MATH 311 - Change of Basis

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The Matrix of a Linear Transformation

Recall (Theorem 20)

A map $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if and only if there exists an $m \times n$ matrix A such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$. Moreover,

$$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \dots & T(\vec{e}_n) \end{bmatrix} .$$

Motivation: Let V and W be finite dimensional vector spaces. Suppose $\dim(V) = n$ and $\dim(W) = m$. Then $V \cong \mathbb{R}^n$ and $W \cong \mathbb{R}^m$. Now, given a linear transformation $T : V \to W$, is there a way to understand T as a linear transformation $\mathbb{R}^n \to \mathbb{R}^m$?

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\cong"'] & W \arrow[d, "\cong"] \\ \mathbb{R}^n \arrow[r, dashed] & \mathbb{R}^m \end{tikzcd}$$

e.g. $T : P_2 \to M_{22}$

$$\begin{tikzcd} P_2 \arrow[r, "T"] \arrow[d, "\cong"'] & M_{22} \arrow[d, "\cong"] \\ \mathbb{R}^3 \arrow[r, dashed] & \mathbb{R}^4 \end{tikzcd}$$

Definition (Ordered Basis)

An ordered basis $\beta = \{\vec{b}_1, \ldots, \vec{b}_n\}$ of a vector space V is a basis for V in which the order of the elements listed in the set is fixed.

For example, $\{\vec{e}_1, \vec{e}_2\}$ and $\{\vec{e}_2, \vec{e}_1\}$ are diferrent ordered bases for $\mathbb{R}^2$. Why does this matter? Well, we'll take a look at **coordinate vectors** next.

Definition (Coordinate Vectors)

If $\beta = \{\vec{b}_1, \ldots, \vec{b}_n\}$ is an ordered basis for V, then for any $\vec{v} = c_1\vec{b}_1 + \ldots + c_n\vec{b}_n \in V$, the **coordinate vector** of $\vec{v}$ with respect to β is

$$\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n$$

denoted by

$$[\vec{v}]_\beta \quad \text{or} \quad C_\beta(\vec{v})\ .$$

1. Let $V = P_2$, $\beta = \{1, x, x^2\}$, $\alpha = \{x+1, x^2, 3\}$.

   (a) Find $[2x^2 + x - 1]_\beta$.

   We get that

$$\begin{aligned}[2x^2 + x - 1]_\beta &= [2(x^2) + 1(x) - 1(1)]_\beta \\ &= [-1(1) + 1(x) + 2 \cdot x^2]_\beta \\ &= \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \\ &\in \mathbb{R}^3.\end{aligned}$$

   (b) Find $[2x^2 + x - 1]_\alpha$. We get that

$$\begin{aligned}[2x^2 + x - 1]_\alpha &= \left[2(x^2) + 1(x+1) - \left(\frac{2}{3}\right) 3\right]_\alpha \\ &= \end{aligned}$$

Theorem 33

Let $\beta = \{\vec{b}_1, \ldots, \vec{b}_n\}$ be an ordered basis for V. The map

$$C_\beta : V \to \mathbb{R}^n \text{ given by } \vec{v} \mapsto [\vec{v}]_\beta$$

is an isomorphism (bijective transformation) with inverse

$$C_\beta^- 1 : \mathbb{R}^n \to V$$

given by

$$C_\beta^- 1 \left(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \right) = a_1\vec{b}_1 + \ldots + a_n\vec{b}_n \ .$$

Note: C_β is also denoted $[\ ]_\beta$. Similarly, C_β^{-1} is also denoted $[\ ]_\beta^{-1}$.

Proof. Suppose $\beta = \{\vec{b}_1, \ldots, \vec{b}_n\}$ is an ordered basis for a vector space V. We show C_β is an isomorphism (linear and bijective) and that C_β^{-1} is given by the formula above.

