MATH 367 - Week 2 Notes

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Derivatives Continued

Recall from last week the derivative.

Definition (Differentiable)

Suppose $\vec{f} : U \to \mathbb{R}^m$ where $U \subseteq \mathbb{R}^n$. We say that f is differentiable at $\vec{x}_0 \in U^{\text{o}}$ if there is an $m \times n$ matrix A such that

$$\lim_{\vec{x}\to\vec{x}_0} \frac{||\vec{f}(\vec{x}) - \vec{f}(\vec{x}_0) - A(\vec{x}-\vec{x}_0)||}{||\vec{x}-\vec{x}_0||} = 0 \ ,$$

where the numerator is the Euclidean distance in $\mathbb{R}^m$ and the denominator is the Euclidean distance in $\mathbb{R}^n$. When this is the case, A must be unique. We call this matrix the derivative of $\vec{f}$ at $\vec{x}_0$ and write it as

$$D\vec{f}(\vec{x}_0) = A \ .$$

We can compare this definition to the single variable derivative (alternate) definition:

$$f'(x_0) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} \ .$$

We can get this limit definition of a derivative of a single-variable in a form similar to that of our definition above.

$$\begin{aligned}
f'(x_0) &= \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} \\
0 &= \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} - f'(x_0) \\
0 &= \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} - \frac{f'(x_0)(x-x_0)}{x-x_0} \\
0 &= \lim_{x\to x_0} \left| \frac{f(x)-f(x_0)-f'(x_0)(x-x_0)}{x-x_0} \right| \ .
\end{aligned}$$

Definition (Directional Derivative)

Let $\vec{f}: U \to \mathbb{R}^m$ and $\vec{x}_0 \in U$. Let $\vec{v}$ be a unit vector. The **directional derivative** of $\vec{f}$ at $\vec{x}_0$ in the direction $\vec{v}$ is

$$D_{\vec{v}}\vec{f}(\vec{x}_0) := \lim_{t\to 0} \frac{\vec{f}(\vec{x}_0 + t\vec{v}) - \vec{f}(\vec{x}_0)}{t} \in \mathbb{R}^m \ .$$

Note that $\vec{f}(\vec{x}_0 + t\vec{v}) - \vec{f}(\vec{x}_0) \in \mathbb{R}^m$.

Special Case of the Directional Derivative when $\mathbf{m = 1}$ (i.e. f is a scalar valued function).

In this case, we have that $D_{\vec{v}}f(\vec{x}_0)$ is a scalar. So, when

$$\vec{v} = \vec{\mathrm{e}}_1, \vec{\mathrm{e}}_2, \ldots, \vec{\mathrm{e}}_n$$

where $\vec{\mathrm{e}}_j = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ with 1 in the j^{th} position, then this derivative is the **partial derivative**. That is,

$$D_{\vec{\mathrm{e}}_j} f(\vec{x}_0) = \frac{\partial f}{\partial x_j}(\vec{x}_0) \ .$$

This is what we expect, since $\dfrac{\partial f}{\partial x_j}$ is the rate of change of $f(\vec{x}_0) = f(x_1, \ldots, x_n)$ with respect to x_j (holding all other variables constant).

For example, if $f(x,y) = xy^2$, then

$$\begin{aligned}
\frac{\partial f}{\partial x}(x,y) &= \frac{\partial}{\partial x} f(x,y) \\
&= \frac{\partial}{\partial x}\left[xy^2\right] \\
&= y^2 \cdot \frac{\partial}{\partial x}[x] \\
&= y^2 \cdot 1 \\
&= y
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial f}{\partial y}(x,y) &= \frac{\partial}{\partial y} f(x,y) \\
&= \frac{\partial}{\partial y}\left[xy^2\right] \\
&= x \cdot \frac{\partial}{\partial y}\left[y^2\right] \\
&= x \cdot 2y \\
&= 2xy\ .
\end{aligned}$$

Returning to the derivative $D\vec{f}$, we first write

$$\vec{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} ,$$

where $f_1, \ldots, f_m$ are scalar functions. Then each $D\vec{f_i}$ are $1 \times n$ matrices (i.e. rows), and so the derivative is

$$D\vec{f}(\vec{x}_0) = \begin{bmatrix} Df_1(\vec{x}_0) \\ \vdots \\ Df_m(\vec{x}_0) \end{bmatrix} .$$

So, it suffices to determine $Df(\vec{x}_0)$ when f is a **scalar-valued** function. In this case,

$$Df(\vec{x}_0) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(\vec{x}_0) & \ldots & \frac{\partial f}{\partial x_n}(\vec{x}_0) \end{bmatrix}$$

Theorem

Let $\vec{f}: U \to \mathbb{R}^m$, $U \subseteq \mathbb{R}^n$, $\vec{x}_0 \in U$, and $\vec{v}$ be a unit vector.

