Stability for anisotropic curvature functionals

Julian Scheuer (Goethe University Frankfurt)
joint work with Xuwen Zhang (Freiburg)

Happy birthday Michael and Guofang

Scan me for the slides!

Closed Soap Bubbles

Isoperimetric problem: Determine properties of area minimising surface, given volume constraint.

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Standard variational methods:

- Minimisers of $\mathcal{R}$ have **constant mean curvature**

$$H = \mathrm{tr}(A) = \sum_{i=1}^{n} \kappa_i$$

(κ_i) are eigenvalues of the Weingarten map A, principal curvatures.

Alexandrov's theorem

Is a closed embedded constant mean curvature (CMC) hypersurface of $\mathbb{R}^{n+1}$ necessarily a sphere?

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$$n - \delta \leq H \leq n + \delta.$$

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Relaxed CMC condition: Suppose for some $\delta > 0$, on a hypersurface M

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Can we conclude

$$\mathrm{dist}(M, S) \leq C\epsilon$$

for the unit sphere S, a constant C and where

$$\lim_{\delta \to 0} \epsilon(\delta) = 0 \ ?$$

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The question of stability

Theorem (Giulio Ciraolo and Luigi Vezzoni)

A sharp quantitative version of Alexandrov's theorem via the method of moving planes, J. Eur. Math. Soc. **20** (2018), no. 2, 261–299.

Let Ω be a smooth domain with connected boundary, then $\partial\Omega$ **lies within an annulus of thickness** $C\,\mathrm{osc}(H)$*. C depends on $|\partial\Omega|$ and a lower bound for interior and exterior balls.*

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Generalization to spaceforms and other curvature functions

$$F = F(\kappa_i)$$

was given by Ciraolo/Roncoroni/Vezzoni.[2]

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The question of stability

Theorem (Rolando Magnanini and Giorgio Poggesi)

On the stability for Alexandrov's soap bubble theorem, J. Anal. Math. **139** (2019), no. 1, 179–205.

Let Ω be a smooth domain with connected boundary, then $\partial\Omega$ lies within an annulus of thickness at most $C\|H - H_0\|^{\tau_n}_{L^1(\partial\Omega)}$, where

$$H_0 = \frac{n}{n+1}\frac{|\partial\Omega|}{|\Omega|},$$

τ_n is a dimensional constant and C depends on few geometric quantities, such as interior and exterior ball conditions.

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- $q(x) = \frac{1}{2}(F^0(x))^2$ is smooth and strictly convex, hence
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  - Φ is the inverse of the Gauss map of $\mathcal{W}$.

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- *Anisotropic volume element* $d\mu = F(\tilde{\nu})d\,\mathrm{vol}_n$ ($\neq$ volume element induced from g).

Level sets

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  - Operator degenerates where $Df = 0$.

Anisotropic level-set stability

Theorem (with Xuwen Zhang)

Stability of the Wulff shape with respect to anisotropic curvature functionals, (2023).

Let $n \geq 2$, $M \subset \mathbb{R}^{n+1}$ *closed hypersurface,* F *an elliptic integrand,* $\mu(M) = 1$. *Let* $\mathcal{U}$ *be one-sided neighbourhood of* M, *given by level sets of* $f \in C^2(\bar{\mathcal{U}})$,

$$\bar{\mathcal{U}} = \bigcup_{0 \leq t \leq \max|f|} M_t, \quad M_t = \{|f| = t\},$$

with $f_{|M} = 0$ *and* $df_{|\bar{\mathcal{U}}} > 0$. *Let* $p > n$ *and* $\max_{0 \leq t \leq \max|f|} \|A\|_{p,M_t} \leq C_0$. *Then*

$$\operatorname{dist}(M, \mathcal{W}) \leq \frac{C(n,p,F,C_0)}{\min(\max|f|, \min|df|)^{\frac{p}{p+1}}} \left(\int_{\mathcal{U}} |\mathring{\nabla}^2_F f|^p \right)^{\frac{1}{p+1}}$$

for the Wulff shape corresponding to F, *provided the RHS is small.*

Some words about the proof

If h is the anisotropic 2^{nd} fundamental form of any regular level set of f, then

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Hence

$$F^2(D^\sharp f)|\mathring{A}|^2 \leq |\mathring{\nabla}^2_F f|^2,$$

where $\mathring{A}$ is the tracefree part of the anisotropic second fundamental form,

$$|\mathring{A}|^2 = c_n \sum_{i<j} (\kappa_i - \kappa_j)^2.$$

Some words about the proof

Hence $\mathring{A}$ can be controlled by $\mathring{\nabla}^2_F f$.

