Stability for the quermassintegral inequalities in the hyperbolic space

Julian Scheuer
(Goethe University Frankfurt)

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Closed Soap Bubbles

Isoperimetric problem: Determine properties of area minimising surface, given volume constraint.

Round spheres in $\mathbb{R}^{n+1}$ are unique closed minimisers of

$$\mathcal{R}(\Omega) = \frac{\mathrm{Area}^{\frac{n+1}{n}}(\partial\Omega)}{\mathrm{Volume}(\Omega)}$$

Standard variational methods:

- Minimisers of $\mathcal{R}$ have **constant mean curvature**

$$H = \mathrm{tr}(A) = \sum_{i=1}^{n} \kappa_i$$

(κ_i) are eigenvalues of the Weingarten map A, principal curvatures.

Is a closed embedded constant mean curvature (CMC) hypersurface of $\mathbb{R}^{n+1}$ necessarily a sphere?

Answer: **YES!** (Alexandrov[1])

- Proof: Reflection across moving planes and the maximum principle.
- We are going to see another elegant proof today.

Relaxed CMC condition: Suppose for some $\delta > 0$, on a hypersurface M

$$n - \delta \leq H \leq n + \delta.$$

Can we conclude

$$\mathrm{dist}(M, S) \leq C\epsilon$$

for the unit sphere S, a constant C and where

$$\lim_{\delta \to 0} \epsilon(\delta) = 0 \ ?$$

[1] *A characteristic property of spheres*, Ann. Mat. Pura Appl. **58** (1962), no. 4, 303–315.

The question of stability

Theorem (Giulio Ciraolo and Luigi Vezzoni)

A sharp quantitative version of Alexandrov's theorem via the method of moving planes, J. Eur. Math. Soc. **20** (2018), no. 2, 261–299.

Let Ω be a smooth domain with connected boundary, then $\partial\Omega$ **lies within an annulus of thickness** $C\,\mathrm{osc}(H)$. *C depends on $|\partial\Omega|$ and a lower bound for interior and exterior balls.*

Generalization to spaceforms and other curvature functions

$$F = F(\kappa_i)$$

was given by Ciraolo/Roncoroni/Vezzoni.[2]

[2] *Quantitative stability for hypersurfaces with almost constant curvature in space forms*, Ann. Mat. Pura Appl. **200** (2021), no. 5, 2043–2083.

The question of stability

Theorem (Rolando Magnanini and Giorgio Poggesi)

On the stability for Alexandrov's soap bubble theorem, J. Anal. Math. **139** (2019), no. 1, 179–205.

Let Ω be a smooth domain with connected boundary, then $\partial\Omega$ lies within an annulus of thickness at most $C\|H - H_0\|^{\tau_n}_{L^1(\partial\Omega)}$, where

$$H_0 = \frac{n}{n+1}\frac{|\partial\Omega|}{|\Omega|},$$

τ_n is a dimensional constant and C depends on few geometric quantities, such as interior and exterior ball conditions.

Integral approach

Key ingredient (A. Ros): Use **Reilly's integral identity**.

For C^2-functions f on a domain $\Omega \subset \mathbb{R}^{n+1}$, with $f_{|\partial\Omega} = \text{const}$:

$$\int_\Omega (\Delta f)^2 - \int_\Omega |\nabla^2 f|^2 = \int_{\partial\Omega} H(\partial_\nu f)^2.$$

Cauchy-Schwarz-deficit:

$$|\mathring{\nabla}^2 f|^2 = |\nabla^2 f|^2 - \tfrac{1}{n+1}(\Delta f)^2,$$

Then

$$\int_\Omega |\mathring{\nabla}^2 f|^2 = \frac{n}{n+1}\int_\Omega (\Delta f)^2 - \int_{\partial\Omega} H(\partial_\nu f)^2.$$

Solve

$$\begin{aligned} \Delta f &= 1 \quad \text{in } \Omega \\ f &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Integral approach