First, we prove that C_β is a linear combination by showing that C_β preserves vector addition and scalar multiplication. Let $\vec{v} = a_1\vec{b}_1 + \ldots + a_n\vec{b}_n$, where $a_1, \ldots, a_n \in \mathbb{R}$, and let $\vec{w} = c_1\vec{b}_1 + \ldots + c_n\vec{b}_n$, where $c_1, \ldots, c_n \in \mathbb{R}$, be arbitrary vectors in V. In other words, $\vec{v}$ and $\vec{w}$ are linear combinations of the vectors that make up the given basis for β. Then we have that

$$\begin{aligned}
C_\beta(\vec{v} + \vec{w}) &= C_\beta((a_1\vec{b}_1 + \ldots + a_n\vec{b}_n) + (c_1\vec{b}_1 + \ldots + c_n\vec{b}_n)) \\
&= C_\beta(a_1\vec{b}_1 + \ldots + a_n\vec{b}_n + c_1\vec{b}_1 + \ldots + c_n\vec{b}_n) \\
&= C_\beta((a_1 + c_1)\vec{b}_1 + \ldots (a_n + c_n)\vec{b}_n) \\
&= \begin{bmatrix} a_1 + c_1 \\ \vdots \\ a_n + c_n \end{bmatrix} \\
&= \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \\
&= C_\beta(a_1\vec{b}_1 + \ldots + a_n\vec{b}_n) + C_\beta(c_1\vec{b}_1 + \ldots + c_n\vec{b}_n) \\
&= C_\beta(\vec{v}) + C_\beta(\vec{w}) \ .
\end{aligned}$$

Hence, C_β preserves vector addition.

Now, let $\vec{v} \in V$ and $k \in \mathbb{R}$. Since $\vec{v} \in V$, this means that $\vec{v} = a_1\vec{b}_1 + \ldots + a_n\vec{b}_n$, where $a_1, \ldots, a_n \in \mathbb{R}$. Then

$$\begin{aligned}
C_\beta(k\vec{v}) &= C_\beta(k(a_1\vec{b}_1 + \ldots + a_n\vec{b}_n)) \\
&= c_\beta(ka_1\vec{b}_1 + \ldots + ka_n\vec{b}_n) \\
&= \begin{bmatrix} ka_1 \\ \vdots \\ ka_n \end{bmatrix} \\
&= k\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \\
&= kC_\beta(a_1\vec{b}_1 + \ldots + a_n\vec{b}_n) \\
&= kC_\beta(\vec{v}) \ .
\end{aligned}$$

Hence, C_β preserves scalar multiplication. Thus, C_β is a linear transformation.

Second, we show that C_β is an isomorphism. To show that C_β is isomorphism, we show that C_β is bijective. However, since $\dim(V) = n = \dim(\mathbb{R}^n)$, it is enough to show that **one of** one-to-one and onto. We will show that C_β is one-to-one by showing $\ker(C_\beta) = \{\vec{0}_V\}$. Let $\vec{v} \in V$. Then this means that $\vec{v} = a_1\vec{b}_1 + \ldots + a_n\vec{b}_n$, where $a_1, \ldots, a_n \in \mathbb{R}$. So,

$$\begin{aligned}
\ker(C_\beta) &= \{\vec{v} \in V : C_\beta(\vec{v}) = \vec{0}_{\mathbb{R}^n}\} \\
&= \{a_1\vec{b}_1 + \ldots + a_n\vec{b}_n : C_\beta(a_1\vec{b}_1 + \ldots + a_n\vec{b}_n) = \vec{0}_{\mathbb{R}^n}\} \\
&= \left\{ a_1\vec{b}_1 + \ldots + a_n\vec{b}_n : \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \right\} \ ,
\end{aligned}$$

which means that $a_1 = \ldots = a_n = 0$. So, we get that

$$\begin{aligned}
\ker(C_\beta) &= \{0\vec{b}_1 + \ldots + 0\vec{b}_n\} \\
&= \{\vec{0}_V\} \ .
\end{aligned}$$

Hence, C_β is one-to-one. Thus, C_β is an isomorphism.

Finally, We check that $C_\beta^{-1}(C_\beta(\vec{v})) = \vec{v}$ for all $\vec{v} \in V$ and $C_\beta(C_\beta^{-1}(\vec{x})) = \vec{x}$ for all $\vec{x} \in \mathbb{R}^n$. Let $\vec{v} \in V$ and $\vec{x} \in \mathbb{R}^n$. Then this means that $\vec{v} = a_1\vec{b}_1 + \ldots + a_n\vec{b}_n$

and $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$. So,

$$\begin{aligned} C_\beta^{-1}(C_\beta(\vec{v})) &= C_\beta^{-1}(C_\beta(a_1\vec{b}_1 + \ldots + a_n\vec{b}_n)) \\ &= C_\beta^{-1}\left(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}\right) \\ &= a_1\vec{b}_1 + \ldots + a_n\vec{b}_n \\ &= \vec{v} \end{aligned}$$

and

$$\begin{aligned} C_\beta(C_\beta^{-1}(\vec{x})) &= C_\beta\left(C_\beta^{-1}\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right)\right) \\ &= C_\beta(x_1\vec{b}_1 + \ldots + x_n\vec{b}_n) \\ &= \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\ &= \vec{x}\ . \end{aligned}$$

$\square$

Let's consider the matrix of $T : V \to W$ with respect to bases α and β.

