MATH 311 - Abstract Vector Spaces

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Examples and Basic Properties

Definition (Vector Space)

A vector space (over $\mathbb{R}$) is a non-empty set V with a rule "+" for addition and a rule "*" for scalar multiplication that satisfies the following:

For all $\vec{u}, \vec{v}, \vec{w} \in V$ and for all $k, l \in \mathbb{R}$,

(1) $\vec{u} + \vec{w} \in V$ (closure under addition)

(2) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (addition is commutative)

(3) $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$ (addition is associative)

(4) $\exists \vec{0} \in V$ such that $\vec{v} + \vec{0} = \vec{v} = \vec{0} + \vec{v}$ (additive identity)

(5) $\exists - \vec{v} \in V$ such that $\vec{v} + (-\vec{v}) = \vec{0} = (-\vec{v}) + \vec{v}$ (additive inverse)

(6) $k\vec{v} \in V$ (closure under scalar multiplication)

(7) $(k + l)\vec{v} = k\vec{v} + l\vec{v}$ (* is distributive)

(8) $k(\vec{v} + \vec{w}) = k\vec{v} + k\vec{w}$ (* is vector distributive)

(9) $(kl)\vec{v} = k(l\vec{v})$ (* is associative)

(10) $1\vec{v} = \vec{v}$ (* identity)

Note: The "*" for scalar multiplication has been omitted above for clarity purposes.

Elements of V are called vectors.

Remark: If the scalars are in another field, say $\mathbb{F}$, then we say the "vector space over $\mathbb{F}$". (e.g. $\mathbb{Q}$, $\mathbb{C}$)

Observation: $\mathbb{R}^n$ with usual addition and usual scalar multiplication is a vector space over $\mathbb{R}$.

1. $V = \{x \in \mathbb{R} \mid x > 0\}$. Define "addition" in V by $x \oplus y = xy$ and "scalar multiplication" by $c \odot x = x^c$ for all $x, y \in \mathbb{R}$, $c \in \mathbb{R}$. Prove that V is a vector space over $\mathbb{R}$ under these operations.

Proof. Let $x, y, z \in V$ and $k, l \in \mathbb{R}$.

(1) We show that $x \oplus y \in V$. Since $x, y \in V$, this means that $x, y \in \mathbb{R}$ such that $x > 0$ and $y > 0$. Then it follows that

$$x \oplus y = xy \ ,$$

where $xy \in \mathbb{R}$ (since $\mathbb{R}$ is closed under multiplication). Now, since $x > 0$ and $y > 0$, we get that $xy > 0$. Hence, since $xy \in \mathbb{R}$ and $xy > 0$, $x \oplus y \in V$.

(2) We show that $x \oplus y = y \oplus x$.

$$x \oplus y = xy = yx = y \oplus x \ .$$

(3) We show that $x \oplus (y \oplus z) = (x \oplus y) \oplus z$.

$$\begin{aligned} x \oplus (y \oplus z) &= x \oplus (yz) \\ &= x(yz) \\ &= (xy)z \\ &= (x \oplus y)z \\ &= (x \oplus y) \oplus z \ . \end{aligned}$$

(4) Claim: $\exists \vec{0} = 1 \in V$ such that $x \oplus 1 = 1 \oplus x$. Proof of Claim: First, $1 \in \mathbb{R}$ and $1 > 0$, so it holds that $1 \in V$. Then

$$x \oplus 1 = x1 = x = 1x = 1 \oplus x \ .$$

(5) Claim: $\forall x \in V$, $\exists \frac{1}{x} \in V$ such that $x \oplus \frac{1}{x} = 1 = \frac{1}{x} \oplus x$. Proof of Claim: Since $x \in V$, $x \in \mathbb{R}$ and $x > 0$. Then it follows that $\frac{1}{x} \in \mathbb{R}$ and $\frac{1}{x} > 0$. So, we have that $\frac{1}{x} \in V$ because

$$\begin{aligned} x \oplus \frac{1}{x} &= x\frac{1}{x} \\ &= 1 \\ &= \frac{1}{x}x \\ &= \frac{1}{x} \oplus x \ . \end{aligned}$$

(6) We prove that $k \odot x \in V$. Since $x \in V$, this means that $x \in \mathbb{R}$ and $x > 0$. . Then $k \odot x = x^k$. Now, since $x \in \mathbb{R}$ and $k \in \mathbb{R}$, it holds that $x^k \in \mathbb{R}$. Also, since $x > 0$, it holds that $x^k > 0$. Hence, since $x^k \in \mathbb{R}$ and $x^k > 0$, $x^k \in V$. Therefore, $k \odot x \in V$.

(7) We show that $(k \oplus l) \odot x = (k \odot x) \oplus (l \odot x)$.

$$\begin{aligned} (k \oplus l) \odot x &= (kl) \odot x \\ &= x^{kl} \\ &= x^k x^l \\ &= x^k \oplus x^l \\ &= (k \odot x) \oplus (l \odot x) \end{aligned}$$

(8) We show that $k \odot (x \oplus y) = (k \odot x) \oplus (k \odot y)$.

$$\begin{aligned} k \odot (x \oplus y) &= k \odot (xy) \\ &= (xy)^k \\ &= x^k y^k \\ &= x^k \oplus y^k \\ &= (k \odot x) \oplus (k \odot y) \ . \end{aligned}$$

(9) We show that $(kl) \odot x = k \odot (l \odot x)$. Note that kl is multiplication of real numbers, so we don't use the $\odot$ rule there.

$$\begin{aligned} (kl) \odot x &= x^{kl} \\ &= x^{lk} \\ &= (x^l)^k \\ &= k \odot (x^l) \\ &= k \odot (l \odot x) \ . \end{aligned}$$

(10) We show that $1 \odot x = x$. Note here that 1 is just the number one. So,

$$1 \odot x = x^1 = x \ .$$

Now, since V satisfies all 10 vector space axioms, we can conclude that V is indeed a vector space (over $\mathbb{R}$) under the operations defined above. $\square$

2. Let $V = \mathbb{R}^2$ and define addition by

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 + 1 \\ 2y_1 + 3y_2 - 4 \end{bmatrix}$$

and define mulitplication by

$$k * \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_1/k \\ y_1 k^2 \end{bmatrix}$$

for all

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \in \mathbb{R}^2$$

and

$$0 \neq k \in \mathbb{R} \ .$$

Is V a vector space (over $\mathbb{R}$) with respect to these operations?

