MATH 311 - Linear Transformations

Jasraj Sandhu

May 2023

Examples and Basic Properties

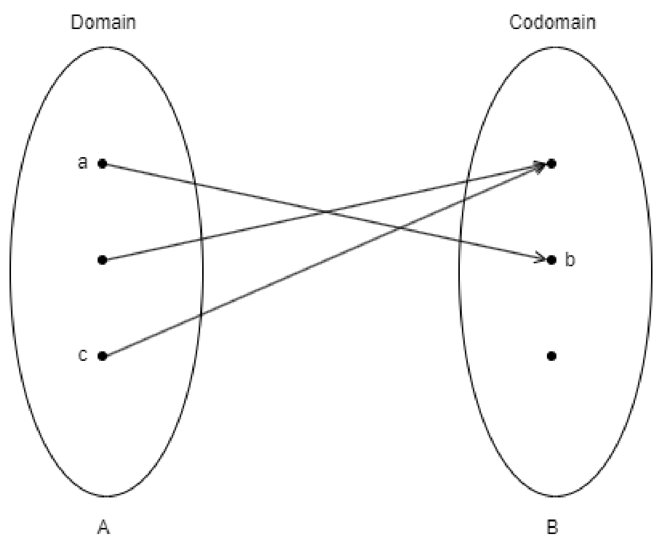
Let A and B be two sets.

Definition (Function)

A function from A to B is a rule $f : A \to B$ so that for every $a \in A$, f assigns to a **exactly one** element $b \in B$. We write $f(a) = b$. Here, b is called the image of a under f.

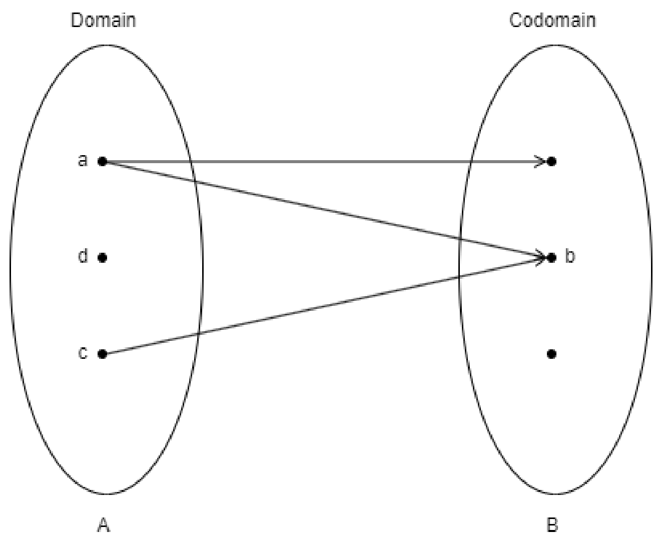
A is the domain of f, and B is the codomain of f.

1. The following is a function:



Note that it is okay if two elements go the same place. What's important is that one element doesn't go to more than one place.

2. The following is NOT a function:



This is not a function because

1. a goes to more than one place.
2. d in the domain is not sent anywhere.

Let V and W be vector spaces.

> **Definition (Linear Transformation)**
>
> A functuon $T : V \to W$ is a linear transformation if
>
> (1) $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v} \in V$.
>
> (2) $T(k\vec{v}) = kT(\vec{v})$ for all $\vec{v} \in V$ and $k \in \mathbb{R}$.

That is, T preserves vector addition and scalar multiplication.

1. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ x \\ 0 \end{bmatrix}$. Prove that T is a linear transformation.

   Proof. Let $\vec{u} = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$, $\vec{v} = \begin{bmatrix} c \\ d \end{bmatrix} \in \mathbb{R}^2$ (the domain) and let $k \in \mathbb{R}$. Then we prove the properties of a linear transformation hold for this particular transformation T. That is, we show that T preserves vector addition and scalar multiplication.

   (1) We show that $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$. So,

   $$\begin{aligned} T(\vec{u} + \vec{v}) &= T\left(\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}\right) \\ &= T\left(\begin{bmatrix} a+c \\ b+d \end{bmatrix}\right) \\ &= \begin{bmatrix} b+d \\ a+c \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} b \\ a \\ 0 \end{bmatrix} + \begin{bmatrix} d \\ c \\ 0 \end{bmatrix} \\ &= T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) + T\left(\begin{bmatrix} c \\ d \end{bmatrix}\right) \\ &= T(\vec{u}) + T(\vec{v}) \ . \end{aligned}$$

   Hence, T preserves vector addition.

(2) We show that $T(k\vec{u}) = kT(k\vec{v})$. So,

$$\begin{aligned} T(k\vec{u}) &= T\left(k\begin{bmatrix} a \\ b \end{bmatrix}\right) \\ &= T\left(\begin{bmatrix} ka \\ kb \end{bmatrix}\right) \\ &= \begin{bmatrix} kb \\ ka \\ 0 \end{bmatrix} \\ &= k\begin{bmatrix} b \\ a \\ 0 \end{bmatrix} \\ &= kT\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) \\ &= kT(\vec{u}) \ . \end{aligned}$$

Hence, T preserves scalar multiplication.

Thus, T is a linear transformation. $\square$

2. Which of the following maps are linear transformations?

   (a) $T : \mathbb{R} \to \mathbb{R}$, where $T : x \mapsto 3x + 4$.

   No, T is NOT a linear transformation.

   Proof. Let $x, y \in \mathbb{R}$ (the domain) and $k \in \mathbb{R}$. To show that T is not a linear transformation, is suffices to show that one of the properties of linear transformations does not hold. That is, we check if T either fails to preserves vector addition or scalar multiplication. However, we'll check both properties!

   (1) We check if T preserves vector addition. So,

$$\begin{aligned} T(x+y) &= 3(x+y)+4 \\ &= 3x+3y+4 \\ &\neq 3x+3y+8 \\ &= (3x+4)+(3y+4) \\ &= T(x)+T(y) \ . \end{aligned}$$

   Hence, T does not preserve vector addition.

   (2) We check if T preserves scalar multiplication. So,

$$\begin{aligned} T(kx) &= 3(kx)+4 \\ &= 3kx+4 \\ &\neq 3kx+4k \\ &= k(3x+4) \\ &= kT(x) \ . \end{aligned}$$

   Hence, T does not preserve scalar multiplication.

   Here we see that T violates both properties required to be a linear transformation. Thus, T is not a linear transformation. $\square$

(b) $L : P_n \to P_{n-1}$, where $L : p(x) \mapsto p(1)$.

Proof. To check if L is a linear transformation, we check if L preservers vector addition and scalar multiplication. Let $f(x), g(x) \in P_n$ and $k \in \mathbb{R}$.

(1) We check if $L(f + g) = L(f) + L(g)$. So,

$$\begin{aligned} L(f(x) + g(x)) &= L((f + g)(x)) \\ &= (f + g)(1) \\ &= f(1) + g(1) \\ &= L(f(x)) + L(g(x)) \ . \end{aligned}$$

Hence, L preserves vector addition.

(2) We check if $L(kf(x)) = kL(f(x))$. So,

$$\begin{aligned} L(kf(x)) &= L((kf)(x)) \\ &= (kf)(1) \\ &= kf(1) \\ &= kL(f(x)) \ . \end{aligned}$$

Hence, L preserves scalar multiplication.

Thus, L is a linear transformation. □

(c) $Q : \mathbb{R}^2 \to \mathbb{R}^2$, where $Q : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x^2 \\ y^2 \end{bmatrix}$.

Proof. To determine if Q is a linear transformation, we check if Q preserves vector addition and scalar multiplication. Let $\vec{u} = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$, $\vec{v} = \begin{bmatrix} c \\ d \end{bmatrix} \in \mathbb{R}^2$, and $k \in \mathbb{R}$.

