MATH 367 - Week 10 Notes

Jasraj Sandhu

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Four Important Differential Operators

(1) **The gradient** of a scalar-function f is

$$\nabla f = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} .$$

We can think of ∇ as a linear operator taking scalar functions to vector fields. Note that we can think of the nabla operator, ∇, as

$$\nabla = \begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{bmatrix} = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k} .$$

(2) **Divergence**

Given a vector field $\vec{F} = \begin{bmatrix} F_1 & F_2 & F_3 \end{bmatrix}^\mathsf{T}$, the **divergence** of $\vec{F}$ is given by

$$\operatorname{div}(\vec{F}) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} .$$

Of course, we can also write it as

$$\operatorname{div}(\vec{F}) = \text{"}\boldsymbol{\nabla} \cdot \left(\vec{F}\right)\text{"} = \begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} .$$

We can think of "$\boldsymbol{\nabla}\cdot$" as an operator from vector fields to scalar fields.

(3) **Curl** accepts a vector field and returns a vector field.

$$\begin{aligned}
\operatorname{curl}(\vec{F}) &= \text{"}\boldsymbol{\nabla} \times \vec{F}\text{"} \\
&= \det\left(\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{bmatrix}\right) \\
&= \begin{bmatrix} \dfrac{\partial F_3}{\partial y} - \dfrac{\partial F_2}{\partial z} \\[2ex] \dfrac{\partial F_1}{\partial z} - \dfrac{\partial F_3}{\partial x} \\[2ex] \dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y} \end{bmatrix}.
\end{aligned}$$

"$\boldsymbol{\nabla}\times$" is a linear operator from **vector fields** to **vector fields**.

Note: If $\vec{F}$ is a **conservative field**, then $\operatorname{curl}(\vec{F}) = 0$.

(4) **Laplacian**

(i) Given a **scalar** f, the **Laplacian** of f is the divergence of the gradient of f:

$$\begin{aligned}
\Delta f &= \boldsymbol{\nabla} \cdot \nabla f \\
&= \begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \\ \dfrac{\partial f}{\partial z} \end{bmatrix} \\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) \\
&= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \\
&= f_{xx} + f_{yy} + f_{zz}
\end{aligned}$$

(**scalar** to **scalar**)

(ii) Given a **vector field** $\vec{F}$,

$$\Delta \vec{F} = \vec{F}_{xx} + \vec{F}_{yy} + \vec{F}_{zz}$$

(**Vector field** to **vector field**)

Remarks

(i) ∇f gives a vector field which points in the direction of greatest rate of change on the hypersurface $w = f(x, y, z)$.

(ii) The divergence of $\vec{F}$ at P_0 is a signed measure of the degree to which P_0 is a **source** or a **sink** of vectors.

Large positive divergence at P_0.

Large negative divergence at P_0.

Zero divergence at P_0.

(iii) Divergence describes **flux density**. At P_0, the flux density (or point flux) is

$$\operatorname{div}(\vec{F}(P_0)) = \lim_{r \to 0} \frac{1}{\operatorname{Vol}(B_r)} \iint_{\mathscr{S}_r(P_0)} \vec{F} \cdot d\vec{S} \,,$$

where

$$\frac{1}{\operatorname{Vol}(B_r)} \iint_{\mathscr{S}_r(P_0)} \vec{F} \cdot d\vec{S}$$

is the average flux across $\mathscr{S}_r$. Here, we let $B_r(P_0)$ be the closed ball of radius r about P_0 and $\mathscr{S}_r(P_0)$ be the sphere of radius r around P_0.

(iv) Suppose we drop a particle into a field $\vec{F}$ (so it follows some field line). If the particle follows any kind of positive curvature path, then $\operatorname{curl}(\vec{F})$ at P_0 gives the axis of rotation for the particle at P_0.

If $\left|\left|\operatorname{curl}(\vec{F})\right|\right|$ is large, then $\vec{F}$ is **turbulent**.

(v) There are **many** identities concerning these 4 operators. The two most important identities are

$$\operatorname{div}(\operatorname{curl}(\vec{F})) = 0$$

as long as $\vec{F}$ is sufficiently smooth, and

$$\operatorname{curl}(\nabla\Phi) = \vec{0}\ ,$$

where Φ is the potential of the vector field $\vec{F}$. In fact, if $\operatorname{curl}(\vec{F}) = \vec{0}$ at every $P_0 \in \mathbb{R}^3$, then $\vec{F}$ must be **conservative**.

Divergence Theorem

Theorem (Divergence Theorem)

Suppose $\mathscr{S}$ is a closed surface with outward facing normals and $\vec{F}$ is a smooth vector field. If D is the interior of $\mathscr{S}$, then

$$\oiint_{\mathscr{S}} \vec{F} \cdot d\vec{S} = \iiint_D \operatorname{div}(\vec{F})\ dV\ .$$

This can be read as "the flux = the integral of flux density".

Examples

1. Compute $\displaystyle\oiint_{\mathscr{S}} \vec{F} \cdot d\vec{S}$, where $\mathscr{S}$ is the sphere of radius 1 with outward normal and $\vec{F}(x,y,z) = \begin{bmatrix} 3x - y^2 \\ 2y - 6xz \\ e^{x+2y} \end{bmatrix}$ is the vector field.

   By the divergence theorem, we have that

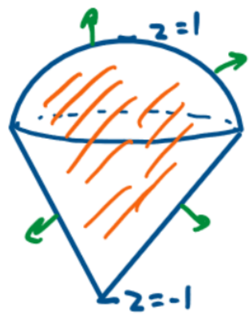
$$\begin{aligned}
\oiint_{\mathscr{S}} \vec{F} \cdot d\vec{S} &= \iiint_{x^2+y^2+z^2 \leq 1} \operatorname{div}(\vec{F})\ dV \\
&= \iiint_{x^2+y^2+z^2 \leq 1} \big((3x - y^2)_x + (2y - 6xz)_y + (e^{x+2y})_z\big)\ dV \\
&= \iiint_{x^2+y^2+z^2 \leq 1} (3 + 2 + 0)\ dV \\
&= \iiint_{x^2+y^2+z^2 \leq 1} 5\ dV \\
&= 5 \iiint_{x^2+y^2+z^2 \leq 1} dV \\
&= 5 \iiint_{x^2+y^2+z^2 \leq 1} 1\ dV \\
&= 5 \cdot \operatorname{Vol}(x^2 + y^2 + z^2 \leq 1) \\
&= 5 \cdot \frac{4}{3}\pi \\
&= \frac{20\pi}{3}\ .
\end{aligned}$$

   Note that the volume of a sphere of radius 1 is

$$\frac{4\pi}{3} r^3 = \frac{4\pi}{3}\ ,$$

   where $r = 1$.

2. Compute $\displaystyle\oiint_{\mathscr{S}} \vec{F} \cdot d\vec{S}$, where $\mathscr{S}$ is the boundary, with outward normal, of the solid determined by $\sqrt{x^2+y^2}-1 \leq z \leq \sqrt{1-x^2-y^2}$ and vector field $\vec{F}(x,y,z) = \big(xy,\ 3z,\ 2y^2\big)$.



Standard approach results in two integrals.