Proof. To show V is a vector space, we show that all 10 axioms hold. To show that V is not a vector space, it suffices to show one axiom that doesn't hold. We will try them all for fun. Suppose $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in V$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in V$, $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in V$, and $k, l \in \mathbb{R}$.

(1) V is closed under addition.

$$\begin{aligned} \vec{u} + \vec{v} &= \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ &= \begin{bmatrix} u_1 - 2v_1 + 1 \\ 2u_2 + 3v_2 - 4 \end{bmatrix} \\ &\in \mathbb{R}^2 \ . \end{aligned}$$

Hence, since $V = \mathbb{R}^2$, $\vec{u} + \vec{v} \in V$.

(2) Is it true that $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (addition is commutative)? We see that

$$\begin{aligned} \vec{u} + \vec{v} &= \begin{bmatrix} u_1 - 2v_2 + 1 \\ 2u_2 + 3v_2 - 4 \end{bmatrix} \\ &\neq \begin{bmatrix} v_1 - 2u_2 + 1 \\ 2v_2 + 3u_2 - 4 \end{bmatrix} \\ &= \vec{v} + \vec{u} \end{aligned}$$

We show a counterexample to prove that addition is not commutative in V.

Choose $\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then

$$\begin{aligned} \vec{u} + \vec{v} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 - 2(0) + 1 \\ 2(0) + 3(1) - 4 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ &\neq \begin{bmatrix} -1 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} 0 - 2(1) + 1 \\ 2(1) + 3(0) - 4 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \vec{v} + \vec{u} \ . \end{aligned}$$

Hence, addition is not commutative in V.

(3) Is it true that $\vec{u}+(\vec{v}+\vec{w})=(\vec{u}+\vec{v})+\vec{w}$ (addtion is associative)?

$$\begin{aligned}
\vec{u}+(\vec{v}+\vec{w}) &= \begin{bmatrix}u_1\\u_2\end{bmatrix}+\left(\begin{bmatrix}v_1\\v_2\end{bmatrix}+\begin{bmatrix}w_1\\w_2\end{bmatrix}\right)\\
&= \begin{bmatrix}u_1\\u_2\end{bmatrix}+\begin{bmatrix}v_1-2w_1+1\\2v_1+3w_2-4\end{bmatrix}\\
&= \begin{bmatrix}u_1-2(v_1-2w_1+1)+1\\2u_2+3(2v_1+3w_2-4)-4\end{bmatrix}\\
&= \begin{bmatrix}u_1-2v_1+4w_1-2+1\\2u_2+6v_1+9w_2-12-4\end{bmatrix}\\
&= \begin{bmatrix}u_1-2v_1+4w_1-1\\2u_2+6v_1+9w_2-16\end{bmatrix}\\
&\neq \begin{bmatrix}u_1-2v_1-2w_1+2\\4u_2+6v_2+3w_2-12\end{bmatrix}\\
&= \begin{bmatrix}u_1-2v_1+1-2w_1+1\\4u_2+6v_2-8+3w_2-4\end{bmatrix}\\
&= \begin{bmatrix}(u_1-2v_1+1)-2(w_1)+1\\2(2u_2+3v_2-4)+3(w_2)-4\end{bmatrix}\\
&= \begin{bmatrix}u_1-2v_1+1\\2u_2+3v_2-4\end{bmatrix}+\begin{bmatrix}w_1\\w_2\end{bmatrix}\\
&= \left(\begin{bmatrix}u_1\\u_2\end{bmatrix}+\begin{bmatrix}v_1\\v_2\end{bmatrix}\right)+\begin{bmatrix}w_1\\w_2\end{bmatrix}\\
&= (\vec{u}+\vec{v})+\vec{w}
\end{aligned}$$

We can see this better with a counterexample. Choose $\vec{u}=\begin{bmatrix}1\\0\end{bmatrix}$, $\vec{v}=\begin{bmatrix}0\\1\end{bmatrix}$, and $\vec{w}=\begin{bmatrix}1\\2\end{bmatrix}$. Then (see next page)

$$
\begin{aligned}
\vec{u} + (\vec{v} + \vec{w}) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right) \\
&= \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 - 2(1) + 1 \\ 2(1) + 3(2) - 4 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 4 \end{bmatrix} \\
&= \begin{bmatrix} 1 - 2(-1) + 1 \\ 2(0) + 3(4) - 4 \end{bmatrix} \\
&= \begin{bmatrix} 4 \\ 8 \end{bmatrix} \\
&\neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 2 - 2(1) + 1 \\ 2(-1) + 3(2) - 4 \end{bmatrix} \\
&= \begin{bmatrix} 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} 1 - 2(0) + 1 \\ 2(0) + 3(1) - 4 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\
&= \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\
&= (\vec{u} + \vec{v}) + \vec{w}
\end{aligned}
$$

Hence, addition is not associative in V.

