omega)}{|\Omega|} \right\|_{1, \partial\Omega}^{\frac{1}{n+2}}$$

for some Wulff sphere  $\mathcal{W} \subset \mathbb{R}^{n+1}$ , provided the RHS is sufficiently small.

- Basically the same proof as in Heintze-Karcher, up to few algebraic modifications. We again estimate  $\int_{\Omega} |\mathring{\nabla}_F^2 f|^2$ .

## Application III: Serrin' problem

- Serrin's overdetermined problem asks which domains  $\Omega$  allow solutions to

$$\begin{aligned}\Delta u &= 1 && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \\ |\nabla u| &= c && \text{on } \partial\Omega.\end{aligned}$$

- ▶ The answer is: Only balls.
- The anisotropic version (yielding equality to the Wulff shape)

$$\begin{aligned}\operatorname{div}(D^\sharp(\tfrac{1}{2}F^2)(D^\sharp f)) &= 1 && \text{in } \Omega \\ f &= 0 && \text{on } \partial\Omega \\ F(D^\sharp f) &= c && \text{on } \partial\Omega\end{aligned}$$

is due to Cianchi/Salani.<sup>4</sup>

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<sup>4</sup>*Overdetermined anisotropic elliptic problems*, Math. Ann. **345** (2009), no. 4, 859–881.

## Application III: Stability in the anisotropic Serrin problem

### Theorem (with Xuwen Zhang, Stability in the anisotropic Serrin problem)

Let  $n \geq 2$ ,  $\alpha > 0$  and  $F$  be an elliptic integrand on  $\mathbb{R}^{n+1}$ . Let  $\Omega \subset \mathbb{R}^{n+1}$  be a bounded domain with connected boundary  $\partial\Omega \in C^{2,\alpha}$  that satisfies a uniform interior Wulff sphere condition with radius  $r$ . Then there exists a positive constant depending only on  $n, \alpha, F, r, \text{diam}(\Omega), \mu(\partial\Omega)$  and  $|\partial\Omega|_{2,\alpha}$ , such that

$$\text{dist}(\partial\Omega, \mathcal{W}) \leq C \left\| F(D^\# f) - \frac{|\Omega|}{\mu(\partial\Omega)} \right\|_{1,\partial\Omega}^{\frac{1}{2(n+2)}}$$

for some Wulff sphere  $\mathcal{W} \subset \mathbb{R}^{n+1}$ , provided the RHS is sufficiently small. Here

$$\begin{aligned} \text{div}(D^\#(\tfrac{1}{2}F^2)(D^\# f)) &= 1 && \text{in } \Omega \\ f &= 0 && \text{on } \partial\Omega. \end{aligned}$$

## Application III: Stability in the anisotropic Serrin problem

- The proof works via the use of a so-called  $P$ -function. In our setting,

$$P = \frac{F^2(D^\sharp f)}{2} - \frac{1}{n+1}f.$$

- ▶ A computation gives

$$\operatorname{div}(\nabla_F^\sharp P) = |\mathring{\nabla}_F^2 f|^2.$$

- The next crucial ingredient is the Pohozaev-type identity

$$\int_{\Omega} P = \frac{1}{2(n+1)} \int_{\partial\Omega} F^2(D^\sharp f) \langle x, \tilde{\nu} \rangle.$$

- Further computations lead to

$$\int_{\Omega} (-f) |\mathring{\nabla}_F^2 f|^2 dx = \frac{1}{2} \int_M \left( F^2(D^\sharp f) - \frac{|\Omega|^2}{\mu(M)^2} \right) \left\langle \bar{\nabla}_F^\sharp f - \bar{\nabla}_F^\sharp \ell, \tilde{\nu} \right\rangle d\tilde{\mu},$$

where  $\ell(x) = (F^0(x))^2/(2(n+1))$ .

Děkuji!