Stability for anisotropic curvature functionals

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Closed Soap Bubbles

Isoperimetric problem: Determine properties of area minimising surface, given volume constraint.

Round spheres in $\mathbb{R}^{n+1}$ are unique closed minimisers of

$$\mathcal{R}(\Omega) = \frac{\text{Area}^{\frac{n+1}{n}}(\partial\Omega)}{\mathrm{Volume}(\Omega)}$$

Standard variational methods:

- Minimisers of $\mathcal{R}$ have **constant mean curvature**

$$H = \mathrm{tr}(A) = \sum_{i=1}^{n} \kappa_i$$

(κ_i) are eigenvalues of the Weingarten map A, principal curvatures.

Alexandrov's theorem

Is a closed embedded constant mean curvature (CMC) hypersurface of $\mathbb{R}^{n+1}$ necessarily a sphere?

Answer: **YES!** (Alexandrov[1])

- Proof: Reflection across moving planes and the maximum principle.
- We are going to see another elegant proof today.

Relaxed CMC condition: Suppose for some $\delta > 0$, on a hypersurface M

$$n - \delta \leq H \leq n + \delta.$$

Can we conclude

$$\mathrm{dist}(M, S) \leq C\epsilon$$

for the unit sphere S, a constant C and where

$$\lim_{\delta \to 0} \epsilon(\delta) = 0 \ ?$$

[1] *A characteristic property of spheres*, Ann. Mat. Pura Appl. **58** (1962), no. 4, 303–315.

The question of stability

Theorem (Giulio Ciraolo and Luigi Vezzoni)

A sharp quantitative version of Alexandrov's theorem via the method of moving planes, J. Eur. Math. Soc. **20** (2018), no. 2, 261–299.

Let Ω be a smooth domain with connected boundary, then $\partial\Omega$ **lies within an annulus of thickness** $C\,\mathrm{osc}(H)$*. C depends on $|\partial\Omega|$ and a lower bound for interior and exterior balls.*

Generalization to spaceforms and other curvature functions

$$F = F(\kappa_i)$$

was given by Ciraolo/Roncoroni/Vezzoni.[2]

[2] *Quantitative stability for hypersurfaces with almost constant curvature in space forms*, Ann. Mat. Pura Appl. **200** (2021), no. 5, 2043–2083.

The question of stability

Theorem (Rolando Magnanini and Giorgio Poggesi)

On the stability for Alexandrov's soap bubble theorem, J. Anal. Math. **139** (2019), no. 1, 179–205.

Let Ω be a smooth domain with connected boundary, then $\partial\Omega$ lies within an annulus of thickness at most $C\|H - H_0\|^{\tau_n}_{L^1(\partial\Omega)}$, where

$$H_0 = \frac{n}{n+1}\frac{|\partial\Omega|}{|\Omega|},$$

τ_n is a dimensional constant and C depends on few geometric quantities, such as interior and exterior ball conditions.

Ambient anisotropic geometry

- $\mathcal{W}$ (aka Wulff shape) smooth boundary of convex body $\mathcal{W}_0$ containing the origin.
  - Minkowski norm

$$F^0(x) = \inf\{s > 0 \colon x \in s\mathcal{W}_0\}$$

  - Then $\mathcal{W} = \{F^0 = 1\}$.
- $q(x) = \frac{1}{2}(F^0(x))^2$ is smooth, convex and hence

$$\bar{g}(x) := D^2 q(x)$$

  is a metric on $\mathbb{R}^{n+1} \backslash \{0\}$.
- Let $F(z) = \sup_{x \in \mathcal{W}} \langle x, z \rangle$ be the support function of $\mathcal{W}$,
  - $\Phi = (D^\sharp F)_{|\mathbb{S}^n} \colon \mathbb{S}^n \to \mathcal{W}$ is an embedding of the Wulff shape.