Let V and W be vector spaces with bases $\alpha = \{\vec{v}_1, \ldots, \vec{v}_n\}$ and $\beta = \{\vec{w}_1, \ldots, \vec{w}_m\}$, respectively. Let $T : V \to W$ be a linear transformation. Consider the commuting diagram below

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, shift left, "C_\alpha"] & W \arrow[d, shift right, "C_\beta"'] \\ \mathbb{R}^n \arrow[u, shift left, "C_\alpha^{-1}"] \arrow[r, dashed] & \mathbb{R}^m \arrow[u, shift right, "C_\beta^{-1}"'] \end{tikzcd}$$

We can use the other notation to get the equivalent diagram

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, shift left, "{[\,]_\alpha}"] & W \arrow[d, shift right, "{[\,]_\beta}"'] \\ \mathbb{R}^n \arrow[u, shift left, "{[\,]_\alpha^{-1}}"] \arrow[r, dashed] & \mathbb{R}^m \arrow[u, shift right, "{[\,]_\beta^{-1}}"'] \end{tikzcd}$$

From the diagram, we get that

$$(C_\beta \circ T \circ C_\alpha^{-1}) : \mathbb{R}^n \to \mathbb{R}^m$$

is a linear map (since the composition of linear maps is also linear). So, this composition has a corresponding matrix, which we'll call $M_{\beta\alpha}$. So,

$$(C_\beta \circ T \circ C_\alpha^{-1})(\vec{x}) = M_{\beta\alpha}(\vec{x})$$

for all $\vec{x} \in \mathbb{R}^n$. Note that the right side of the equation is matrix multiplication. This is analagous to

$$T(\vec{x}) = A\vec{x}$$

for all $\vec{x} \in \mathbb{R}^n$ which we discussed earlier during the topic of linear transformations.

Question: How can we find the corresponding matrix $M_{\beta\alpha}$?

Answer: Take $\vec{x} = C_\alpha(\vec{v}) = [\vec{v}]_\alpha \in \mathbb{R}^n$ for $\vec{v} \in V$. Then

$$\begin{aligned}
M_{\beta\alpha}\vec{x} &= M_{\beta\alpha}C_\alpha(\vec{v}) \\
&= (C_\beta \circ T \circ C_\alpha^{-1})(C_\alpha(\vec{v})) \\
&= (C_\beta \circ T \circ C_\alpha^{-1} \circ C_\alpha)(\vec{v}) \\
&= C_\beta(T(C_\alpha^{-1}(C_\alpha(\vec{v})))) \\
&= C_\beta(T(\vec{v})) && \text{(since } C_\alpha^{-1}(C_\alpha(\vec{v})) = \vec{v}\text{)} \\
&= [T(\vec{v})]_\beta \ .
\end{aligned}$$

That is, the matrix $M_{\beta\alpha}$ is the coordinate vector of the transformation applied to $\vec{v}$ with respect to the basis β. Woah!!!!

In particular,

$$\begin{aligned}
[T(\vec{v}_i)]_\beta &= M_{\beta\alpha}C_\alpha(\vec{v}_i) \\
&= M_{\beta\alpha}[\vec{v}_i]_\alpha \\
&= M_{\beta\alpha}\vec{e}_i \ .
\end{aligned}$$

So the i^{th} column of $M_{\beta\alpha}$ is $[T(\vec{v}_i)]_\beta$:

$$M_{\beta\alpha} = \begin{bmatrix} | & | & & | \\ [T(\vec{v}_1)]_\beta & [T(\vec{v}_2)]_\beta & \dots & [T(\vec{v}_n)]_\beta \\ | & | & & | \end{bmatrix}$$

Remark: If ε is the standard basis, then $[\vec{v}]_\varepsilon = \vec{v}$.

