MATH 367 - Week 5 Notes

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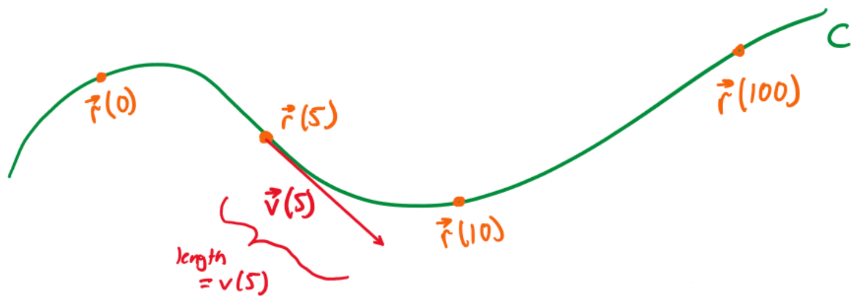
Parametric Curves Continued

Basic Definitions

The standard physical interpretation of a curve C is the motion of a particle along a curve C with position function $\vec{r}(t)$.



Definition (Velocity)

The **velocity** at t is

$$\vec{v}(t) = \vec{r}\,'(t) = D\vec{r}(t) = \begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} .$$

Definition (Speed)

The **speed** is given by

$$v(t) = ||\vec{v}(t)|| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} .$$

Definition (Arc-Length)

The arc-length from $t = t_0$ to $t = t_1$ is given by

$$\int_{t_0}^{t_1} v(t)\ dt .$$

This is essentially the total **distance** travelled by the particle from t_0 to

t_1. From initial time t_0, this can be written as

$$s(\mathsf{T}) = \int_{t_0}^{\mathsf{T}} v(t)\ dt\ .$$

Definition (Tangent Vector)

The **tangent vector at t**

$$\vec{T}(t) = \frac{\vec{v}(t)}{v(t)} = \frac{1}{v(t)}\vec{v}(t)\ .$$

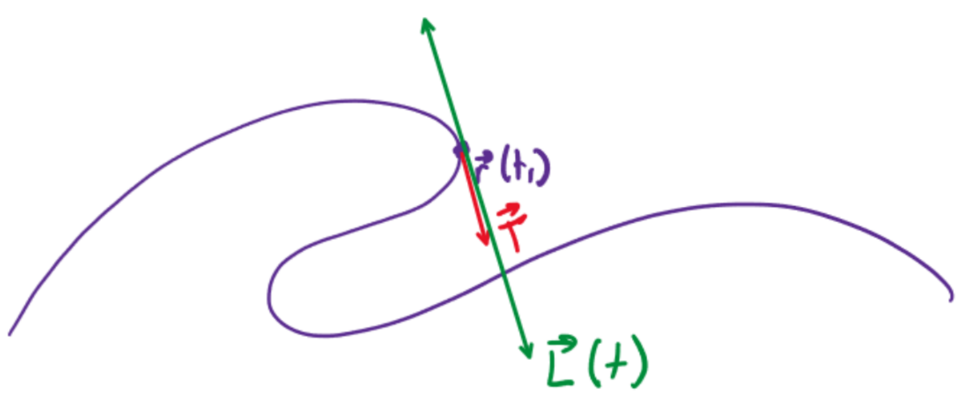
is a **unit vector** expressing the direction of travel. Notice that this is the normalized veclocity vector!

Definition (Tangent Line)

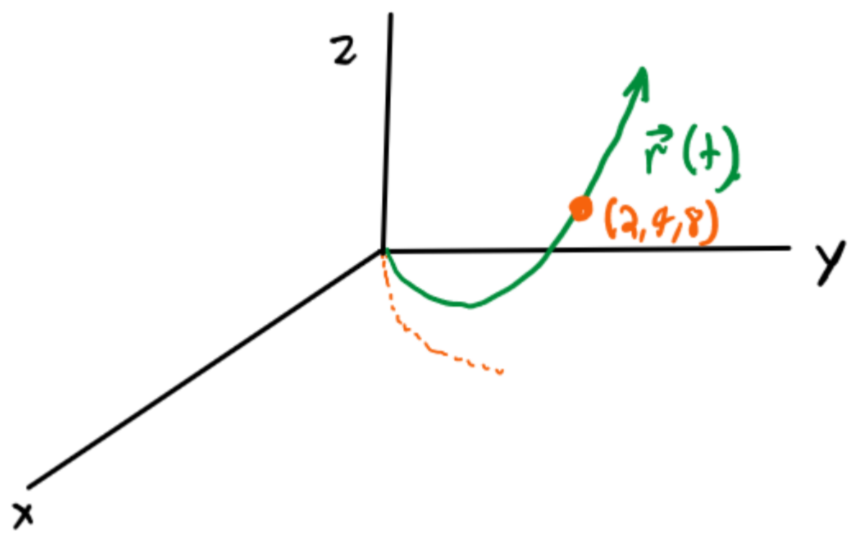
The **tangent line at $\mathbf{t_1}$** is the line with direction $\vec{T}(t_1)$ containing the point $\vec{r}(t_1)$, given by

