MATH 361 - Week 2 Notes

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Spectral Theory Continued

Definition (Algebraic and Geometric Multiplicity)

Suppose A has eigenvalue λ. λ has **algebraic multiplicity** $k =: a(\lambda)$ if k is the largest integer so that

$$C_A(z) = (z-\lambda)^k p(z)$$

for some polynomial p.

The **geometric multiplicity** of A is

$$g(\lambda) := \dim(\ker(A - \lambda I)) \ .$$

That is, the geometric multiplicity is the number of parameters in the solution to $(A - \lambda I)X = 0$.

- For example, the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has only one eigenvalue $\lambda = 1$. Here $a(\lambda) = 2$ and $g(\lambda) = 1$. Note that $\ker(A - \lambda I)$ is the eigenspace of λ. We can check this. We have that

$$A - \lambda I = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix} \ ,$$

and so

$$\ker(A - \lambda I) = \{\vec{x} : (A - \lambda I)\vec{x} = \vec{0}\}$$

Proposition

For an eigenvalue λ of A, the algebraic multiplicity is at least the geometric multiplicity. That is,

$$a(\lambda) \geq g(\lambda) \ .$$

Note: Since $(A - \lambda I)X = 0$ has a non-trivial solution, this means that $g(\lambda) \geq 1$ (since a non-trivial solution involves at least one parameter).

Proof. Suppose A is $n \times n$ and $k = g(\lambda) \leq n$. Let $\{\vec{v}_1, \ldots, \vec{v}_k\}$ be a basis for $\ker(A - \lambda I)$. Extend to a basis $\{\vec{v}_1, \ldots, \vec{v}_k, \vec{w}_{k+1}, \ldots, \vec{w}_n\}$ of V. The matrix representation for A with respect to this basis is

$$\begin{bmatrix} \lambda I_k & B \\ 0 & C \end{bmatrix} \ .$$

Then $c_A(z) = c_{\lambda I_k}(z) c_C(z) = (z - \lambda)^k c_C(z)$. By definition of $a(\lambda)$, $k = g(\lambda) \leq a(\lambda)$. $\square$

Corollary

A is diagonalizable if and only if $a(\lambda) = g(\lambda)$ for all eigenvalues λ of A.

Proof. The sum of the $a(\lambda)$ is n, so if $g(\lambda) = a(\lambda)$ for all λ, we have n linearly independent eigenvectors, i.e. a basis of eigenvectors. Hence, A is diagonalizable by Theorem 15 from MATH 311. Conversely, if A is diagonalizable, then A must have n linearly independent eigenvectors. This only happens when $g(\lambda)$ is as large as possible (i.e. equal to $a(\lambda)$). □

Corollary

Suppose A is diagonalizable. Then $C_A(A) = 0$ (the zero matrix).

Note: If $p(z)$ is a polynomial $p(z) = a_0 + a_1 z + \ldots + a_n z^n$, then

$$p(A) := a_0 I + a_1 A + \ldots + a_n A^n \ .$$

Proof. Suppose A is diagonalizable. Then there exist matrices S and D such that

$$S^{-1}AS = D \ .$$

We write

$$C_A(z) = \alpha(z - \lambda_1)(z - \lambda_2)\ldots(z - \lambda_n)$$

and so

$$D = \mathrm{diag}\{\lambda_1, \ldots, \lambda_n\}.$$

So,

$$\begin{aligned}
&C_A(D) \\
&= \alpha(D - \lambda_1 I)(D - \lambda_2 I)\ldots(D - \lambda_n I) \\
&= \alpha \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 - \lambda_1 & 0 & \ldots & 0 \\ 0 & 0 & \lambda_3 - \lambda_1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \lambda_n - \lambda_1 \end{bmatrix} \begin{bmatrix} \lambda_1 - \lambda_2 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & \lambda_3 - \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \lambda_n - \lambda_2 \end{bmatrix} \ldots \\
&= \alpha 0
\end{aligned}$$

$= 0 \ ,$

where 0 is the zero matrix. Thus,

$$\begin{aligned} C_A(A) &= C_A(SDS^{-1}) \\ &= SC_A(D)S^{-1} \\ &= S0S^{-1} \\ &= 0 \ . \end{aligned}$$

$\square$

<u>**Wrong proof:**</u> $C_A(A) = \det(AI - A) = \det(0) = 0$. This doesn't make sense since det returns a real number, not a matrix.