Note that $\sqrt{1-x^2-y^2}$ is the upper unit hemisphere (radius of 1). Indeed,

$$\begin{aligned} z &= \sqrt{1-x^2-y^2} \\ z^2 &= 1-x^2-y^2 \\ x^2+y^2+z^2 &= 1 \end{aligned}$$

is the equation of a sphere of radius 1, and the positive square root gives the upper half of the sphere. Also,

$$\begin{aligned} z &= \sqrt{x^2+y^2} \\ z^2 &= x^2+y^2 \\ \frac{z^2}{1^2} &= \frac{x^2}{1} + \frac{y^2}{1} \end{aligned}$$

is a cone (actually more of an hour glass shape with two cones) whose tip is at the origin. The positive square root gives the upper cone of the hour glass. Then

$$z = \sqrt{x^2+y^2} - 1$$

is simply the upper cone of the hour glass shifted such that it's tip is at $(0,0,-1)$. So, the surface of interest is the portion between the upper cone that starts with its tip at $z=-1$ and the upper hemisphere of radius 1. This surface is essentially an ice cream cone. Since this surface is closed and has outward facing normal, we can apply the divergence theorem. So,

$$\oiint_{\mathscr{S}} \vec{F} \cdot d\vec{S} = \iiint_D \operatorname{div}(\vec{F})\ dV\ ,$$

where

$$\begin{aligned}
\operatorname{div}(\vec{F}) &= \nabla \cdot \vec{F} \\
&= \begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} xy \\ 3z \\ 2y^2 \end{bmatrix} \\
&= \frac{\partial}{\partial x}\,[xy] + \frac{\partial}{\partial y}\,[3z] + \frac{\partial}{\partial z}\,[2y^2] \\
&= (xy)_x + (3z)_y + (2y^2)_z \\
&= y + 0 + 0 \\
&= y \ .
\end{aligned}$$

So,

$$\begin{aligned}
\oiint_{\mathscr{S}} \vec{F} \cdot d\vec{S} &= \iiint_D y \ dV \\
&= \iiint_{\sqrt{x^2+y^2}-1 \le z \le \sqrt{1-x^2-y^2}} y \ dV \ .
\end{aligned}$$

This will be difficult to evaluate. So, we can convert to cylindrical coordinates. Then

$$\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta \\
r &= \sqrt{x^2+y^2} \ ,
\end{aligned}$$

and so

$$\vec{r}(r,\theta,z) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ z \end{bmatrix}$$

For the restriction on z, we get that

$$\begin{aligned}
\sqrt{x^2+y^2} - 1 \le z &\le \sqrt{1-x^2-y^2} \\
r - 1 \le z &\le \sqrt{1-r^2} \ .
\end{aligned}$$

As usual, the restriction on θ is

$$0 \le \theta \le 2\pi \ .$$

Lastly, we consider the restriction on r. Clearly the largest value that r can take on for the upper hemisphere is 1 as that is its radius. Consequently, the largest value for r that the cone portion can take on is also $r = 1$ as that is the point where the hemisphere and cone meet. Indeed, the surfaces meet when

$$\begin{aligned} r - 1 &= \sqrt{1 - r^2} \\ (r-1)^2 &= 1 - r^2 \\ r^2 - 2r + 1 &= 1 - r^2 \\ 2r^2 - 2r &= 0 \\ 2r(r-1) &= 0 \ . \end{aligned}$$

So, since the positive solution to the above equation is $r = 1$, the restriction on r is

$$0 \leq r \leq 1 \ .$$

Then noting that $dV = r \ dz \ dr \ d\theta$, we get that

$$\begin{aligned} \oiint_{\mathscr{S}} \vec{F} \cdot d\vec{S} &= \iiint_D y \ dV \\ &= \iiint_{\sqrt{x^2+y^2}-1 \leq z \leq \sqrt{1-x^2-y^2}} y \ dV \\ &= \int_0^{2\pi} \int_0^1 \int_{r-1}^{\sqrt{1-r^2}} r \sin\theta \cdot r \ dz \ dr \ d\theta \\ &= \int_0^{2\pi} \int_0^1 \int_{r-1}^{\sqrt{1-r^2}} r^2 \sin\theta \ dz \ dr \ d\theta \\ &= \int_0^{2\pi} \int_0^1 \left(\int_{r-1}^{\sqrt{1-r^2}} r^2 \sin\theta \ dz \right) \ dr \ d\theta \\ &= \int_0^{2\pi} \int_0^1 \left(r^2 \sin\theta \int_{r-1}^{\sqrt{1-r^2}} \ dz \right) \ dr \ d\theta \\ &= \int_0^{2\pi} \int_0^1 \left(r^2 \sin\theta \ \Big[z \Big]_{z=r-1}^{\sqrt{1-r^2}} \right) \ dr \ d\theta \\ &= \int_0^{2\pi} \int_0^1 \left(r^2 \sin\theta \cdot \left(\sqrt{1-r^2} - (r-1) \right) \right) \ dr \ d\theta \\ &= \int_0^{2\pi} \int_0^1 \left(r^2 \sin\theta \cdot \left(\sqrt{1-r^2} - r + 1 \right) \right) \ dr \ d\theta \\ &= \int_0^{2\pi} \int_0^1 \left(r^2 \sqrt{1-r^2} \sin\theta - r^3 \sin\theta + r^2 \sin\theta \right) \ dr \ d\theta \\ &= \int_0^{2\pi} \left(\int_0^1 \left(r^2 \sqrt{1-r^2} \sin\theta - r^3 \sin\theta + r^2 \sin\theta \right) \ dr \right) \ d\theta \\ &= \int_0^{2\pi} \left(\int_0^1 r^2 \sqrt{1-r^2} \sin\theta \ dr - \int_0^1 r^3 \sin\theta \ dr + \int_0^1 r^2 \sin\theta \ dr \right) \ d\theta \end{aligned}$$

$$
\begin{aligned}
&= \int_0^{2\pi} \left(\sin\theta \int_0^1 r^2\sqrt{1-r^2}\ dr - \sin\theta \int_0^1 r^3\ dr + \sin\theta \int_0^1 r^2\ dr \right)\ d\theta \\
&= \int_0^{2\pi} \left(\sin\theta \int_0^1 r^2\sqrt{1-r^2}\ dr - \sin\theta\ \left[\frac{r^4}{4}\right]_0^1 + \sin\theta\ \left[\frac{r^3}{3}\right]_0^1 \right)\ d\theta \\
&= \int_0^{2\pi} \left(\sin\theta \int_0^1 r^2\sqrt{1-r^2}\ dr - \sin\theta \cdot \frac{1}{4} + \sin\theta \cdot \frac{1}{3} \right)\ d\theta \\
&= \int_0^{2\pi} \left(\sin\theta \int_0^1 r^2\sqrt{1-r^2}\ dr - \frac{\sin\theta}{4} + \frac{\sin\theta}{3} \right)\ d\theta \\
&= \int_0^{2\pi} \left(\sin\theta \int_0^1 r^2\sqrt{1-r^2}\ dr - \frac{3\sin\theta}{12} + \frac{4\sin\theta}{12} \right)\ d\theta \\
&= \int_0^{2\pi} \left(\sin\theta \int_0^1 r^2\sqrt{1-r^2}\ dr + \frac{\sin\theta}{12} \right)\ d\theta\ .
\end{aligned}
$$

Now, let $u = 1 - r^2$. Then $du = -2r\ dr$, $dr = -\dfrac{du}{2r}$, $r = \sqrt{1-u}$, and the new bounds of integration are

$$
\begin{aligned}
u(1) &= 1 - (1)^2 = 0 \\
u(0) &= 1 - (0)^2 = 1\ .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\oiint_{\mathscr{S}} \vec{F} \cdot d\vec{S} &= \iiint_D y\ dV \\
&= \int_0^{2\pi} \left(\sin\theta \int_0^1 r^2\sqrt{1-r^2}\ dr + \frac{\sin\theta}{12} \right)\ d\theta \\
&= \int_0^{2\pi} \left(\sin\theta \int_0^1 -\frac{1}{2} \cdot (-2) \cdot r \cdot r\sqrt{1-r^2}\ dr + \frac{\sin\theta}{12} \right)\ d\theta \\
&= \int_0^{2\pi} \left(-\frac{1}{2}\sin\theta \int_0^1 r\sqrt{1-r^2} \cdot (-2r)\ dr + \frac{\sin\theta}{12} \right)\ d\theta \\
&= \int_0^{2\pi} \left(-\frac{1}{2}\sin\theta \int_1^0 \sqrt{1-u} \cdot \sqrt{1-u^2}\ du + \frac{\sin\theta}{12} \right)\ d\theta\ .
\end{aligned}
$$

This is unfortunately a tough integral. Let's use the fact that $\displaystyle\int_0^{2\pi} \sin\theta = 0$. Thus,

$$
\begin{aligned}
\oiint_{\mathscr{S}} \vec{F} \cdot d\vec{S} &= \int_0^{2\pi} \left(-\frac{1}{2}\sin\theta \int_1^0 \sqrt{1-u} \cdot \sqrt{1-u^2}\ du + \frac{\sin\theta}{12} \right)\ d\theta \\
&= 0 + 0 \\
&= 0\ .
\end{aligned}
$$

3. Suppose $\mathscr{S}$ is any closed surface with outward normal and $\vec{F} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is constant. Show that $\displaystyle\iint_{\mathscr{S}} \vec{F} \cdot d\vec{S} = 0$.