Induced anisotropic geometry

- Let $x\colon M^n \to \mathbb{R}^{n+1}$ embedding with M closed manifold, $\tilde{\nu}\colon M \to \mathbb{S}^n$ its normal vector field aka Gauss map.
  - *Anisotropic normal* $\nu = \Phi \circ \tilde{\nu} =$ "position $\nu(x)$ on the Wulff shape, at which the normal of the Wulff shape equals $\tilde{\nu}(x)$"
- Tangent space coincidence: $T_x M = T_{\tilde{\nu}(x)}\mathbb{S}^n = T_{\nu(x)}\mathcal{W}$.
  - Homogeneity of F^0 implies for all tangent vectors $V \in T_x M$,

$$\bar{g}_\nu(\nu, \nu) = 1, \quad g_\nu(\nu, V) = 0$$

  - *Induced anisotropic metric* and *second fundamental form*:

$$g_{ij}(x) = \bar{g}_{\nu(x)}(\partial_i x, \partial_j x), \quad h_{ij} = \bar{g}_{\nu(x)}(\partial_i \nu, \partial_j x).$$

  - *Anisotropic mean curvature* $H = g^{ij} h_{ij} \equiv \operatorname{tr}(A)$.
- *Anisotropic volume element* $d\mu = F(\tilde{\nu})d\operatorname{vol}_n$ ($\neq$ volume element induced from g).

Level sets

- For a function f on $\mathbb{R}^{n+1}$, define the *F-gradient* by

$$\nabla^{\sharp}_F f = F(D^{\sharp} f) D^{\sharp} F(D^{\sharp} f).$$

  - On a regular level set M of f this coincides with the definition

$$\bar{g}_{\nu(x)}(\nabla^{\sharp}_F f, V) = Df(x)V \quad \forall V \in T_x M.$$

- Define the *F-Hessian endomorphism* by

$$\nabla^2_F f = D^2(\tfrac{1}{2}F^2)(D^{\sharp} f) \circ D^2 f$$

  - On a regular level set M of f this coincides with

$$\nabla^2_F f = \mathrm{tr}(\bar{g}^{-1}_{\nu(x)} \circ D^2 f)$$

  - Operator degenerates where $Df = 0$.

Anisotropic level-set stability

Theorem (with Xuwen Zhang)

Stability of the Wulff shape with respect to anisotropic curvature functionals, (2021), arxiv:2308.15999.

Let $n \geq 2$, $M \subset \mathbb{R}^{n+1}$ *closed hypersurface,* F *an elliptic integrand,* $\mu(M) = 1$. *Let* $\mathcal{U}$ *be one-sided neighbourhood of* M, *given by level sets of* $f \in C^2(\bar{\mathcal{U}})$,

$$\bar{\mathcal{U}} = \bigcup_{0 \leq t \leq \max|f|} M_t, \quad M_t = \{|f| = t\},$$

with $f_{|M} = 0$ *and* $df_{|\bar{\mathcal{U}}} > 0$. *Let* $p > n$ *and* $\max_{0 \leq t \leq \max|f|} \|A\|_{p,M_t} \leq C_0$. *Then*

$$\operatorname{dist}(M, \mathcal{W}) \leq \frac{C(n, p, F, C_0)}{\min(\max|f|, \min|df|)^{\frac{p}{p+1}}} \left(\int_{\mathcal{U}} |\mathring{\nabla}^2_F f|^p \right)^{\frac{1}{p+1}}$$

for the Wulff shape corresponding to F, *provided the RHS is small.*

Some words about the proof

If h is the anisotropic 2^{nd} fundamental form of any regular level set of f, then

$$\nabla^2 f_{|M} = F(D^\sharp f)h.$$

Hence

$$F^2(D^\sharp f)|\mathring{A}|^2 \le |\mathring{\nabla}^2_F f|^2,$$

where $\mathring{A}$ is the tracefree part of the anisotropic second fundamental form,

$$|\mathring{A}|^2 = c_n \sum_{i<j} (\kappa_i - \kappa_j)^2.$$

Some words about the proof

Hence $\mathring{A}$ can be controlled by $\mathring{\nabla}^2_F f$.