Definition (Internal Direct Sum of Subspaces)

Let W be a $\mathbb{C}$-vector space with subspaces $V_1, V_2, \ldots, V_k$. We say W is the **direct sum** of $V_1, \ldots, V_k$ and write

$$W = V_1 \oplus V_2 \oplus \ldots \oplus V_k = \bigoplus_{j=1}^{k} V_j$$

if for all $\vec{w} \in W$, there are unique $\vec{v}_j \in V_j$ so that

$$\vec{w} = \vec{v}_1 + \ldots + \vec{v}_k \ .$$

1. If $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ is a basis for W, then

$$W = \mathrm{span}\{\vec{w}_1\} \oplus \mathrm{span}\{\vec{w}_2\} \oplus \mathrm{span}\{\vec{w}_k\} \ .$$

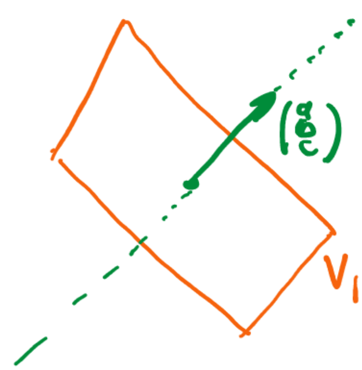
2. In $\mathbb{R}^3$, let

$$V_1 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : ax + by + cz = 0 \right\}$$

(a plane through the origin) and

$$V_2 = \mathrm{span} \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\}$$

(a normal line through the origin).



Then $\mathbb{R}^3 = V_1 \oplus V_2$.

Theorem

Suppose A has distinct eigenvalues $\{\lambda_1, \ldots, \lambda_k\}$. Then A is diagonalizable if and only if

$$V = \ker(A - \lambda_i I) \oplus \ldots \oplus \ker(A - \lambda_k I) \ ,$$

where $\ker(A - \lambda_i I)$ are the eigenspaces of each λ_i.

Inner Product Spaces

- Recall in $\mathbb{R}^n$, we have the **dot product**:

$$\vec{x} \cdot \vec{y} = \sum_{j=1}^{n} x_j y_j \ ,$$

where $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$.

- Familiar properties:

  (i) Orthogonality: $\vec{x} \cdot \vec{y} = 0$ means $\vec{x}$ is perpendicular to $\vec{y}$.

  (ii) Cauchy-Schwarz inequality:

$$|\vec{x} \cdot \vec{y}| \leq ||\vec{x}|| \ ||\vec{y}||$$

  Triangle inequality:

$$||\vec{x} + \vec{y}|| \leq ||\vec{x}|| + ||\vec{y}||$$

  Pythagorean law:

$$\vec{x} \cdot \vec{y} = 0 \implies ||\vec{x}||^2 + ||\vec{y}||^2 = ||\vec{x} + \vec{y}||^2$$

- We want to replicate this geometric structure in an arbitrary complex finite dimensional vector space V.

Definition (Complex Inner Product Space)

Suppose V is a $\mathbb{C}$-vector space (not necessarily finite dimensional). We say that V is an **inner product space** if there is a map

$$\langle \ , \ \rangle : V \times V \to \mathbb{C}$$

called an **inner product**, satisfying

(1) Positive semidefinitiveness and non-degeneracy:

$$\langle \vec{x}, \vec{x} \rangle \geq 0 \quad \forall \vec{x} \in V \ ,$$

and

$$\langle \vec{x}, \vec{x} \rangle = 0$$

if and only if $\vec{x} = \vec{0}$.