By the divergence theorem, we know that

$$\iint_{\mathscr{S}} \vec{F} \cdot d\vec{S} = \iiint_D \operatorname{div}(\vec{F})\ dV\ .$$

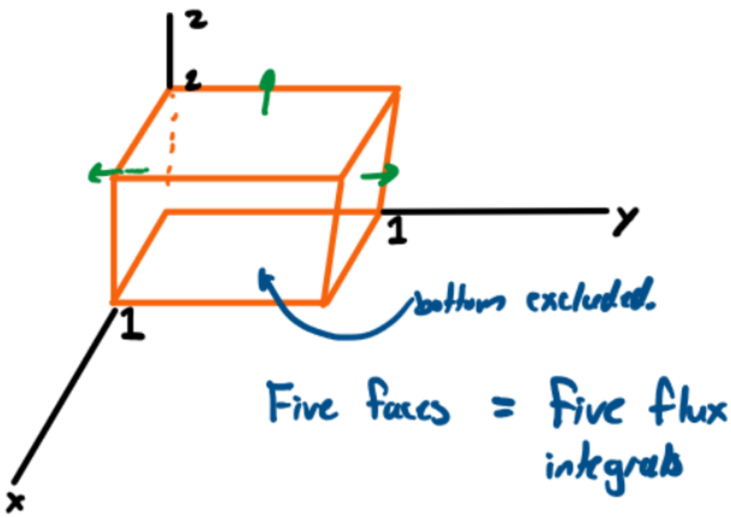
Then noting that a, b, and c are constants, we get that

$$\begin{aligned}
\operatorname{div}(\vec{F}) &= \nabla \cdot \vec{F} \\
&= \begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} \\
&= \frac{\partial a}{\partial} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z} \\
&= 0 + 0 + 0 \\
&= 0\ .
\end{aligned}$$

Thus,

$$\begin{aligned}
\iint_{\mathscr{S}} \vec{F} \cdot d\vec{S} &= \iiint_D \operatorname{div}(\vec{F})\ dV \\
&= \iiint_D 0\ dV \\
&= 0 \iiint_D dV \\
&= 0\ .
\end{aligned}$$

4. Compute $\displaystyle\iint_B \vec{F}\cdot d\vec{S}$ where B is the unit box $[0,1]^3$ with the bottom face removed, $\vec{F} = \big(xy^3,\ -3x,\ 2z+y\big)$, and the normal is outward.



There is a shortcut method we can use. Let B_1 be the entire box (with the bottom included). Then by the Divergence theorem, we get that

$$\begin{aligned}\iint_{B_1} \vec{F}\cdot d\vec{S} &= \iiint_D \operatorname{div}(\vec{F})\ dV \\ &= \int_0^1\int_0^1\int_0^1 \operatorname{div}(\vec{F})\ dV\ .\end{aligned}$$

So,

$$\begin{aligned}\operatorname{div}(\vec{F}) &= \nabla\cdot\vec{F} \\ &= \begin{bmatrix}\dfrac{\partial}{\partial x}\\ \dfrac{\partial}{\partial y}\\ \dfrac{\partial}{\partial z}\end{bmatrix}\cdot\begin{bmatrix}xy^3\\ -3x\\ 2z+y\end{bmatrix} \\ &= \frac{\partial}{\partial x}\left[xy^3\right] + \frac{\partial}{\partial y}\left[-3x\right] + \frac{\partial}{\partial z}\left[2z+y\right] \\ &= y^3 + 0 + 2 \\ &= y^3 + 2\ .\end{aligned}$$

Then

$$\iint_{B_1} \vec{F}\cdot d\vec{S} = \iiint_D \operatorname{div}(\vec{F})\ dV$$

$$
\begin{aligned}
&= \int_0^1 \int_0^1 \int_0^1 \operatorname{div}(\vec{F})\ dV \\
&= \int_0^1 \int_0^1 \int_0^1 (y^3 + 2)\ dx\ dy\ dz \\
&= \int_0^1 \int_0^1 \left(\int_0^1 (y^3 + 2)\ dx \right)\ dy\ dz \\
&= \int_0^1 \int_0^1 \left(\int_0^1 y^3\ dx + \int_0^1 2\ dx \right)\ dy\ dz \\
&= \int_0^1 \int_0^1 \left(y^3 \int_0^1 dx + 2 \int_0^1 dx \right)\ dy\ dz \\
&= \int_0^1 \int_0^1 \left(y^3 \Big[x \Big]_{x=0}^{1} + 2 \Big[x \Big]_{x=0}^{1} \right)\ dy\ dz \\
&= \int_0^1 \int_0^1 \big(y^3(1-0) + 2(1-0) \big)\ dy\ dz \\
&= \int_0^1 \int_0^1 \big(y^3 + 2 \big)\ dy\ dz \\
&= \int_0^1 \left(\int_0^1 \big(y^3 + 2 \big)\ dy \right)\ dz \\
&= \int_0^1 \left(\int_0^1 y^3\ dy + \int_0^1 2\ dy \right)\ dz \\
&= \int_0^1 \left(\int_0^1 y^3\ dy + 2 \int_0^1 dy \right)\ dz \\
&= \int_0^1 \left(\left[\frac{y^4}{4} \right]_{y=0}^{1} + 2 \Big[y \Big]_{y=0}^{1} \right)\ dz \\
&= \int_0^1 \left(\frac{1}{4} + 2 \right)\ dz \\
&= \int_0^1 \frac{1}{4}\ dz + \int_0^1 2\ dz \\
&= \frac{1}{4} \int_0^1 dz + 2 \int_0^1 dz \\
&= \frac{1}{4} \Big[z \Big]_{z=0}^{1} + 2 \Big[z \Big]_{z=0}^{1} \\
&= \frac{1}{4} + 2 \\
&= \frac{1}{4} + \frac{8}{4} \\
&= \frac{9}{4}\ .
\end{aligned}
$$

Now, let B_2 be the bottom face of the box. Then

$$\iint_B \vec{F} \cdot d\vec{S} = \iint_{B_1} \vec{F} \cdot d\vec{S} - \iint_{B_2} \vec{F} \cdot d\vec{S} \ .$$

The bottom face of the box is given by $z = 0$ with $0 \leq x \leq 1$ and $0 \leq y \leq 1$. Then the bottom face is given by

$$\vec{r}(x,y) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

for $0 \leq x \leq 1$ and $0 \leq y \leq 1$. So, $\vec{n} = (0,0,-1)$ and so

$$\begin{aligned}
\vec{F}(\vec{r}(x,y)) \cdot \vec{n} &= \vec{F}\left(\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}\right) \cdot \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} xy^3 \\ -3x \\ y \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \\
&= 0 + 0 - y \\
&= -y \ .
\end{aligned}$$