(1) We check if $Q(\vec{u} + \vec{v}) + Q(\vec{u}) + Q(\vec{v})$. So,

$$\begin{aligned} Q(\vec{u} + \vec{v}) &= Q\left(\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}\right) \\ &= Q\left(\begin{bmatrix} a+c \\ b+d \end{bmatrix}\right) \\ &= \begin{bmatrix} (a+c)^2 \\ (b+d)^2 \end{bmatrix} \\ &\neq \begin{bmatrix} a^2 + c^2 \\ b^2 + d^2 \end{bmatrix} \\ &= \begin{bmatrix} a^2 \\ b^2 \end{bmatrix} + \begin{bmatrix} c^2 \\ d^2 \end{bmatrix} \\ &= Q\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) + Q\left(\begin{bmatrix} c \\ d \end{bmatrix}\right) \\ &= Q(\vec{u}) + Q(\vec{v}) \ . \end{aligned}$$

Hence, Q does not preserve vector addition.

(2) We check if $Q(k\vec{u}) = kQ(\vec{u})$. So,

$$\begin{aligned} Q(k\vec{u}) &= Q\left(k \begin{bmatrix} a \\ b \end{bmatrix}\right) \\ &= Q\left(\begin{bmatrix} ka \\ kb \end{bmatrix}\right) \\ &= \begin{bmatrix} (ka)^2 \\ (kb)^2 \end{bmatrix} \\ &\neq \begin{bmatrix} ka^2 \\ kb^2 \end{bmatrix} \\ &= k \begin{bmatrix} a^2 \\ b^2 \end{bmatrix} \\ &= kQ\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) \\ &= kQ(\vec{u}) \ . \end{aligned}$$

Hence, Q does not preserve scalar multiplication.

Thus, Q is not a linear transformation. $\square$

(d) $\mathrm{tr} : M_{22} \to \mathbb{R}$, where $\mathrm{tr} : A \mapsto \mathrm{tr}(A)$.

Proof. To check if tr is a linear transformation, we check if tr preserves vector addition and scalar multiplication. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{22}$, $B = \begin{bmatrix} x & y \\ w & z \end{bmatrix} \in M_{22}$, and $k \in \mathbb{R}$.

(1) We check if $\mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B)$. So,

$$\begin{aligned} \mathrm{tr}(A+B) &= \mathrm{tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} x & y \\ w & z \end{bmatrix}\right) \\ &= \mathrm{tr}\left(\begin{bmatrix} a+x & b+y \\ c+w & d+z \end{bmatrix}\right) \\ &= (a+x) + (d+z) \\ &= a+x+d+z \\ &= (a+d) + (x+z) \\ &= \mathrm{tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) + \mathrm{tr}\left(\begin{bmatrix} x & y \\ w & z \end{bmatrix}\right) \\ &= \mathrm{tr}(A) + \mathrm{tr}(B) \ . \end{aligned}$$

Hence, tr preserves vector addition.

(2) We check if $\mathrm{tr}(kA) = k \cdot \mathrm{tr}(A)$. So,

$$\begin{aligned} \mathrm{tr}(kA) &= \mathrm{tr}\left(k\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \\ &= \mathrm{tr}\left(\begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}\right) \\ &= ka + kd \\ &= k(a+d) \\ &= k \cdot \mathrm{tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \\ &= k \cdot \mathrm{tr}(A) \ . \end{aligned}$$

Hence, tr preserves scalar multiplication.

Thus, tr is a linear transformation. $\square$

(e) $\det : M_{nn} \to \mathbb{R}$, where $\det : A \mapsto \det(A)$ and $n > 1$.

Proof. To check if det is a linear transformation, we check if det preserves vector addition and scalar multiplication. Recall from the properties of determinants that $\det(A + B) \neq \det(A) + \det(B)$ in general. With this, we can tell that det is not a linear transformation. Furthermore, we can show that det does not preserve scalar multiplication. Let $A = I_n \in M_{nn}$ and $k \in \mathbb{R}$. Then

$$\begin{aligned}
\det(kA) &= \det(kI_n) \\
&= k^n \det(I_n) \\
&= k^n \cdot 1 \\
&= k^n \\
&\neq k \\
&= k \cdot 1 \\
&= k \det(I_n) \\
&= k \det(A) \ .
\end{aligned}$$

Thus, since det doesn't preserve scalar multiplication, det is not a linear transformation. □

Recall: For any $A \in M_{mn}$, we can make $T_A : \mathbb{R}^n \to \mathbb{R}^n$, $T_A(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}$.

Theorem 20:

A map $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if and only if there exists an $m \times n$ matrix A such that $T(\vec{x}) = A\vec{x}$ for all $x \in \mathbb{R}^n$.

Proof. ($\Longleftarrow$) Suppose there exists $A \in M_{mn}$ such that we have a map $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\vec{x}) = A\vec{x}$. We show that T is a linear transformation. To show that T is a linear transformation, we show that T preserves vector addition and scalar multiplication. So, let $\vec{u}, \vec{v} \in \mathbb{R}^n$ (the domain) and let $k \in \mathbb{R}$. Then $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$, and $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$. (See next page.)

(1) We show that $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$. So,

$$
\begin{aligned}
T(\vec{u}+\vec{v}) &= T\left(\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}\right) \\
&= T\left(\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ \vdots \\ u_n+v_n \end{bmatrix}\right) \\
&= A\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ \vdots \\ u_n+v_n \end{bmatrix} \\
&= \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ \vdots \\ u_n+v_n \end{bmatrix} \\
&= \begin{bmatrix} a_{11}(u_1+v_1) + a_{12}(u_2+v_2) + \ldots + a_{1n}(u_n+v_n) \\ a_{21}(u_1+v_1) + a_{22}(u_2+v_2) + \ldots + a_{2n}(u_n+v_n) \\ \vdots \\ a_{n1}(u_1+v_1) + a_{n2}(u_2+v_2) + \ldots + a_{nn}(u_n+v_n) \end{bmatrix} \\
&= \begin{bmatrix} a_{11}u_1 + a_{12}u_2 + \ldots + a_{1n}u_n \\ a_{11}u_1 + a_{12}u_2 + \ldots + a_{1n}u_n \\ \vdots \\ a_{11}u_1 + a_{12}u_2 + \ldots + a_{1n}u_n \end{bmatrix} + \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \ldots + a_{1n}v_n \\ a_{11}v_1 + a_{12}v_2 + \ldots + a_{1n}v_n \\ \vdots \\ a_{11}v_1 + a_{12}v_2 + \ldots + a_{1n}v_n \end{bmatrix} \\
&= \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \\
&= A\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + A\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \\
&= T\left(\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}\right) + T\left(\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}\right)
\end{aligned}
$$

$$= T(\vec{u}) + T(\vec{v}) \ .$$

Hence, T preserves vector addition.

(Continued on next page.)

(2) We show that $T(k\vec{u}) = kT(\vec{u})$. So,

$$
\begin{aligned}
T(k\vec{u}) &= T\left(k\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}\right) \\
&= T\left(\begin{bmatrix} ku_1 \\ ku_2 \\ \vdots \\ ku_n \end{bmatrix}\right) \\
&= A\begin{bmatrix} ku_1 \\ ku_2 \\ \vdots \\ ku_n \end{bmatrix} \\
&= \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}\begin{bmatrix} ku_1 \\ ku_2 \\ \vdots \\ ku_n \end{bmatrix} \\
&= \begin{bmatrix} a_{11}ku_1 + a_{12}ku_2 + \ldots + a_{1n}ku_n \\ a_{21}ku_1 + a_{22}ku_2 + \ldots + a_{2n}ku_n \\ \vdots \\ a_{n1}ku_1 + a_{n2}ku_2 + \ldots + a_{nn}ku_n \end{bmatrix} \\
&= k\begin{bmatrix} a_{11}u_1 + a_{12}u_2 + \ldots + a_{1n}u_n \\ a_{21}u_1 + a_{22}u_2 + \ldots + a_{2n}u_n \\ \vdots \\ a_{n1}u_1 + a_{n2}u_2 + \ldots + a_{nn}u_n \end{bmatrix} \\
&= k\left(\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}\right) \\
&= k \cdot A\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \\
&= kT\left(\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}\right)
\end{aligned}
$$

$$= kT(\vec{u}) \ .$$

Hence, T preserves scalar multiplication.