(2) $\langle \vec{x}, \vec{y} \rangle = \overline{\langle \vec{y}, \vec{x} \rangle}$ for all $\vec{x}, \vec{y} \in V$.

(3) Sesquilinearity ("linear and a half"):

$$\langle \alpha\vec{x} + \beta\vec{y}, \vec{z} \rangle = \alpha\langle \vec{x}, \vec{z} \rangle + \beta\langle \vec{y}, \vec{z} \rangle$$

and

$$\langle \vec{x}, \alpha\vec{y} + \beta\vec{z} \rangle = \bar{\alpha}\langle \vec{x}, \vec{y} \rangle + \bar{\beta}\langle \vec{x}, \vec{z} \rangle$$

for all $\vec{x}, \vec{y}, \vec{z} \in V$ and $\alpha, \beta \in \mathbb{C}$. So, $\langle \ , \ \rangle$ is linear in the first variable and "conjugate-linear" in the second variable.

Note: Another acceptable notation for the inner product is $\langle \ | \ \rangle$.

1. $\mathbb{C}^n$ with the complex inner product

$$\langle \vec{z}, \vec{w} \rangle = \langle \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}, \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \rangle := \sum_{j=1}^{n} z_j \bar{w}_j$$

is an inner product space. We can verify this:

(i) We have that

$$\langle \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}, \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \rangle = \sum_{j=1}^{n} z_j \bar{z}_j = \sum_{j=1}^{n} |z_j|^2 \geq 0 \ .$$

Indeed, for complex numbers $z_j = a_j + b_j i$ and $\bar{z}_j = a_j - b_j i$, we have that

$$\begin{aligned} z_j \bar{z}_j &= (a_j + b_j i)(a_j - b_j i) \\ &= a_j^2 - b_j^2 i^2 \\ &= a_j^2 - b_j^2(-1) \\ &= a_j^2 + b_j^2 \\ &= \left(\sqrt{a_j^2 + b_j^2} \right)^2 \\ &= |z_j|^2 \ . \end{aligned}$$

(ii) Let $\vec{x} = \begin{bmatrix} z_1 & \dots & z_n \end{bmatrix}^T$ and $\vec{y} = \begin{bmatrix} w_1 & \dots & w_n \end{bmatrix}^T$ be in $\mathbb{C}^n$. We have that

$$\begin{aligned} \overline{\langle \vec{x}, \vec{y} \rangle} &= \overline{\sum_{j=1}^{n} z_j \bar{w}_j} \\ &= \sum_{j=1}^{n} \overline{z_j \bar{w}_j} \\ &= \sum_{j=1}^{n} \bar{z}_j \bar{\bar{w}}_j \\ &= \sum_{j=1}^{n} \bar{z}_j w_j \\ &= \sum_{j=1}^{n} w_j \bar{z}_j \\ &= \langle \vec{y}, \vec{x} \rangle \ . \end{aligned}$$

(iii) Let $\vec{x} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T, \vec{y} = \begin{bmatrix} y_1 & \dots & y_n \end{bmatrix}^T, \vec{z} = \begin{bmatrix} z_1 & \dots & z_n \end{bmatrix}^T \in \mathbb{C}^n$ and $\alpha, \beta \in \mathbb{C}$. For convenience, let

$$\vec{u} = \alpha\vec{x} + \beta\vec{y} = \begin{bmatrix} \alpha x_1 \\ \vdots \\ \alpha x_1 \end{bmatrix} + \begin{bmatrix} \beta y_1 \\ \vdots \\ \beta y_n \end{bmatrix} = \begin{bmatrix} \alpha x_1 + \beta y_1 \\ \vdots \\ \alpha x_n + \beta y_n \end{bmatrix} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

and

$$\vec{v} = \alpha\vec{y} + \beta\vec{z} = \begin{bmatrix} \alpha y_1 \\ \vdots \\ \alpha y_1 \end{bmatrix} + \begin{bmatrix} \beta z_1 \\ \vdots \\ \beta z_n \end{bmatrix} = \begin{bmatrix} \alpha y_1 + \beta z_1 \\ \vdots \\ \alpha y_n + \beta z_n \end{bmatrix} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