1. Consider $T : P_2 \to \mathbb{R}^3$ given by $T(a + bx + cx^2) = \begin{bmatrix} a-c \\ b \\ 2a-c \end{bmatrix}$. Fix $\alpha = \{1, x, x^2\}$ and $\beta = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ as bases for P_2 and $\mathbb{R}^3$, respectively.

(a) Find the matrix $M_{\beta\alpha}$ associated to T.

Let's draw a commuting diagram.

$$\begin{tikzcd} P_2 \arrow[rr, "T"] \arrow[d, shift left, "C_\alpha"] & & \mathbb{R}^3 \arrow[d, shift right, "C_\beta"'] \\ \mathbb{R}^3 \arrow[u, shift left, "C_\alpha^{-1}"] \arrow[rr, dashed] & & \mathbb{R}^3 \arrow[u, shift right, "C_\beta^{-1}"'] \end{tikzcd}$$

First, find $T(\vec{v}_i)$ for all $1 \le i \le n$. Then find $[T(\vec{v}_i)]_\beta$ for all $1 \le i \le n$. So,

$$T(1) = T(1 + 0x + 0x^2) = \begin{bmatrix} 1-0 \\ 0 \\ 2(1)-0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix},$$
$$T(x) = T(0 + x + 0x^2) = \begin{bmatrix} 0-0 \\ 1 \\ 2(0)-0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$
$$T(x^2) = T(0 + 0x + x^2) = \begin{bmatrix} 0-1 \\ 0 \\ 2(0)-1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$$

Now,

$$T(1) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (-2) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$
$$T(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = (-1) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$
$$T(x^2) = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} = (-1) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + (-1) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Then this means that

$$[T(1)]_\beta = \left[\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}\right]_\beta = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix},$$

$$[T(x)]_\beta = \left[\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right]_\beta = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix},$$

$$[T(x^2)]_\beta = \left[\begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}\right]_\beta = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}.$$

Hence, the matrix $M_{\beta\alpha}$ associated with with the transformation T is

$$\begin{aligned} M_{\beta\alpha} &= \begin{bmatrix} | & | & | \\ [T(1)]_\beta & [T(x)]_\beta & [T(x^2)]_\beta \\ | & | & | \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 & -1 \\ -2 & 1 & 1 \\ 2 & 0 & -1 \end{bmatrix}. \end{aligned}$$

Indeed,

$$\begin{aligned} M_{\beta\alpha}(a + bx + cx^2) &= \begin{bmatrix} 1 & -1 & -1 \\ -2 & 1 & 1 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \\ &= \begin{bmatrix} a - b - c \\ -2a + b + c \\ 2a - c \end{bmatrix} \end{aligned}$$

and so

$$\begin{bmatrix} a - b - c \\ -2a + b + c \\ 2a - c \end{bmatrix} = (a - c) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (-b) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + (2a - c) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(b) If $\gamma = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$, then find $M_{\gamma\alpha}$.

Let's consider the commuting diagram.

$$\begin{tikzcd}
P_2 \arrow[rr, "T"] \arrow[d, shift left=2, "C_\alpha"] & & \mathbb{R}^3 \arrow[d, shift right=2, "C_\gamma"'] \\
\mathbb{R}^3 \arrow[u, shift left=2, "C_\alpha^{-1}"] \arrow[rr, dashed] & & \mathbb{R}^3 \arrow[u, shift right=2, "C_\gamma^{-1}"']
\end{tikzcd}$$

First, find $T(\vec{v}_i)$ for all $1 \leq i \leq 3$. Then find $[T(\vec{v}_i)]_\gamma$ for $1 \leq i \leq 3$. From part (a), we got that

$$T(1) = T(1 + 0x + 0x^2) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix},$$

$$T(x) = T(0 + x + 0x^2) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

$$T(x^2) = T(0 + 0x + x^2) = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}.$$

Then, expressing these as linear combination of the basis vectors in γ, we get that

$$T(1) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$T(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$T(x^2) = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} = (-1) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + (-1) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

So, the coefficients of these linear transformations of the basis vectors give us the coordinates of each transformation with respect to γ. That is,

$$
\begin{aligned}
[T(1)]_\gamma &= \left[\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}\right]_\gamma = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} , \\
[T(x)]_\gamma &= \left[\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right]_\gamma = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} , \\
[T(x^2)]_\gamma &= \left[\begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}\right]_\gamma = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} .
\end{aligned}
$$