Then

$$
\begin{aligned}
\int_\Omega |\mathring{\nabla}^2 f|^2 &= \frac{n}{(n+1)\operatorname{vol}(\Omega)}\left(\int_\Omega \Delta f\right)^2 - \int_{\partial\Omega} H(\partial_\nu f)^2\\
&= \frac{n}{(n+1)\operatorname{vol}(\Omega)}\left(\int_{\partial\Omega} \partial_\nu f\right)^2 - \int_{\partial\Omega} H(\partial_\nu f)^2\\
&\leq \frac{n}{(n+1)}\frac{\operatorname{Area}(\partial\Omega)}{\operatorname{vol}(\Omega)}\int_{\partial\Omega}(\partial_\nu f)^2 - \int_{\partial\Omega} H(\partial_\nu f)^2\\
&\equiv \int_{\partial\Omega}(H_0 - H)(\partial_\nu f)^2.
\end{aligned}
$$

The function $f - q$ is harmonic, where

$$
q = \frac{1}{2(n+1)}|x-c|^2 - R.
$$

Magnanini/Poggesi: Good estimates for $f - q$.

New general shortcut

Theorem (Level set stability)

Stability from rigidity via umbilicity, (2021), arxiv:2103.07178.

Let $n \geq 2$, $M \subset \mathbb{R}^{n+1}$ *closed hypersurface,* $|M| = 1$*. Let* $\mathcal{U}$ *be one-sided neighbourhood of* M*, foliated by level sets of* $f \in C^2(\bar{\mathcal{U}})$,

$$\bar{\mathcal{U}} = \bigcup_{0 \leq t \leq \max|f|} M_t, \quad M_t = \{|f| = t\},$$

with $f_{|M} = 0$ *and* $|\nabla f|_{|\bar{\mathcal{U}}} > 0$*. Let* $p > n$ *and* $\max_{0 \leq t \leq \max|f|} \|A\|_{p,M_t} \leq C_0$*. Then*

$$\mathrm{dist}(M, \mathcal{S}) \leq \frac{C(n,p,C_0)}{\min(\max|f|, \min|\nabla f|)^{\frac{p}{p+1}}} \left(\int_{\mathcal{U}} |\mathring{\nabla}^2 f|^p\right)^{\frac{1}{p+1}}$$

for a sphere $\mathcal{S}$*, provided the RHS is small.*

Some Remarks

This also works in **conformally flat Riemannian manifolds**.

The exponent $1/(p+1)$ is worse than the constant τ_n in Magnanini/Poggesi's CMC stability problem.

But, the result is universal among many geometric problems.

Note in particular, that f **is not assumed to solve any PDE.**

Some words about the proof: Almost umbilicity I

If h is the 2^{nd} fundamental form of the boundary, then.

$$\nabla^2 f_{|M} = -|\nabla f| h.$$

Hence

$$|\nabla f|^2 |\mathring{A}|^2 = |\mathring{\nabla}^2 f_{|TM}|^2 - \frac{1}{n}(\mathring{\nabla}^2 f(\nu,\nu))^2,$$

where $\mathring{A}$ is the tracefree part of the second fundamental form,

$$|\mathring{A}|^2 = c_n \sum_{i<j} (\kappa_i - \kappa_j)^2.$$

Some words about the proof: Almost umbilicity I

Hence $\mathring{A}$ can be controlled by $\mathring{\nabla}^2 f$.

Classical result from hypersurface theory, due to Darboux:

$$\mathring{A} = 0 \quad \Rightarrow \quad M = \text{Sphere}.$$

Stability versions available and due to De Lellis/Müller, Topping, Grosjean and recently...

Some words about the proof: Almost umbilicity I

Theorem (Antonio De Rosa, Stefano Gioffré)

Absence of bubbling phenomena for non-convex anisotropic nearly umbilical and quasi-Einstein hypersurfaces, J. Reine Angew. Math. **780** (2021), 1–40.