$$\vec{L}(t) = \vec{r}(t_1) + t\vec{T}(t_1)\ .$$

(Notice that this is essentially the vector form equation of a line!)



1. Find the velocity, speed, tangent vector, and tangent line for $\vec{r}(t) = \begin{bmatrix} t \\ t^2 \\ t^3 \end{bmatrix}$ at the point $(2, 4, 8)$. Note that the point $(2, 4, 8)$ corresponds to when $t = 2$.



Let $x(t) = t$, $y(t) = t^2$, and $z(t) = t^3$. Then $\vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$. So, the velocity vector is

$$\begin{aligned} \vec{v}(t) &= \vec{r}\,'(t) \\ &= \begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{d}{dt}[x(t)] \\ \dfrac{d}{dt}[y(t)] \\ \dfrac{d}{dt}[z(t)] \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{d}{dt}[t] \\ \dfrac{d}{dt}\left[t^2\right] \\ \dfrac{d}{dt}\left[t^3\right] \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} 1 \\ 2t \\ 3t^2 \end{bmatrix} .$$

Then the velocity at $t = 2$ is

$$\vec{v}(2) = \begin{bmatrix} 1 \\ 2(2) \\ 3(2)^2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 12 \end{bmatrix} .$$

So, the speed at $t = 2$ is given by

$$\begin{aligned} v(2) &= ||\vec{v}(2)|| \\ &= \sqrt{1^2 + 4^2 + 12^2} \\ &= \sqrt{1 + 16 + 144} \\ &= \sqrt{161} . \end{aligned}$$

Now, the tangent vector at $t = 2$ is

$$\begin{aligned} \vec{T}(2) &= \frac{\vec{v}(2)}{v(2)} \\ &= \frac{1}{v(2)} \cdot \vec{v}(2) \\ &= \frac{1}{\sqrt{161}} \begin{bmatrix} 1 \\ 4 \\ 12 \end{bmatrix} . \end{aligned}$$

Hence, the tangent line at $t = 2$ is given by

$$\begin{aligned} \vec{L}(t) &= \vec{r}(2) + l\vec{T}(2) \\ &= \begin{bmatrix} 2 \\ 2^2 \\ 2^3 \end{bmatrix} + l \left(\frac{1}{\sqrt{161}} \begin{bmatrix} 1 \\ 4 \\ 12 \end{bmatrix} \right) \\ &= \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix} + \frac{l}{\sqrt{161}} \begin{bmatrix} 1 \\ 4 \\ 12 \end{bmatrix} , \end{aligned}$$

where $l \in \mathbb{R}$.

2. Find the arc-length for $\vec{r}(t) = \begin{bmatrix} \frac{1}{3}t^3 & \sqrt{2}t & -\frac{1}{t} \end{bmatrix}^\top$ from $t = 1$ to $t = 2$.

Recall that the arc-length from t_0 to t_1 is given by

$$\int_{t_0}^{t_1} v(t)\ dt\ .$$

We have that the velocity

$$\vec{v}(t) = \vec{r}\,'(t) = \begin{bmatrix} t^2 \\ \sqrt{2} \\ \frac{1}{t^2} \end{bmatrix}\ ,$$

which means that the speed is

$$\begin{aligned} v(t) &= ||\vec{v}(t)|| \\ &= \sqrt{\left(t^2\right)^2 + \left(\sqrt{2}\right)^2 + \left(\frac{1}{t^2}\right)^2} \\ &= \sqrt{t^4 + 2 + \frac{1}{t^4}}\ . \end{aligned}$$