Thus,

$$
\begin{aligned}
M_{\gamma\alpha} &= \begin{bmatrix} | & | & | \\ [T(1)]_\gamma & [T(x)]_\gamma & [T(x^2)]_\gamma \\ | & | & | \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix} .
\end{aligned}
$$

2. Let $\alpha = \{1, x, x^2\}$ and $\beta = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ be ordered bases for P_2 and $\mathbb{R}^2$, respectively. Suppose $T : P_2 \to \mathbb{R}^2$ has an associated matrix $M_{\beta\alpha} = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 0 & 1 \end{bmatrix}$. Find $T(a + bx + cx^2)$ where $a, b, c \in \mathbb{R}$.

We have the following commuting diagram:

$$\begin{tikzcd}
P_2 \arrow[rr, "T"] \arrow[d, shift left, "C_\alpha"] & & \mathbb{R}^2 \arrow[d, shift right, "C_\beta"'] \\
\mathbb{R}^3 \arrow[u, shift left, "C_\alpha^{-1}"] \arrow[rr, dashed] & & \mathbb{R}^2 \arrow[u, shift right, "C_\beta^{-1}"']
\end{tikzcd}$$

We know that

$$(C_\beta \circ T)(\vec{v}) = C_\beta(T(\vec{v})) = [T(\vec{v})]_\beta = M_{\beta\alpha}(\vec{v})$$

for all $\vec{v} = a + bx + cx^2 \in P_2$. So,

$$\begin{aligned}
C_\beta(T(\vec{v})) &= [T(\vec{v})]_\beta \\
&= M_{\beta\alpha}(\vec{v}) \\
&= \begin{bmatrix} 1 & 2 & -1 \\ -1 & 0 & 1 \end{bmatrix} (a + bx + cx^2) \\
&= \begin{bmatrix} 1 & 2 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \\
&= \begin{bmatrix} a + 2b - c \\ -a + c \end{bmatrix}.
\end{aligned}$$

Hence,

$$\begin{aligned}
T(\vec{v}) &= T(a + bx + cx^2) \\
&= (a + 2b - c) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (-a + c) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} a + 2b - c \\ a + 2b - c \end{bmatrix} + \begin{bmatrix} 0 \\ -a + c \end{bmatrix} \\
&= \begin{bmatrix} a + 2b - c \\ 2b \end{bmatrix}
\end{aligned}$$

is the transformation associated with the matrix $M_{\beta\alpha}$.

Theorem 34

Let U, V, and W be vector spaces with ordered bases α, β, and γ, respectively. Given the composition of linear maps

$$U \xrightarrow{T} V \xrightarrow{S} W \ ,$$

we have

$$M_{\gamma\alpha}(S \circ T) = M_{\gamma\beta}(S) M_{\beta\alpha}(T) \ .$$

$$(M_{\gamma\alpha} = M_{\gamma\beta} M_{\beta\alpha})$$

Proof. Suppose U, V, and W are vector spaces. Suppose α is an ordered basis for U, β is an ordered basis for V, and γ is an ordered basis for W. Then

$$M_{\gamma\alpha}(S \circ T)$$

□

Theorem 35

Let $T : V \to W$ be a linear transformation where $\dim(V) = \dim(W) < \infty$. Then the following are equivalent:

(1) T is an isomorphism.

(2) $M_{\beta\alpha}$ is invertible for **all** bases α and β of V and W, respectively.

(3) $M_{\beta\alpha}$ is invertible for **some** bases α and β of V and W, respectively.

Moreover, $(M_{\beta\alpha}(T))^{-1} = M_{\alpha\beta}(T^{-1})$, where $M_{\alpha\beta}$ is the matrix of $T^{-1} : W \to V$.

Proof. See Theorem 9.1.4 from textbook. $\square$

1. Let $\alpha = \{x^3, x^2, x, 1\}$ and $\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ be ordered bases for P_3 and M_{22}, respectively. Consider $T : P_3 \to M_{22}$ given by

$$T(ax^3 + bx^2 + cx + d) = \begin{bmatrix} a+d & b-c \\ b+c & a-d \end{bmatrix} .$$

(a) Find $M_{\beta\alpha}(T)$.

