MATH 361 - Week 6 Tutorial

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2.1. True or false:

(a) Every unitary operator $U : X \to X$ is normal.

Answer: True. Suppose U is unitary. Then from property (i) of unitaries, $U^*U = I = UU^*$. Thus, since $U^*U = UU^*$, U is normal.

(b) A matrix is unitary if and only if it is invertible.

Answer: False. The ($\Longrightarrow$) direction holds by the definition of unitary, However, the ($\Longleftarrow$) direction does not hold. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. Then A is invertible since $\det A = 2 \cdot 1 = 2 \neq 0$. But, A is not a unitary since it is not an isometry. Indeed, from property (i) of unitaries, A is not an isometry since

$$A^* = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^* = \begin{bmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \neq \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = A^{-1} \ .$$

Alternatively, using the corollary of the proposition for isometries, A is not an isometry since

$$A^*A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^* \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \neq I \ ,$$

So, A is not a unitary (since it's not an isometry).

(c) If two matrices are unitarily equivalent, then they are also similar.

Answer: True. Suppose A and B are unitarily equivalent, denoted $A \sim_U B$. Then there exists a unitary U such that $U^*AU = B$. Since U is unitary, it follows from property (i) of unitaries that $U^* = U^{-1}$. So, we get that

$$\begin{aligned} & U^*AU = B \\ \Longrightarrow \ & U^{-1}AU = B \ . \end{aligned}$$

Thus, by the definition of similarity, A is similar to B.

(d) The sum of self-adjoint operators is self-adjoint.

Answer: True. Suppose $A, B : V \to W$ and $A^*, B^* : W \to V$ with $\dim V = n$ and $\dim W = m$. Suppose $A = A^*$ and $B = B^*$. Note that A and B must have the same matrix dimensions since we are going to take their sum. Denote

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{bmatrix} .$$

Then

$$\begin{aligned}
&(A^* + B^*)^* \\
&= (A + B)^* \\
&= \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \dots & a_{mn} + b_{mn} \end{bmatrix}^* \\
&= \begin{bmatrix} \overline{a_{11} + b_{11}} & \overline{a_{21} + b_{21}} & \dots & \overline{a_{m1} + b_{m1}} \\ \overline{a_{12} + b_{12}} & \overline{a_{22} + b_{22}} & \dots & \overline{a_{m2} + b_{m2}} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{a_{1n} + b_{1n}} & \overline{a_{2n} + b_{2n}} & \dots & \overline{a_{mn} + b_{mn}} \end{bmatrix} \\
&= \begin{bmatrix} \overline{a_{11}} & \overline{a_{21}} & \dots & \overline{a_{m1}} \\ \overline{a_{12}} & \overline{a_{22}} & \dots & \overline{a_{m2}} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{a_{1n}} & \overline{a_{2n}} & \dots & \overline{a_{mn}} \end{bmatrix} + \begin{bmatrix} \overline{b_{11}} & \overline{b_{21}} & \dots & \overline{b_{m1}} \\ \overline{b_{12}} & \overline{b_{22}} & \dots & \overline{b_{m2}} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{b_{1n}} & \overline{b_{2n}} & \dots & \overline{b_{mn}} \end{bmatrix} \\
&= \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}^* + \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{bmatrix} \\
&= A^* + B^* .
\end{aligned}$$

So, $A^* + B^*$ is self-adjoint. Of course, we could have simply used property (i) of the self-adjoint: $(A + B)^* = A^* + B^*$. With this, we would have gotten

$$(A^* + B^*) = (A + B)^* = A^* + B^* .$$

Thus, it is true that the sum of self-adjoint operators is self-adjoint.

(e) The adjoint of a unitary operator is unitary.

Answer: True. Let U be unitary. Then

$$U^*(U^*)^* = U^*U = I \ .$$

Since U is unitary, U is invertible. So,

$$U = U^{**} = (U^*)^{-1} \ .$$

(f) The adjoint of a normal operator is normal.