Thus, T is a linear transformation.

$(\implies)$ Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear. Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \in \mathbb{R}^n$. Then $\vec{u} = u_1\vec{e_1} + u_2\vec{e_2} + \ldots + u_n\vec{e_n}$, where $\{e_1, e_2, \ldots, e_n\}$ is the standard basis for $\mathbb{R}^n$. Then since T is linear,

$$\begin{aligned} T(\vec{u}) &= T(u_1\vec{e_1} + u_2\vec{e_2} + \ldots + u_n\vec{e_n}) \\ &= T(u_1\vec{e_1}) + T(u_2\vec{e_2}) + \ldots + T(u_n\vec{e_n}) \\ &= u_1T(\vec{e_1}) + u_2T(\vec{e_2}) + \ldots + u_nT(\vec{e_n}) \\ &= T(\vec{e_1})u_1 + T(\vec{e_2})u_2 + \ldots + T(\vec{e_n})u_n \\ &= \begin{bmatrix} T(\vec{e_1}) & T(\vec{e_2}) & \ldots & T(\vec{e_n}) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \\ &= A\vec{u} \ . \end{aligned}$$

Hence, $T(\vec{u}) = A\vec{u}$ for all $\vec{u} \in \mathbb{R}^n$, where A is the matrix whose ith column is $T(\vec{e_i})$ for $1 \le i \le n$. $\square$

Note: The linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ given by $T_A(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$ is called the linear transformation induced by A.

1. Find the matrix associated with the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x - z \\ z - y \\ x + y - 3z \end{bmatrix}, \qquad \forall \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$$

Solution: First, we find $T(\vec{e}_1)$, $T(\vec{e}_2)$, and $T(\vec{e}_3)$. So,

$$\begin{aligned}
T(\vec{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) &= \begin{bmatrix} 2(1) - 0 \\ 0 - 0 \\ 1 + 0 - 3(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \\
T(\vec{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) &= \begin{bmatrix} 2(0) - 0 \\ 0 - 1 \\ 0 + 1 - 3(0) \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \\
T(\vec{e}_3) = T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) &= \begin{bmatrix} 2(0) - 1 \\ 1 - 0 \\ 0 + 0 - 3(1) \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}.
\end{aligned}$$

Hence,

$$\begin{aligned}
A &= \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) \end{bmatrix} \\
&= \begin{bmatrix} 2 & 0 & -1 \\ 0 & -1 & 1 \\ 1 & 1 & -3 \end{bmatrix}.
\end{aligned}$$

Note that A is 3×3 since $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is 3×1.

Proposition 21 (Properties of Linear Transformations)

Let $T : V \to W$ be a linear transformation. Then for all $\vec{u}, \vec{v} \in V$ and $k \in \mathbb{R}$,

(1) $T(\vec{0}_V) = \vec{0}_W$

(2) $T(-\vec{v}) = -T(\vec{v})$

(3) $T(\vec{u} - \vec{v}) = T(\vec{u}) - T(\vec{v})$

(4) T is linear $\iff T(\vec{u} + k\vec{v}) = T(\vec{u}) + kT(\vec{v})$

(5) The composition of two linear maps is linear. (Recall that $(f \circ g)(x) = f(g(x))$.)

Proof. Suppose $T : V \to W$ is a linear transformation and suppose that $\vec{u}, \vec{v} \in V$ and $k \in \mathbb{R}$. We show the five properties above hold. Since, T is a linear transformation, it holds that T preserves vector addition and scalar multiplication. That is, $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ and $T(k\vec{u}) = kT(\vec{u})$.

(1) We show that $T(\vec{0}_V) = \vec{0}_W$. Let $\vec{v} \in V$. Then

$$\begin{aligned} T(\vec{0}_V) &= T(\vec{v} - \vec{v}) \\ &= T(\vec{v} + (-\vec{v})) \\ &= T(\vec{v}) + T(-\vec{v}) \\ &= T(\vec{v}) + T(-1\vec{v}) \\ &= T(\vec{v}) + (-1)T(\vec{v}) \\ &= T(\vec{v}) - T(\vec{v}) \\ &= \vec{0}_W \ . \end{aligned}$$

(2) We show that $T(-\vec{v}) = -T(\vec{v})$. So,

$$T(-\vec{v}) = T(-1\vec{v}) = (-1)T(\vec{v}) = -T(\vec{v}) \ .$$

(3) We show that $T(\vec{u} - \vec{v}) = T(\vec{u}) - T(\vec{v})$. So,

$$\begin{aligned} T(\vec{u} - \vec{v}) &= T(\vec{u} + (-\vec{v})) \\ &= T(\vec{u}) + T(-\vec{v}) \\ &= T(\vec{u}) + T(-1\vec{v}) \\ &= T(\vec{u}) + (-1)T(\vec{v}) \\ &= T(\vec{u}) - T(\vec{v}) \ . \end{aligned}$$

(4) We show that T is linear $\iff T(\vec{u}+k\vec{v}) = T(\vec{u}) + kT(\vec{v})$.

($\implies$) Suppose T is linear. Then T preserves vector addition and scalar multiplication. So,

$$\begin{aligned} T(\vec{u}+k\vec{v}) &= T(\vec{u}) + T(k\vec{v}) \\ &= T(\vec{u}) + kT(\vec{v}) \ . \end{aligned}$$

(5) Let $T : U \to V$ and $S : V \to W$ be linear maps. We show that $(S \circ T)$ is linear by showing that the composition preserves vector addition and scalar multiplication. Let $\vec{u}_1, \vec{u}_2 \in U$ and $k \in \mathbb{R}$.

(i) We show that $(S \circ T)(\vec{u}_1 + \vec{u}_2) = (S \circ T)(\vec{u}_1) + (S \circ T)(\vec{u}_2)$. So,

$$\begin{aligned} (S \circ T)(\vec{u}_1 + \vec{u}_2) &= S(T(\vec{u}_1 + \vec{u}_2)) \\ &= S(T(\vec{u}_1) + T(\vec{u}_2)) \\ &= S(T(\vec{u}_1)) + S(T(\vec{u}_2)) \\ &= (S \circ T)(\vec{u}_1)) + (S \circ T)(\vec{u}_2) \ . \end{aligned}$$

Hence $(S \circ T)$ preserves vector addition.

(ii) We show that $(S \circ T)(k\vec{u}_1) = k(S \circ T)(\vec{u}_1)$. So,

$$\begin{aligned} (S \circ T)(k\vec{u}_1) &= S(T(k\vec{u}_1)) \\ &= S(kT(\vec{u}_1)) \\ &= kS(T(\vec{u}_1)) \\ &= k(S \circ T)(\vec{u}_1) \ . \end{aligned}$$

Hence, $(S \circ T)$ preserves scalar multiplication.

So, the composition of linear maps is linear.

Thus, all five properties of Proposition 21 hold. $\square$

Theorem 22

Let V and W be two vector spaces and let $B = \{\vec{v}_1, \ldots, \vec{v}_n\}$ be a basis for V. Given a collection of vectors $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n\} \subseteq W$ (they need not be distinct), there exists a unique linear transformation $T : V \to W$ such that $T(\vec{v}_i) = \vec{w}_i$ for all $1 \leq i \leq n$. Moreover, if $\vec{v} = a_1\vec{v}_1 + \ldots + a_n\vec{v}_n$, then $T(\vec{v}) = a_1\vec{w}_1 + \ldots + a_n\vec{w}_1$.