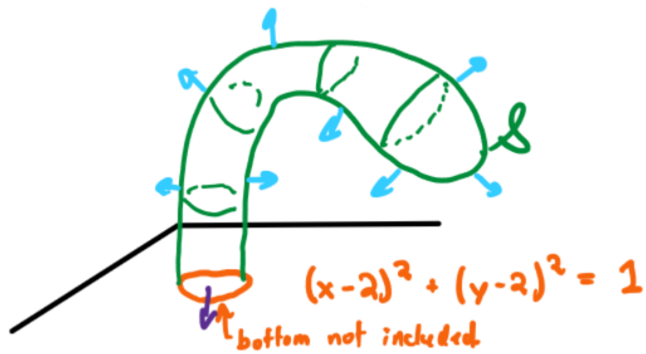
Then

$$\begin{aligned}
\iint_{B_2} \vec{F} \cdot d\vec{S} &= \iint_D \vec{F}(\vec{r}(x,y)) \cdot \vec{n} \ dA \\
&= \int_0^1 \int_0^1 -y \ dy \ dx \\
&= -\int_0^1 \left(\int_0^1 y \ dy\right) \ dx \\
&= -\int_0^1 \left(\left[\frac{y^2}{2}\right]_{y=0}^1\right) \ dx \\
&= -\int_0^1 \frac{1}{2} \ dx \\
&= -\frac{1}{2}\int_0^1 dx \\
&= -\frac{1}{2}\left[x\right]_{x=0}^1 \\
&= -\frac{1}{2}(1) \\
&= -\frac{1}{2} \ .
\end{aligned}$$

Thus

$$\begin{aligned}
\iint_B \vec{F} \cdot d\vec{S} &= \iint_{B_1} \vec{F} \cdot d\vec{S} - \iint_{B_2} \vec{F} \cdot d\vec{S} \\
&= \frac{9}{4} - \left(-\frac{1}{2}\right) \\
&= \frac{9}{4} + \frac{1}{2} \\
&= \frac{9}{4} + \frac{2}{4} \\
&= \frac{11}{4} \ .
\end{aligned}$$

5. Compute $\iint_{\mathscr{S}} \vec{F} \cdot d\vec{S}$ where $\vec{F} = \big(2z,\ 2xz,\ 3xy^2\big)$.



Let $\mathscr{S}_2$ be the bottom circle of the surface $\mathscr{S}$. Let $\mathscr{S}_1 = \mathscr{S} \cup \mathscr{S}_2$. Then $\mathscr{S}_1$ is a closed surface. Since $\mathscr{S}_1$ is closed and has outward normals, we can apply the Divergence theorem. So,

$$\iint_{\mathscr{S}_1} \vec{F} \cdot d\vec{S} = \iiint_D \operatorname{div}(\vec{F})\ dV\ .$$

Then

$$\begin{aligned}
\operatorname{div}(\vec{F}) &= \nabla \cdot \vec{F} \\
&= \begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} 2z \\ 2xz \\ 3xy^2 \end{bmatrix} \\
&= \frac{\partial}{\partial x}[2z] + \frac{\partial}{\partial y}[2xz] + \frac{\partial}{\partial z}\left[3xy^2\right] \\
&= 0 + 0 + 0 \\
&= 0\ .
\end{aligned}$$

So,

$$\begin{aligned}
\iint_{\mathscr{S}_1} \vec{F} \cdot d\vec{S} &= \iiint_D \operatorname{div}(\vec{F})\ dV \\
&= \iiint_D 0\ dV \\
&= 0 \iiint_D dV \\
&= 0\ .
\end{aligned}$$

Then

$$\begin{aligned}\iint_{\mathscr{S}} \vec{F} d\vec{S} &= \iint_{\mathscr{S}_1} \vec{F} \cdot d\vec{S} - \iint_{\mathscr{S}_2} \vec{F} \cdot d\vec{S} \\ &= 0 - \iint_{\mathscr{S}_2} \vec{F} \cdot d\vec{S} \\ &= -\iint_{\mathscr{S}_2} \vec{F} \cdot d\vec{S} \ .\end{aligned}$$

Now, to evaluate this surface integral, we do it the ordinary way. The bottom circle $\mathscr{S}_2$ is parameterized in shifted polar coordinates by

$$\vec{r}(r,\theta) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\cos\theta + 2 \\ r\sin\theta + 2 \\ 0 \end{bmatrix} \ .$$

for $0 \le r \le 1$ (since this circle given by $(x-2)^2 + (y-2)^2 = 1$ is a circle of radius 1, and $0 \le \theta \le 2\pi$. Note that $z = 0$ since we're talking about a circle on the xy-plane. The normal of $\mathscr{S}_2$ is then

$$\begin{aligned}\vec{n} &= \vec{r}_r \times \vec{r}_\theta \\ &= \begin{bmatrix} (r\cos\theta+2)_r \\ (r\sin\theta+2)_r \\ (0)_r \end{bmatrix} \times \begin{bmatrix} (r\cos\theta+2)_\theta \\ (r\sin\theta+2)_\theta \\ (0)_\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} \times \begin{bmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \end{bmatrix} \\ &= (-r\cos\theta)\,\hat{\imath} - (r\sin\theta)\,\hat{\jmath} + \left(r\cos^2\theta + r\sin^2\theta\right)\hat{k} \\ &= (-r\cos\theta)\,\hat{\imath} + (-r\sin\theta)\,\hat{\jmath} + (r)\,\hat{k} \\ &= \begin{bmatrix} -r\cos\theta \\ -r\sin\theta \\ r \end{bmatrix} \ .\end{aligned}$$

Also,

$$\begin{aligned}\vec{F}(\vec{r}(r,\theta)) &= \vec{F}\left(\begin{bmatrix} r\cos\theta+2 \\ r\sin\theta+2 \\ 0 \end{bmatrix}\right) \\ &= \begin{bmatrix} 2(0) \\ 2(r\cos\theta+2)(0) \\ 3(r\cos\theta+2)(r\sin\theta+2)^2 \end{bmatrix}\end{aligned}$$

$$= \begin{bmatrix} 0 \\ 0 \\ (3r\cos\theta + 6)(r^2\sin^2\theta + 4r\sin\theta + 4) \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ 0 \\ 3r^3\sin^2\theta\cos\theta + 12r^2\sin\theta\cos\theta + 12r\sin\theta + 6r^2\sin^2\theta + 24r\sin\theta + 24 \end{bmatrix}$$

Then we get that

$$\begin{aligned} &\vec{F}(\vec{r}(r,\theta)) \cdot \vec{n} \\ &= \vec{F}(\vec{(}r,\theta)) \cdot \begin{bmatrix} -r\cos\theta \\ -r\sin\theta \\ r \end{bmatrix} \\ &= 3r^3\sin^2\theta\cos\theta + 12r^2\sin\theta\cos\theta + 12r\sin\theta + 6r^2\sin^2\theta + 24r\sin\theta + 24 \end{aligned}$$

Green's Theorem

Green's Theorem

Suppose C is a closed curve surrounding a region D in $\mathbb{R}^2$ (oriented counterclockwise, meaning D is to the left of the curve while moving along C). For a vector field $\vec{F} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$, we have

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D \left((F_2)_x - (F_1)_y \right) \, dA \ .$$

Note that $(F_2)_x - (F_1)_y = 0$ when $\vec{F}$ is conservative (this is due to equality of mixed partials).

Remarks:

(1) If we regard $\vec{F} = \begin{bmatrix} F_1 \\ F_2 \\ 0 \end{bmatrix}$, then

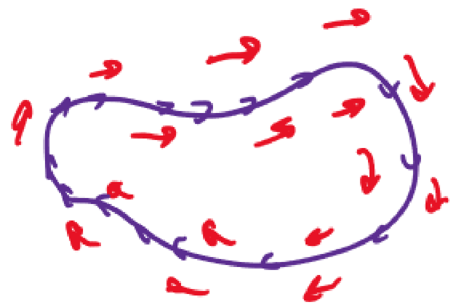
$$\begin{aligned} \operatorname{curl}(\vec{F}) &= \nabla \times \vec{F} \\ &= \det\left(\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & 0 \end{bmatrix} \right) \\ &= \left(-\frac{\partial F_2}{\partial z} \right) \hat{i} - \left(\frac{\partial F_1}{\partial z} \right) \hat{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \hat{k} \\ &= 0\hat{i} - 0\hat{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \hat{k} \ , \end{aligned}$$

where $\dfrac{\partial F_2}{\partial z} = 0$ and $\dfrac{\partial F_1}{\partial z} = 0$ since $\vec{F}$ only depends on x and y; that is, $\vec{F} = \vec{F}(x, y)$. Now, recall that the $\boldsymbol{\nabla} \times \left(\vec{F} \right)$ gives the rotation axes for $\vec{F}$. So,

$$\iint_D \left((F_2)_x - (F_1)_y \right) \, dA$$

is the integral over the z-component of curl.