(4) Is it true that $\exists\vec{0}$ such that $\vec{u}+\vec{0}=\vec{0}+\vec{u}$. Yes. This identity in V in this case would be $\vec{0}=\begin{bmatrix}0\\0\end{bmatrix}$. So,

$$\begin{aligned}
\vec{u}+\vec{0} &= \begin{bmatrix}u_1\\u_2\end{bmatrix}+\begin{bmatrix}0\\0\end{bmatrix}\\
&= \begin{bmatrix}u_1-2(0)+1\\2u_2+3(0)-4\end{bmatrix}\\
&= \begin{bmatrix}u_1+1\\2u_2-4\end{bmatrix}\\
&\neq \begin{bmatrix}-2u_1+1\\3u_2-4\end{bmatrix}\\
&= \begin{bmatrix}0-2(u_1)+1\\2(0)+3(u_2)-4\end{bmatrix}\\
&= \begin{bmatrix}0\\0\end{bmatrix}+\begin{bmatrix}u_1\\u_2\end{bmatrix}\\
&= \vec{0}+\vec{u}
\end{aligned}$$

(5) Is it true that $\exists(-\vec{u}) \in V$ such that $\vec{u} + (-\vec{u}) = \vec{0} = (-\vec{u}) + \vec{u}$?

$$\begin{aligned}
\vec{u} + (-\vec{u}) &= \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \left(- \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right) \\
&= \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} -u_1 \\ -u_2 \end{bmatrix} \\
&= \begin{bmatrix} u_1 - 2(-u_1) + 1 \\ 2u_2 + 3(-u_2) - 4 \end{bmatrix} \\
&= \begin{bmatrix} u_1 + 2u_1 + 1 \\ 2u_2 - 3u_2 - 4 \end{bmatrix} \\
&\neq \begin{bmatrix} -u_1 - 2u_1 + 1 \\ -2u_2 + 3u_2 - 4 \end{bmatrix} \\
&= \begin{bmatrix} -u_1 - 2u_1 + 1 \\ 2(-u_2) + 3u_2 - 4 \end{bmatrix} \\
&= \begin{bmatrix} -u_1 \\ -u_2 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\
&= \left(- \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right) + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\
&= (-\vec{u}) + \vec{u} \ .
\end{aligned}$$

(6) Is V closed under multiplication; that is, $k * \vec{u} \in V$?

$$k * \vec{u} = \begin{bmatrix} u_1/k \\ u_2k^2 \end{bmatrix} \in \mathbb{R}^2 = V \ .$$

where $\frac{u_1}{k}, u_2k^2 \in \mathbb{R}$. Hence, V is closed under multiplication.

(7) Is it true that $(k + l) * \vec{u} = (k * \vec{u}) + (k * \vec{l})$ (* is distributive)? Note here that the "+" in $k+l$ is the usual addition between real numbers.

$$\begin{aligned}
(k + l) * \vec{u} &= (k + l) * \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\
&= \begin{bmatrix} u_1/(k + l) \\ u_2(k + l)^2 \end{bmatrix} \\
&\neq \begin{bmatrix} u_1/k + u_1/l \\ u_2k^2 + u_2l^2 \end{bmatrix} \\
&= \begin{bmatrix} u_1/k \\ u_2k^2 \end{bmatrix} + \begin{bmatrix} u_1/l \\ u_2l^2 \end{bmatrix} \\
&= \left(k * \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) + \left(l * \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) \\
&= (k * \vec{u}) + (l * \vec{u}) \ .
\end{aligned}$$

For example, choose $k = 1$, $l = 2$, and $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Then

$$\begin{aligned}
(k + l) * \vec{u} &= (1 + 2) * \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\
&= 3 * \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} 1/3 \\ 2(3)^2 \end{bmatrix} \\
&= \begin{bmatrix} 1/3 \\ 18 \end{bmatrix} \\
&\neq \begin{bmatrix} 3/2 \\ 10 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1/2 \\ 8 \end{bmatrix} \\
&= \begin{bmatrix} 1/1 \\ 2(1)^2 \end{bmatrix} + \begin{bmatrix} 1/2 \\ 2(2)^2 \end{bmatrix} \\
&= \left(1 * \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) + \left(2 * \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) \\
&= (k * \vec{u}) + (l * \vec{u})
\end{aligned}$$

Hence, * is not distributive.

(8) Is it true that $k * (\vec{u} + \vec{v}) = (k * \vec{u}) + (k * \vec{v})$ (* is vector distributive)?

$$\begin{aligned}
k * (\vec{u} + \vec{v}) &= k * \left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right) \\
&= k * \begin{bmatrix} u_1 - 2v_1 + 1 \\ 2u_2 + 3v_2 - 4 \end{bmatrix} \\
&= \begin{bmatrix} (u_1 - 2v_1 + 1)/k \\ (2u_2 + 3v_2 - 4)k^2 \end{bmatrix} \\
&\neq \begin{bmatrix} (u_1 + v_1)/k \\ (u_2 + v_2)k^2 \end{bmatrix} \\
&= \begin{bmatrix} u_1/k \\ u_2 k^2 \end{bmatrix} + \begin{bmatrix} v_1/k \\ v_2 k^2 \end{bmatrix} \\
&= \left(k * \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right) + \left(k * \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right) \\
&= (k * \vec{u}) + (k * \vec{v})
\end{aligned}$$

For example, choose $k = 2$, $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Then

$$\begin{aligned}
k * (\vec{u} + \vec{v}) &= 2 * \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right) \\
&= 2 * \begin{bmatrix} 1 - 2(1) + 1 \\ 2(1) + 3(2) - 4 \end{bmatrix} \\
&= 2 * \begin{bmatrix} 0 \\ 4 \end{bmatrix} \\
&= \begin{bmatrix} 0/2 \\ 4(2)^2 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 16 \end{bmatrix} \\
&\neq \begin{bmatrix} 1/2 \\ 22 \end{bmatrix} \\
&= \begin{bmatrix} 1/2 - 2(1/2) + 1 \\ 2(1) + 3(8) - 4 \end{bmatrix} \\
&= \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1/2 \\ 8 \end{bmatrix} \\
&= \begin{bmatrix} 1/2 \\ 1(2)^2 \end{bmatrix} + \begin{bmatrix} 1/2 \\ 2(2)^2 \end{bmatrix} \\
&= \left(2 * \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) + \left(2 * \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)
\end{aligned}$$

$$= (k * \vec{u}) + (k * \vec{v}) \ .$$

Hence, * is not vector distributive.