Proof. Suppose V and W are vector spaces and suppose $B = \{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis for V. Let $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n\} \subseteq W$. $\square$

Observation: A linear map is completely determined by what it does to the elements in a basis for its domain.

1. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation, where $T\left(\begin{bmatrix} 4 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

   (a) Find $T\left(\begin{bmatrix} 11 \\ 7 \end{bmatrix}\right)$.

   Observe that $\left\{\begin{bmatrix} 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$ is a basis for $\mathbb{R}^2$. (This can be shown by showing this set is linearly independent and spans $\mathbb{R}^2$.) So, we first find $a, b \in \mathbb{R}$ such that

$$\begin{bmatrix} 11 \\ 7 \end{bmatrix} = a\begin{bmatrix} 4 \\ 2 \end{bmatrix} + b\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

   by solving the augmented matrix corresponding to this equation. Then we'll find that $a = 3$ and $b = -1$. Now, taking the transformation of both sides, we get that

$$\begin{aligned} T\left(\begin{bmatrix} 11 \\ 7 \end{bmatrix}\right) &= T\left(a\begin{bmatrix} 4 \\ 2 \end{bmatrix} + b\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) \\ &= T\left(a\begin{bmatrix} 4 \\ 2 \end{bmatrix}\right) + T\left(b\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) \\ &= aT\left(\begin{bmatrix} 4 \\ 2 \end{bmatrix}\right) + bT\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) \\ &= a\begin{bmatrix} 1 \\ 3 \end{bmatrix} + b\begin{bmatrix} -2 \\ 1 \end{bmatrix} \\ &= 3\begin{bmatrix} 1 \\ 3 \end{bmatrix} + (-1)\begin{bmatrix} -2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ 9 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 5 \\ 8 \end{bmatrix}. \end{aligned}$$

   Hence, $T\left(\begin{bmatrix} 11 \\ 7 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$.

(b) If $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$ (the domain), find $T(\vec{v})$.

Similarly, find $a, b \in \mathbb{R}$ such that

$$\begin{bmatrix} x \\ y \end{bmatrix} = a \begin{bmatrix} 4 \\ 2 \end{bmatrix} + b \begin{bmatrix} 1 \\ -1 \end{bmatrix} .$$

So, we get that

$$\left[\begin{array}{rr|r} 4 & 1 & x \\ 2 & -1 & y \end{array}\right]$$

$$\xrightarrow{R1<-->R2} \left[\begin{array}{rr|r} 2 & -1 & y \\ 4 & 1 & x \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R1} \left[\begin{array}{rr|r} 1 & -\frac{1}{2} & \frac{y}{2} \\ 4 & 1 & x \end{array}\right]$$

$$\xrightarrow{-4R1+R2} \left[\begin{array}{rr|r} 1 & -\frac{1}{2} & \frac{y}{2} \\ 0 & 3 & x-2y \end{array}\right]$$

$$\xrightarrow{\frac{1}{3}R2} \left[\begin{array}{rr|r} 1 & -\frac{1}{2} & \frac{y}{2} \\ 0 & 1 & \frac{x-2y}{3} \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R2+R1} \left[\begin{array}{rr|r} 1 & 0 & \frac{x+y}{6} \\ 0 & 1 & \frac{x-2y}{3} \end{array}\right] .$$

Here we get that $a = \frac{x+y}{6}$ and $b = \frac{x-2y}{3}$. So, we have that

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{x+y}{6} \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \frac{x-2y}{3} \begin{bmatrix} 1 \\ -1 \end{bmatrix} .$$

Then taking the transformation on both sides gives us (see next page)

$$
\begin{aligned}
T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= T\left(\frac{x+y}{6}\begin{bmatrix} 4 \\ 2 \end{bmatrix} + \frac{x-2y}{3}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) \\
&= T\left(\frac{x+y}{6}\begin{bmatrix} 4 \\ 2 \end{bmatrix}\right) + T\left(\frac{x-2y}{3}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) \\
&= \frac{x+y}{6} \cdot T\left(\begin{bmatrix} 4 \\ 2 \end{bmatrix}\right) + \frac{x-2y}{3} \cdot T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) \\
&= \frac{x+y}{6} \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \frac{x-2y}{3} \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix} \ .
\end{aligned}
$$

(c) Find the matrix associated with T.

Here we find the matrix A such that $T(\vec{v}) = A\vec{v}$. The matrix is

$$A = \begin{bmatrix} T(\vec{e_1}) & T(\vec{e_2}) \end{bmatrix} \ .$$

So,

$$
\begin{aligned}
T(\vec{e_1}) &= T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \\
&= \frac{1}{6}\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \frac{1}{6}\begin{bmatrix} -2 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1/6 \\ 1/2 \end{bmatrix} + \begin{bmatrix} -2/3 \\ 1/3 \end{bmatrix} \\
&= \begin{bmatrix} -1/2 \\ 5/6 \end{bmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
T(\vec{e_2}) &= T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \\
&= \frac{1}{6}\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \left(-\frac{2}{3}\right)\begin{bmatrix} -2 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1/6 \\ 1/2 \end{bmatrix} + \begin{bmatrix} 4/3 \\ -2/3 \end{bmatrix} \\
&= \begin{bmatrix} 3/2 \\ -1/6 \end{bmatrix} \ .
\end{aligned}
$$

Thus, $A = \begin{bmatrix} T(\vec{e_1}) & T(\vec{e_2}) \end{bmatrix} = \begin{bmatrix} -1/2 & 3/2 \\ 5/6 & -1/6 \end{bmatrix}$.

2. Let $D : P_n \to P_{n-1}$ given by $D(p(x)) = p'(x)$ for all $p(x) \in P_n$.

   (a) Show that D is linear.

   Proof. To show that D is a linear transformation, we need to show that D preserves vector addition and scalar multiplication. Let $f, g \in P_n$ (the domain) and $k \in \mathbb{R}$. Then $f(x) = a_0 + a_1x + \ldots + a_nx^n$ and $g(x) = b_0 + b_1x + \ldots + b_nx^n$. We can write these as $f = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}$ and $g = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{bmatrix}$.

   (1) We show that $T(f+g) = T(f) + T(g)$. So,

$$\begin{aligned}
T(f+g) &= T\left(\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{bmatrix}\right) \\
&= T\left(\begin{bmatrix} a_0 + b_0 \\ a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix}\right) \\
&= T((a_0 + b_0) + (a_1 + b_1)x + \ldots + (a_n + b_n)x^n) \\
&= (a_1 + b_1) + \ldots + n(a_n + b_n)x^{n-1} \\
&= \begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix} \\
&= (a_1 + b_1) + \ldots + n(a_n + b_n)x^{n-1} \\
&= (a_1 + \ldots + n(a_n)x^{n-1}) + (b_1 + \ldots + n(b_n)x^{n-1}) \\
&= T(a_0 + a_1x + \ldots + a_nx^n) + T(b_0 + b_1x + \ldots + b_nx^n) \\
&= T\left(\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}\right) + T\left(\begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{bmatrix}\right) \\
&= T(f) + T(g)\ .
\end{aligned}$$

   Hence, T preserves vector addition.

(2) We show that $T(kf) = k(T(f))$. So,

$$\begin{aligned}
T(kf) &= T\left(k\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}\right) \\
&= T\left(\begin{bmatrix} ka_0 \\ ka_1 \\ \vdots \\ ka_n \end{bmatrix}\right) \\
&= T((ka_0) + (ka_1)x + \ldots + (ka_n)x^n) \\
&= (ka_1) + n(ka_n)^{n-1} \\
&= \ldots \\
&= k(a_1 + \ldots n(a_n)x^{n-1}) \\
&= kT(a_0 + a_1x + \ldots + a_nx^n) \\
&= kT\left(\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}\right) \\
&= kT(f)\ .
\end{aligned}$$

$\square$

Kernel and Image

Let V and W be vector spaces and $T : V \to W$ be a linear transformation.