[3] Yijun He and Haizhong Li: *Integral formula of Minkowski type and new characterization of the Wulff shape*, Acta Math. Sin. **24** (2008), no. 4, 697–704.

Some words about the proof

Hence $\mathring{A}$ can be controlled by $\mathring{\nabla}_F^2 f$.

Classical result from (isotropic) hypersurface theory, due to Darboux:

$$\mathring{A} = 0 \quad \Rightarrow \quad M = \text{Sphere}.$$

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- In the anisotropic world there is a recent one...

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Some words about the proof

Theorem (Antonio De Rosa, Stefano Gioffré)

Absence of bubbling phenomena for non-convex anisotropic nearly umbilical and quasi-Einstein hypersurfaces, J. Reine Angew. Math. **780** (2021), 1–40.

Let $M \subset \mathbb{R}^{n+1}$ be a closed hypersurface, $\mathcal{W}$ a Wulff shape, $p > n$ and

$$|M| = |\mathcal{W}|, \quad \|A\|_{L^p(M)} \leq c_0.$$

Then there exist $C = C(n, p, F, c_0) > 0$, such that: if

$$\|\mathring{A}\|_{L^p(M)} \leq \frac{1}{C},$$

then there exists $c \in \mathbb{R}^{n+1}$ and a parametrization $\psi\colon \mathcal{W} \to M$, such that

$$\|\psi - \mathrm{id} - c\|_{W^{2,p}(\mathcal{W})} \leq C\|\mathring{A}\|_{L^p(M)}.$$

Application I: Anisotropic Heintze-Karcher

- **Stability of the domain in the Heintze-Karcher inequality**. In the Euclidean space $\mathbb{R}^{n+1}$, for every domain Ω with mean-convex $\partial\Omega$:

$$\int_{\partial\Omega} \frac{n}{H} \geq (n+1)\,\mathrm{vol}(\Omega)$$

with equality precisely when Ω is a ball.

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  - The same holds in the anisotropic setting, if we integrate w.r.t. the anisotropic area measure.

Application I: Anisotropic Heintze-Karcher

Theorem (with Xuwen Zhang, Stability in the anisotropic Heintze-Karcher)

Let $n \geq 2$, $\alpha > 0$ and F be an elliptic integrand on $\mathbb{R}^{n+1}$. Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded domain with connected F-mean convex $C^{2,\alpha}$-boundary that satisfies a uniform interior Wulff sphere condition with radius r. Then there exists a positive constant C depending only on $n, \alpha, F, r, \mu(\partial\Omega)$ and $|\partial\Omega|_{2,\alpha}$, such that

$$\mathrm{dist}(\partial\Omega, \mathcal{W}) \leq C \left(\int_{\partial\Omega} \frac{1}{H} \, d\mu - \frac{n+1}{n} |\Omega| \right)^{\frac{1}{n+2}}$$

for some Wulff sphere $\mathcal{W} \subset \mathbb{R}^{n+1}$, provided the RHS is sufficiently small.

Proof of Heintze-Karcher stability

- The key for stability is the following estimate

$$\int_\Omega |\mathring{\nabla}^2_F f|^2 \, dx \leq \left(\frac{n}{n+1}\right)^2 \int_{\partial\Omega} \frac{1}{H} \, d\mu - \frac{n+1}{n} |\Omega|,$$

where

$$\begin{aligned} \operatorname{div}(D^\sharp(\tfrac{1}{2}F^2)(D^\sharp f)) =: \Delta_F f &= 1 \quad \text{in } \Omega \\ f &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

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  - f shall serve as the foliation function in a neighbourhood of $\partial\Omega$.
  - For this we need a lower gradient bound of f on $\partial\Omega$.