So, the arc-length of $\vec{r}(t)$ from $t = 1$ to $t = 2$ is

$$\begin{aligned} \int_1^2 v(t)\ dt &= \int_1^2 \sqrt{t^4 + 2 + \frac{1}{t^4}}\ dt \\ &= \int_1^2 \sqrt{\left(t^2 + \frac{1}{t^2}\right)^2}\ dt \\ &= \int_1^2 \left(t^2 + \frac{1}{t^2}\ dt\right)\ . \end{aligned}$$

Now, since

$$\begin{aligned} \int \left(t^2 + \frac{1}{t^2}\ dt\right) &= \int t^2\ dt + \int \frac{1}{t^2}\ dt \\ &= \left(\frac{t^{2+1}}{2+1} + C_1\right) + \left(\frac{t^{-2+1}}{-2+1} + C_2\right) \\ &= \left(\frac{t^3}{3} + C_1\right) + \left(\frac{t^{-1}}{-1} + C_2\right) \\ &= \left(\frac{1}{3}t^3 + C_1\right) + \left(-t^{-1} + C_2\right) \\ &= \left(\frac{1}{3}t^3 + C_1\right) + \left(-\frac{1}{t} + C_2\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{3}t^3 + C_1 - \frac{1}{t} - C_2 \\
&= \frac{1}{3}t^3 - \frac{1}{t} + C_1 - C_2 \\
&= \frac{1}{3}t^3 - \frac{1}{t} + C \ ,
\end{aligned}
$$

where $C = C_1 - C_2$ is some constant. Thus,

$$
\begin{aligned}
\int_1^2 v(t)\ dt &= \int_1^2 \left(t^2 + \frac{1}{t^2}\ dt\right) \\
&= \left[\frac{1}{3}t^3 - \frac{1}{t} + C\right]_{t=1}^2 \\
&= \left(\frac{1}{3}(2)^3 - \frac{1}{2} + C\right) - \left(\frac{1}{3}(1)^3 - \frac{1}{1} + C\right) \\
&= \left(\frac{8}{3} - \frac{1}{2} + C\right) - \left(\frac{1}{3} - 1 + C\right) \\
&= \left(\frac{16}{6} - \frac{3}{6} + C\right) - \left(\frac{1}{3} - \frac{3}{3} + C\right) \\
&= \left(\frac{13}{6} + C\right) - \left(-\frac{2}{3} + C\right) \\
&= \frac{13}{6} + C + \frac{2}{3} - C \\
&= \frac{13}{6} + \frac{2}{3} + C - C \\
&= \frac{13}{6} + \frac{4}{6} \\
&= \frac{17}{6}
\end{aligned}
$$

is the arc length of $\vec{r}(t)$ from $t = 1$ to $t = 2$.

Definition (Acceleration)

The **acceleration** of a parametric curve $\vec{r}(t)$ is

$$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)\ .$$

Product Rule for Curves

Suppose $\vec{r}_1$ and $\vec{r}_2$ are parametric curves and f is a scalar function. Then

(i) $(f(t)\vec{r}_1(t))' = f'(t)\vec{r}_1(t) + f(t)\vec{r}_1{}'(t)$

(ii) $(\vec{r}_1 \cdot \vec{r}_2)' = \vec{r}_1{}' \cdot \vec{r}_2 + \vec{r}_1 \cdot \vec{r}_2{}'$

Note that (ii) is the dot product of two vectors!

1. Show that $\vec{T}$ is orthogonal to $\vec{T}\,'$. (Note that it is not always the case that $\vec{v} \perp \vec{a}$.)

   Recall that $\vec{T} = \dfrac{\vec{v}}{v}$ is the tangent vector. So, $\vec{T}$ is a unit vector, which means that

   $$1 = ||\vec{T}||^2 = \left(\sqrt{\vec{T} \cdot \vec{T}}\,\right)^2 = \vec{T} \cdot \vec{T}\ .$$

   Then differentiating both sides with respect to t gives

   $$\begin{aligned} 0 &= \vec{T} \cdot \vec{T}\,' + \vec{T}\,' \cdot \vec{T} \\ &= \vec{T} \cdot \vec{T}\,' + \vec{T} \cdot \vec{T}\,' \\ &= 2\left(\vec{T} \cdot \vec{T}\,'\right)\ , \end{aligned}$$

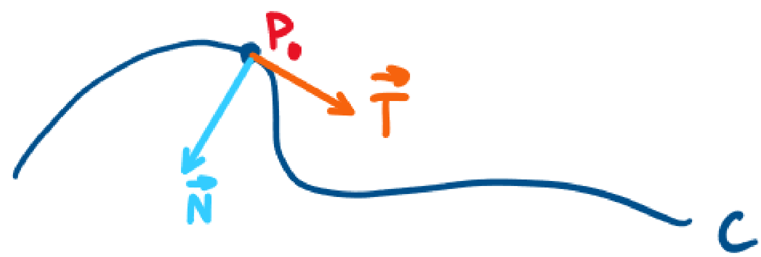
   which means that $\vec{T} \cdot \vec{T}\,' = 0$.