Classical result from (isotropic) hypersurface theory, due to Darboux:

$$\mathring{A} = 0 \quad \Rightarrow \quad M = \text{Sphere}.$$

- A similar result holds in the anisotropic world[3]

Stability versions in the isotropic case are plentyfull and due to De Lellis/Müller, Topping, Grosjean.

- In the anisotropic world there is a recent one...

[3] Yijun He and Haizhong Li: *Integral formula of Minkowski type and new characterization of the Wulff shape*, Acta Math. Sin. **24** (2008), no. 4, 697–704.

Some words about the proof

Theorem (Antonio De Rosa, Stefano Gioffré)

Absence of bubbling phenomena for non-convex anisotropic nearly umbilical and quasi-Einstein hypersurfaces, J. Reine Angew. Math. **780** (2021), 1–40.

Let $M \subset \mathbb{R}^{n+1}$ be a closed hypersurface, $\mathcal{W}$ a Wulff shape, $p > n$ and

$$|M| = |\mathcal{W}|, \quad \|A\|_{L^p(M)} \leq c_0.$$

Then there exist $C = C(n, p, F, c_0) > 0$, such that: if

$$\|\mathring{A}\|_{L^p(M)} \leq \tfrac{1}{C},$$

then there exists $c \in \mathbb{R}^{n+1}$ and a parametrization $\psi\colon \mathcal{W} \to M$, such that

$$\|\psi - \mathrm{id} - c\|_{W^{2,p}(\mathcal{W})} \leq C\|\mathring{A}\|_{L^p(M)}.$$

Application I: Anisotropic Heintze-Karcher

- **Stability of the domain in the Heintze-Karcher inequality**. In the Euclidean space $\mathbb{R}^{n+1}$, for every domain Ω with mean-convex $\partial\Omega$:

$$\int_{\partial\Omega} \frac{n}{H} \geq (n+1)\,\mathrm{vol}(\Omega)$$

  with equality precisely when Ω is a ball.
  - The same holds in the anisotropic setting, if we integrate w.r.t. the anisotropic area measure.

Application I: Anisotropic Heintze-Karcher

Theorem (with Xuwen Zhang, Stability in the anisotropic Heintze-Karcher)

Let $n \geq 2$, $\alpha > 0$ *and* F *be an elliptic integrand on* $\mathbb{R}^{n+1}$. *Let* $\Omega \subset \mathbb{R}^{n+1}$ *be a bounded domain with connected* F*-mean convex* $C^{2,\alpha}$*-boundary that satisfies a uniform interior Wulff sphere condition with radius* r. *Then there exists a positive constant* C *depending only on* $n, \alpha, F, r, \mu(\partial\Omega)$ *and* $|\partial\Omega|_{2,\alpha}$, *such that*

$$\operatorname{dist}(\partial\Omega, \mathcal{W}) \leq C \left(\int_{\partial\Omega} \frac{1}{H} \, d\mu - \frac{n+1}{n} |\Omega| \right)^{\frac{1}{n+2}}$$

for some Wulff sphere $\mathcal{W} \subset \mathbb{R}^{n+1}$, *provided the RHS is sufficiently small.*

Proof of Heintze-Karcher stability

- The key for stability is the following estimate

$$\int_\Omega |\mathring{\nabla}_F^2 f|^2\, dx \le \left(\frac{n}{n+1}\right)^2 \int_{\partial\Omega} \frac{1}{H}\, d\mu - \frac{n+1}{n}|\Omega|,$$

where

$$\begin{aligned} \operatorname{div}(D^\sharp(\tfrac{1}{2}F^2)(D^\sharp f)) =: \Delta_F f &= 1 \quad \text{in } \Omega \\ f &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

  - Follows from divergence theorem type argument and Hölder's inequality.
  - f shall serve as the foliation function in a neighbourhood of $\partial\Omega$.
  - For this we need a lower gradient bound of f on $\partial\Omega$.