Proof of Heintze-Karcher stability

Lemma (Gradient bound on $\partial\Omega$)

Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded domain with C^2-boundary that satisfies the uniform interior Wulff sphere condition with radius r and let $f \in C^{1,\beta}(\overline{\Omega}) \cap W^{2,2}(\Omega)$ for some $\beta \in (0,1)$ be a solution of

$$\begin{aligned} \operatorname{div}(D^\sharp(\tfrac{1}{2}F^2)(D^\sharp f)) &= 1 \quad \text{in } \Omega \\ f &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

then

$$|D^\sharp f| \geq C(n,F)r \quad \text{on } \partial\Omega.$$

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Lemma (Gradient bound on $\partial\Omega$)

Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded domain with C^2-boundary that satisfies the uniform interior Wulff sphere condition with radius r and let $f \in C^{1,\beta}(\overline{\Omega}) \cap W^{2,2}(\Omega)$ for some $\beta \in (0,1)$ be a solution of

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- From here, higher regularity in a controlled neighbourhood of $\partial\Omega$ follows from Schauder theory.
- The level set stability theorem completes the proof of the anisotropic Heintze-Karcher stability.

Application II: Stability in the anisotropic soap bubble theorem

Theorem (with Xuwen Zhang, Stability in the anisotropic soap bubble theorem)

Let $n \geq 2$, $\alpha > 0$ and F be an elliptic integrand on $\mathbb{R}^{n+1}$. Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded domain with connected boundary $\partial\Omega \in C^{2,\alpha}$ that satisfies a uniform interior Wulff sphere condition with radius r. Then there exists a positive constant C depending only on $n, \alpha, F, r, \mu(\partial\Omega)$, such that

$$\mathrm{dist}(\partial\Omega, \mathcal{W}) \leq C \left\| H - \frac{n}{n+1} \frac{\mu(\partial\Omega)}{|\Omega|} \right\|_{1,\partial\Omega}^{\frac{1}{n+2}}$$

for some Wulff sphere $\mathcal{W} \subset \mathbb{R}^{n+1}$, provided the RHS is sufficiently small.

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Application III: Serrin' problem

- Serrin's overdetermined problem asks which domains Ω allow solutions to

$$\begin{aligned} \Delta u &= 1 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \\ |\nabla u| &= c \quad \text{on } \partial\Omega. \end{aligned}$$

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- The anisotropic version (yielding equality to the Wulff shape)

$$\begin{aligned} \operatorname{div}(D^\sharp(\tfrac{1}{2}F^2)(D^\sharp f)) &= 1 \quad \text{in } \Omega \\ f &= 0 \quad \text{on } \partial\Omega \\ F(D^\sharp f) &= c \quad \text{on } \partial\Omega \end{aligned}$$

  is due to Cianchi/Salani.[4]

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$$\mathrm{dist}(\partial\Omega, \mathcal{W}) \leq C \left\| F(D^\sharp f) - \tfrac{|\Omega|}{\mu(\partial\Omega)} \right\|_{1,\partial\Omega}^{\frac{1}{2(n+2)}}$$

for some Wulff sphere $\mathcal{W} \subset \mathbb{R}^{n+1}$, provided the RHS is sufficiently small. Here

$$\begin{aligned} \mathrm{div}(D^\sharp(\tfrac{1}{2}F^2)(D^\sharp f)) &= 1 \quad \text{in } \Omega \\ f &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

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- Further computations lead to

$$\int_\Omega (-f)|\mathring{\bar{\nabla}}_F^2 f|^2 \, dx = \frac{1}{2} \int_M \left(F^2(D^\sharp f) - \frac{|\Omega|^2}{\mu(M)^2} \right) \left\langle \bar{\nabla}_F^\sharp f - \bar{\nabla}_F^\sharp \ell, \tilde{\nu} \right\rangle d\tilde{\mu},$$

where $\ell(x) = (F^0(x))^2/(2(n+1))$.

Thank you!