We know that

$$M_{\beta\alpha} = \begin{bmatrix} | & | & | & | \\ [T(x^3)]_\beta & [T(x^2)]_\beta & [T(x)]_\beta & [T(1)]_\beta \\ | & | & | & | \end{bmatrix}$$

First, we find the transformations. So,

$$\begin{aligned} T(x^3) &= T(x^3 + 0x^2 + 0x + 0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} , \\ T(x^2) &= T(0x^3 + x^2 + 0x + 0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \\ T(x) &= T(0x^3 + 0x^2 + x + 0) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} , \\ T(1) &= T(0x^3 + 0x^2 + 0x + 1) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} . \end{aligned}$$

Then, we can express each of these transformations as linear combinations of the basis vectors of β (since we're trying to find the coordinates with respect to β). So,

$$\begin{aligned} T(x^3) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 1 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} , \\ T(x^2) &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 0 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 1 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 1 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} , \\ T(x) &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = 0 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (-1) \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 1 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} , \\ T(1) &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = 1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + (-1) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} . \end{aligned}$$

From this, we can get the coordinates of each transformation from the coefficients of the linear combinations. So,

$$[T(x^3)]_\beta = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} ,$$

$$[T(x^2)]_\beta = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} ,$$

$$[T(x)]_\beta = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} ,$$

$$[T(1)]_\beta = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} .$$

Hence,

$$\begin{aligned} M_{\beta\alpha} &= \begin{bmatrix} | & | & | & | \\ [T(x^3)]_\beta & [T(x^2)]_\beta & [T(x)]_\beta & [T(1)]_\beta \\ | & | & | & | \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} . \end{aligned}$$

(b) Verify that T is an isomorphism by proving that $M_{\beta\alpha}$ is invertible.

To check that T is an isomorphism, we check that the columns are orthogonal, and hence linearly independent.

(c) Find $(M_{\beta\alpha}(T))^{-1}$ and use this to find a formula for $T^{-1} : M_{22} \to P_3$

Operators and Similarity

Let V be a vector space.

Definition (Linear Operator)

A linear transformation $T : V \to V$ is called a **linear operator on V**. The set of all linear operators on V is denoted by $\mathcal{L}(\vec{v})$.

Let β be an ordered basis for V.

Definition (β-matrix of T)

If $T : V \to V$ is a linear operator, define $M_\beta(T) = M_{\beta\beta}(T)$ and call this the β-matrix of T. We have

$$M_\beta(T) = \begin{bmatrix} | & & | \\ [T(\vec{v}_1)]_\beta & \dots & [T(\vec{v}_n)]_\beta \\ | & & | \end{bmatrix} ,$$

where $\beta = \{\vec{v}_1, \dots, \vec{v}_n\}$ and $[T(\vec{v})]_\beta = M_\beta[\vec{v}]_\beta$.

1. Consider the linear operator $T : P_2 \to P_2$ given by $T(ax^2 + bx + c) = a - bx + cx^2$. Find the β-matrix of T with respect to $\beta = \{x^2, x, 1\}$.

First, apply the transformation to each element of β.

$$\begin{aligned} T(x^2) &= T(x^2 + 0x + 0) = 1 - 0x + 0x^2 = 1 \ , \\ T(x) &= T(0x^2 + x + 0) = 0 - x + 0x^2 = -x \ , \\ T(1) &= T(0x^2 + 0x + 1) = 0 - 0x + x^2 = x^2 \ . \end{aligned}$$

Then, express each transformation above as a vector, for convenience.

$$T(x^2) = 1 = 0x^2 - 0x + 1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \ ,$$

$$T(x) = -x = 0x^2 - x + 0 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \ ,$$

$$T(1) = x^2 = x^2 - 0x + 0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \ .$$

Now. express these transformations as linear combinations of the elements of β (the elements of β are also expressed as vectors).

$$T(x^2) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

where $a = 0$, $b = 0$, and $c = 1$,

$$T(x) = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = d \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + e \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + f \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

where $d = 0$, $e = -1$, and $f = 0$, and

$$T(1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = g \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + h \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

where $g = 1$, $h = 0$, and $i = 0$. So,

$$[T(x^2)]_\beta = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \ , \ [T(x)]_\beta = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \ , \ [T(1)]_\beta = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \ .$$

Thus,

$$
\begin{aligned}
M_{\beta\beta} &= M_\beta \\
&= \begin{bmatrix} | & | & | \\ [T(x^2)]_\beta & [T(x)]_\beta & [T(1)]_\beta \\ | & | & | \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

is the β-matrix of T with respect to $\beta = \{x^2, x, 1\}$.