(1) If $\vec{f}$ is differentiable at $\vec{x}_0$, then all partials for $\vec{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}$ exist, and

$$D\vec{f}(\vec{x}_0) = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & & \frac{\partial f_m}{\partial x_n} \end{bmatrix}\right|_{\vec{x}_0} \tag{*}$$

(2) If $\vec{f}$ is differentiable at $\vec{x}_0$, then all directional derivatives exist and

$$D_{\vec{v}}\vec{f}(\vec{x}_0) = D\vec{f}(\vec{x}_0)\vec{v} \in \mathbb{R}^m \ , \tag{**}$$

where $D\vec{f}(\vec{x}_0)$ is $m \times n$ and $\vec{v}$ is $n \times 1$

Remarks

(i) (*) is often called the **Jacobian Matrix**.

(ii) For a scalar function f (codomain is $\mathbb{R}$), the vector

$$Df(\vec{x}_0)^{\mathsf{T}} = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix} \in \mathbb{R}^n$$

is called the **gradient** of f at $\vec{x}_0$, written as $\nabla f(\vec{x}_0)$.

(iii) The converses to (1) and (2) need not hold. Partial/directional derivatives **may** exist even if $\vec{f}$ is not differentiable.

> **Theorem**
>
> If all partials are themselves continuous, then $\vec{f}$ is differentiable, and therefore (*) and (**) hold.

Ex: Let $\vec{f}(x, y, z) = \begin{pmatrix} x^2 - yz \\ y - z^2 \end{pmatrix}$.

Note that the codomain is $\mathbb{R}^2$ since $m = 2$ (2D output). Let $\vec{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$, where

$$\begin{aligned} f_1 &= f_1(x, y, z) = x^2 - yz \\ f_2 &= f_2(x, y, z) = y - z^2 \ . \end{aligned}$$

Note that f_1 and f_2 are scalar-valued functions! Then it follows that

$$Df_1(x, y, z) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} & \dfrac{\partial f_1}{\partial z} \end{bmatrix} = \begin{bmatrix} 2x & -z & -y \end{bmatrix}$$

and

$$Df_2(x, y, z) = \begin{bmatrix} \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} & \dfrac{\partial f_2}{\partial z} \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2z \end{bmatrix} \ .$$

Then using the theorem, since all of these partials are continuous, this means that $\vec{f}$ is differentiable. Thus,

$$\begin{aligned} D\vec{f}(x, y, z) &= \begin{bmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} & \dfrac{\partial f_1}{\partial z} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} & \dfrac{\partial f_2}{\partial z} \end{bmatrix} \\ &= \begin{bmatrix} 2x & -z & -y \\ 0 & 1 & -2z \end{bmatrix} \ . \end{aligned}$$

For a directional derivative, suppose that

$$\vec{v} = \begin{pmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ -1/\sqrt{3} \end{pmatrix}$$

and $\vec{x}_0 = (3, 1, 4)$. (Note that $|\vec{v}|| = 1$.) Then by (**),

$$\begin{aligned}
D_{\vec{v}}\vec{f}(\vec{x}_0) &= D_{\vec{v}}\vec{f}(3,1,4) \\
&= D\vec{f}(3,1,4)\ \vec{v} \\
&= \begin{bmatrix} 2x & -z & -y \\ 0 & 1 & -2z \end{bmatrix}\Bigg|_{(3,1,4)} \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix} \\
&= \begin{bmatrix} 6 & -4 & -1 \\ 0 & 1 & -8 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix} \\
&= \begin{bmatrix} 11/\sqrt{3} \\ 7/\sqrt{3} \end{bmatrix}\ .
\end{aligned}$$

Remark: When $m = 1$,

$$\begin{aligned}
D_{\vec{v}}f(\vec{x}_0) &= Df(\vec{x}_0)\vec{v} \\
&= Df(\vec{x}_0)^{\mathsf{T}} \cdot \vec{v} \\
&= \nabla f(\vec{x}_0) \cdot \vec{v}\ .
\end{aligned}$$

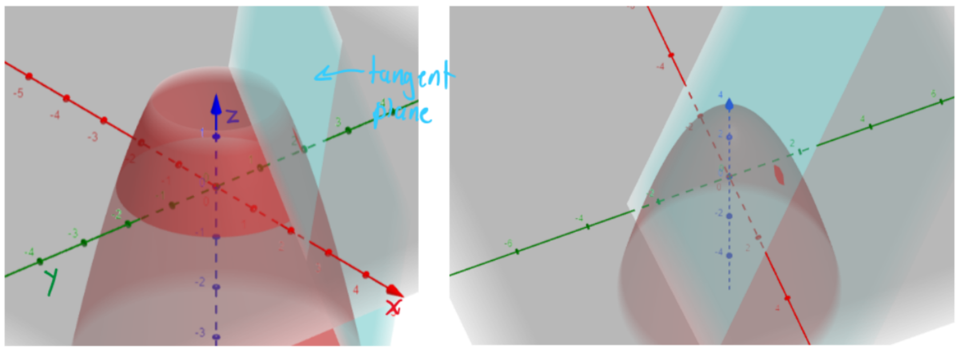
- Recall in the classic setting, the tangent line
$$L(x) = f(x_0) + f'(x_0) \cdot (x - x_0)$$
is the best possible approximation of $f(x)$ by a line near $x = x_0$.

- In the general case, the function $L : U \to \mathbb{R}^m$ given by
$$\vec{L}(\vec{x}) = \vec{f}(\vec{x}_0) + D\vec{f}(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0)$$
is the best possible (affine) approximation to $\vec{f}$ at $\vec{x}_0$. L is called the **tangent plane** or **linear approximation** or **linearization** for $\vec{f}$ at $\vec{x}_0$.