(9) Is it true that $(kl) * \vec{u} = k * (l * \vec{u})$. Note that kl is just the usual multiplication of two real numbers.

$$
\begin{aligned}
(kl) * \vec{u} &= (kl) * \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\
&= \begin{bmatrix} u_1/(kl) \\ u_2(kl)^2 \end{bmatrix} \\
&= \begin{bmatrix} u_1/lk \\ u_2 l^2 k^2 \end{bmatrix} \\
&= k * \begin{bmatrix} u_1/l \\ u_2 l^2 \end{bmatrix} \\
&= k * \left(l * \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right) \\
&= k * (l * \vec{u}) \ .
\end{aligned}
$$

Hence, * is associative.

(10) Is it true that $1 * \vec{u} = \vec{u}$ (* identity)?

$$
\begin{aligned}
1 * \vec{u} &= 1 * \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\
&= \begin{bmatrix} u_1/1 \\ u_2(1)^2 \end{bmatrix} \\
&= \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\
&= \vec{u} \ .
\end{aligned}
$$

Hence, V has a * identity.

Thus, since we have shown that at least one of these 10 axioms does not hold, we can conclude that V is not a vector space. $\square$

Theorem 18

The following sets are vector spaces (over $\mathbb{R}$).

(1) The set of all $m \times n$ matrices over $\mathbb{R}$, $M_{m\times n}(\mathbb{R})$, with the usual addition and scalar multiplication of matrices.

(2) The set $F(-\infty, \infty)$ of all real-valued functions on $\mathbb{R} = (-\infty, \infty)$ with the usual addition and scalar multiplication of functions. Here $f + g$ and kf are defined by

$$(f + g)(x) = f(x) + g(x)$$

and

$$(kf)(x) = kf(X)$$

for all $x \in \mathbb{R}$, $f, g \in F(-\infty, \infty)$, and $k \in \mathbb{R}$.

Remark: The definitions of span, linear independence, subspace, basis, and dimension are all exactly the same if you replace $\mathbb{R}$ with "a vector space V".

Let V be a vector space (over $\mathbb{R}$).

Definition (Subspace)

A subset U of the vector space V is called a subspace of V if U itself is a vector space under the addition and scalar multiplication on V.

Theorem (Subspace Test)

A subset U of a vector space V is a subspace of V iff

(1) U contains the zero vector.

(2) U is closed under addition.

(3) U is closed under scalar multiplication.

1. The set of $P_n(\mathbb{R})$ of polynomials in one variable with coefficients in $\mathbb{R}$ and with degree at most n is a vector space under the usual addition and scalar multiplication of polynomials.

   E.g. $P_n(\mathbb{R})$ is a subspace of $F(-\infty,\infty)$, the space of real valued functions on $(-\infty,\infty)$.

   Show that $P_2(\mathbb{R})$ is a subspace of $F(-\infty,\infty)$ using the subspace test.

   Solution: Note that $P_2(\mathbb{R})$ is the set of polynomials of degree at most 2 with coefficients in $\mathbb{R}$. That is, $P_2(\mathbb{R}) = \{a+bx+cx^2 \mid a,b,c \in \mathbb{R}^2\}$. We show that $P_2(\mathbb{R})$ contains the zero vector (in this case the zero polynomial $0+0x+0x^2$), $P_2(\mathbb{R})$ is closed under addition, and $P_2(\mathbb{R})$ is closed under scalar multiplication.

   (1) $0 \in P_2(\mathbb{R})$, since there exist $a,b,c \in \mathbb{R}$ such that $a+bx+cx^2 = 0$. In this case, $a=b=c=0$, as $a+bx+cx^2 = 0+0x+0x^2 = 0$.

   (2) Suppose $u,v \in P_2(\mathbb{R})$. Then this means that $u = a+bx+cx^2$ and $v = r+sx+tx^2$, where $a,b,c,r,s,t \in \mathbb{R}$. We want to show that $u+v \in P_2(\mathbb{R})$. Then

   $$\begin{aligned} u+v &= (a+bx+cx^2)+(r+sx+tx^2) \\ &= a+r+bx+sx+cx^2+tx^2 \\ &= (a+r)+(b+s)x+(c+t)x^2 \ , \end{aligned}$$

   where $a+r, b+s, c+t \in \mathbb{R}$. So, $u+v \in P_2(\mathbb{R})$, and hence $P_2(\mathbb{R})$ is closed under the usual addition.

   (3) Suppose $u \in P_2(\mathbb{R})$ and $k \in \mathbb{R}$. We want to show that $ku \in P_2(\mathbb{R})$. Since $u \in P_2(\mathbb{R})$, this means that $u = a+bx+cx^2$ for some $a,b,c \in \mathbb{R}$. Then

   $$\begin{aligned} ku &= k(a+bx+cx^2) \\ &= ka+kbx+kcx^2 \\ &= (ka)+(kb)x+(kc)x^2 \ , \end{aligned}$$

   where $ka, kb, kc \in \mathbb{R}$. So, $ku \in P_2(\mathbb{R})$, and hence $P_2(\mathbb{R})$ is closed under scalar multiplication.

   Therefore, $P_2(\mathbb{R})$ is a subspace of $F(-\infty,\infty)$. $\square$

2. Which of the following is a subspace of $M_{n\times n}(\mathbb{R})$?

   (a) The set of all $n \times n$ symmetric matrices.

   Yes. Let $W = \{A \in M_{n\times n}(\mathbb{R}) \mid A^{\mathsf{T}} = A\}$ be the set of all $n \times n$ symmetric matrices. We prove that W is a subspace using the subspace test.

   (1) $0 \in W$ because $0^{\mathsf{T}} = 0$.