2. Let $\beta = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ be an ordered basis for $\mathbb{R}^2$. Find $M_\beta = M_{\beta\beta}(T)$ for

$$T\left(\begin{bmatrix} a \\ b \end{bmatrix} \right) = \begin{bmatrix} a+b \\ a-b \end{bmatrix}$$

Change of Basis

Question: Say we have two ordered bases B and D for a vector space V. Can we change from B-coordinates to D-coordinates? How? For all $\vec{v} \in V$, can we find a matrix P_{DB} such that

$$[\vec{v}]_D = P_{DB}[\vec{v}]_D \ ?$$

> **Definition (Identity Operator)**
>
> The identity operator on V is the linear transformation given by
>
> $$1_V : V \to V \ , \ 1_V(\vec{v}) = \vec{v}$$
>
> for all $\vec{v} \in V$.

> **Definition (Change of Basis Matrix)**
>
> Let B and D be ordered bases for a vector space V. Then the change of basis matrix from B to D is $P_{D \leftarrow B}$, or P_{DB}, is $M_{DB}(\vec{1_V})$.

1. Let $B = \{\vec{e}_1, \vec{e}_2\}$ and $D = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ be ordered bases for $\mathbb{R}^2$. Find the change of basis matrix from B to D. That is, find P_{DB}.

   We have that

$$\begin{aligned} P_{DB} &= M_{DB}(1_{\mathbb{R}^2}) \\ &= \begin{bmatrix} | & | \\ [1_{\mathbb{R}^2}(\vec{e}_1)]_D & [1_{\mathbb{R}}(\vec{e}_2)] \\ | & | \end{bmatrix} . \end{aligned}$$

   Fron here, we find $1_{\mathbb{R}^2}(\vec{e}_1)$ and $1_{\mathbb{R}^2}(\vec{e}_2)$. So,

$$\begin{aligned} 1_{\mathbb{R}^2}(\vec{e}_1) &= \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} , \\ 1_{\mathbb{R}^2}(\vec{e}_2) &= \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} . \end{aligned}$$

   Then we express these transformations as linear combinations of the elements of the basis D.

$$1_{\mathbb{R}^2}(\vec{e}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

   where $a = \frac{1}{2}$ and $b = \frac{1}{2}$, and

$$1_{\mathbb{R}^2}(\vec{e}_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = c \begin{bmatrix} 1 \\ 1 \end{bmatrix} + d \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

   where $c = \frac{1}{2}$ and $d = -\frac{1}{2}$. So, we get that

$$\begin{aligned} [1_{\mathbb{R}^2}(\vec{e}_1)]_D &= \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} , \\ [1_{\mathbb{R}^2}(\vec{e}_2)]_D &= \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix} . \end{aligned}$$

   Hence,

$$\begin{aligned} P_{DB} &= M_{DB}(1_{\mathbb{R}^2}) \\ &= \begin{bmatrix} | & | \\ [1_{\mathbb{R}^2}(\vec{e}_1)]_D & [1_{\mathbb{R}}(\vec{e}_2)] \\ | & | \end{bmatrix} \\ &= \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \end{aligned}$$

   is the change of basis matrix from B to D.

2. Let $B = \{1, x, x^2\}$ and $D = \{1, 1-x, 1-x^2\}$ be ordered bases for P_2. Find P_{DB} and use it to express $p(x) = a + bx + cx^2$ as a linear combination of vectors in D.