Definition (Kernel and Image)

The kernel of T is

$$\ker(T) = \{\vec{v} \in V \mid T(\vec{v}) = \vec{0}\} \subseteq V \ .$$

The image of T is

$$\mathrm{Im}(T) = \{T(\vec{v}) \in W \mid \vec{v} \in V\} \subseteq W \ .$$

In other words,

- The kernel of a transformation is the set of all vectors from the domain that map to the zero vector in the codomain.
- The image of a transformation is the set of those vectors in the codomain that get mapped to by something from the domain. One could say this is the range.

Note: $\mathrm{Im}(T) = \{\vec{w} \in W \mid \exists \vec{v} \in V \text{ s.t. } T(\vec{v}) = \vec{w}\}$

Recall from Vector Spaces in $\mathbb{R}^n$ the definitions of image space and null space.

Image Space

The image space of an $m \times n$ matrix A is defined as

$$\mathrm{Im}(A) = \{A\vec{x} \in \mathbb{R}^m \mid \vec{x} \in \mathbb{R}^n\} \ .$$

Definition (Null Space)

The nullspace of an $m \times n$ matrix A is defined by

$$\mathrm{null}(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\} \ .$$

1. Let $T_A : \mathbb{R}^n \to \mathbb{R}^m$ be given by $\vec{x} \mapsto A\vec{x}$ for $A \in M_{mn}$. In other words, $T_A(\vec{x}) = A\vec{x}$.

   Let $\vec{x} \in \mathbb{R}^n$ (the domain). Then

$$\begin{aligned}
\ker(T_A) &= \{\vec{x} \in \mathbb{R}^n \mid T_A(\vec{x}) = \vec{0}\} \\
&= \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\} && \text{(since } T_A(\vec{x}) = A\vec{x}\text{)} \\
&= \mathrm{null}(A) \ . && \text{(by definition of null space)}
\end{aligned}$$

   and

$$\begin{aligned}
\mathrm{Im}(T_A) &= \{T_A(\vec{x}) \in \mathbb{R}^m \mid \vec{x} \in \mathbb{R}^n\} \\
&= \{A\vec{x} \in \mathbb{R}^m \mid \vec{x} \in \mathbb{R}^n\} && \text{(since } T_A(\vec{x}) = A\vec{x}\text{)} \\
&= \mathrm{Im}(A) \\
&= \mathrm{Col}(A) \ .
\end{aligned}$$

2. Let $S : P_1 \to \mathbb{R}^2$ be given by $S(a + bx) = \begin{bmatrix} a \\ a+b \end{bmatrix}$ for all $a + bx \in P_1$ (the domain). Find $\ker(S)$ and $\mathrm{Im}(S)$.

   Here we have the transformation given by $S(a+bx) = S\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a \\ a+b \end{bmatrix}$. Now, let $u \in P_1$. Then $u = a + bx = \begin{bmatrix} a \\ b \end{bmatrix}$, and so

$$\begin{aligned}
\ker(S) &= \{u \in P_1 : S(u) = \vec{0}\} \\
&= \{a + bx \in P_1 : S(a + bx) = \vec{0}\} \\
&= \left\{a + bx \in P_1 : S\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\} \\
&= \left\{a + bx \in P_1 : \begin{bmatrix} a \\ a+b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}
\end{aligned}$$

   So, if we have some $u = a + bx \in P_1$ and this same element u is in $\ker(T)$, this means that $\begin{bmatrix} a \\ a+b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, which implies that $a = 0$ and $a + b = 0$. Hence,

$$\ker(S) = \{\vec{0}_{P_1}\} = \{0 + 0x\} \ .$$

Then for $\operatorname{Im}(S)$, we have that

$$\begin{aligned}
\operatorname{Im}(S) &= \{S(u) \in \mathbb{R}^2 : u \in P_1\} \\
&= \{S(a+bx) \in \mathbb{R}^2 : a+bx \in P_1\} \\
&= \left\{ S\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) \right\} \\
&= \left\{ \begin{bmatrix} a \\ a+b \end{bmatrix} \right\} \\
&= \left\{ \begin{bmatrix} a \\ a \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} \right\} \\
&= \left\{ a\begin{bmatrix} 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \\
&= \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \\
&= \mathbb{R}^2 \ .
\end{aligned}$$

Theorem 23

Let V and W be vector spaces and $T : V \to W$ be linear. Then

(i) $\ker(T)$ is a subspace of V

(ii) $\mathrm{Im}(T)$ is a subspace of W

Proof. Suppose V and W are vector spaces. Suppose $T : V \to W$ is a linear transformation. We show that $\ker(T)$ is a subspace of V and $\mathrm{Im}(T)$ is a subspace of W. we use the subspace test.

First, we show that $\ker(T)$ is a subspace of V. Note here that $ker(T) = \{\vec{v} \in V : T(\vec{v}) = \vec{0}_W\}$.

(1) Is is true that the zero vector is in $\ker(T)$? Yes, $\vec{0}_V \in \ker(T)$. This is because $T(\vec{0}_V) = \vec{0}_W$ (by the first property from Proposition 21).

(2) Is it true that if $\vec{x}, \vec{y} \in \ker(T)$, then $\vec{x}+\vec{y} \in \ker(T)$? Suppose $\vec{x}, \vec{y} \in \ker(T)$. Since $\vec{x}, \vec{y} \in \ker(T)$, this means that $T(\vec{x}) = \vec{0}_W$ and $T(\vec{y}) = \vec{0}_W$. So,

$$\begin{aligned} T(\vec{x}+\vec{y}) &= T(\vec{x}) + T(\vec{y}) \\ &= \vec{0}_W + \vec{0}_W \\ &= \vec{0}_W \ . \end{aligned}$$

So, because $T(\vec{x}+\vec{y}) = \vec{0}_W$, this means that $\vec{x}+\vec{y} \in \ker(T)$. Hence, $\ker(T)$ is closed under vector addition.

(3) Is is true that if $\vec{x} \in \ker(T)$ and $k \in \mathbb{R}$, then $k\vec{x} \in \ker(T)$? Suppose $\vec{x} \in \ker(T)$ and $k \in \mathbb{R}$. Since $\vec{x} \in \ker(T)$, this means that $T(\vec{x}) = \vec{0}_W$. So,

$$\begin{aligned} T(k\vec{x}) &= k(T\vec{x}) \\ &= k \cdot \vec{0}_W \\ &= \vec{0}_W \ . \end{aligned}$$

So, because $T(k\vec{x}) \in \ker(T)$, this means that $k\vec{x} \in \ker(T)$. Hence, $\ker(T)$ is closed under scalar multiplication.

Thus, by the subspace test, $\ker(T)$ is a subspace of V.

Now, we show that $\operatorname{Im}(T)$ is a subspace of W. Note that $\operatorname{Im}(T) = \{T(\vec{v}) \in W : \vec{u} \in V\}$.

(1) Is is true that $\operatorname{Im}(T)$ contains the zero vector. Yes! Indeed,

$\square$

Injective and Surjective

Let $T : V \to W$ be linear.

Definition (Surjective & Injective)

T is surjective (onto) iff $\forall \vec{w} \in W$, $\exists \vec{v} \in V$ such that $T(\vec{v}) = \vec{w}$.

T is injective (one-to-one) iff $\forall \vec{u}, \vec{v} \in V$, if $T(\vec{u}) = T(\vec{v})$ then $\vec{u} = \vec{v}$.

Note: The contrapositive of injective is: $\forall \vec{u}, \vec{v} \in V$, if $\vec{u} \neq \vec{v}$ then $T(\vec{u}) \neq T(\vec{v})$.