Ex: Let $f(x, y) = \sin\left(xy^2\right)$. Estimate $f(0.1, -0.2)$ using linear approximation.

Note that f is a scalar-valued function. Let $\vec{x}_0 = (0, 0)$ (this is close to $(0.1, -0.2)$). Then
$$\begin{aligned} L(\vec{x}) &= L(x, y) \\ &= f(\vec{x}_0) + Df(\vec{x}_0)(\vec{x} - \vec{x}_0) \\ &= f(0, 0) + Df(0, 0)((x, y) - (0, 0)) \\ &= f(0, 0) + Df(0, 0) \begin{bmatrix} x \\ y \end{bmatrix} . \end{aligned}$$
Now, we have that
$$Df(x, y) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} y^2 \cos\left(xy^2\right) & 2xy \cos\left(xy^2\right) \end{bmatrix} ,$$
which means that
$$Df(0, 0) = \begin{bmatrix} 0 & 0 \end{bmatrix} .$$
Thus,
$$\begin{aligned} L(\vec{x}) &= L(x, y) \\ &= f(0, 0) + Df(0, 0) \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \sin(0) + \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= 0 + 0 \\ &= 0 . \end{aligned}$$
In this case, $f(0.1, -0.2) \approx L(0.1, -0.2) = 0$. ($L$ is constant, unfortunately).

Now, let's consider the geometric meaning of the tangent plane when $n = 2$ and $m = 1$. That is, the function $f : U \to \mathbb{R}$, where $U \subseteq \mathbb{R}^2$.



In this case, we have that $z = f(x, y)$, and the gradient of f is given by

$$\nabla f = \nabla f(x,y) = \begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{bmatrix} .$$

The tangent plane at (x_0, y_0) is given by

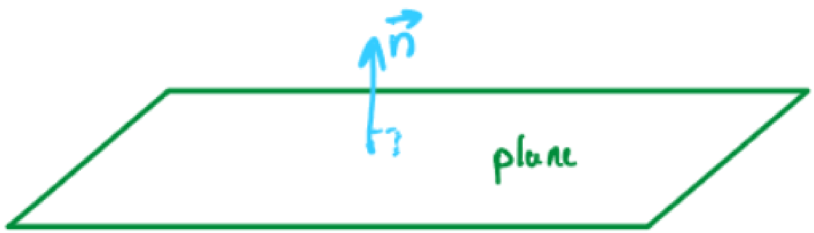
$$z = f(x_0, y_0) + \begin{bmatrix} f_x & f_y \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} .$$

This gives us the equation

$$\boxed{z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}$$

Recall that the normal to a plane of the form $ax + by + cz = d$ is

$$\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$



For the tangent plane, we can rearrange the equation for $f(x_0, y_0)$, which gives us

$$\begin{aligned} -f(x_0, y_0) &= f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - z \\ f(x_0, y_0) &= -f_x(x_0, y_0)(x - x_0) - f_y(x_0, y_0)(y - y_0) + z \ . \end{aligned}$$

This gives us the normal for a tangent plane:

$$\vec{n} = \begin{bmatrix} f_x \\ f_y \\ -1 \end{bmatrix} \text{ or } \begin{bmatrix} -f_x \\ -f_y \\ 1 \end{bmatrix}$$

1. Compute the partials at $(0,0)$ for

$$f(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0) \\ \dfrac{x^3}{x^2+y^2} & \text{otherwise} \end{cases} .$$

For which directions do the directional derivatives exist? Is f differentiable?

Answer: For the partials, we have that

$$\begin{aligned} f_x(0,0) &= \lim_{t\to 0} \frac{f(0+t,0) - f(0,0)}{t} \\ &= \lim_{t\to 0} \frac{f(t,0) - 0}{t} \\ &= \lim_{t\to 0} \frac{\frac{t^3}{t^2+0^2}}{t} \\ &= \lim_{t\to 0} \frac{\frac{t^3}{t^2}}{t} \\ &= \lim_{t\to 0} \frac{t^3}{t^3} \\ &= \lim_{t\to 0} 1 \\ &= 1 \ . \end{aligned}$$

Note: this is the definition of the directional derivative of f with direction vector $\vec{v} = (1,0)$ at $\vec{x}_0 = (0,0)$. Indeed, we can check this.

$$\begin{aligned} D_{\vec{v}}\vec{f}(0,0) &= \lim_{t\to 0} \frac{f((0,0)+t(1,0)) - \vec{f}(0,0)}{t} \\ &= \lim_{t\to 0} \frac{f((0,0)+(t,0)) - 0}{t} \\ &= \lim_{t\to 0} \frac{f(t,0)}{t} \\ &= \lim_{t\to 0} \frac{\frac{t^3}{t^2+0^2}}{t} \\ &= \lim_{t\to 0} \frac{\frac{t^3}{t^2}}{t} \\ &= \lim_{t\to 0} \frac{t^3}{t^3} \\ &= \lim_{t\to 0} 1 \end{aligned}$$

$$= 1 \ .$$

Also,

$$\begin{aligned} f_y(0,0) &= \lim_{t\to 0} \frac{f(0,0+t) - f(0,0)}{t} \\ &= \lim_{t\to 0} \frac{f(0,t) - 0}{t} \\ &= \lim_{t\to 0} \frac{\dfrac{0^3}{0^2+t^2}}{t} \\ &= \lim_{t\to 0} \frac{0}{t} \\ &= \lim_{t\to 0} 0 \\ &= 0 \ . \end{aligned}$$