Definition (Normal Vector)

For a parametric curve $\vec{r}(t)$, the **normal vector** $\vec{N}$ is given by

$$\vec{N} = \frac{\vec{T}\,'}{||\vec{T}\,'||} \ .$$

(That is, $\vec{N}$ is the normalization of $\vec{T}\,'$). $\vec{N}$ is perpendicular to $\vec{T}$ and points in the direction of "turn" of a particle along the curve.



Remark: At a fixed point P_0 on a curve C, $\vec{T}$ and $\vec{N}$ do not depend on the choice of $\vec{r}(t)$.

Definition (Curvature and Radius of Curvature)

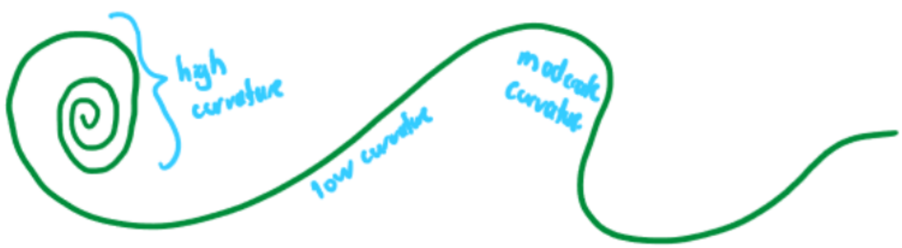
Given a curve C, the **curvature** at some point P_0 is given by

$$\kappa := \left\| \frac{d\vec{T}}{ds} \right\| \ .$$

In other words, the curvature at some point P_0 is the magnitude of the derivative of the tangent vector with respect to the arc length, where the arc-length from, say t_0 to t, is given by

$$s(t) = \int_{t_0}^{t} v(\ell)\ d\ell \ .$$

- Eg. the absolute rate of change of the turn of the tangent vector with respect to position on C.



- Note the following:
  - The curvature $\kappa \geq 0$.
  - $\kappa > 0$ when the curve is **non-linear**.
  - The radius of curvature is $\rho = \dfrac{1}{\kappa}$.

- To find the a formula for κ, we can use the chain rule:

$$\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds} \cdot \frac{ds}{dt}$$
$$\implies \left\| \frac{d\vec{T}}{dt} \right\| = \left\| \frac{d\vec{T}}{ds} \right\| v \ ,$$

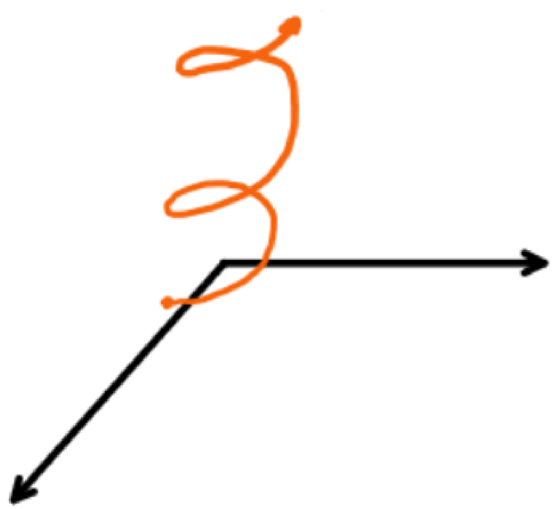
where $\frac{d\vec{T}}{dt}$ and $\frac{d\vec{T}}{ds}$ are vectors, and $\frac{ds}{dt}$ is a scalar function of time (aka the speed). Now, since $\kappa := \left|\left|\frac{d\vec{T}}{ds}\right|\right|$, it follows that

$$\begin{aligned}
\kappa &= \left|\left|\frac{d\vec{T}}{ds}\right|\right| \\
&= \frac{1}{v}\left|\left|\frac{d\vec{T}}{dt}\right|\right| \\
&= \frac{1}{v}\left|\left|\frac{d\left(\frac{\vec{v}}{v}\right)}{dt}\right|\right| \\
&= \frac{1}{v}\left|\left|\frac{d}{dt}\left(\frac{\vec{v}}{v}\right)\right|\right| \\
&= \frac{1}{v}\cdot\left|\left|\frac{\vec{v}\,'v-\vec{v}v'}{v^2}\right|\right| && \text{(quotient rule)} \\
&= \frac{1}{v^3}\cdot||\vec{a}v-\vec{v}v'|| && \text{(since } \vec{a}=\vec{v}\,'\text{)} \\
&= \text{(some algebra happens)} \\
&= \frac{||\vec{v}\times\vec{a}||}{v^3}\ .
\end{aligned}$$