Then

$$\begin{aligned}
\langle \alpha\vec{x} + \beta\vec{y}, \vec{z} \rangle &= \langle \vec{u}, \vec{z} \rangle \\
&= \sum_{j=1}^{n} u_j \bar{z}_j \\
&= \sum_{j=1}^{n} (\alpha x_j + \beta y_j)\bar{z}_j \\
&= \sum_{j=1}^{n} (\alpha x_j \bar{z}_j + \beta y_j \bar{z}_j) \\
&= \sum_{j=1}^{n} \alpha x_j \bar{z}_j + \sum_{j=1}^{n} \beta y_j \bar{z}_j \\
&= \alpha \sum_{j=1}^{n} x_j \bar{z}_j + \beta \sum_{j=1}^{n} y_j \bar{z}_j \\
&= \alpha \sum_{j=1}^{n} x_j \bar{z}_j + \beta \sum_{j=1}^{n} y_j \bar{z}_j \\
&= \alpha \langle \vec{x}, \vec{z} \rangle + \beta \langle \vec{y}, \vec{z} \rangle \ .
\end{aligned}$$

Similarly,

$$\begin{aligned}
\langle \vec{x}, \alpha\vec{y} + \beta\vec{z} \rangle &= \langle \vec{x}, \vec{v} \rangle \\
&= \sum_{j=1}^{n} x_j \bar{v}_j \\
&= \sum_{j=1}^{n} x_j \overline{(\alpha y_j + \beta z_j)} \\
&= \sum_{j=1}^{n} x_j (\overline{\alpha y_j} + \overline{\beta z_j})
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{n} x_j(\bar{\alpha}\bar{y}_j + \bar{\beta}x_j\bar{z}_j) \\
&= \sum_{j=1}^{n}(\bar{\alpha}x_j\bar{y}_j + \bar{\beta}x_j\bar{z}_j) \\
&= \sum_{j=1}^{n} \bar{\alpha}x_j\bar{y}_j + \sum_{j=1}^{n} \bar{\beta}x_j\bar{z}_j \\
&= \bar{\alpha}\sum_{j=1}^{n} x_j\bar{y}_j + \bar{\beta}\sum_{j=1}^{n} x_j\bar{z}_j \\
&= \bar{\alpha}\langle\vec{x},\vec{y}\rangle + \bar{\beta}\langle\vec{x},\vec{z}\rangle \ .
\end{aligned}
$$

2. Let V be the set of complex-valued (codomain is $\mathbb{C}$) continuous functions on $[0,1]$. Define

$$\langle f, g\rangle := \int_0^1 f(x)\overline{g(x)}\ dx\ .$$

We can show that this is an inner product space.

(i) Let $f \in V$. Then

$$\langle f, f\rangle = \int_0^1 f(x)\overline{f(x)}\ dx = \int_0^1 |f(x)|^2\ dx \geq 0\ ,$$

where we used the fact that the integral of a non-negative value is non-negative. Any non-negative continuous function integrating to 0 is 0 on that interval (follows from FTOC). So, if

$$\int_0^1 |f(x)|^2\ dx = 0\ ,$$

then

$$|f(x)|^2 = 0 \iff f(x) = 0$$

for $x \in [0,1]$.