We have that

$$\begin{aligned} P_{DB} &= M_{DB}(1_{P_2}) \\ &= \begin{bmatrix} | & | & | \\ [1_{P_2}(1)]_D & [1_{P_2}(x)]_D & [1_{P_2}(x^2)]_D \\ | & | & | \end{bmatrix}. \end{aligned}$$

First, we find $1_{P_2}(1)$, $1_{P_2}(x)$, and $1_{P_2}(x^2)$.

$$\begin{aligned} 1_{P_2}(1) = 1 &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \\ 1_{P_2}(x) = x &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \\ 1_{P_2}(x^2) = x^2 &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \end{aligned}$$

Then we express these transformations as linear combinations of the elements of the basis D. Note that $1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $1 - x = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, and $1 - x^2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. So,

$$1_{P_2}(1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

where $a = 1$, $b = 0$, and $c = 0$,

$$1_{P_2}(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = d \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + e \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + f \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

where $d = 1$, $e = -1$, and $f = 0$, and

$$1_{P_2}(x^2) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = g \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + h \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

where $g = 1$, $h = 0$, and $i = -1$.

Then this means that

$$[1_{P_2}(1)]_D = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} ,$$

$$[1_{P_2}(x)]_D = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} ,$$

$$[1_{P_2}(x^2)]_D = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} .$$

So, we get that

$$\begin{aligned} P_{DB} &= M_{DB}(1_{P_2}) \\ &= \begin{bmatrix} | & | & | \\ [1_{P_2}(1)]_D & [1_{P_2}(x)]_D & [1_{P_2}(x^2)]_D \\ | & | & | \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \end{aligned}$$

is the change of basis matrix from B to D. Now, we got from before the following linear transformations.

$$\begin{aligned} 1 &= 1(1) + 0(1-x) + 0(1-x^2) , \\ x &= 1(1) + (-1)(1-x) + 0(1-x^2) , \\ x^2 &= 1(1) + 0(1-x) + (-1)(1-x^2) \end{aligned}$$

So, for $p(x) = a + bx + cx^2$, where $a, b, c \in \mathbb{R}$, we have that $[p(x)]_\beta = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

Then $[p(x)]_D = P_{DB}[p(x)]_B$. Hence,

$$\begin{aligned} [p(x)]_D &= \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \\ &= \begin{bmatrix} a+b+c \\ -b \\ -c \end{bmatrix} . \end{aligned}$$

Question: Let $T : V \to V$ be a linear operator on V and let B and D be ordered bases for V. How are $M_B(T)$ and $M_D(T)$ related?

1. Let

$$B = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\}$$

and

$$D = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

be ordered bases for M_{22}. Find $M_B(T)$ and $M_D(T)$ for

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} a+d & b+c \\ a+c & b+d \end{bmatrix} .$$

For $M_B(T)$, We have that

$$T\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 1 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + 1 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + 0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 0 \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} ,$$

$$T\left(\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 1 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + 1 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + 0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 0 \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} ,$$

$$T\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} = 2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + 3 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + 0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (-2) \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} ,$$

$$T\left(\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} = 0 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + 1 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + 1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 2 \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} .$$

So, this means that

$$M_B(T) = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 1 & 1 & 3 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & 2 \end{bmatrix}$$

Now, for $M_D(T)$ we have that

$$T\left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\right) = \begin{bmatrix}1 & 0\\1 & 0\end{bmatrix} = 1\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + 0\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} + 0\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} + 0\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\ ,$$

$$T\left(\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\right) = \begin{bmatrix}0 & 1\\0 & 1\end{bmatrix} = 0\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + 1\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} + 0\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} + 1\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\ ,$$

$$T\left(\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}\right) = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} = 0\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + 1\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} + 1\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} + 0\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\ ,$$

$$T\left(\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right) = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = 1\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + 0\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} + 0\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} + 1\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\ .$$

So, this means that

$$\begin{aligned} M_D(T) &= \begin{bmatrix} | & | & | & | \\ \left[T\left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\right)\right]_D & \left[T\left(\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\right)\right]_D & \left[T\left(\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}\right)\right]_D & \left[T\left(\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right)\right]_D \\ | & | & | & | \end{bmatrix} \\ &= \begin{bmatrix}1 & 0 & 0 & 1\\0 & 1 & 1 & 0\\0 & 0 & 1 & 0\\0 & 1 & 0 & 1\end{bmatrix}\ . \end{aligned}$$

Observation: $(P_{DB})^{-1} = P_{BD}$.

Theorem 36

Let B and D be ordered bases for V and let $T : V \to V$ be a linear operator. Then $M_B(T)$ and $M_D(T)$ are similar matrices. Moreover, we have

$$M_D(T) = (P_{DB})^{-1} M_D(T) P_{DB}$$

Proof. See Theorem 9.2.3. □