(2) $\oint_C \vec{F} \cdot d\vec{r}$ is a measure of the tendency of $\vec{F}$ to "circulate" about C.



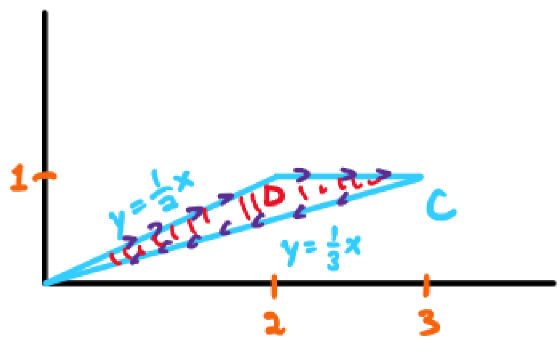
High positive circulation.

Green's theorem says that

$$\text{total circulation around } C = \iint_D \text{local circulation } dA\ .$$

Examples

1. Compute $\oint_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \begin{bmatrix} x - y^2 \\ 3xy \end{bmatrix}$ and C is given below.



If we directly computed this line integral, we would need to evaluate 3 seperate integrals. Let's instead utilize Green's theorem! First, we can describe D. The restriction on y is

$$0 \le y \le 1 \ .$$

Then since $y = \frac{1}{3}x \implies x = 3y$ and $y = \frac{1}{2}x \implies x = 2y$, the restriction on x is

$$2y \le x \le 3y \ .$$

Thus, by Green's theorem,

$$\begin{aligned}
\oint_C \vec{F} \cdot d\vec{r} &= \iint_D \left((3xy)_x - (x - y^2)_y\right) \ dA \\
&= \int_0^1 \int_{2y}^{3y} (3y - (-2y)) \ dx \ dy \\
&= \int_0^1 \int_{2y}^{3y} (3y + 2y) \ dx \ dy \\
&= \int_0^1 \int_{2y}^{3y} 5y \ dx \ dy \\
&= 5 \int_0^1 \int_{2y}^{3y} y \ dx \ dy \\
&= 5 \int_0^1 \left(y \int_{2y}^{3y} dx \right) \ dy \\
&= 5 \int_0^1 \left(y \Big[x \Big]_{x=2y}^{3y} \right) \ dy
\end{aligned}$$

$$
\begin{aligned}
&= 5 \int_0^1 y(3y - 2y)\ dy \\
&= 5 \int_0^1 y \cdot y\ dy \\
&= 5 \int_0^1 y^2\ dy \\
&= 5 \left[\frac{y^3}{3} \right]_0^1 \\
&= 5 \left(\frac{1}{3} \right) \\
&= \frac{5}{3}\ .
\end{aligned}
$$

2. Compute $\oint_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \begin{bmatrix} e^{x^2} - 2y \\ \cos(y^2) + 3x^2 \end{bmatrix}$ is the vector field and C is $(x-1)^2 + (y+1)^2 = 4$ is the curve oriented counterclockwise.

Note that the curve is the circle of radius 2 centered at $(1, -1)$. By Green's theorem,

$$\begin{aligned} \oint_C \vec{F} \cdot d\vec{r} &= \iint_D \left[\left(\cos(y^2) + 3x^2\right)_x - \left(e^{x^2} - 2y\right)_y \right] \, dA \\ &= \iint_{(x-1)^2+(y+1)^2 \leq 4} (6x - (-2)) \, dA \\ &= \iint_{(x-1)^2+(y+1)^2 \leq 4} (6x + 2) \, dA \ . \end{aligned}$$

Now, we switch to translated polar coordinates (translated since the circle is not centered at the origin). Here, $a = 1$ is the x-coordinate of the center of the circle, and $b = -1$ is the y-coordinate of the center of the circle. So, we get that

$$\begin{aligned} x &= r\cos\theta + a = \cos\theta + 1 \\ y &= r\sin\theta + b = \sin\theta - 1 \\ r &= \sqrt{x^2 + y^2} \end{aligned}$$

where $0 \leq r \leq 2$ (since the circle has a radius of 2) and $0 \leq \theta \leq 2\pi$. Note that we chose $y = r\sin\theta + a$ to have a positive value for $\sin\theta$ because of **counter-clockwise** motion along the curve. If the motion along the curve were clockwise, then $\sin\theta$ would be negative. Now, noting that $dA = r \ dr \ d\theta$, subsituting in the shifted polar coordinates gives us

$$\begin{aligned} \oint_C \vec{F} \cdot d\vec{r} &= \iint_{(x-1)^2+(y+1)^2 \leq 4} (6x + 2) \, dA \\ &= \int_0^{2\pi} \int_0^2 (6(r\cos\theta + 1) + 2) \cdot r \ dr \ d\theta \\ &= \int_0^{2\pi} \int_0^2 (6r\cos\theta + 6 + 2) \cdot r \ dr \ d\theta \\ &= \int_0^{2\pi} \int_0^2 (6r\cos\theta + 8) \cdot r \ dr \ d\theta \\ &= \int_0^{2\pi} \int_0^2 (6r^2\cos\theta + 8r) \ dr \ d\theta \\ &= \int_0^{2\pi} \left(\int_0^2 (6r^2\cos\theta + 8r) \ dr \right) \ d\theta \\ &= \int_0^{2\pi} \left(\int_0^2 6r^2\cos\theta \ dr + \int_0^2 8r \ dr \right) \ d\theta \end{aligned}$$

$$
\begin{aligned}
&= \int_0^{2\pi} \left(6\cos\theta \int_0^2 r^2 \ dr + 8 \int_0^2 r \ dr \right) \ d\theta \\
&= \int_0^{2\pi} \left(6\cos\theta \ \left[\frac{r^3}{3} \right]_0^2 + 8 \left[\frac{r^2}{2} \right]_0^2 \right) \ d\theta \\
&= \int_0^{2\pi} \left(6\cos\theta \left(\frac{2^3}{3} - \frac{0^3}{3} \right) + 8 \left(\frac{2^2}{2} - \frac{0^2}{2} \right) \right) \ d\theta \\
&= \int_0^{2\pi} \left(6\cos\theta \left(\frac{8}{3} \right) + 8\,(2) \right) \ d\theta \\
&= \int_0^{2\pi} (16\cos\theta + 16) \ d\theta \\
&= \int_0^{2\pi} 16\cos\theta \ d\theta + \int_0^{2\pi} 16 \ d\theta \\
&= 16 \int_0^{2\pi} \cos\theta \ d\theta + 16 \int_0^{2\pi} d\theta \\
&= 16 \Big[\sin\theta \Big]_0^{2\pi} + 16 \Big[\theta \Big]_0^{2\pi} \\
&= 16(\sin(2\pi) - \sin(0)) + 16(2\pi - 0) \\
&= 16(0 - 0) + 32\pi \\
&= 32\pi \ .
\end{aligned}
$$

3. Show that $\oint_C \vec{F} \cdot d\vec{r} = \text{Area}(D)$ where $\vec{F} = \frac{1}{2}\begin{bmatrix} -y \\ x \end{bmatrix} = \begin{bmatrix} -y/2 \\ x/2 \end{bmatrix}$.