So, if f were differentiable at $(0,0)$, we must then have that

$$Df(0,0) = \begin{bmatrix} f_x(0,0) & f_y(0,0) \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \ .$$

Now, for directional derivatives, we must use the limit definition since we do not know if f is differentiable. So, let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be a unit vector such that $v_1^2 + v_2^2 = 1$. Then

$$\begin{aligned} D_{\vec{v}} f(0,0) &= \lim_{t\to 0} \frac{f((0,0) + t\vec{v}) - f(0,0)}{t} \\ &= \lim_{t\to 0} \frac{f((0,0) + t(v_1, v_2)) - 0}{t} \\ &= \lim_{t\to 0} \frac{f((0,0) + (tv_1, tv_2))}{t} \\ &= \lim_{t\to 0} \frac{f(tv_1, tv_2)}{t} \\ &= \lim_{t\to 0} \frac{\dfrac{(tv_1)^3}{(tv_1)^2 + (tv_2)^2}}{t} \\ &= \lim_{t\to 0} \frac{\dfrac{t^3 v_1^3}{t^2 v_1^2 + t^2 v_2^2}}{t} \\ &= \lim_{t\to 0} \frac{t^3 v_1^3}{t\left(t^2 v_1^2 + t^2 v_2^2\right)} \\ &= \lim_{t\to 0} \frac{t^3 v_1^3}{t^3 v_1^2 + t^3 v_2^2} \\ &= \lim_{t\to 0} \frac{t^3 v_1^3}{t^3\left(v_1^2 + v_2^2\right)} \end{aligned}$$

$$
\begin{aligned}
&= \lim_{t\to 0} \frac{v_1^3}{v_1^2+v_2^2} \\
&= \lim_{t\to 0} \frac{v_1^3}{1} \\
&= \lim_{t\to 0} v_1^3 \\
&= v_1^3 \ .
\end{aligned}
$$

So, we get that $D_{\vec{v}} f(0,0)$ exists for any choice of $\vec{v}$. Now, since all directional derivatives exist, it **may** be the case that f is differentiable at $(0,0)$. If f were differentiable at $(0,0)$, then we would have that

$$
\begin{aligned}
0 &= \lim_{(x,y)\to(0,0)} \frac{\left| f(x,y) - f(0,0) - Df(0,0)\begin{bmatrix} x-0 \\ y-0 \end{bmatrix} \right|}{\sqrt{x^2+y^2}} \\
&= \lim_{(x,y)\to(0,0)} \frac{\left| f(x,y) - f(0,0) - \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \right|}{\sqrt{x^2+y^2}} \\
&= \lim_{(x,y)\to(0,0)} \frac{\left| \dfrac{x^3}{x^2+y^2} - 0 - (x+0y) \right|}{\sqrt{x^2+y^2}} \\
&= \lim_{(x,y)\to(0,0)} \frac{\left| \dfrac{x^3}{x^2+y^2} - x \right|}{\sqrt{x^2+y^2}} \\
&= \lim_{(x,y)\to(0,0)} \frac{\left| \dfrac{x^3}{x^2+y^2} - \dfrac{x(x^2+y^2)}{x^2+y^2} \right|}{\sqrt{x^2+y^2}} \\
&= \lim_{(x,y)\to(0,0)} \frac{\left| \dfrac{x^3}{x^2+y^2} - \dfrac{x^3+xy^2}{x^2+y^2} \right|}{\sqrt{x^2+y^2}} \\
&= \lim_{(x,y)\to(0,0)} \frac{\left| \dfrac{-xy^2}{x^2+y^2} \right|}{\left(x^2+y^2\right)^{1/2}} \\
&= \lim_{(x,y)\to(0,0)} \frac{\left|-xy^2\right|}{\left(x^2+y^2\right)^{3/2}} \\
&= \lim_{(x,y)\to(0,0)} \frac{|x|\, y^2}{\left(x^2+y^2\right)^{3/2}} \ ,
\end{aligned}
$$

and so no matter which path to $(0,0)$ we take, the answer should be 0. Along $y=0$ and $x \to 0^+$, we get that

$$
\lim_{x\to 0^+} \frac{|x|\cdot 0}{\left(x^2\right)^{3/2}} = 0 \ .
$$

Along $x = y$ and $x \to 0^+$, we get that

$$\begin{aligned}
\lim_{x \to 0^+} \frac{|x| \cdot x^2}{\left(x^2 + x^2\right)^{3/2}} &= \lim_{x \to 0^+} \frac{x \cdot x^2}{\left(2x^2\right)^{3/2}} \\
&= \lim_{x \to 0^+} \frac{x^3}{\left(2x^2\right)^{3/2}} \\
&= \lim_{x \to 0^+} \frac{x^3}{2^{3/2} \cdot x^3} \\
&= \lim_{x \to 0^+} \frac{1}{2^{3/2}} \\
&\neq 0 \ .
\end{aligned}$$

Thus, since this is not 0, f is not differentiable at $(0, 0)$.