The curvature κ and the radius of curvature ρ do not depend on the parameterization used.

$$\boxed{\kappa = \frac{||\vec{v}\times\vec{a}||}{v^3}}$$

1. Compute $\vec{T}$, $\vec{N}$, and κ for $\vec{r}(t) = \begin{bmatrix} \cos(e^t) \\ \sin(e^t) \\ e^t \end{bmatrix}$, where $t \geq 0$. Note that is the parameterization of a helix.



We have that the velocity is

$$\vec{v}(t) = \vec{r}\,'(t) = \begin{bmatrix} -e^t \sin(e^t) \\ e^t \cos(e^t) \\ e^t \end{bmatrix}$$

and the acceleration is

$$\vec{a}(t) = \vec{v}\,'(t) = \begin{bmatrix} -e^t \sin(e^t) - e^t \cdot e^t \cos(e^t) \\ e^t \cos(e^t) - e^t \cdot e^t \sin(e^t) \\ e^t \end{bmatrix}.$$

So, the speed is

$$\begin{aligned} v &= ||\vec{v}|| \\ &= \sqrt{(-e^t \sin(e^t))^2 + (e^t \cos(e^t))^2 + (e^t)^2} \\ &= \sqrt{e^{2t} \sin^2(e^t) + e^{2t} \cos^2(e^t) + e^{2t}} \\ &= \sqrt{e^{2t} \left(\sin^2(e^t) + \cos^2(e^t)\right) + e^{2t}} \\ &= \sqrt{e^{2t} \cdot 1 + e^{2t}} \\ &= \sqrt{e^{2t} + e^{2t}} \\ &= \sqrt{2e^{2t}} \end{aligned}$$

$$\begin{aligned}
&= \sqrt{2} \cdot \sqrt{e^{2t}} \\
&= \sqrt{2} \cdot \left(e^{2t}\right)^{\frac{1}{2}} \\
&= \sqrt{2}e^t \ .
\end{aligned}$$

Then the tangent vector is

$$\begin{aligned}
\vec{T} &= \frac{\vec{v}}{v} \\
&= \frac{1}{v} \cdot \vec{v} \\
&= \frac{1}{\sqrt{2}e^t} \begin{bmatrix} -e^t \sin(e^t) \\ e^t \cos(e^t) \\ e^t \end{bmatrix} \\
&= \frac{1}{\sqrt{2}e^t} \cdot e^t \begin{bmatrix} -\sin(e^t) \\ \cos(e^t) \\ 1 \end{bmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{bmatrix} -\sin(e^t) \\ \cos(e^t) \\ 1 \end{bmatrix}
\end{aligned}$$

and its derivative is

$$\vec{T}' = \frac{1}{\sqrt{2}} \begin{bmatrix} -e^t \cos(e^t) \\ -e^t \sin(e^t) \\ 0 \end{bmatrix} \ .$$

So, we get that the normal vector is

$$\begin{aligned}
\vec{N} &= \frac{\vec{T}'}{||\vec{T}'||} \\
&= \frac{1}{||\vec{T}'||} \cdot \vec{T} \\
&= \frac{1}{e^t/\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} -e^t \cos(e^t) \\ -e^t \sin(e^t) \\ 0 \end{bmatrix} \\
&= \frac{\sqrt{2}}{e^t} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} -e^t \cos(e^t) \\ -e^t \sin(e^t) \\ 0 \end{bmatrix} \\
&= \frac{1}{e^t} \begin{bmatrix} -e^t \cos(e^t) \\ -e^t \sin(e^t) \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -\cos(e^t) \\ -\sin(e^t) \\ 0 \end{bmatrix} \ .
\end{aligned}$$