   By Green's theorem,

$$\begin{aligned}
\oint_C \vec{F} \cdot d\vec{r} &= \iint_D \left[\left(\frac{x}{2}\right)_x - \left(-\frac{y}{2}\right)_y \right] \, dA \\
&= \iint_D \left(\frac{1}{2} - \left(-\frac{1}{2}\right)\right) \, dA \\
&= \iint_D 1 \; dA \\
&= \text{Area}(D) \; .
\end{aligned}$$

   In fact, this is true for **any** $\vec{F}$ with $(F_2)_x - (F_1)_y = 1$. For example, the same is true if $\vec{F} = \begin{bmatrix} 0 \\ x \end{bmatrix}$ or $\vec{F} = \begin{bmatrix} -y \\ 0 \end{bmatrix}$.

4. Find the area of $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \leq 1$, where $\vec{F}$ is the same as in Problem 3.

We have that

$$\iint_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\leq 1} 1 \; dA = \oint_{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1} \vec{F} \cdot d\vec{r}$$

Note that his is just reading Green's theorem from right to left! Now, the ellipse given by $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ has an x-radius of a and y-radius of b. So, the ellipse can be described by the paramaterization

$$\vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} a\cos(t) \\ b\sin(t) \end{bmatrix}$$

for $0 \leq t \leq 2\pi$. So, we want to evaluate.

$$\begin{aligned} \iint_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\leq 1} 1 \; dA &= \oint_{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1} \vec{F} \cdot d\vec{r} \\ &= \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{v}(t) \; dt \; . \end{aligned}$$

We get that

$$\vec{v}(t) = \begin{bmatrix} -a\sin(t) \\ b\cos(t) \end{bmatrix}$$

and

$$\begin{aligned} \vec{F}(\vec{r}(t)) &= \vec{F}\left(\begin{bmatrix} a\cos(t) \\ b\sin(t) \end{bmatrix}\right) \\ &= \vec{F}\left(\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}\right) \\ &= \frac{1}{2}\begin{bmatrix} -y(t) \\ x(t) \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} -b\sin(t) \\ a\cos(t) \end{bmatrix} \; . \end{aligned}$$

Then

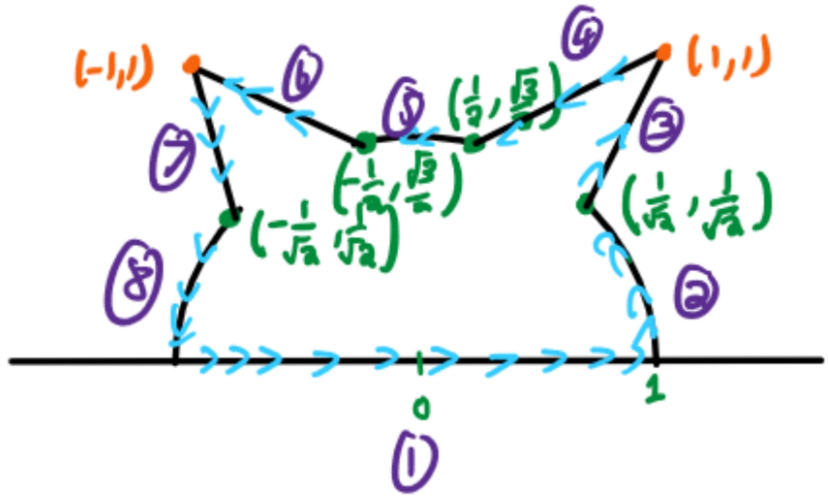
$$\begin{aligned} \vec{F}(\vec{r}(t)) \cdot \vec{v}(t) &= \left(\frac{1}{2}\begin{bmatrix} -b\sin(t) \\ a\cos(t) \end{bmatrix}\right) \cdot \begin{bmatrix} -a\sin(t) \\ b\cos(t) \end{bmatrix} \\ &= \frac{1}{2}\left(\begin{bmatrix} -b\sin(t) \\ a\cos(t) \end{bmatrix} \cdot \begin{bmatrix} -a\sin(t) \\ b\cos(t) \end{bmatrix}\right) \\ &= \frac{1}{2}\left(ab\sin^2(t) + ab\cos^2(t)\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\left(ab(\sin^2(t) + \cos^2(t))\right) \\
&= \frac{1}{2}\left(ab \cdot 1\right) \\
&= \frac{1}{2}\left(ab\right) \\
&= \frac{ab}{2} \ .
\end{aligned}
$$

Thus, the area of the of the ellipse is

$$
\begin{aligned}
\iint_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\leq 1} 1 \ dA &= \oint_{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1} \vec{F} \bullet d\vec{r} \\
&= \int_0^{2\pi} \vec{F}(\vec{r}(t)) \bullet \vec{v}(t) \ dt \\
&= \int_0^{2\pi} \frac{ab}{2} \ dt \\
&= \frac{ab}{2} \int_0^{2\pi} dt \\
&= \frac{ab}{2} \left[t \right]_{t=0}^{2\pi} \\
&= \frac{ab}{2}(2\pi - 0) \\
&= \frac{ab}{2} \cdot 2\pi \\
&= \pi ab \ .
\end{aligned}
$$

5. Let $\vec{F}$ be the same as in Problem 3. Find the area of this thing:



- For (1), this is a line going from $(-1, 0)$ to $(1, 0)$, and so a parameterization is given by

$$\begin{aligned} \vec{r}_1(t) &= (1-t)\begin{bmatrix}-1\\0\end{bmatrix} + t\begin{bmatrix}1\\0\end{bmatrix} \\ &= \begin{bmatrix}-1+t\\0\end{bmatrix} + \begin{bmatrix}t\\0\end{bmatrix} \\ &= \begin{bmatrix}2t-1\\0\end{bmatrix} \quad , \quad 0 \le t \le 1 \ . \end{aligned}$$

- For (2), this is the part of the unit circle, and so a parameterization is given by

$$\vec{r}_2(t) = \begin{bmatrix}\cos(t)\\ \sin(t)\end{bmatrix} \quad , \quad 0 \le t \le \frac{\pi}{4} \ .$$

- For (3), this is a line segment going from $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ to $(1, 1)$, and so a parameterization is given by

$$\begin{aligned} \vec{r}_3(t) &= (1-t)\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix} + t\begin{bmatrix}1\\1\end{bmatrix} \\ &= (1-t)\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix} + \begin{bmatrix}t\\t\end{bmatrix} \end{aligned}$$

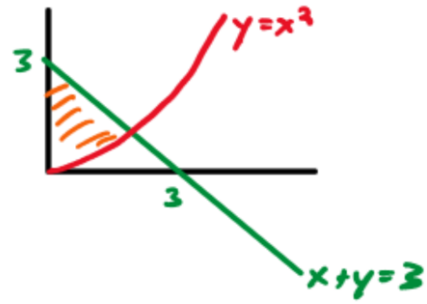
$$= \dots$$

Thus,

$$\begin{aligned}
\text{Area} &= \sum_{n=1}^{8} \oint_{\text{n}} \vec{F} \cdot d\vec{r} \\
&= \sum_{n=1}^{8} \oint_{\text{n}} \vec{F}(\vec{r}(x,y)) \cdot \vec{v}(t)\ dt \\
&= \dots
\end{aligned}$$

6. Compute $\oint_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \begin{bmatrix} 3x^2ye^{x^3} \\ e^{x^3} \end{bmatrix}$ where C is the counterclockwise oriented boundary for

$$D: \begin{cases} x^2 \leq y \\ x+y \leq 3 \\ 0 \leq x \end{cases}$$



C consists of 3 pieces.