2. Compute the partials at $(0,0)$ for

$$f(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0) \\ \dfrac{x^2y^2}{\sqrt{x^2+y^2}} & \text{otherwise} \end{cases}$$

and conclude that f is differentiable.

Answer: We have that

$$\begin{aligned} f_x(0,0) &= \lim_{t\to 0} \frac{f(0+t,0) - f(0,0)}{t} \\ &= \lim_{t\to 0} \frac{f(t,0) - 0}{t} \\ &= \lim_{t\to 0} \frac{\frac{t^2 \cdot 0^2}{\sqrt{t^2+0^2}}}{t} \\ &= \lim_{t\to 0} \frac{0}{t} \\ &= \lim_{t\to 0} 0 \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} f_y(0,0) &= \lim_{t\to 0} \frac{f(0,0+t) - f(0,0)}{t} \\ &= \lim_{t\to 0} \frac{f(0,t) - 0}{t} \\ &= \lim_{t\to 0} \frac{\frac{0^2 \cdot t^2}{\sqrt{0^2+t^2}}}{t} \\ &= \lim_{t\to 0} \frac{0}{t} \\ &= \lim_{t\to 0} 0 \\ &= 0\ . \end{aligned}$$

So, if f were differentiable at $(0,0)$, then we must have that

$$Df(0,0) = \begin{bmatrix} f_x(0,0) & f_y(0,0) \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}\ .$$

Now, we check if $Df(0,0) = \begin{bmatrix} 0 & 0 \end{bmatrix}$ exists.

$$\begin{aligned} 0 &= \lim_{(x,y)\to(0,0)} \frac{|f(x,y) - f(0,0) - Df(0,0)(x-0,y-0)|}{||(x-0,y-0)||} \\ &= \lim_{(x,y)\to(0,0)} \frac{|f(x,y) - 0 - Df(0,0)(x,y)|}{||(x,y)||} \end{aligned}$$

$$
\begin{aligned}
&= \lim_{(x,y)\to(0,0)} \frac{\left| f(x,y) - \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right|}{\sqrt{x^2+y^2}} \\
&= \lim_{(x,y)\to(0,0)} \frac{|f(x,y) - (0x+0y)|}{\sqrt{x^2+y^2}} \\
&= \lim_{(x,y)\to(0,0)} \frac{|f(x,y)|}{\sqrt{x^2+y^2}} \\
&= \lim_{(x,y)\to(0,0)} \frac{\left| \frac{x^2y^2}{\sqrt{x^2+y^2}} \right|}{\sqrt{x^2+y^2}} \\
&= \lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2} \ .
\end{aligned}
$$

Now, one tool we can use is the inequality

$$
\begin{aligned}
(x-y)^2 &\geq 0 \\
x^2 - 2xy + y^2 &\geq 0 \\
x^2 + y^2 &\geq 2xy \\
\frac{x^2+y^2}{2} &\geq xy \ .
\end{aligned}
$$

With this, we get that

$$
\begin{aligned}
0 &= \lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2} \\
&= \lim_{(x,y)\to(0,0)} \frac{(xy)^2}{x^2+y^2} \\
&\leq \lim_{(x,y)\to(0,0)} \frac{\left(\frac{x^2+y^2}{2}\right)^2}{x^2+y^2} \\
&= \lim_{(x,y)\to(0,0)} \frac{\frac{\left(x^2+y^2\right)^2}{4}}{x^2+y^2} \\
&= \lim_{(x,y)\to(0,0)} \frac{\left(x^2+y^2\right)^2}{4\left(x^2+y^2\right)} \\
&= \frac{1}{4} \cdot \lim_{(x,y)\to(0,0)} \left(x^2+y^2\right) \\
&= 0 \ .
\end{aligned}
$$

By the squeeze theorem, this limit is 0.

Directions of Min/Max Growth

Suppose $f : U \to \mathbb{R}$ where U is either in $\mathbb{R}^2$ or $\mathbb{R}^3$ (note that f is a scalar function). Assume f is differentiable. Let the gradient of f be

$$\nabla f = \begin{bmatrix} f_x \\ f_y \end{bmatrix} \quad (\text{in } \mathbb{R}^2)$$

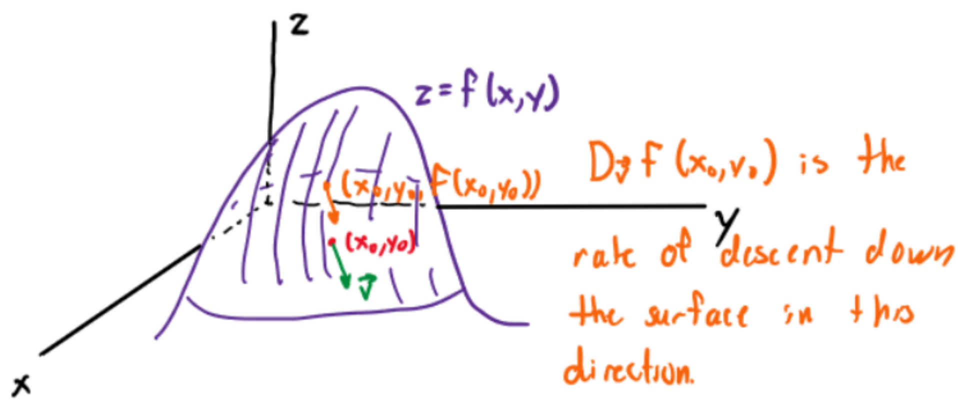
or

$$\nabla f = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} \quad (\text{in } \mathbb{R}^3)\ .$$