(ii) Let $f, g \in V$. Using the fact that

$$\overline{\int_0^1 f(x)\ dx} = \int_0^1 \overline{f(x)}\ dx\ ,$$

we get that

$$\begin{aligned}
\overline{\langle f, g\rangle} &= \overline{\int_0^1 f(x)\overline{g(x)}\ dx} \\
&= \int_0^1 \overline{f(x)\overline{g(x)}}\ dx \\
&= \int_0^1 \overline{f(x)}\ \overline{\overline{g(x)}}\ dx \\
&= \int_0^1 \overline{f(x)}g(x)\ dx \\
&= \int_0^1 g(x)\overline{f(x)}\ dx \\
&= \langle g, f\rangle\ .
\end{aligned}$$

(iii) Let $f, g, h \in V$. Then using the fact that

$$\int_0^1 (\alpha f(x) + \beta g(x))\ dx = \alpha \int_0^1 f(x)\ dx + \beta \int_0^1 g(x)\ dx\ ,$$

we get that

$$\begin{aligned}
\langle \alpha f + \beta g, h \rangle &= \int_0^1 (\alpha f(x) + \beta g(x))\overline{h(x)}\ dx \\
&= \int_0^1 (\alpha f(x)\overline{h(x)} + \beta g(x)\overline{h(x)})\ dx \\
&= \int_0^1 \alpha f(x)\overline{h(x)}\ dx + \int_0^1 \beta g(x)\overline{h(x)})\ dx \\
&= \alpha \int_0^1 f(x)\overline{h(x)}\ dx + \beta \int_0^1 g(x)\overline{h(x)})\ dx \\
&= \alpha\langle f, h\rangle + \beta\langle g, h\rangle\ .
\end{aligned}$$

Similarly,

$$\begin{aligned}
\langle f, \alpha g + \beta h \rangle &= \int_0^1 f(x)\overline{(\alpha g(x) + \beta h(x))}\ dx \\
&= \int_0^1 f(x)(\overline{\alpha g(x)} + \overline{\beta h(x)})\ dx \\
&= \int_0^1 f(x)(\bar{\alpha}\overline{g(x)} + \bar{\beta}\overline{h(x)})\ dx \\
&= \int_0^1 (\bar{\alpha} f(x)\overline{g(x)} + \bar{\beta} f(x)\overline{h(x)})\ dx \\
&= \int_0^1 \bar{\alpha} f(x)\overline{g(x)}\ dx + \int_0^1 \bar{\beta} f(x)\overline{h(x)}\ dx \\
&= \bar{\alpha} \int_0^1 f(x)\overline{g(x)}\ dx + \bar{\beta} \int_0^1 f(x)\overline{h(x)}\ dx \\
&= \bar{\alpha}\langle f, g\rangle + \bar{\beta}\langle f, h\rangle
\end{aligned}$$

3. Any finite dimensional $\mathbb{C}$-vector space has an inner product. Let $\{\vec{b}_1, \ldots, \vec{b}_n\}$ be a basis for V and declare

$$\langle \vec{b}_i, \vec{b}_j \rangle := \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} .$$

Extend to arbitary vector using (iii):

$$\langle \alpha_1 \vec{b}_1 + \ldots + \alpha_n \vec{b}_n, \beta_1 \vec{b}_1 + \ldots + \beta_n \vec{b}_n \rangle = \sum_{j=1}^{n} \alpha_j \bar{\beta}_j$$

4. (Tracial Inner Product on $M_n(\mathbb{C})$)

$M_n(\mathbb{C})$ is a $\mathbb{C}$-vector space. Define

$$\langle A, B \rangle := \operatorname{tr}(AB^*) \ ,$$

where B^* denotes the **conjugate transpose** (adjoint/adjugate) of B. Note that if $A = [a_{ij}]$, then $\operatorname{tr}(A) := a_{11} + a_{22} + \ldots + a_{nn}$. $A \mapsto \operatorname{tr} A$ is a $\mathbb{C}$-linear map, so

$$\operatorname{tr}(A + cB) = \operatorname{tr} A + c \operatorname{tr} B$$

and

$$\operatorname{tr}(AB) = \operatorname{tr}(BA) \ .$$

Also note that we assuming that A and B are square matrices, as the trace is only defined for square matrices. Now, we show that this an inner product space.