By Green's theorem,

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D \left[\left(e^{x^3}\right)_x - \left(3x^2ye^{x^3}\right)_y \right] \, dA \; .$$

So, we get that

$$\begin{aligned} \left(e^{x^3}\right)_x &= \frac{\partial}{\partial x}\left[e^{x^3}\right] \\ &= \frac{\partial\left(e^{x^3}\right)}{\partial(x^3)} \cdot \frac{\partial(x^3)}{\partial x} \\ &= e^{x^3} \cdot 3x^2 \\ &= 3x^2e^{x^3} \; . \end{aligned}$$

and

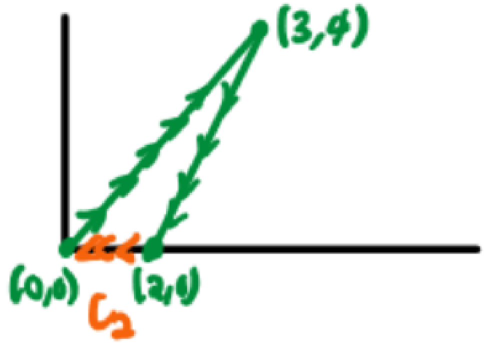
$$\begin{aligned} \left(3x^2ye^{x^3}\right)_y &= \frac{\partial}{\partial y}\left[3x^2ye^{x^3}\right] \\ &= 3x^2e^{x^3} \cdot \frac{\partial}{\partial y}[y] \\ &= 3x^2e^{x^3} \cdot 1 \\ &= 3x^2e^{x^3} \; . \end{aligned}$$

Thus,

$$\begin{aligned} \oint_C \vec{F} \cdot d\vec{r} &= \iint_D \left[\left(e^{x^3}\right)_x - \left(3x^2ye^{x^3}\right)_y \right] \, dA \\ &= \iint_D \left(3x^2e^{x^3} - 3x^2e^{x^3}\right) \, dA \end{aligned}$$

$$
\begin{aligned}
&= \iint_D 0 \, dA \\
&= 0 \, .
\end{aligned}
$$

7. Compute $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \begin{bmatrix} y^2 e^{xy^2} + y \\ 2xye^{xy^2} \end{bmatrix}$ and C is the pair of line segments connecting $(0,0)$ to $(3,4)$ to $(2,0)$.



Note that there is no segment connecting $(0,0)$ to $(2,0)$. However, we can pretend as if there was a segment connected $(2,0)$ to $(0,0)$, which we'll denote as the curve C_2 (note that the direction is important). Let C_1 denote the boundary of the triangle D with vertices $(0,0)$, $(3,4)$, and $(2,0)$. We can apply Green's theorem on C_1. So, applying Green's theorem on C_1 gives us

$$\oint_{C_1} \vec{F} \cdot d\vec{r} = \iint_D \left[\left(2xye^{xy^2}\right)_x - \left(y^2 e^{xy^2} + y\right)_y \right] \, dA \ .$$

From this, we get that the partials are

$$\begin{aligned}
\left(2xye^{xy^2}\right)_x &= \frac{\partial}{\partial x}\left[2xye^{xy^2}\right] \\
&= 2y \cdot \frac{\partial}{\partial x}\left[xe^{xy^2}\right] \\
&= 2y\left(\frac{\partial}{\partial x}[x] \cdot e^{xy^2} + x \cdot \frac{\partial}{\partial x}\left[e^{xy^2}\right]\right) \\
&= 2y\left(1 \cdot e^{xy^2} + x\left(\frac{\partial\left(e^{xy^2}\right)}{\partial(xy^2)} \cdot \frac{\partial(xy^2)}{\partial x}\right)\right) \\
&= 2y\left(e^{xy^2} + x\left(e^{xy^2} \cdot y^2\right)\right) \\
&= 2y\left(e^{xy^2} + xy^2 e^{xy^2}\right) \\
&= 2ye^{xy^2} + 2xy^3 e^{xy^2}
\end{aligned}$$

and

$$\left(y^2 e^{xy^2} + y\right)_y = \frac{\partial}{\partial y}\left[y^2 e^{xy^2} + y\right]$$

$$
\begin{aligned}
&= \frac{\partial}{\partial y}\left[y^2 e^{xy^2}\right] + \frac{\partial}{\partial y}[y] \\
&= \left(\frac{\partial}{\partial y}\left[y^2\right] \cdot e^{xy^2} + y^2 \cdot \frac{\partial}{\partial y}\left[e^{xy^2}\right]\right) + 1 \\
&= \left(2y \cdot e^{xy^2} + y^2\left(\frac{\partial\left(e^{xy^2}\right)}{\partial(xy^2)} \cdot \frac{\partial(xy^2)}{\partial y}\right)\right) + 1 \\
&= \left(2ye^{xy^2} + y^2\left(e^{xy^2} \cdot 2xy\right)\right) + 1 \\
&= \left(2ye^{xy^2} + y^2\left(2xye^{xy^2}\right)\right) + 1 \\
&= \left(2ye^{xy^2} + 2xy^3e^{xy^2}\right) + 1 \\
&= 2ye^{xy^2} + 2xy^3e^{xy^2} + 1\ .
\end{aligned}
$$

Then

$$
\begin{aligned}
\oint_{C_1} \vec{F} \cdot d\vec{r} &= \iint_D \left[\left(2xye^{xy^2}\right)_x - \left(y^2e^{xy^2} + y\right)_y\right]\, dA \\
&= \iint_D \left(2ye^{xy^2} + 2xy^3e^{xy^2} - \left(2ye^{xy^2} + 2xy^3e^{xy^2} + 1\right)\right)\, dA \\
&= \iint_D \left(2ye^{xy^2} + 2xy^3e^{xy^2} - 2ye^{xy^2} - 2xy^3e^{xy^2} - 1\right)\, dA \\
&= \iint_D -1\, dA \\
&= -\text{Area}(D) \\
&= -\frac{1}{2} \cdot 2 \cdot 4 \\
&= -4\ .
\end{aligned}
$$

Note that the area of D is simply the area of a triangle. In this case, the base is 2 and the height is 4. Now, to find $\oint_C \vec{F} \cdot d\vec{r}$, we can simply subtract out the line integral of $\vec{F}$ over C_2 from our above result. That is,

$$
\oint_C \vec{F} \cdot d\vec{r} = \oint_{C_1} \vec{F} \cdot d\vec{r} - \oint_{C_2} \vec{F} \cdot d\vec{r}\ .
$$

Since C_2 is just a line segment from $(2,0)$ to $(0,0)$, C_2 can be described by the parameterization

$$
\begin{aligned}
\vec{r}_2(t) &= (1-t)\begin{bmatrix}2\\0\end{bmatrix} + t\begin{bmatrix}0\\0\end{bmatrix} \\
&= \begin{bmatrix}2-2t\\0\end{bmatrix} + \begin{bmatrix}0\\0\end{bmatrix}
\end{aligned}
$$

$$= \begin{bmatrix} 2-2t \\ 0 \end{bmatrix} \ , \quad 0 \le t \le 1 \ .$$

Then

$$\oint_{C_2} \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{v}(t) \ dt \ .$$

We get that

$$\vec{v}(t) = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

and

$$\begin{aligned}
\vec{F}(\vec{r}(t)) &= \vec{F}\left(\begin{bmatrix} 2-2t \\ 0 \end{bmatrix}\right) \\
&= \vec{F}\left(\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}\right) \\
&= \begin{bmatrix} y(t)^2 \cdot e^{x(t)\cdot y(t)^2} + y(t) \\ 2 \cdot x(t) \cdot y(t) \cdot e^{x(t)\cdot y(t)^2} \end{bmatrix} \\
&= \begin{bmatrix} (0)^2 \cdot e^{(2-2t)\cdot(0)^2} + 0 \\ 2 \cdot (2-2t) \cdot 0 \cdot e^{(2-2t)\cdot(0)^2} \end{bmatrix} \\
&= \begin{bmatrix} 0 \cdot e^0 + 0 \\ 2 \cdot (2-2t) \cdot 0 \cdot e^0 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \ .
\end{aligned}$$