Since f is differentiable, we know that for any direction $\vec{v}$, the directional derivative is given by

$$D_{\vec{v}} f(x, y) = \nabla f \cdot \vec{v}\ .$$

(Gives the rate of change of $f(x, y)$ with respect to the direction $\vec{v}$, along the surface of the function.)

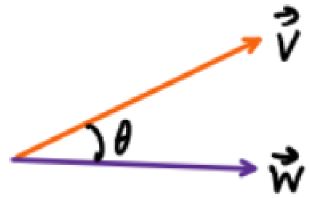


This can be thought of as descending down the "mountain" in the direction $\vec{v}$.

Question: What are the maximal/minimal possible values for $\nabla f \cdot \vec{v}$ at (x_0, y_0) (or (x_0, y_0, z_0) in the case of 3-dimensions) by allowing free selection for $\vec{v}$?

Recall that given vectors $\vec{v}$ and $\vec{w}$, the induced angle θ separating them is given by

$$\cos\theta = \frac{\vec{v}\cdot\vec{w}}{||\vec{v}||\cdot||\vec{w}||}$$



- $\cos(\theta)$ is maximized when $\theta = 0$ (so $\vec{v}$ and $\vec{w}$ are parallel in the same direction). In this case, $\cos\theta = 1$.

- $\cos\theta$ is minimized when $\theta = \pi$ (i.e. $\cos\theta = -1$). So, $\vec{v}$ and $\vec{w}$ are parallel in opposite directions (aka antiparallel).

- If $\vec{v}$ is a unit vector (that is, $|\vec{v}| = 1$), then
$$\frac{\nabla f\cdot\vec{v}}{||\nabla f||\cdot||\vec{v}||} = \frac{\nabla f\cdot\vec{v}}{||\nabla f||\cdot 1} = \frac{\nabla f\cdot\vec{v}}{||\nabla f||} = 1$$
is maximized when ∇f and $\vec{v}$ are in the same direction.

- So $\vec{v}$ is a positive multiple of the gradient ∇f.

- From this, we get that
$$||\nabla f|| = \nabla f\cdot\vec{v}$$
(so the maximal direction is given by $||\nabla f||$).

- Moreover, since $\vec{v} = c\nabla f$, ($\vec{v}$ is some positive multiple of the gradient), we get that
$$\begin{aligned} ||\nabla f|| &= \nabla f\cdot c\nabla f \\ &= c\,(\nabla f\cdot\nabla f) \\ &= c||\nabla f||^2\ , \end{aligned}$$
which implies that
$$c = \frac{||\nabla f||}{||\nabla f||^2} = \frac{1}{\nabla f}\ .$$

In conclusion,

- The maximal directional derivative is $||\nabla f||$ and it occurs in the direction $\frac{\nabla f}{||\nabla f||}$ (the normalized gradient).

- So the gradient itself represents the direction of maximal ascent on the graph of f at (x_0, y_0).

- Similarly at $\theta = \pi$,

  - The minimal directional derivative is $-||\nabla f||$ occuring at $\vec{v} = -\frac{\nabla f}{||\nabla f||}$.

1. Find the maximal/minimal directional derivatives for

$$f(x,y,z) = \frac{x+y+z}{x^2+y^2+z^2}$$

at $(x_0, y_0, z_0) = (1, -1, 0)$.

Answer: For the partial of f with respect to x, we have that

$$\begin{aligned}
f_x &= \frac{\partial f}{\partial x} \\
&= \frac{\partial}{\partial x}\left[f(x,y,z)\right] \\
&= \frac{\partial}{\partial x}\left[\frac{x+y+z}{x^2+y^2+z^2}\right] \\
&= \frac{1\cdot\left(x^2+y^2+z^2\right) - \left((x+y+z)\cdot 2x\right)}{\left(x^2+y^2+z^2\right)^2} \\
&= \frac{\left(x^2+y^2+z^2\right) - 2x(x+y+z)}{\left(x^2+y^2+z^2\right)^2}
\end{aligned}$$

Similarly,

$$f_y = \frac{\partial f}{\partial y} = \frac{\left(x^2+y^2+z^2\right) - 2y(x+y+z)}{\left(x^2+y^2+z^2\right)^2}$$

and

$$f_z = \frac{\partial f}{\partial z} = \frac{\left(x^2+y^2+z^2\right) - 2z(x+y+z)}{\left(x^2+y^2+z^2\right)^2}\ .$$