Let $M \subset \mathbb{R}^{n+1}$ be a closed hypersurface, $p > n$ and

$$|M| = |\mathbb{S}^n|, \quad \|A\|_{L^p(M)} \leq c_0.$$

Then there exist $C = C(n, p, c_0) > 0$, such that: if

$$\|\mathring{A}\|_{L^p(M)} \leq \frac{1}{C},$$

then there exists $c \in \mathbb{R}^{n+1}$, such that $M - c$ is a graph over the sphere,

$$\psi \colon \mathbb{S}^n \to M, \quad \psi(x) = e^{f(x)}x + c,$$

$$\|f\|_{W^{2,p}(\mathbb{S}^n)} \leq C\|\mathring{A}\|_{L^p(M)}.$$

Almost umbilicity II: Quermassintegral inequalities

Classical Steiner formula for convex $\Omega \subset \mathbb{R}^{n+1}$:

$$|\bar{\Omega} + \epsilon B| = \sum_{k=0}^{n+1} \binom{n+1}{k} W_k(\Omega)\epsilon^k \quad \forall \epsilon \geq 0.$$

Here

$$W_0(\Omega) = |\Omega|, \quad W_k(\Omega) = \frac{1}{n+1} \int_{\partial\Omega} H_{k-1} \quad 1 \leq k \leq n;$$

H_k are the normalized elementary symmetric polynomials and $H_0 = 1$. The classical Alexandrov-Fenchel inequalities say that

$$\frac{W_{k+1}(\Omega)}{W_{k+1}(B)} \geq \left(\frac{W_k(\Omega)}{W_k(B)}\right)^{\frac{n-1-k}{n-k}}, \quad 0 \leq k \leq n-1,$$

with equality precisely if Ω is a ball.

Almost umbilicity II: Quermassintegral inequalities

P. Guan/J. Li[3]: Generalization of the **QM-inequalities to starshaped and k-convex hypersurfaces**, i.e.

$$\text{principal curvatures} \in \Gamma_k = \{H_1 > 0, \ldots, H_k > 0\}.$$

For this result we get stability:

[3] *The quermassintegral inequalities for k-convex starshaped domains*, Adv. Math. **221** (2009), no. 5, 1725–1732.

Almost umbilicity II: Quermassintegral inequalities

Theorem (Stability in non-convex QM-inequalities)

Stability from rigidity via umbilicity, (2021), arxiv:2103.07178.

Let $n \geq 2$ and $1 \leq k \leq n$. Let $M^n \subset \mathbb{M}_0$ be a closed and starshaped C^2-hypersurface with interior ρ-ball condition such that $\kappa \in \Gamma_k$. Then there exists a constant C, depending on n, $|M|$ and on **upper bounds for** ρ^{-1} *and* $(\min_M \operatorname{dist}(\kappa, \partial\Gamma_k))^{-1}$, *such that*

$$\frac{W_{k+1}(\Omega)}{W_{k+1}(B)} - \left(\frac{W_k(\Omega)}{W_k(B)}\right)^{\frac{n-k}{n-k+1}} < C^{-1}$$

implies, for a sphere S,

$$\operatorname{dist}(M, S) \leq C \left(\frac{W_{k+1}(\Omega)}{W_{k+1}(B)} - \left(\frac{W_k(\Omega)}{W_k(B)}\right)^{\frac{n-k}{n-k+1}}\right)^{\frac{1}{2(n+1)}}.$$

Almost umbilicity II: Quermassintegral inequalities

We use the **curvature flow**

$$\partial_t x = \left(\frac{H_{k-1}}{H_k} - u\right)\nu.$$

Convergence to a round sphere settled by Claus Gerhardt[4].

This flow preserves $\tilde{W}_k(\Omega)$ (normalized W_k) and decreases $\tilde{W}_{k+1}(\Omega)$.