Proof of Heintze-Karcher stability

Lemma (Gradient bound on $\partial\Omega$)

Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded domain with C^2-boundary that satisfies the uniform interior Wulff sphere condition with radius r and let $f \in C^{1,\beta}(\overline{\Omega}) \cap W^{2,2}(\Omega)$ for some $\beta \in (0,1)$ be a solution of

$$\begin{aligned} \operatorname{div}(D^\sharp(\tfrac{1}{2}F^2)(D^\sharp f)) &= 1 \quad \text{in } \Omega \\ f &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

then

$$|D^\sharp f| \geq C(n,F)r \quad \text{on } \partial\Omega.$$

- From here, higher regularity in a controlled neighbourhood of $\partial\Omega$ follows from Schauder theory.
- The level set stability theorem completes the proof of the anisotropic Heintze-Karcher stability.

Application II: Stability in the anisotropic soap bubble theorem

Theorem (with Xuwen Zhang, Stability in the anisotropic soap bubble theorem)

Let $n \geq 2$, $\alpha > 0$ and F be an elliptic integrand on $\mathbb{R}^{n+1}$. Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded domain with connected boundary $\partial\Omega \in C^{2,\alpha}$ that satisfies a uniform interior Wulff sphere condition with radius r. Then there exists a positive constant C depending only on $n, \alpha, F, r, \mu(\partial\Omega)$, such that

$$\mathrm{dist}(\partial\Omega, \mathcal{W}) \leq C \left\| H - \frac{n}{n+1}\frac{\mu(\partial\Omega)}{|\Omega|} \right\|_{1,\partial\Omega}^{\frac{1}{n+2}}$$

for some Wulff sphere $\mathcal{W} \subset \mathbb{R}^{n+1}$, provided the RHS is sufficiently small.

- Basically the same proof as in Heintze-Karcher, up to few algebraic modifications. We again estimate $\int_\Omega |\mathring{\nabla}^2_F f|^2$.

Application III: Stability in the anisotropic Serrin problem

Theorem (with Xuwen Zhang, Stability in the anisotropic Serrin problem)

Let $n \geq 2$, $\alpha > 0$ and F be an elliptic integrand on $\mathbb{R}^{n+1}$. Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded domain with connected boundary $\partial\Omega \in C^{2,\alpha}$ that satisfies a uniform interior Wulff sphere condition with radius r. Then there exists a positive constant depending only on $n, \alpha, F, r, \mathrm{diam}(\Omega)$, $\mu(\partial\Omega)$ and $|\partial\Omega|_{2,\alpha}$, such that

$$\mathrm{dist}(\partial\Omega, \mathcal{W}) \leq C \left\| F(D^\sharp f) - \tfrac{|\Omega|}{\mu(\partial\Omega)} \right\|_{1,\partial\Omega}^{\frac{1}{2(n+2)}}$$

for some Wulff sphere $\mathcal{W} \subset \mathbb{R}^{n+1}$, provided the RHS is sufficiently small. Here

$$\begin{aligned} \mathrm{div}(D^\sharp(\tfrac{1}{2}F^2)(D^\sharp f)) &= 1 \quad \text{in } \Omega \\ f &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Application III: Stability in the anisotropic Serrin problem

- The proof works via the use of a so-called P-function. In our setting,

$$P = \frac{F^2(D^\sharp f)}{2} - \frac{1}{n+1} f.$$

  - A computation gives

$$\operatorname{div}(\nabla_F^\sharp P) = |\mathring{\nabla}_F^2 f|^2.$$

- The next crucial ingredient is the Pohozaev-type identity

$$\int_\Omega P = \frac{1}{2(n+1)} \int_{\partial\Omega} F^2(D^\sharp f) \langle x, \tilde{\nu} \rangle .$$

- Further computations lead to

$$\int_\Omega (-f) |\mathring{\bar{\nabla}}_F^2 f|^2 \, dx = \frac{1}{2} \int_M \left(F^2(D^\sharp f) - \frac{|\Omega|^2}{\mu(M)^2} \right) \left\langle \bar{\nabla}_F^\sharp f - \bar{\nabla}_F^\sharp \ell, \tilde{\nu} \right\rangle \, d\tilde{\mu},$$

where $\ell(x) = (F^0(x))^2 / (2(n+1))$.

Dêkuji!