Then

$$\nabla f(x,y,z) = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \begin{bmatrix} \dfrac{\left(x^2+y^2+z^2\right) - 2x(x+y+z)}{\left(x^2+y^2+z^2\right)^2} \\ \dfrac{\left(x^2+y^2+z^2\right) - 2y(x+y+z)}{\left(x^2+y^2+z^2\right)^2} \\ \dfrac{\left(x^2+y^2+z^2\right) - 2z(x+y+z)}{\left(x^2+y^2+z^2\right)^2} \end{bmatrix}$$

So,

$$\nabla f(1,-1,0) = \begin{bmatrix} \dfrac{\left((1)^2+(-1)^2+(0)^2\right) - 2(1)(1+(-1)+0)}{\left((1)^2+(-1)^2+(0)^2\right)^2} \\ \dfrac{\left((1)^2+(-1)^2+(0)^2\right) - 2(-1)(1+(-1)+0)}{\left((1)^2+(-1)^2+(0)^2\right)^2} \\ \dfrac{\left((1)^2+(-1)^2+(0)^2\right) - 2(0)(1+(-1)+0)}{\left((1)^2+(-1)^2+(0)^2\right)^2} \end{bmatrix}$$

$$
\begin{aligned}
&= \begin{bmatrix} \dfrac{(1+1+0)-2(0)}{(1+1+0)^2} \\ \dfrac{(1+1+0)+2(0)}{(1+1+0)^2} \\ \dfrac{(1+1+0)-0}{(1+1+0)^2} \end{bmatrix} \\
&= \begin{bmatrix} \dfrac{2-0}{(2)^2} \\ \dfrac{2+0}{(2)^2} \\ \dfrac{2-0}{(2)^2} \end{bmatrix} \\
&= \begin{bmatrix} 2/4 \\ 2/4 \\ 2/4 \end{bmatrix} \\
&= \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad .
\end{aligned}
$$

So, the maximal rate of change of f at $(1, -1, 0)$ is

$$
\begin{aligned}
||\nabla f(1,-1,0)|| &= \frac{1}{2} \left|\left| \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right|\right| \\
&= \frac{1}{2} \cdot \sqrt{1+1+1} \\
&= \frac{1}{2} \cdot \sqrt{3} \\
&= \frac{\sqrt{3}}{2}
\end{aligned}
$$

in the direction

$$
\begin{aligned}
\frac{\nabla f(1,-1,0)}{||\nabla f(1,-1,0)||} &= \frac{1}{||\nabla f(1,-1,0)||} \cdot \nabla f(1,-1,0) \\
&= \frac{1}{\sqrt{3}/2} \cdot \left(\frac{1}{2} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) \\
&= \frac{2}{\sqrt{3}} \cdot \frac{1}{2} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}
\end{aligned}
$$

$$= \frac{1}{\sqrt{3}} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} .$$

The minimal rate of change of f at $(1, -1, 0)$ is

$$-||\nabla f(1, -1, 0)|| = -\frac{\sqrt{3}}{2}$$

in the direction

$$-\left(\frac{\nabla f(1, -1, 0)}{||\nabla f(1, -1, 0)||} \right) = - \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = \begin{bmatrix} -1/\sqrt{3} \\ -1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix} .$$

2. Suppose the maximal rate of change for $f(x, y, z)$ at a point (x_0, y_0, z_0) is $3\sqrt{6}$ and occurs in the direction $\vec{v} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$. (Note that $\vec{v}$ is NOT a unit vector, as $||\vec{v}|| \neq 1$.)

   (a) Find $\nabla f(x_0, y_0, z_0)$.

   We know that the maximal rate of change for f at (x_0, y_0, z_0) is

   $$||\nabla f(x_0, y_0, z_0)|| = 3\sqrt{6} \ .$$

   Since the maximal rate of change occurs in the direction given by $\vec{v}$, and the gradient ∇f is the direction of steepest ascent, it follows that

   $$\frac{\vec{v}}{||\vec{v}||} = \frac{\nabla f}{||\nabla f||} \ .$$

   So, we have that

   $$||\vec{v}|| = \sqrt{4 + 1 + 1} = \sqrt{6} \ .$$

   Now, since $||\nabla f|| = 3\sqrt{6}$, it follows that

   $$\begin{aligned} \nabla f &= \frac{||\nabla f(x_0, y_0, z_0)|| \cdot \vec{v}}{||\vec{v}||} \\ &= \frac{||\nabla f(x_0, y_0, z_0)||}{||\vec{v}||} \cdot \vec{v} \\ &= \frac{3\sqrt{6}}{\sqrt{6}} \cdot \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \\ &= 3 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 6 \\ -3 \\ 3 \end{bmatrix} \ . \end{aligned}$$

(b) Find $D_{\vec{v}} f(x_0, y_0, z_0)$ for $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

Note that we do not need to know (x_0, y_0, z_0). Also note that $\vec{v}$ is NOT a unit vector. So, using the normalized version of $\vec{v}$, we get that

$$\begin{aligned}
D_{\vec{v}} f(x_0, y_0, z_0) &= \nabla f(x_0, y_0, z_0) \cdot \frac{\vec{v}}{||\vec{v}||} \\
&= \nabla f(x_0, y_0, z_0) \cdot \left(\frac{1}{||\vec{v}||} \cdot \vec{v} \right) \\
&= \begin{bmatrix} 6 \\ -3 \\ 3 \end{bmatrix} \cdot \left(\frac{1}{\sqrt{1+1+0}} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right) \\
&= \begin{bmatrix} 6 \\ -3 \\ 3 \end{bmatrix} \cdot \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right) \\
&= \begin{bmatrix} 6 \\ -3 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \\
&= \frac{6}{\sqrt{2}} + \left(-\frac{3}{\sqrt{2}} \right) + 0 \\
&= \frac{6}{\sqrt{2}} - \frac{3}{\sqrt{2}} \\
&= \frac{3}{\sqrt{2}} \,.
\end{aligned}$$