   (2) Suppose $A, B \in W$. Then this means that A and B are $n \times n$ matrices where $A^{\mathsf{T}} = A$ and $B^{\mathsf{T}} = B$ (symmetric). We want to show that $(A+B) \in W$; that is, we show that $(A+B)^{\mathsf{T}} = A+B$. So,

   $$(A+B)^{\mathsf{T}} = A^{\mathsf{T}} + B^{\mathsf{T}} = A + B \ .$$

   Here we get that $A+B$ is symmetric, and so $A+B \in W$ (we also know that $A+B$ is $n \times n$ due to how the usual matrix addition works). Hence, W is closed under addition.

   (3) Suppose $A \in W$ and $k \in \mathbb{R}$. We show that $kA \in W$; that is, we show that $(kA)^{\mathsf{T}} = kA$. Since $A \in W$, this means that A is an $n \times n$ matrix where $A^{\mathsf{T}} = A$. Then

   $$(kA)^{\mathsf{T}} = k(A^{\mathsf{T}}) = k(A) = kA \ .$$

   So, we get that $kA \in W$. Hence, W is closed under scalar multiplication.

   Therefore, the set of all $n\times n$ symmetric matrices W is a subspace. $\square$

(b) The set of all $n \times n$ invertible matrices.

No, this set is not a subspace. Denote

$$V = \{A \in M_{n\times n}(\mathbb{R}) \mid \exists B \in M_{n\times n}(\mathbb{R}) \text{ s.t. } AB = BA = I_n\}$$

as the set of all $n \times n$ invertible matrices. To show that V is not a subspace, it suffices to show one axiom of the subspace test does not hold. However, we will try all axioms and see which ones hold and don't hold.

(1) $0 \notin V$ because 0 is not invertible. Note that we know 0 is the "zero vector" (in this case the zero matrix) because $0 + A = A$ for any $A \in V$.

(2) Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ be 2×2 matrices. Then $A, B \in V$ since A and B are invertible (we can check this with either the definition specified in our set V or by checking that their determinants are non-zero). However,

$$A + B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} .$$

Here we see that $A + B$ is not invertible (again, we can check this by using the definition as specified in V or by taking its determinant). Hence, $A + B$ is not closed under addition.

(3) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $k = 0$. Then $A \in V$ since A is invertible. However

$$kA = k \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} .$$

By Axiom 1, $0 \notin V$, and so $kA \notin V$. Hence, V is not closed under scalar multiplication.

So, we see that our set of $n \times n$ invertible matrices V doesn't satisfy any of the axioms needed to be a subsapce. Therefore, V is not a subspace.

3. Determine if $W = \{r(1+x^2) \mid r \in \mathbb{R}\}$ is a subspace of $P_2(\mathbb{R})$.

Note here that $P_2(\mathbb{R})$ is the set of all polynomials of degree at most 2 with coefficients in $\mathbb{R}$. That is, $P_2(\mathbb{R}) = \{a + bx + cx^2 \mid a, b, c \in \mathbb{R}\}$. We show that W is a subspace $P_2(\mathbb{R})$ using the subspace test.

(1) $0 \in W$ because $0(1+x^2) = 0 \in W$, where $r = 0$.

(2) Suppose $u, v \in W$. We prove that $u + v \in W$. Since $u, v \in W$, this means that $u = r(1+x^2)$ and $v = s(1+x^2)$, where $r, s \in \mathbb{R}$. Then

$$\begin{aligned} u + v &= r(1+x^2) + s(1+x^2) \\ &= r + rx^2 + s + sx^2 \\ &= r + s + rx^2 + sx^2 \\ &= (r+s) + (r+s)x^2 \\ &= (r+s)\left(1+x^2\right) \ , \end{aligned}$$

where $r + s \in \mathbb{R}$. So, $u + v \in W$. Hence W is closed under addition.

(3) Suppose $u \in W$ and $k \in \mathbb{R}$. We show that $ku \in W$. Since $u \in W$, this means that $u = r(1+x^2)$ where $r \in \mathbb{R}$. Then

$$\begin{aligned} ku &= k(r(1+x^2)) \\ &= k(r + rx^2) \\ &= kr + krx^2 \\ &= kr(1+x^2) \ , \end{aligned}$$

where $kr \in \mathbb{R}$. So, we get that $ku \in W$. Hence, W is closed under scalar multiplication.

Therefore, W is a subspace of $P_2(\mathbb{R})$. $\square$

Definition (Span)

Let V be a vector space. The span of a set of vectors $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is the set of all linear combinations of $\vec{v}_1, \ldots, \vec{v}_k$. That is,

$$\text{span}\{\vec{v}_1, \ldots, \vec{v}_k\} = \{a_1\vec{v}_1 + \ldots + a_k\vec{v}_k \mid a_1, \ldots, a_k \in \mathbb{R}\} \ .$$

For example, $\text{span}\{1, x, x^2\} = \{a + bx + cx^2\} = P_2(\mathbb{R})$, where $a, b, c \in \mathbb{R}$.

1. Let $T = \{x^2 + x + 1, x^2 + x, x + 1\}$. What is span T?

   <u>Solution:</u> Let $u \in \text{span } T$. Then u can be expressed as a linear combination of the elements in T. That is,

   $$\begin{aligned} u &= a(x^2 + x + 1) + b(x^2 + x) + c(x + 1) \\ &= ax^2 + ax + a + bx^2 + bx + cx + c \\ &= (a + b)x^2 + (a + b + c)x + (a + c) \cdot 1 \\ &= (a + c) \cdot 1 + (a + b + c)x + (a + b)x^2 \ . \end{aligned}$$

   Notice how similar this is to $\text{span}\{1, x, x^2\}$ from the above example.

   <u>Claim:</u> span $T = \text{span}\{1, x, x^2\}$.

   <u>Proof of Claim:</u> ($\subseteq$) We already showed that for any $u \in \text{span } T$, $u \in \text{span}\{1, x, x^2\}$ above.