Theorem 24

A linear transformation $T : V \to W$ is one-to-one if and only if $\ker(T) = \{\vec{0}_V\}$.

Proof. Suppose V and W are vector spaces and suppose $T : V \to W$ is linear.

$(\implies)$ Suppose T is one-to-one. We show that $\ker(T) = \{\vec{0}_V\}$. (Note that $\vec{0_V}$ is the zero vector of the domain V.) Suppose $\vec{v} \in \ker(T) = \{\vec{v} \in V : T(\vec{v}) = \vec{0}_W\}$. Then this means that $T(\vec{v}) = \vec{0}_W$. So,

$$\begin{aligned} T(\vec{v}) &= \vec{0}_W \\ &= T(\vec{0}_V) \ . && \text{(by Proposition 21)} \end{aligned}$$

Now, since T is one-to-one (injective), this means that it must be the case that $\vec{v} = \vec{0_V}$. Hence,

$$\begin{aligned} \ker(T) &= \{\vec{v} \in V : T(\vec{v}) = \vec{0}_W\} \\ &= \{\vec{0}_V\} \ . \end{aligned}$$

$(\impliedby)$ Suppose $\ker(T) = \{\vec{0}_V\}$. More specifically,

$$\ker(T) = \{\vec{v} \in V : T(\vec{x}) = \vec{0}_W\} = \{\vec{0}_V\} \ .$$

We show that T is one-to-one. By the definition of one-to-one, suppose $\vec{u}, \vec{v} \in W$ and suppose $T(\vec{u}) = T(\vec{v})$. We show that $\vec{u} = \vec{v}$. Now, since $T(\vec{u}) = T(\vec{v})$, we get that $T(\vec{u}) - T(\vec{v}) = \vec{0}_W$. Then by Proposition 21, $T(\vec{u} - \vec{v}) = \vec{0}_W$. From this we get that $\vec{u} - \vec{v} \in \ker(T)$ because $T(\vec{u} - \vec{v}) = \vec{0}_W$. So, we get that $\vec{u} - \vec{v} = \vec{0}_V$. Hence, $\vec{u} = \vec{v}$. Thus, T is one-to-one. $\square$

Theorem

A linear transformation $T : V \to W$ is onto if and only if $\mathrm{Im}(T) = W$.

1. Let $T : P_2 \to \mathbb{R}^4$ be defined by $a + bx + cx^2 \mapsto \begin{bmatrix} a \\ b \\ c \\ a \end{bmatrix}$. Is T onto? Is T one-to-one?

   T is one-to-one. To show this, we prove that $\ker(T) = \{\vec{0}_{P_2}\}$. Suppose $\vec{v} = a + bx + cx^2 \in \ker(T) = \{\vec{v} \in P_2 : T(\vec{v}) = \vec{0}_{\mathbb{R}^4}\}$. Then

$$\begin{aligned} T(\vec{v}) &= T(a + bx + cx^2) \\ &= \begin{bmatrix} a \\ b \\ c \\ a \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ &= \vec{0}_{\mathbb{R}^4} \ . \end{aligned}$$

   From this we get that $a = b = c = 0$. Hence,

$$\begin{aligned} \ker(T) &= \{\vec{v} \in P_2 : T(\vec{v}) = \vec{0}_{\mathbb{R}^4}\} \\ &= \{a + bx + cx^2 : T(a + bx + cx^2) = \vec{0}_{\mathbb{R}^4}\} \\ &= \{\vec{0}_{P_2}\} \ . \end{aligned}$$

Rank and Nullity

Definition

Let $T : V \to W$ be linear.

(1) $\text{nullity}(T) = \dim(\ker(T))$.

(2) $\text{rank}(T) = \dim(\text{Im}(T))$

Observe:

- T is one-to-one iff $\text{nullity}(T) = 0$.
- T is onto iff $\text{rank}(T) = \dim(W)$.

Theorem (Rank-Nullity or Dimension Theorem)

Let $T : V \to W$ be linear and $\dim(V) = n$. Then

$$n = \text{rank}(T) + \text{nullity}(T) = \dim(\text{Im}(T)) + \dim(\ker(T)) \ .$$

1. If $T : P_3 \to M_{23}$ is one-to-one, then what is $\text{rank}(T)$.

   Since T is one-to-one, this means that $\ker(T) = \{\vec{0}_{P_2}\}$. Then this implies that $\text{nullity}(T) = \dim(\ker(T)) = 0$. We also know that $n = \dim(P_3) = 4$. So, by the Rank-Nullity Theorem,

   $$n = 4 = 0 + \text{rank}(T) = \text{nullity}(T) + \text{rank}(T) \ .$$

   Hence, because $4 = 0 + \text{rank}(T)$, this means that $\text{rank}(T) = 4$.

2. If $S : M_{23} \to P_2$ is onto, what is $\text{nullity}(A)$?

   Since S is onto, this means that $\text{Im}(S) = P_2$. Then this implies that $\text{rank}(S) = \dim(\text{Im}(S)) = \dim(P_2) = 3$. We also know that $n = \dim(M_{23}) = 2 \cdot 3 = 6$. So, by the Rank-Nullity Theorem,

   $$\begin{aligned} \text{nullity}(S) &= n - \text{rank}(S) \\ &= 6 - 3 \\ &= 3 \end{aligned}$$

Isomorphisms

<u>Review of Inverse:</u> Let A and B be sets.

Definition (Inverse)

If $f : A \to B$ is both injective and surjective, also called bijective, then the inverse of f is the function $f^{-1} : B \to A$ such that

$$(f^{-1} \circ f)(a) = a$$

for all $a \in A$ and

$$(f \circ f^{-1})(b) = b$$

for all $b \in B$.

Definition (Isomorphism)

A linear transformation $T : V \to W$ is called an isomorphism if and only if it is bijective (both injective and surjective).

Definition (Isomorphic)

Two vector spaces V and W are called isomorphic if and only if there exists an isomorphism $T : V \to W$. In this case, we write $V \cong W$.

WARNING: Being isomorphic doesn't mean "same". It means "same structure".

1. The map $\mathbb{R}^3 \to P_2$, defined by $T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = a+bx+cx^2$ is an isomorphism.

Proof. First, we check that T is linear. To check if T is linear, we check if T preserves vector addition and scalar multiplication.

(1) Suppose $\vec{u}, \vec{v} \in \mathbb{R}^3$. We show that $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$. So,

$$\begin{aligned}
T(\vec{u}+\vec{v}) &= T\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) \\
&= T\left(\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{bmatrix}\right) \\
&= (u_1+v_1) + (u_2+v_2)x + (u_3+v_3)x^2 \\
&= u_1 + v_1 + u_2x + v_2x + u_3x^2 + v_3x^2 \\
&= (u_1 + u_2x + u_3x^2) + (v_1 + v_2x + v_3x^2) \\
&= T\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) + T\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) \\
&= T(\vec{u}) + T(\vec{v})\ .
\end{aligned}$$

Hence, T preserves vector addition.

(2) Suppose $\vec{u} \in \mathbb{R}^3$ and $k \in \mathbb{R}$. We show that $T(k\vec{u}) = kT(\vec{u})$. So,

$$\begin{aligned}
T(k\vec{u}) &= T\left(k\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) \\
&= T\left(\begin{bmatrix} ku_1 \\ ku_2 \\ ku_3 \end{bmatrix}\right) \\
&= (ku_1) + (ku_2)x + (ku_3)x^2 \\
&= k(u_1 + u_2x + u_3x^2) \\
&= kT\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) \\
&= kT(\vec{u})\ .
\end{aligned}$$

Hence, T preserves scalar multiplication.

Thus, T is a linear transformation.