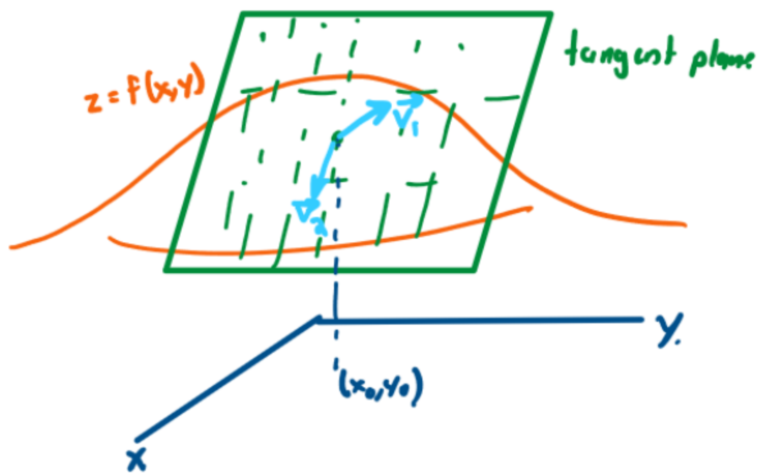
3. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable and let

$$D_{\vec{v}_1} f(x_0, y_0) = \sqrt{5} \quad \text{and} \quad D_{\vec{v}_2} f(x_0, y_0) = -2 \ ,$$

where

$$\vec{v}_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} -3 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \ .$$

Determine $D_{\vec{v}} f(x_0, y_0)$, where $\vec{v} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$



We have the following property of directional derivatives:

$$\begin{aligned} D_{c_1\vec{v}_1 + c_2\vec{v}_2} f(x_0, y_0) &= \nabla f(x_0, y_0) \cdot (c_1\vec{v}_1 + c_2\vec{v}_2) \\ &= c_1 \nabla f(x_0, y_0) \cdot \vec{v}_1 + c_2 \nabla f(x_0, y_0) \cdot \vec{v}_2 \\ &= c_1 \left(\nabla f(x_0, y_0) \cdot \vec{v}_1 \right) + c_2 \left(\nabla f(x_0, y_0) \cdot \vec{v}_2 \right) \\ &= c_1 \left(D_{\vec{v}_1} f(x_0, y_0) \right) + c_2 \left(D_{\vec{v}_2} f(x_0, y_0) \right) \\ &= c_1 D_{\vec{v}_1} f(x_0, y_0) + c_2 D_{\vec{v}_2} f(x_0, y_0) \ . \end{aligned}$$

Now, if we can find c_1 and c_2 such that

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 \ ,$$

then we are done. That is, we solve the system of equations for c_1 and c_2 given by

$$\begin{aligned} \vec{v} &= c_1\vec{v}_1 + c_2\vec{v}_2 \\ \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} &= c_1 \begin{bmatrix} -3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix} + c_2 \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \ . \end{aligned}$$

So, we have that

$$\left[\begin{array}{cc|c} -3/\sqrt{10} & 1/\sqrt{2} & 1/\sqrt{5} \\ 1/\sqrt{10} & -1/\sqrt{2} & 2/\sqrt{5} \end{array}\right]$$

$$\xrightarrow{R1<-->R2} \left[\begin{array}{cc|c} 1/\sqrt{10} & -1/\sqrt{2} & 2/\sqrt{5} \\ -3/\sqrt{10} & 1/\sqrt{2} & 1/\sqrt{5} \end{array}\right]$$

$$\xrightarrow{\sqrt{10}R1} \left[\begin{array}{cc|c} 1 & -\sqrt{10}/\sqrt{2} & 2\sqrt{10}/\sqrt{5} \\ -3/\sqrt{10} & 1/\sqrt{2} & 1/\sqrt{5} \end{array}\right]$$

$$\vdots$$

So, we get that $c_1 = -\frac{3}{\sqrt{2}}$ and $c_2 = -\frac{7}{\sqrt{10}}$ are solutions to this system of equations. Then plugging these back in to the derived directional derivative formula above, we get that

$$\begin{aligned} D_{c_1\vec{v}_1+c_2\vec{v}_2} f(x_0, y_0) &= c_1 D_{\vec{v}_1} f(x_0, y_0) + c_2 D_{\vec{v}_2} f(x_0, y_0) \\ &= -\frac{3}{\sqrt{2}} \cdot \sqrt{5} + \left(\left(-\frac{7}{\sqrt{10}} \right) \cdot (-2) \right) \\ &= -\frac{3\sqrt{5}}{\sqrt{2}} + \frac{14}{\sqrt{10}} . \end{aligned}$$