(i) If we have two matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \quad \text{and} \quad A^* = \begin{bmatrix} \overline{a_{11}} & \overline{a_{21}} & \ldots & \overline{a_{n1}} \\ \overline{a_{12}} & \overline{a_{22}} & \ldots & \overline{a_{n2}} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{a_{1n}} & \overline{a_{2n}} & \ldots & \overline{a_{nn}} \end{bmatrix} \ ,$$

then

$$\begin{aligned} AA^* &= \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} \overline{a_{11}} & \overline{a_{21}} & \ldots & \overline{a_{n1}} \\ \overline{a_{12}} & \overline{a_{22}} & \ldots & \overline{a_{n2}} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{a_{1n}} & \overline{a_{2n}} & \ldots & \overline{a_{nn}} \end{bmatrix} \\ &= \begin{bmatrix} a_{11}\overline{a_{11}} + a_{12}\overline{a_{12}} + \ldots + a_{1n}\overline{a_{1n}} & \rule{2cm}{0.4pt} & \rule{2cm}{0.4pt} \\ \vdots & \ddots & \vdots \\ \rule{2cm}{0.4pt} & \rule{2cm}{0.4pt} & a_{n1}\overline{a_{n1}} + a_{12}\overline{a_{12}} + \ldots + a_{1n}\overline{a_{1n}} \end{bmatrix} \ , \end{aligned}$$

and so

$$\operatorname{tr}(AA*) = \sum_{i,j=1}^{n} |a_{ij}|^2 \geq 0 \ .$$

Note that $\operatorname{tr}(AA^*) = 0$ iff every $a_{ij} = 0$; i.e. $A = 0$ (the zero matrix).

Definition

For an inner space, define

$$||\vec{x}|| := \sqrt{\langle \vec{x}, \vec{x} \rangle} \ .$$

Note that this is always real and ≥ 0.

Theorem (Cauchy-Schwarz Inequality)

$$|\langle \vec{x}, \vec{y} \rangle| \leq ||\vec{x}|| \ ||\vec{y}|| \ .$$

Proof. For any scalar t,

$$\begin{aligned} 0 &\leq ||\vec{x} - t\vec{y}||^2 \\ &= \left(\sqrt{\langle \vec{x} - t\vec{y}, \vec{x} - t\vec{y} \rangle} \right)^2 \\ &= \langle \vec{x} - t\vec{y}, \vec{x} - t\vec{y} \rangle \\ &= \ldots \end{aligned}$$

□

Normed Space

Definition (Normed Space)

A $\mathbb{C}$-vector space V is called a **normed space** if there exists a map

$$|| \; || : V \to \mathbb{R}^+ = [0, \infty)$$

called a **norm** satisfying

(i) $||\vec{v}|| \geq 0$ and $||\vec{v}|| = 0$ if and only if $\vec{v} = 0$ (Non- degeneracy)

(ii) $||\alpha \vec{v}|| = |\alpha| \; ||\vec{v}||$ for all $\vec{v} \in V$, $\alpha \in \mathbb{C}$ (homogeneity)

(iii) $||\vec{v} + \vec{w}|| \leq ||\vec{v}|| + ||\vec{w}||$ (triangle inequality)

1. (Every inner product space is a normed space)

Let V be a complex inner product space. Recall that $||\vec{x}|| = \sqrt{\langle \vec{x}, \vec{x} \rangle}$.

(i) Since V is an inner product space, this means that for any $\vec{x} \in V$, $\langle \vec{x}, \vec{x} \rangle \geq 0$. This implies that

$$||\vec{x}|| = \sqrt{\langle \vec{x}, \vec{x} \rangle} \geq 0 \ .$$

Further more, if $\vec{x} = 0$, then $\langle \vec{x}, \vec{x} \rangle = 0$, and so

$$||\vec{x}|| = 0 \ .$$

(ii) Let $\vec{x} \in V$ and $\alpha \in \mathbb{C}$. Then since V is an inner product space, we get that

$$\begin{aligned} ||\alpha\vec{x}|| &= \sqrt{\langle \alpha\vec{x}, \alpha\vec{x} \rangle} \\ &= \sqrt{\alpha\bar{\alpha}\langle \vec{x}, \vec{x} \rangle} \\ &= \sqrt{|\alpha|^2\langle \vec{x}, \vec{x} \rangle} \\ &= \sqrt{|\alpha|^2}\sqrt{\langle \vec{x}, \vec{x} \rangle} \\ &= |\alpha| \ ||\vec{x}|| \ . \end{aligned}$$

