Chapter 1, problem #7

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our moving average process: $x_t = w_{t-1} + 2w_t + w_{t+1}$
w_t are independent w/ zero means and variance σ_w^2

autocovariance and autocorrelation functions?

$\gamma(t,s) = \text{cov}(x_t, x_s)$ and
$\text{cov}(x,x) = \text{var}(x)$
$\text{cov}(x+y, z) = \text{cov}(x,z) + \text{cov}(y,z)$
$\text{var}(cX) = c^2\,\text{var}(X),\ c \in \mathbb{R}^+$
$\text{cov}(cx_1, x_2) = c \cdot \text{cov}(x_1, x_2)$

- when $t=s$ (no lag, $h=0$)

$$\begin{aligned}\gamma(t,t) &= \text{cov}(w_{t-1} + 2w_t + w_{t+1},\ w_{t-1} + 2w_t + w_{t+1}) \\ &= \text{cov}(w_{t-1}, w_{t-1}) + \text{cov}(2w_t, 2w_t) + \text{cov}(w_{t+1}, w_{t+1}) \\ &= \text{var}(w_{t-1}) + \text{var}(2w_t) + \text{var}(w_{t+1}) \\ &= \sigma_w^2 + 4\sigma_w^2 + \sigma_w^2 = \boxed{6\sigma_w^2}\end{aligned}$$

- when $s = t+1$ (lag $= h = 1$)

$$\begin{aligned}\gamma(t,s) &= \text{cov}(w_{t-1} + 2w_t + w_{t+1},\ w_t + 2w_{t+1} + w_{t+2}) \\ &= \text{cov}(2w_t, w_t) + \text{cov}(w_{t+1}, 2w_{t+1}) \\ &= 2\,\text{var}(w_t) + 2\,\text{var}(w_{t+1}) = \boxed{4\sigma_w^2}\end{aligned}$$

Following this pattern, we get the autocovariance function:

$$\boxed{\gamma(h) = \begin{cases} 6\sigma_w^2 & h=0 \\ 4\sigma_w^2 & h=1 \\ \sigma_w^2 & h=2 \\ 0 & h>2 \end{cases}}$$

$*\ h = |s-t|$

Ch. 1, #7 continued

ACF is given by $\rho(s,t) = \dfrac{\gamma(s,t)}{\sqrt{\gamma(s,s)\,\gamma(t,t)}}$

- when $s=t$, $\gamma(s,t) = \gamma(0) = 6\sigma_w^2 = \gamma(s,s) = \gamma(t,t)$

$$\rho(s,t) = \frac{6\sigma_w^2}{\sqrt{6\sigma_w^2 \cdot 6\sigma_w^2}} = \frac{6\sigma_w^2}{6\sigma_w^2} = 1$$

this makes sense since the time series is not shifted

- when $s=t+1$, we use $\gamma(h=1) = 4\sigma_w^2$

$$\rho(t+1,t) = \frac{4\sigma_w^2}{6\sigma_w^2} = \frac{4}{6}$$

Following this pattern, our ACF becomes:

$$\boxed{\rho(h) = \begin{cases} 1 & h=0 \\ \frac{4}{6} & h=1 \\ \frac{1}{6} & h=2 \\ 0 & h>2 \end{cases}}$$