Answer: True. Suppose $N : V \to V$ is normal. Then $N^*N = NN^*$. We show that N^* is normal. That is, we show that $(N^*)^*N^* = N^*(N^*)^*$. Note here that $N^* : V \to V$. So,

$$\begin{aligned}(N^*)^*N^* &= NN^* \\ &= N^*N \\ &= N^*(N^*)^* \ .\end{aligned}$$

Thus, N^* is normal.

(g) If all eigenvalues of a linear operator are 1, then the operator must be unitary or orthogonal (aka isometry).

Answer: False. We prove there exists a linear operator $A : V \to W$ with all eigenvalues equal to 1 such that A is neither unitary nor an isometry. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then $\dim V = 2 = \dim W$, and so

$$A^*A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^* \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq I \ ,$$

which means that A is not an isometry. Since A is not an isometry it can't be unitary. This comes from the definition of unitary: A is unitary if it is an invertible **isometry**. We can even see that A is not invertible since $\det A = 0$.

(h) If all eigenvalues of a normal operator are 1, then the operator is the identity.

Answer: True. Normals are diagonalizable. Suppose V is an inner product space and $N : V \to V$ is normal with all eigenvalues equal to 1. Let $\dim V = n$. Then N is $n \times n$ with eigenvalues $\{\lambda_1, \dots, \lambda_n\}$, where $\lambda_j = 1$ for $1 \leq j \leq n$. Since V is an inner product space and N is normal, we can apply the **Spectral Theorem for Normal Operators**. By the spectral theorem, N is unitarily equivalent to a diagonal, denoted $N \sim_U D$. That is, there exists a unitary U such that

$$U^* N U = D \ .$$

Here,

$$D = \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix} = \begin{bmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{bmatrix} = I \ .$$

In other words, $U^* N U = D = I$. Then since U is unitary, it holds that $U^* U = U U^* = I$, and so

$$\begin{aligned} U^* N U &= I \\ U U^* N U &= U I \\ I N U &= U \\ N U &= U \\ N U U^* &= U U^* \\ N I &= I \\ N &= I \ . \end{aligned}$$

Thus, N itself is the identity.

(i) A linear operator may preserve norm, but not the inner product.

<u>**Answer:**</u> False. The original statement can be rewritten as follows: In general, for any linear operator $A : V \to W$, if $\vec{v} \in V$ and $\vec{w} \in W$, then $||A\vec{v}|| = ||\vec{v}||$ but $\langle A\vec{v}, A\vec{w}\rangle \neq \langle \vec{v}, \vec{w}\rangle$. This does not hold. Recall that A is an isometry if it preserves the norm (aka "distance"); that is, $||A\vec{v}|| = ||\vec{v}||$ for $\vec{v} \in V$. We also covered the proposition that A is an isometry if and only if A preserves the inner product; that is, $\langle A\vec{v}, A\vec{w}\rangle = \langle \vec{v}, \vec{w}\rangle$ for $\vec{v} \in V$ and $\vec{w} \in W$. Thus, A must preserve both the norm and inner product, as the preservation of one of them implies that A is an isometry.

2.2. True or false: The sum of normal operators is normal. Justify your conclusion.

Answer: We can first try to prove it is true. Let M and N be normal operators. Then $MM^* = M^*M$ and $NN^* = N^*N$. We want to show that $M + N$ is normal. That is, we want to show that

$$(M + N)(M + N)^* = (M + N)^*(M + N) \ .$$

So,

$$\begin{aligned}(M + N)(M + N)^* &= (M + N)(M^* + N^*) \\ &= MM^* + MN^* + NM^* + NN^* \\ &= MM^* + N^*M + M^*N + NN^* \\ &= M^*M + M^*N + N^*M + N^*N \\ &= (M^* + N^*)(M + N) \\ &= (M + N)^*(M + N) \ .\end{aligned}$$

Note that M ... So, it's probably false. Let's take a counter example. Let

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

be normal operators. Note that A and B are self-adjoint. Indeed,

$$\ldots$$

Also note that **normal** $\implies$ **diagonalizable**. Equivalently, **not diagonalizable** $\not\Longleftarrow$ **not normal**. Now,

$$\begin{aligned}(A + B)(A + B)^* &= \ldots \\ &\neq \ldots \\ &= (A + B)^*(A + B) \ .\end{aligned}$$

2.6. Orthogonally diagonalize the matrix

$$A = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix} .$$

That is, represent it as $A = UDU^*$, where D is diagonal and U is unitary. Note: among all square roots of A, i.e. among all matrices B such that $B^2 = A$, find one that has positive eigenvalues. You can leave B as a product.