Second, we check that T is injective (one-to-one). We can use Theorem 24, by showing that $\ker(T) = \{\vec{0}_{\mathbb{R}^3}\}$. We know that $\ker(T) = \{\vec{v} \in \mathbb{R}^3 : T(\vec{v}) = \vec{0}_{P_2}\}$. Suppose $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \in \ker(T)$. Then this means that $T(\vec{u}) = \vec{0}_{P_2}$. So,

$$\begin{aligned} \ker(T) &= \{\vec{u} \in \mathbb{R}^3 : T(\vec{u}) = \vec{0}_{P_2}\} \\ &= \left\{ \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \in \mathbb{R}^3 : T\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \right) = \vec{0}_{P_2} \right\} . \end{aligned}$$

So we have that $\vec{0}_{P_2} = T\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \right) = u_1 + u_2 x + u_3 x^2$. From this we get that $u_1 + u_2 x + u_3 x^2 = \vec{0}_{P_2} = 0 + 0x + 0x^2$.

Finally, we show that T is surjective. Since T is one-to-one, $\text{nullity}(T) = \dim(\ker(T)) = 0$ (since $\ker(T) = \{\vec{0}_{P_2}\}$). We also know that $\dim(\mathbb{R}^3) = 3$. Then by Rank-Nullity, we have that

$$\begin{aligned} \text{rank}(T) &= n - \text{nullity}(T) \\ &= \dim(\mathbb{R}^3) - \text{nullity}(T) \\ &= 3 - 0 \\ &= 3 . \end{aligned}$$

Now, since $\dim(P_2) = 3 = \dim(\text{Im}(T)) = \text{rank}(T)$, this means that $\text{Im}(T) = P_2$. Hence, T is surjective. $\square$

Theorem 25

Let V and W be finite dimensional vector spaces. Let $V \to W$ be linear. The following are equivalent:

(1) T is an isomorphism

(2) If $\alpha = \{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis for V, then $\beta = \{T(\vec{v}_1), \ldots, T(\vec{v}_n)\}$ is a basis for W.

Proof. ($\implies$) Suppose $V \cong W$. Then there exists an isomorphism $T : V \to W$. Let $\alpha = \{\vec{v}_1, \ldots, \vec{v}_n\}$ be a basis for V. By $\square$

1. Let $T : P_2 \to \mathbb{R}^3$ be a linear transformation defined by $T(a + bx + cx^2) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Let's note some things about this linear transformation.

   - The matrix A associated with T is

$$\begin{aligned} A &= \begin{bmatrix} T(\vec{e_1}) & T(\vec{e_2}) & T(\vec{e_3}) \end{bmatrix} \\ &= \begin{bmatrix} T(1) & T(x) & T(x^2) \end{bmatrix} \\ &= \begin{bmatrix} T(1 + 0x + 0x^2) & T(0 + x + 0x^2) & T(0 + 0x + x^2) \end{bmatrix} \\ &= \begin{bmatrix} T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) & T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) & T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

   - Note that since $T(\vec{v}) = A\vec{v}$, Then solving $A\vec{v} = \vec{0}_{\mathbb{R}^3}$, we get that

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

     has the solution $a = 0$, $b = 0$, $c = 0$. So,

$$\vec{v} = a + bx + cx^2 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 + 0x + 0x^2 = \vec{0}_{P_2} \ .$$

   - T is injective (one-to-one). This is because

$$\begin{aligned} \ker(T) &= \{\vec{v} \in P_2 : T(\vec{v}) = \vec{0}_{\mathbb{R}^3}\} \\ &= \{\vec{v} \in P_2 : A\vec{v} = \vec{0}_{\mathbb{R}^3}\} \\ &= \{\vec{0}_{P_2}\} \ . \end{aligned}$$

     Then by Theorem 24, since $\ker(T) = \{\vec{0}_{P_2}\}$, this means that T is one-to-one. Also note that $\text{nullity}(T) = \dim(\ker(T)) = 0$.

- Note that

$$
\begin{aligned}
T(\vec{v}) &= A(\vec{v}) \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \\
&= \begin{bmatrix} a \\ b \\ c \end{bmatrix} \\
&= \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix} \\
&= a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\end{aligned}
$$

- T is surjective (onto). This is because

$$
\begin{aligned}
\text{Im}(T) &= \{T(\vec{v}) \in \mathbb{R}^3 : \vec{v} \in P_2\} \\
&= \{A\vec{v} \in \mathbb{R}^3 : \vec{v} \in P_2\} \\
&= \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3 : \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in P_2 \right\} \\
&= \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right\} \\
&= \left\{ \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix} \right\} \\
&= \left\{ a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \\
&= \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} .
\end{aligned}
$$

  Note that $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $\text{Im}(T)$, and so $\dim(\text{Im}(T)) = 3$. So, $\text{Im}(T) = \mathbb{R}^3$ (the codomain). Indeed, $\dim(\text{Im}(T)) = \text{rank}(T) = 3 = \dim(\mathbb{R}^3)$. Then by Theorem 25, since $\text{Im}(T) = \mathbb{R}^3$ (the codomain), T is surjective (onto).

- So, because T is injective AND surjective, this means that T is bijective. Since T is bijective, this means that T is an isomorphism.

- We have that $\alpha = \{1, x, x^2\} = \{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is a basis for P_2 (the domain). Then it holds that

$$\begin{aligned}
\beta &= \{T(\vec{v_1}), T(\vec{v_2}), T(\vec{v_3})\} \\
&= \{T(1), T(x), T(x^2)\} \\
&= \{T(1 + 0x + 0x^2), T(0 + x + 0x^2), T(0 + 0x + x^2)\} \\
&= \left\{ T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right), T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right), T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \right\} \\
&= \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}
\end{aligned}$$

  is indeed a basis for $\mathbb{R}^3$ (the codomain). In fact, this holds for **any** basis of P_2.

Corollary 26

If V and W are finite dimensional vector spaces, then $V \cong W$ if and only if $\dim(V) = \dim(W)$.

For example, $P_1 \cong \mathbb{R}^2$ because $\dim(P_1) = 2 = \dim(\mathbb{R}^2)$.

Proof. Suppose V and W are finite dimensional vector spaces.

($\implies$) Suppose $V \cong W$. Then this means that there exists an isomorphism $T : V \to W$, where T is some linear transformation; that is, T is bijective (both injective and surjective). Let $\alpha = \{\vec{v_1}, \ldots, \vec{v_n}\}$ be a basis for V. Then $\beta = \{T(\vec{v_1}), \ldots, T(\vec{v_n})\}$ is a basis for W. Hence, since both basis vectors have n elements, $\dim(V) = n = \dim(W)$.

($\impliedby$) Suppose $\dim(V) = \dim(W)$. Let $B_1 = \{\vec{v_1}, \ldots, \vec{v_n}\}$ be a basis for V and $B_2 = \{\vec{w_1}, \ldots, \vec{w_n}\}$ be a basis for W. We want to show that $V \cong W$; that is, there exists some linear transformation $T : V \to W$. So, let $T : V \to W$ be a linear map, where $T(\vec{v_i}) = w_i$ for each $1 \leq i \leq n$. Then by Theorem 25, T is an isomorphism, and hence $V \cong W$. $\square$

Corollary 27

If $\dim(V) = \dim(W)$, then a linear transformation $T : V \to W$ is an isomorphism if it is either one-to-one or onto.

Proof. Suppose $T : V \to W$ is linear and suppose $\dim(V) = \dim(W)$. Let $B_1 = \{\vec{v_1}, \ldots, \vec{v_n}\}$ be a basis for V.

(1) We show that if T is one-to-one, then T is an isomorphism. Suppose T is one-to-one. Then by Theorem 24, this means that $\ker(T) = \{\vec{0_V}\}$. This implies that $\dim(\ker(T)) = 0$. So, by Rank Nullity,

$$\begin{aligned}\operatorname{rank}(T) &= n - \operatorname{nullity}(T)\\ &= \dim(W) - \dim(\ker(T))\\ &= \dim(W) - 0\\ &= \dim(W)\ .\end{aligned}$$

So, we get that $\dim(\operatorname{Im}(T)) = \operatorname{rank}(T) = \dim(W)$, which means that T is onto. Thus, since T is one-to-one and onto, T is bijective and hence T is an isomorphism.