We estimate a particular space-time integral:

$$\begin{aligned}
\int_0^\infty \int_{M_t} \left(\frac{H_{k+1}H_{k-1}}{H_k} - H_k\right) dt &= \int_0^\infty \int_{M_t} \left(\frac{H_{k+1}H_{k-1}}{H_k} - uH_{k+1}\right) dt \\
&= c_{n,k} \int_0^\infty \frac{d}{dt} \tilde{W}_{k+1}(\Omega_t) dt \\
&= 1 - \tilde{W}_{k+1}(\Omega) \\
&= \tilde{W}_k(\Omega)^{\frac{n-k-1}{n-k}} - \tilde{W}_{k+1}(\Omega).
\end{aligned}$$

[4] *Flow of nonconvex hypersurfaces into spheres*, J. Differ. Geom. **32** (1990), no. 1, 299–314.

Almost umbilicity II: Quermassintegral inequalities

Hence the right hand side controls the *Newton-MacLaurin-deficit*:

Recall that

$$H_{k+1}H_{k-1} \leq H_k^2$$

with equality precisely at umbilic points.

Apply the associated quantitative estimates:

$$\frac{|\mathring{A}|^2}{c(n, 1/H_k)} \leq H_k^2 - H_{k+1}H_{k-1} \quad \forall 1 \leq k \leq n-2.$$

Further elementary arguments give L^2-control on $|\mathring{A}|^2$.

Almost-umbilicity finishes the proof.

Almost umbilicity III: Quermassintegrals in hyperbolic space

For a bounded domain Ω in hyperbolic space $\mathbb{H}^{n+1}$ with C^2 boundary $M = \partial\Omega$, the k^{th} quermassintegrals W_k is defined inductively:

$$W_{k+1}(\Omega) = \frac{1}{n+1}\int_M H_k(\kappa)d\mu - \frac{k}{n+2-k}W_{k-1}(\Omega)$$

where

$$W_0(\Omega) = |\Omega|, \quad W_1(\Omega) = \frac{1}{n+1}|M|.$$

The extra term comes from ambient hyperbolic curvature, which effects the evolution of the outward parallel bodies.

Under a normal variation $\partial_t x = f\nu$, these quantities behave just as in the Euclidean space (the effect of the extra term in their definition):

$$\partial_t W_k(\Omega) = c_{n,k}\int_M H_k f \, d\mu.$$

Wang and Xia proved corresponding quermassintegral inequalities:

Almost umbilicity III: Stability in hyperbolic space

Theorem (Guofang Wang and Chao Xia)

Isoperimetric type problems and Alexandrov-Fenchel type inequalities in the hyperbolic space, Adv. Math. **259** *(2014), 532–556.*

Let $n \geq 2$, $\Omega \subset \mathbb{H}^{n+1}$ be a horo-convex domain, and $1 \leq m \leq n-1$.

$$W_{m+1}(\Omega) \geq f_{m+1} \circ f_m^{-1}(W_m(\Omega)),$$

with equality iff Ω is a geodesic ball. Here $f_m(r) = W_m(B_r)$.

Associated with this we get a stability estimate:

Almost umbilicity III: Stability in hyperbolic space

Theorem (with Prachi Sahjwani)

Stability of the quermassintegral inequalities in hyperbolic space, J. Geom. Anal. **34** *(2024), art. 13.*

Let $n \geq 2$, $\Omega \subset \mathbb{H}^{n+1}$ *be a horo-convex domain, and* $1 \leq m \leq n-1$. *Then there exists a constant* $C = C\left(n, \rho_-(\Omega), \max_{\partial\Omega} H_m/H_{m-1}\right)$ *and a geodesic sphere* $S_{\mathbb{H}}$ *such that*

$$\mathrm{dist}(\partial\Omega, S_{\mathbb{H}}) \leq C\left(W_{m+1}(\Omega) - f_{m+1} \circ f_m^{-1}\left(W_m(\Omega)\right)\right)^{\frac{1}{m+2}}.$$

The dependence on $H_m \backslash H_{m-1}$ allows for curvature blow-up.

This is in contrast to the Euclidean case.

On the other hand, in the Euclidean case I considered more general hypersurfaces.

Çok teşekkür ederim!