Answer: Notice that

$$A^* = \dots$$

From the Porism of Schur's theorem, we know that $A - UTU^*$. Since $A = A^*$, it follows that $A = UDU^*$. Here,

$$U = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \dots \end{bmatrix}$$

is the matrix of eigenvectors as columns and D is the matrix of eigenvalues.

2.8. Let A be an $m \times n$ matrix. Prove that

(a) A^*A is self-adjoint.

Answer: Let V and W be finite dimensional inner product spaces with $\dim V = n$ and $\dim W = m$. Then $A : V \to W$ satisfisfies the condition that A is an $m \times n$ matrix. Then $A^* : W \to V$ is $n \times m$. We show that A^*A is self-adjoint by showing $(A^*A)^* = A^*A$. So,

$$(A^*A)^* = A^*A^{**} = A^*A \ .$$

Thus, A^*A is self-adjoint.

(b) All eigenvalues of A^*A are non-negative.

Answer: Let V and W be finite dimensional inner product spaces and $A : V \to W$. Since eigenvalues exist only for square matrices, we can assume that $\dim V = \dim W = n$. So, A is an $n \times n$ matrix. Let $\{\lambda_1, \dots, \lambda_n\}$ be the eigenvalues of A. We assume that A is unitarily equivalent to D; that is, there exists a unitary U such that $U^*AU = D$, where

$$D = \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix} .$$

Rearranging for A gives $A = UDU^*$. Then

$$\begin{aligned} A^*A &= (UDU^*)^*UDU^* \\ &= U^{**}D^*U^*UDU^* \\ &= UD^*U^*UDU^* \\ &= UD^*IDU^* \\ &= UD^*DU^* \\ &= U(D^*D)U^* \ . \end{aligned}$$

Now, this tells us that A^*A is unitarily equivalent to D^*D, where

$$\begin{aligned} D^*D &= \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix}^* \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix} \\ &= \begin{bmatrix} \overline{\lambda_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \overline{\lambda_n} \end{bmatrix} \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix} \end{aligned}$$

$$
= \begin{bmatrix} \overline{\lambda_1}\lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \overline{\lambda_n}\lambda_n \end{bmatrix}
$$

$$
= \begin{bmatrix} |\lambda_1|^2 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & |\lambda_n|^2 \end{bmatrix} .
$$

So, we have that the eigenvalue of A^*A are $\{|\lambda_1|^2, \dots, |\lambda_n|^2\}$. Thus, the eigenvalues of A^*A are non-negative.

(c) $A^*A + I$ is invertible.

Answer: Let V be a finite dimensional inner product space and $A : V \to V$. Then A is $n \times n$. Since $A : V \to V$, we have that $A^* : V \to V$, and so A^* is also $n \times m$. Now, let λ be an eigenvalue of $A^*A + I$. Note that $A^*A : V \to V$ and so I must be $n \times n$. So,

$$
(A^*A + I)\vec{v} = \lambda\vec{v} ,
$$

where $\vec{v} \in V$ is non-zero. From this we get that

$$
\begin{aligned}
(A^*A + I)\vec{v} &= \lambda\vec{v} \\
(A^*A)\vec{v} + I\vec{v} &= \lambda\vec{v} \\
A^*A\vec{v} + \vec{v} &= \lambda\vec{v} \\
A^*A\vec{v} &= \lambda\vec{v} - \vec{v} \\
A^*A\vec{v} &= (\lambda - 1)\vec{v} .
\end{aligned}
$$

This tells us that A^*A has eigenvalue $\lambda - 1$ associated with eigenvector $\vec{v}$. From part (b), we know that the eigenvalues of A^*A are **non-negative**, which means that $(\lambda - 1) \geq 0$.