So,

$$\begin{aligned}
\vec{F}(\vec{r}(t)) \cdot \vec{v}(t) &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 0 \end{bmatrix} \\
&= 0 + 0 \\
&= 0 \ .
\end{aligned}$$

and so,

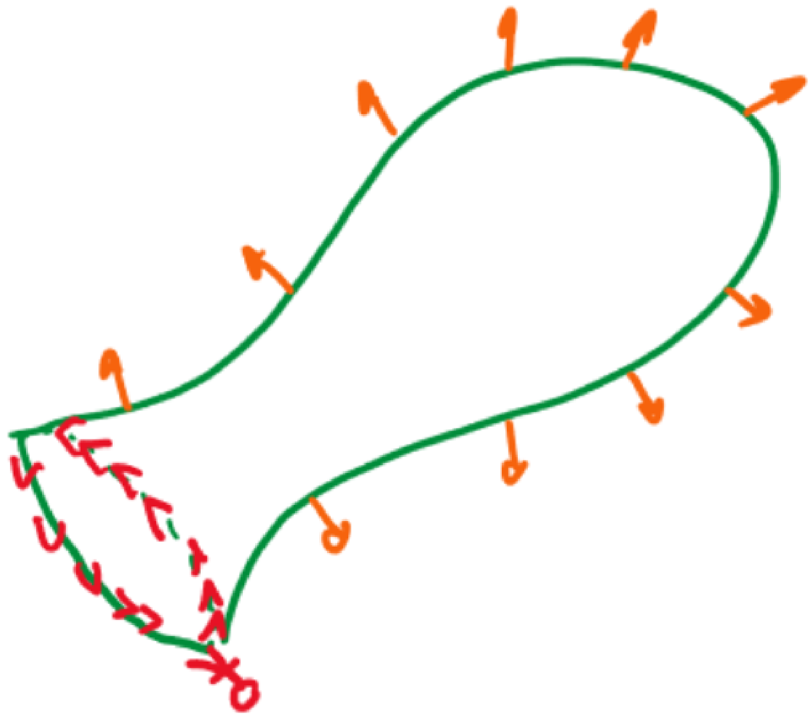
$$\oint_{C_2} \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{v}(t) \ dt = \int_0^1 0 \ dt = 0 \ .$$

Thus,

$$\begin{aligned}
\oint_C \vec{F} \cdot d\vec{r} &= \oint_{C_1} \vec{F} \cdot d\vec{r} - \oint_{C_2} \vec{F} \cdot d\vec{r} \\
&= -4 - 0 \\
&= -4 \ .
\end{aligned}$$

Stokes' Theorem

Recall that for an orientable surface $\mathscr{S}$ with a single bondary curve C (e.g. C is the border of a hole in $\mathscr{S}$), the orientation of $\mathscr{S}$ (the direction of its normals) induces a direction on the boundary curve C:



Stokes' Theorem

Let $\mathscr{S}$ be an orientable surface with boundary curve C. Then

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_{\mathscr{S}} \operatorname{curl}(\vec{F}) \cdot d\vec{S}$$

Remarks

(i) If $\vec{F} = \begin{bmatrix} F_1 \\ F_2 \\ 0 \end{bmatrix}$, this is simply Green's theorem, since

$$\begin{aligned}
\operatorname{curl}(\vec{F}) &= \nabla \times \vec{F} \\
&= \begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{bmatrix} \times \begin{bmatrix} F_1 \\ F_2 \\ 0 \end{bmatrix} \\
&= \det\left(\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & 0 \end{bmatrix} \right) \\
&= \left(0 - \frac{\partial F_2}{\partial z}\right)\hat{\imath} - \left(0 - \frac{\partial F_1}{\partial z}\right)\hat{\jmath} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k} \\
&= 0\hat{\imath} + 0\hat{\jmath} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k} \\
&= 0\hat{\imath} + 0\hat{\jmath} + \left((F_2)_x - (F_1)_y\right)\hat{k} \\
&= \begin{bmatrix} 0 \\ 0 \\ (F_2)_x - (F_1)_y \end{bmatrix},
\end{aligned}$$

where $\mathscr{S}$ is the region in $\mathbb{R}^2$ enclosed by the planar curve C. Note that

$$\frac{\partial F_2}{\partial x} = 0 \quad \text{and} \quad \frac{\partial F_1}{\partial y} = 0$$

since $\vec{F}$ only depends on x and y, not z.

(ii) If $\mathscr{S}$ is closed, we know from the divergence theorem that

$$\begin{aligned}\oiint_{\mathscr{S}} \operatorname{curl}(\vec{F}) \cdot d\vec{S} &= \iiint_D \operatorname{div}(\operatorname{curl}(\vec{F}))\ dV \\ &= \iiint_D 0\ dV \\ &= 0\ .\end{aligned}$$

Note that $\operatorname{div}(\operatorname{curl}(\vec{F})) = 0$. In this case, $\mathscr{S}$ has no boundary curve, so the LHS of Stokes' theorem is 0 by convention!

(iii) ★ If $\mathscr{S}_1$ and $\mathscr{S}_2$ both have the same boundary curve C, then

$$\iint_{\mathscr{S}_1} \operatorname{curl}(\vec{F}) \cdot d\vec{S} = \iint_{\mathscr{S}_2} \operatorname{curl}(\vec{F}) \cdot d\vec{S} \qquad \left(= \oint_C \vec{F} \cdot d\vec{r} \right)$$

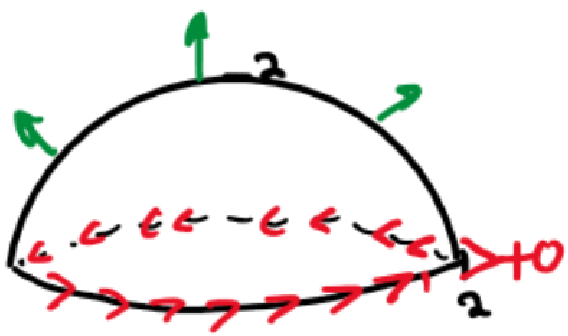
(iv) Geometric Intuition:

$$\iint_{\mathscr{S}} \operatorname{curl}(\vec{F}) \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{S}\ .$$

- $\operatorname{curl}(\vec{F})$ is the localized rotation.
- The LHS is a measure of the tendency for $\vec{F}$ to rotate around $\vec{n}$.
- The RHS is a measure of circulation around the hole.

Examples

1. Let $\mathscr{S}$ be the upper hemisphere of radius 2 with outward normals and $\vec{F} = \left(z^2 - 1,\ z + xy^3,\ 6\right)$. Find $\displaystyle\iint_{\mathscr{S}} \operatorname{curl}(\vec{F}) \cdot d\vec{S}$.



By Stokes' theorem, we have that

$$\iint_{\mathscr{S}} \operatorname{curl}(\vec{F}) \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r}\ .$$

We can view the curve C from an overhead view.

Note that we are assuming the hemisphere is centered at the origin. Now, we have that a parameterization for C is given by

$$\vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} 2\cos(t) \\ 2\sin(t) \\ 0 \end{bmatrix} \quad , \quad 0 \le t \le 2\pi\ .$$

Then

$$\vec{F}(\vec{r}(t)) = \vec{F}\left(\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} \right)$$

$$= \begin{bmatrix} z(t)^2 - 1 \\ z + x(t) \cdot y(t)^2 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} (0)^2 - 1 \\ 0 + 2\cos(t) \cdot (2\sin(t))^3 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 2\cos(t) \cdot 8\sin^3(t) \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 16\sin^3(t)\cos(t) \\ 6 \end{bmatrix}$$

and

$$\vec{v}(t) = \begin{bmatrix} -2\sin(t) \\ 2\cos(t) \\ 0 \end{bmatrix} .$$

So,

$$\begin{aligned} \vec{F}(\vec{r}(t)) \bullet \vec{v}(t) &= \begin{bmatrix} -1 \\ 16\sin^3(t)\cos(t) \\ 6 \end{bmatrix} \bullet \begin{bmatrix} -2\sin(t) \\ 2\cos(t) \\ 0 \end{bmatrix} \\ &= 2\sin(t) + 32\sin^3(t)\cos^2(t) + 0 \\ &= 2\sin(t) + 32\sin^3(t)\cos^2(t) . \end{aligned}$$

Hence,

$$\iint_{\mathscr{S}} \mathrm{curl}(\vec{F})$$