(2) We show that if T is onto, then T is an isomorphism. Suppose T is onto. Since T is onto, this means that $\operatorname{Im}(T) = W$. Then $\dim(\operatorname{Im}(T)) = \dim(W)$. Then by Rank-Nullity,

$$\begin{aligned}\operatorname{nullity}(T) &= n - \operatorname{rank}(T)\\ &= \dim(W) - \dim(\operatorname{Im}(T))\\ &= \dim(W) - \dim(W)\\ &= 0\ .\end{aligned}$$

So, we get that $\operatorname{nullity}(T) = \dim(\ker(T)) = 0$, which implies that $\ker(T) = \{\vec{0_V}\}$. Then by Theorem 24, since $\ker(T) = \{\vec{0_V}\}$, we can conclude that T is one-to-one. Thus, since T is both onto and one-to-one, T is bijective and hence T is an isomorphism.

$\square$

1. Which of the following are isomorphisms.

    (a) $T : M_{mn} \to M_{mn}$, where $x \mapsto AXB$ where $A \in M_{mm}$, $B \in M_{nn}$ fixed invertible matrices.

        First, we show that T is a linear transformation by showing that T preserves vector addition and scalar multiplication.

        (1) Suppose $C, D \in M_{mn}$. We show that $T(C+D) = T(C) + T(D)$. So,

$$
\begin{aligned}
T(C+D) &= T\left(\begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & & \ddots & \\ c_{m1} & c_{m2} & \dots & c_{mn} \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} & \dots & d_{1n} \\ d_{21} & d_{22} & \dots & d_{2n} \\ \vdots & & \ddots & \\ d_{m1} & d_{m2} & \dots & d_{mn} \end{bmatrix}\right) \\
&= T\left(\begin{bmatrix} c_{11}+d_{11} & c_{12}+d_{12} & \dots & c_{1n}+d_{1n} \\ c_{21}+d_{21} & c_{22}+d_{22} & \dots & c_{2n}+d_{2n} \\ \vdots & & \ddots & \\ c_{m1}+d_{m1} & c_{m2}+d_{m2} & \dots & c_{mn}+d_{mn} \end{bmatrix}\right) \\
&= A\begin{bmatrix} c_{11}+d_{11} & c_{12}+d_{12} & \dots & c_{1n}+d_{1n} \\ c_{21}+d_{21} & c_{22}+d_{22} & \dots & c_{2n}+d_{2n} \\ \vdots & & \ddots & \\ c_{m1}+d_{m1} & c_{m2}+d_{m2} & \dots & c_{mn}+d_{mn} \end{bmatrix} B \\
&= \begin{bmatrix} a_{11} & \dots & a_{1m} \\ \vdots & \ddots & \\ a_{mm} & \dots & a_{mm} \end{bmatrix} \begin{bmatrix} c_{11}+d_{11} & \dots & c_{1n}+d_{1n} \\ \vdots & \ddots & \\ c_{m1}+d_{m1} & \dots & c_{mn}+d_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \\ b_{n1} & \dots & b_{nn} \end{bmatrix} \\
&= \dots \\
&= \begin{bmatrix} a_{11}b_{11}c_{11} + \dots a_{m1}b_{11}c_{m1} & a_{11} \end{bmatrix} \\
&= A\begin{bmatrix} c_{11} & \dots & c_{1n} \\ \vdots & \ddots & \\ c_{m1} & \dots & c_{mn} \end{bmatrix} B + A\begin{bmatrix} d_{11} & \dots & d_{1n} \\ \vdots & \ddots & \\ d_{m1} & \dots & d_{mn} \end{bmatrix} B \\
&= T\left(\begin{bmatrix} c_{11} & \dots & c_{1n} \\ \vdots & \ddots & \\ c_{m1} & \dots & c_{mn} \end{bmatrix}\right) + T\left(\begin{bmatrix} d_{11} & \dots & d_{1n} \\ \vdots & \ddots & \\ d_{m1} & \dots & d_{mn} \end{bmatrix}\right) \\
&= T(C) + T(D) \ .
\end{aligned}
$$

(2) Suppose $X \in M_{mn}$ and $k \in \mathbb{R}$. We show that $T(kX) = kT(X)$.

2. $S : M_{mn} \to M_{mn}$, where $A \mapsto A^2$.

A^2 is not even possible, since $A \in M_{mn}$; that is, you can't multiply an $m \times n$ matrix with an $m \times n$ matrix. How about when $m = n$? Is $S : M_{nn} \to M_{nn}$ an isomorphism. Well, S here would not be linear. For, let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Then

$$\begin{aligned}
T(A+B) &= T\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) \\
&= T\left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right) \\
&= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \\
&\neq \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^2 + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}^2 \\
&= T\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) \\
&= T(A) + T(B)\ .
\end{aligned}$$

Hence, S does not preserve vector addition, and so S is not linear. Since S is not even linear, S is definitely NOT an isomorphism.

3. $R : P_2 \to \mathbb{R}^3$, where $a + bx + cx^2 \mapsto \begin{bmatrix} b - c \\ a \\ 2a + b - c \end{bmatrix}$.

Observation: Given $A \in M_{nn}$, consider $T_A : \mathbb{R}^n \to \mathbb{R}^n$ given by $T_A(\vec{x}) = A\vec{x}$. Then T is an isomorphism if and only if A is invertible.

Proof. Suppose $A \in M_{nn}$ and suppose $T_A : \mathbb{R}^n \to \mathbb{R}^n$ is a transformation given by $T_A(\vec{x}) = A\vec{x}$.

($\implies$) Suppose T is an isomorphism. Then this means that T is injective and surjective (bijective). Since T is injective, this means that $\text{nullity}(T) = 0$ $\square$

Definition (Invertible Map)

Let $T : V \to W$ be a linear transformation between finite dimensional vector spaces. A map $T^{-1} : W \to V$ is called the inverse of T if $T(T^{-1}(\vec{w})) = \vec{w}$ for all $\vec{w} \in W$ and $T^{-1}(T(\vec{v})) = \vec{v}$ for all $\vec{v} \in V$. If T has an inverse, then T is called invertible.

Prove that $R : V \to W$ is invertible if and only if T is an isomorphism.

Proof. ($\implies$) Suppose $R : V \to W$ is invertible. We show that T is an isomorphism. Since $R : V \to W$ is invertible, then we have a map $R^{-1} : W \to V$, where $R(R^{-1}(\vec{w})) = \vec{w}$ for all $\vec{w} \in W$ and $R^{-1}(R(\vec{v})) = \vec{v}$ for all $\vec{v} \in V$. $\square$

Lemma

If $T : V \to W$ is linear and invertible, then $T^{-1} : W \to V$ is linear. Morever, T^{-1} is unique.

Proof. Suppose $T : V \to W$ is linear and invertible. Since T is linear, this means that T preserves vector addition and scalar multiplication; that is, $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ and $T(k\vec{v}) = kT(\vec{v})$ for all $\vec{u}, \vec{v} \in V$ and $k \in \mathbb{R}$. Since T is invertible, this means that T has an inverse map $T^{-1} : W \to V$. $\square$

1. Find **<u>the</u>** inverse of the map $T : P_1 \to P_1$, where $a + bx \mapsto (a - b) + ax$.

   We want to solve $T(a + bx) = c + dx$ for a and b. Solve for a and b in $(a - b) + ax = c + dx$. So, $a - b = c$ and $a = d$ which implies that $b = d - c$. So,

   $$T^{-1}(c + dx) = d + (d - c)x$$

   since $T^{-1}(c + dx) = T^{-1}(T(a + bx)) = a + bx$.