(iii) Let $\vec{x}, \vec{y} \in V$. Then

$$\begin{aligned} ||\vec{x} + \vec{y}||^2 &= \left(\sqrt{\langle \vec{x} + \vec{y}, \vec{x} + \vec{y} \rangle}\right)^2 \\ &= \langle \vec{x} + \vec{y}, \vec{x} + \vec{y} \rangle \\ &= \langle 1\vec{x} + 1\vec{y}, \vec{x} + \vec{y} \rangle \\ &= 1\langle \vec{x}, \vec{x} + \vec{y} \rangle + 1\langle \vec{y}, \vec{x} + \vec{y} \rangle \\ &= \langle \vec{x}, 1\vec{x} + 1\vec{y} \rangle + \langle \vec{y}, 1\vec{x} + 1\vec{y} \rangle \\ &= \left(\bar{1}\langle \vec{x}, \vec{x} \rangle + \bar{1}\langle \vec{x}, \vec{y} \rangle\right) + \left(\bar{1}\langle \vec{y}, \vec{x} \rangle + \bar{1}\langle \vec{y}, \vec{y} \rangle\right) \\ &= 1\langle \vec{x}, \vec{x} \rangle + 1\langle \vec{x}, \vec{y} \rangle + 1\langle \vec{y}, \vec{x} \rangle + 1\langle \vec{y}, \vec{y} \rangle \\ &= \langle \vec{x}, \vec{x} \rangle + \langle \vec{x}, \vec{y} \rangle + \langle \vec{y}, \vec{x} \rangle + \langle \vec{y}, \vec{y} \rangle \\ &= ||\vec{x}||^2 + \langle \vec{x}, \vec{y} \rangle + \langle \vec{y}, \vec{x} \rangle + ||\vec{y}||^2 \\ &= \ldots \\ &\leq ||\vec{x}||^2 + 2||\vec{x}||||\vec{y}|| + ||\vec{y}||^2 \qquad \text{(Cauchy-Schwarz)} \\ &= (||\vec{x}|| + ||\vec{y}||)^2 \ . \end{aligned}$$

Thus, the inner product spaces $\mathbb{C}^n$, $M_n(\mathbb{C})$, and the continuous functions on $[0, 1]$ are all normed spaces.

2. On $\mathbb{C}^n$, for any $1 \leq p < \infty$, the quantity

$$||\vec{x}||_p = \left[\sum_{j=1}^{n} |x_i|^p\right]^{1/p}$$

defines a norm. For $p = 1$, we have

$$||\vec{x}||_1 = \left[\sum_{j=1}^{n} |x_i|^1\right]^{1/1} = \sum_{j=1}^{n} |x_i| \ .$$

For $p = 2$, we get the Euclidean norm coming from the complex inner product:

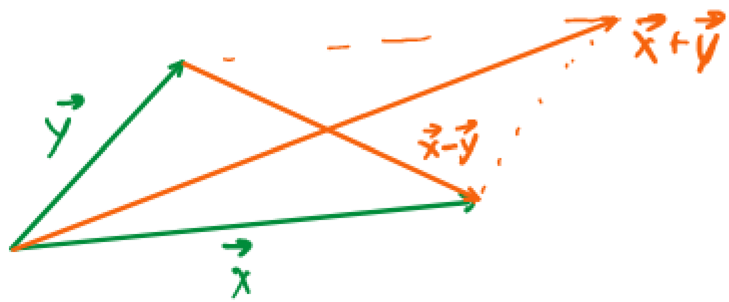
$$||\vec{x}||_2 = \left[\sum_{j=1}^{n} |x_j|^2\right]^{1/2} = \sqrt{\sum_{j=1}^{n} |x_j|^2} = \sqrt{\sum_{j=1}^{n} x_j \bar{x}_j} = \sqrt{\langle \vec{x}, \vec{x} \rangle} \ .$$