   ($\supseteq$) It is enough to prove that $1, x, x^2 \in \text{span } T$. Observe that

   $$\begin{aligned} 1 &= 0 + 0 + 1 \\ &= x^2 = x^2 + x - x + 1 \\ &= x^2 = x^2 + x - x + 1 \\ &= x^2 + x + 1 - x^2 - x \\ &= (x^2 + x + 1) - (x^2 + x) + 0 \\ &= (x^2 + x + 1) - (x^2 + x) + 0(x + 1) \\ &= 1(x^2 + x + 1) - 1(x^2 + x) + 0(x + 1) \\ &\in \text{span } T \ , \end{aligned}$$

   $$\begin{aligned} x &= 0 + x + 0 \\ &= -x^2 + x^2 - x + x + x - 1 + 1 \\ &= -x^2 - x - 1 + x^2 + x + x + 1 \end{aligned}$$

$$
\begin{aligned}
&= -(x^2 + x + 1) + (x^2 + x) + (x + 1) \\
&= -1(x^2 + x + 1) + 1(x^2 + x) + 1(x + 1) \\
&\in \text{span } T \ ,
\end{aligned}
$$

$$
\begin{aligned}
x^2 &= x^2 + 0 + 0 \\
&= x^2 + x - x + 1 - 1 \\
&= x^2 + x + 1 - x - 1 \\
&= (x^2 + x + 1) + 0 - (x + 1) \\
&= 1(x^2 + x + 1) + 0(x^2 + x) - 1(x + 1) \\
&\in \text{span } T \ .
\end{aligned}
$$

Therefore, span $T = \text{span}\{1, x, x^2\} = P_2(\mathbb{R})$.

2. Find a set that spans the subspace $W = \{A \in M_{2\times 2}(\mathbb{R}) \mid A^{\mathsf{T}} = A\}$ of $M_{2\times 2}(\mathbb{R})$.

   Note that W is the set of all 2×2 symmetric matrices. Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in W$. Then it holds that

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = A^{\mathsf{T}} .$$

   From this, we see that $b = c$. So,

$$\begin{aligned} A &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \begin{bmatrix} a & b \\ b & d \end{bmatrix} \\ &= \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix} \\ &= a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} . \end{aligned}$$

   Thus, the set $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ spans W.

Theorem 1*

Let V be a vector space. If $\vec{v}_1, \dots, \vec{v}_k \in V$, then $W = \operatorname{span}\{\vec{v}_1, \dots, \vec{v}_k\}$ is a subspace of V.

Proof. Suppose V is a vector space and suppose $\vec{v}_1, \dots, \vec{v}_k \in V$. We want to show that $W = \operatorname{span}\{\vec{v}_1, \dots, \vec{v}_k\} = \{a_1\vec{v}_1 + \ldots + a_k\vec{v}_k \mid a_1, \dots, a_k \in \mathbb{R}\}$ is a subspace of V. We prove this with the subspace test.

(1) $\vec{0} \in W$ because $a_1\vec{v}_1 + \ldots + a_k\vec{v}_k = 0 \cdot \vec{v}_1 + \ldots + 0 \cdot \vec{v}_k = \vec{0}$, where $a_1 = \ldots = a_k = 0$.

(2) Suppose $\vec{u}, \vec{v} \in W$. We show that $\vec{u} + \vec{v} \in W$. Since $\vec{u}, \vec{v} \in W$, this means that $\vec{u}$ and $\vec{v}$ can be expressed as a linear combination of the vectors $\vec{v}_1, \dots, \vec{v}_k$. That is, $\vec{u} = a_1\vec{v}_1 + \ldots + a_k\vec{v}_k$ and $\vec{v} = b_1\vec{v}_1 + \ldots + b_k\vec{v}_k$, where $a_1, \dots, a_k, b_1, \dots, b_k \in \mathbb{R}$. Now,

$$\begin{aligned}\vec{u} + \vec{v} &= (a_1\vec{v}_1 + \ldots + a_k\vec{v}_k) + (b_1\vec{v}_1 + \ldots + b_k\vec{v}_k) \\ &= (a_1 + b_1)\vec{v}_1 + \ldots + (a_k + b_k)\vec{v}_k \ ,\end{aligned}$$

where $a_1 + b_1, \dots, a_k + b_k \in \mathbb{R}$. So, we get that $\vec{u} + \vec{v} \in W$. Hence, W is closed under addition.

(3) Suppose $\vec{u} \in W$ and $t \in \mathbb{R}$. We show that $t\vec{u} \in W$. Since $\vec{u} \in W$, this means that $\vec{u}$ can be expressed as linear combination of the vectors $\vec{v}_1, \dots, \vec{v}_k$. That is, $\vec{u} = a_1\vec{v}_1 + \ldots + a_k\vec{v}_k$, where $a_1, \dots, a_k \in \mathbb{R}$. Now,

$$\begin{aligned}t\vec{u} &= t(a_1\vec{v}_1 + \ldots + a_k\vec{v}_k) \\ &= ta_1\vec{v}_1 + \ldots + ta_k\vec{v}_k \\ &= (ta_1)\vec{v}_1 + \ldots + (ta_k)\vec{v}_k \ ,\end{aligned}$$

where $ta_1, \dots, ta_k \in \mathbb{R}$. So, we get that $t\vec{u} \in W$. Hence, W is closed under scalar multiplication.

Therefore, W is a subspace of the vector space V. □

Linear Independence

Definition (Linear Independence)

A set of vectors $\vec{v}_1, \ldots, \vec{v}_k$ in a vector space V is linearly independent if

$$a_1\vec{v}_1 + \ldots + a_k\vec{v}_k = \vec{0} \ ,$$

where $a_1, \ldots, a_k \in \mathbb{R}$, has only the trivial solution $a_1 = \ldots = a_k = 0$.

1. Which of the following sets are linearly independent?

   (a) $\{1, x, x^2, \ldots, x^{n-1}, x^n\} \subseteq P_n(\mathbb{R})$.