Proposition

Let V be an inner product space. Then

$$||\vec{x}+\vec{y}||^2 + ||\vec{x}-\vec{y}||^2 = 2||\vec{x}||^2 + 2||\vec{y}||^2$$

Note the Parallelogram Law



Proof. Suppose V is an inner product space. Then

$$\begin{aligned}
&||\vec{x}+\vec{y}||^2 + ||\vec{x}-\vec{y}||^2 \\
&= \langle \vec{x}+\vec{y}, \vec{x}+\vec{y} \rangle + \langle \vec{x}-\vec{y}, \vec{x}-\vec{y} \rangle \\
&= \langle \vec{x}, \vec{x} \rangle + 2\,\mathrm{Re}\{\langle \vec{x}, \vec{y} \rangle\} + \langle \vec{y}, \vec{y} \rangle + \langle \vec{x}, \vec{x} \rangle - 2\,\mathrm{Re}\{\langle \vec{x}, \vec{y} \rangle\} + \langle \vec{y}, \vec{y} \rangle \\
&= 2\langle \vec{x}, \vec{x} \rangle + 2\langle \vec{y}, \vec{y} \rangle \\
&= 2\left(\sqrt{\langle \vec{x}, \vec{x} \rangle}\right)^2 + 2\left(\sqrt{\langle \vec{y}, \vec{y} \rangle}\right)^2 \\
&= 2||\vec{x}||^2 + 2||\vec{y}||^2 \ .
\end{aligned}$$

$\square$

1. The 1-norm $||\vec{x}||_1$ on $\mathbb{C}^n$ does not come from an inner product. Otherwise, if there were an inner product such that $\langle \vec{x}, \vec{x} \rangle = ||\vec{x}||_1^2$, then the parallelogram law would hold. Indeed, if we were to let

$$\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \ , \ \vec{v} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{C}^n \ ,$$

then

$$\begin{aligned}
||\vec{u} + \vec{v}||_1^2 + ||\vec{u} - \vec{v}||_1^2 &= \left|\left| \begin{bmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right|\right|_1^2 + \left|\left| \begin{bmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right|\right|_1^2 \\
&= \left|\left| \begin{bmatrix} 0 \\ 2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right|\right|_1^2 + \left|\left| \begin{bmatrix} 2 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right|\right|_1^2 \\
&= (|0| + |2| + |0| + \ldots + |0|)^2 + (|2| + |0| + |0| + \ldots + |0|)^2 \\
&= (0 + 2 + 0 + \ldots + 0)^2 + (2 + 0 + 0 + \ldots + 0)^2 \\
&= 2^2 + 2^2 \\
&= 4 + 4 \\
&= 8 \ .
\end{aligned}$$

However, the previous proposition doesn't hold since

$$\begin{aligned}
||\vec{u} + \vec{v}||_1^2 + ||\vec{u} - \vec{v}||_1^2 &= 8 \\
&\neq 16 \\
&= 8 + 8 \\
&= 2(4) + 2(4) \\
&= 2(2)|^2 + 2(2)^2 \\
&= 2(1 + 1 + 0 \ldots + 0)|^2 + 2(1 + 1 + 0 + \ldots + 0)^2 \\
&= 2(|1| + |1| + |0| \ldots + |0|)|^2 + 2(| - 1| + |1| + |0| + \ldots + |0|)^2 \\
&= 2||\vec{u}||_1^2 + 2||\vec{v}||_1^2 \ .
\end{aligned}$$