   Yes, this set is linearly independent. In fact, this is a basis for $P_n(\mathbb{R})$ (more on that later).

   (b) $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix} \right\} \subseteq M_{2\times 2}(\mathbb{R})$.

   No, this set is not linearly independent (it is linearly dependent). From observation, we can see that the first vector can be expressed as a linear combination of the second and third vector.

   (c) $\{1 - x, 5 + 3x - 2x^2, 1 + 3x - x^2\} \subseteq P_2(\mathbb{R})$.

   No, this set is not linearly indepdenent (it is linearly dependent). Indeed, after solving the linear system of equations, you'll get a non-trivial solution.

   (d) $\{\sin x, \cos x\} \subseteq F(-\infty, \infty)$.

   Yes, this set is linearly independent.

Basis and Dimension

Let V be a vector space over $\mathbb{R}$.

Definition (Basis)

A set $\{\vec{v}_1, \ldots, \vec{v}_k\} \in V$ is a basis for V if

(1) $V = \text{span}\{\vec{v}_1, \ldots, \vec{v}_k\}$

(2) $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is linearly independent.

Definition (Dimension)

The dimension of a vector space V, denoted $\dim V$, is the number of elements in a basis for V.

1. Find a basis for the following vector spaces and specify their dimension.

   (a) $P_n(\mathbb{R})$.

   A basis for $P_n(\mathbb{R})$ would be $\{1, x, x^2, \ldots, x^{n-1}, x^n\}$, where $\dim P_n(\mathbb{R}) = n + 1$.

   (b) $M_{m\times n}(\mathbb{R})$.

   Note that this is just the set of all $m \times n$ matrices over the real numbers. A basis for this set would be

   $$\{E_{ij} \in M_{m\times n}(\mathbb{R}) \mid 1 \leq i \leq m, 1 \leq j \leq n\} \ ,$$

   with $\dim M_{m\times n}(\mathbb{R}) = mn$.

   (c) 2×2 symmetric matrices.

   A basis for this subspace would be

   $$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\} \ ,$$

   with dimension 3.

Compare this to $M_{2\times 2}(\mathbb{R})$, which has a basis

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

with $\dim M_{2\times 2}(\mathbb{R}) = 4 = 2 \cdot 2 = mn$.

Remarks:

1) All bases of a vector space V have the same number of elements. So, $\dim V$ is well-defined.
2) Let $\{\vec{v}_1, \dots, \vec{v}_k\}$ be a basis for a vector space V. Suppose $S = \{\vec{v}_1, \dots, \vec{v}_k\} \subseteq V$.
   (i) If $k > n$, then S is linearly dependent.
   (ii) If $k < n$, then S cannot span V.

So, a basis is a minimal spanning set and a maximal linearly independent set.

1. Suppose V is a vector space with $\dim V = k$. Determine which of the following are true or false.

   (a) If $S = \{\vec{v}_1, \ldots, \vec{v}_k\} \subseteq V$, then S is a basis for V.

   True. Suppose $S = \{\vec{v}_1, \ldots, \vec{v}_k\}$ and suppose $S \subseteq V$. Here we see that S consists of exactly $\dim V = k$ vectors.

   (b) If $S = \{\vec{v}_1, \ldots, \vec{v}_k\} \subseteq V$ spans V, then S is a basis for V.

   True.

Theorem (Independence Lemma)

Let $\{\vec{v}_1, \ldots, \vec{v}_k\} \subseteq V$ be a linearly independent set. If $\vec{w} \in V$ but $\vec{w} \notin \operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_k\}$ (that is, $\vec{w}$ cannot be expressed as a linear combination of $\vec{v}_1, \ldots, \vec{v}_k$), then $\{\vec{w}, \vec{v}_1, \ldots, \vec{v}_k\}$ is also linearly independent.

Proof. Suppose $\{\vec{v}_1, \ldots, \vec{v_k}\}$ is a linearly independent set. Suppose $\vec{w} \in V$ and $\vec{w} \notin \operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_k\}$. We want to prove that $\{\vec{w}, \vec{v}_1, \ldots, \vec{v}_k\}$ is linearly independent; that is, $c_0\vec{w} + c_1\vec{v}_1 + \ldots + c_k\vec{v}_k = \vec{0}$, where $c_0, c_1, \ldots, c_k \in \mathbb{R}$ and $c_0 = c_1 = \ldots = c_k = 0$. We prove by contradiction. Suppose $\{\vec{w}, \vec{v}_1, \ldots, \vec{v}_k\}$ is linearly dependent. Then this means that

$$a_0\vec{w} + a_1\vec{v}_1 + \ldots + a_k\vec{v}_k = \vec{0} \ ,$$

where $a_0, a_1, \ldots, a_k \in \mathbb{R}$ and at least one of $a_0, a_1, \ldots, a_k$ are nonzero. We consider two cases.

Case 1: If $a_0 \neq 0$, then rearranging $a_0\vec{w} + a_1\vec{v}_1 + \ldots + a_k\vec{v}_k = \vec{0}$ for $\vec{w}$ gives us

$$\begin{aligned} \vec{w} &= \frac{-a_1\vec{v}_1 - \ldots - a_k\vec{v}_k}{a_0} \\ &= -\frac{a_1}{a_0}\vec{v}_1 - \ldots - \frac{a_k}{a_0}\vec{v}_k \ , \end{aligned}$$

and so $\vec{w}$ is indeed a linear combination of $\vec{v}_1, \ldots, \vec{v}_k$. So, $\vec{w} \in \operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_k\}$, which contradicts our assumption that $w \notin \operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_k\}$. Hence, $\{\vec{w}, \vec{v}_1, \ldots, \vec{v}_k\}$ is linearly independent.

Case 2: If $a_0 = 0$, then we get that

$$0 \cdot \vec{w} + a_1\vec{v}_1 + \ldots + a_k\vec{v}_k = 0 \ .$$

Now, since $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is linearly independent, we get that $a_1 = \ldots = a_k = 0$, and so

$$0 \cdot \vec{w} + 0 \cdot \vec{v}_1 + \ldots + 0 \cdot \vec{v}_k = 0 \ .$$

This implies that $\{\vec{w}, \vec{v}_1, \ldots, \vec{v}_k\}$ is linearly independent.

$\square$

Theorem 19

Let V be a vector space with $\dim(V) < \infty$ (finite dimension) and let $U \subseteq V$ be a subspace of V. Then

(i) $\dim(U) \leq \dim(V)$

(ii) If $U \neq V$, then $\dim(U) < \dim(V)$

Proof. (i) Suppose $\dim(U) = m$. Then a spanning set for U has at least $(\geq)$ m elements. Therefore a spanning set for V also has at least $(\geq)$ m elements. Hence, $m \leq \dim(V)$.

(ii) Suppose $U \neq V$. Then there exists $\vec{v} \in V$ such tbat $\vec{v} \notin U$ (this comes from the definition of set minus). Let $B = \{\vec{x}_1, \ldots, \vec{x}_m\}$ be a basis for U. Then since $\vec{v} \notin U$, $\vec{v} \notin \text{span}\{\vec{x}_1, \ldots, \vec{x}_m\} = U$. So because $\vec{v} \in V$ but $\vec{v} \notin U$, by the Independence Lemma, the set $\{\vec{v}, \vec{x}_1, \ldots, \vec{x}_m\}$ is linearly independent in V. Hence, $\dim(U) = m < m + 1 \leq \dim(V)$. $\square$

1. Let $V = P_2(\mathbb{R})$ and $U = \{p(x) \in V \mid p(1) = 0\}$.

   (a) Show that U is a subspace of V.

   Proof. To show that U is a subspace of V, we use the subspace test. That is, we show U contains 0, and U is closed under both addition and scalar multiplication.

   (1) $0 \in U$ because $p(1) = 0 \in U$.

   (2) Suppose $f, g \in U$. We show that $f + g \in U$. Since $f, g \in U$, this implies that $f(1) = 0$ and $g(1) = 0$. Then

   $$(f + g)(1) = f(1) + g(1) = 0 + 0 = 0 \in U \ .$$

   Hence, U is closed under addition.

   (3) Suppose $f \in V$ and $k \in \mathbb{R}$. We show that $kf \in U$. Since $f \in U$, this implies that $f(1) = 0$. So,

   $$(kf)(1) = kf(1) = k \cdot 0 = 0 \in U \ .$$

   Hence, U is closed under scalar multiplication.

   Thus, by the subspace test, U is a subspace of V. □

   (b) $\text{span}\{x-1, x(x-1)\}$ is a two-dimensional subspace of U, which means that $2 \leq \dim(U)$.

   (c) $U \neq V$ since $x - 2 \in V$ but $x - 2 \notin U$, because $1 - 2 = -1 \neq 0$. So, $\dim(U) < \dim(V) = 3$. Therefore, $2 \leq \dim(U) < 3$, and we get $\dim(U) = 2$.

   So, because $\big| \, \{x - 1, x(x - 1)\} \, \big| = 2 = \dim(U)$ and because $\{x - 1, x(x - 1)\}$ is linearly independent, $\{x - 1, x(x - 1)\}$ is a basis for $U$and $U = \text{span}\{x - 1, x(x - 1)\}$.

2. Let $U = \{x^2 - 3x + 2, 1 - 2x, 2x^2 + 1, 2x^2 - x - 3\}$.

   (a) Prove that U spans $P_2(\mathbb{R})$.

   To show that U spans $\mathbb{R}^2$, we must show that $\forall p(x) \in P_2(\mathbb{R})$, $p(x)$ is a linear combination of elements in U.

   Proof. Let $p(x) = ax^2 + bx + c \in P_2(\mathbb{R})$ where $a, b, c \in \mathbb{R}$. We want to show that $\exists t_1, t_2, t_3 \in \mathbb{R}$ such that

$$\begin{aligned} p(x) &= ax^2 + bx + c \\ &= t_1(x^2 - 3x + 2) + t_2(1 - 2x) + t_3(2x^2 + 1) + t_4(2x^2 - x - 3)\ . \end{aligned}$$

   Then it follows that

$$\begin{aligned} p(x) &= (2t_1 + t_2 + t_3 - 3t_4) + (-3t_1 - 2t_2 - t_4)x + (t_1 + 2t_3 + 2t_4)x^2 \\ &\iff \begin{bmatrix} 2 & 1 & 1 & -3 \\ -3 & -2 & 0 & -1 \\ 1 & 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} \\ &= A \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \end{aligned} \qquad (*)$$

   From this we get that

$$\mathrm{REF}(A) = \begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & -3 & -7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

   has 3 leading ones. So, $\mathrm{rank}(A) = 3$. Hence, the system has a solution for all $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3$. Thus, U spans $P_2(\mathbb{R})$. $\square$

   (b) Find a basis for $P_2(\mathbb{R})$ consisting of elements in U.

   Notice that U is linearly dependent. We remove the vector corresponding to the third column of A. So, we remove $2x^2 + 1$ from the set. Then $\{x^2 - 3x + 2, 1 - 2x, 2x^2 - x - 3\}$ is a basis for $P_2(\mathbb{R})$.

   <u>**Observation:**</u> The polynomial $1 + 2x^2$ is a vector in U. This vector, with respect to the standard basis of $P_2(\mathbb{R})$, has coordinates

$\begin{bmatrix}1\\0\\2\end{bmatrix}$. The standard basis for $P_2(\mathbb{R})$ is $\{1, x, x^2\}$. In general, we regard $\begin{bmatrix}a\\b\\c\end{bmatrix} \in \mathbb{R}^3$ as the vector of coefficients for the polynomial $a + bx + cx^2 \in P_2(\mathbb{R})$.