Now, recall that for two vector $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$, the cross product is

$$\begin{aligned}
\vec{u} \times \vec{v} &= \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} \\
&= (u_2 v_3 - u_3 v_2)\, \hat{\imath} - (u_1 v_3 - u_3 v_1)\, \hat{\jmath} + (u_1 v_2 - u_2 v_1)\, \hat{k} \\
&= (u_2 v_3 - u_3 v_2)\, \hat{\imath} + (-u_1 v_3 + u_3 v_1)\, \hat{\jmath} + (u_1 v_2 - u_2 v_1)\, \hat{k} \\
&= (u_2 v_3 - u_3 v_2)\, \hat{\imath} + (u_3 v_1 - u_1 v_3)\, \hat{\jmath} + (u_1 v_2 - u_2 v_1)\, \hat{k} \\
&= \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix} ,
\end{aligned}$$

where $\hat{\imath} = (1, 0, 0)$, $\hat{\jmath} = (0, 1, 0)$, and $\hat{k} = (0, 0, 1)$. are the basis vectors of $\mathbb{R}^3$, and the vector resulting from $\vec{u} \times \vec{v}$ is perpendicular to both $\vec{u}$ and $\vec{v}$. Then

$$\begin{aligned}
\vec{v} \times \vec{a} &= \dots \\
&= e^{3t} \begin{bmatrix} \sin(e^t) \\ -\cos(e^t) \\ 1 \end{bmatrix} .
\end{aligned}$$

Hence, the curvature is

$$\begin{aligned}
\kappa &= \frac{||\vec{v} \times \vec{a}||}{v^3} \\
&= \dots \\
&= \frac{\sqrt{2}}{\left(\sqrt{2}\,\right)^3} \\
&= \frac{1}{2} .
\end{aligned}$$

2. Calculate κ for a planar function $y = f(x)$.

Recall that the curvature κ is defined as

$$\kappa = \frac{||\vec{v} \times \vec{a}||}{v^3} \ ,$$

which a vector in $\mathbb{R}^3$. In fact, the cross product only works for vector in $\mathbb{R}^3$. We can embed $y = f(x)$ into $\mathbb{R}$. Recalling the method of parameterization for "function types", we get that the position vector is

$$\vec{r}(t) = \begin{bmatrix} t \\ f(t) \\ 0 \end{bmatrix} \ ,$$

the velocity vector is

$$\vec{v}(t) = \vec{r}\,'(t) = \begin{bmatrix} 1 \\ f'(t) \\ 0 \end{bmatrix} \ ,$$

and tbe acceleration vector is

$$\vec{a}(t) = \vec{v}\,'(t) = \begin{bmatrix} 0 \\ f''(t) \\ 0 \end{bmatrix} \ .$$

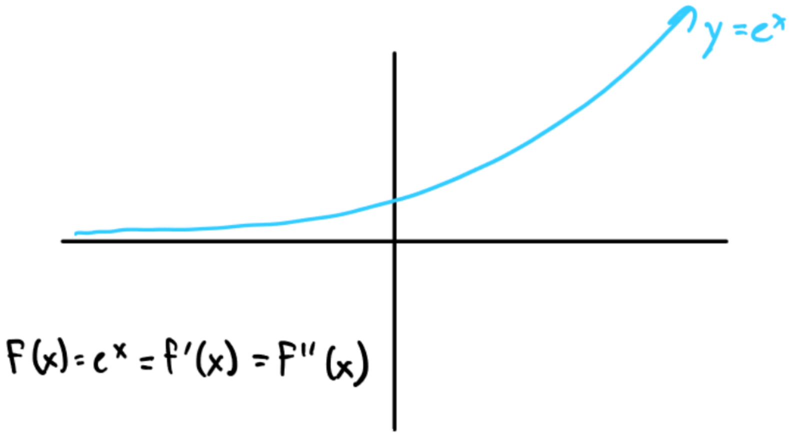
Then

$$\begin{aligned}
\vec{v} \times \vec{a} &= \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & f'(t) & 0 \\ 0 & f''(t) & 0 \end{bmatrix} \\
&= (0-0)\,\hat{i} - (0-0)\,\hat{j} + (f''(t) - 0)\,\hat{k} \\
&= 0\hat{i} - 0\hat{j} + f''(t)\hat{k} \\
&= \begin{bmatrix} 0 \\ 0 \\ f''(t) \end{bmatrix} \ .
\end{aligned}$$

Hence, the curvature for any planar function $y = f(x)$ is

$$\begin{aligned}
\kappa &= \frac{||\vec{v} \times \vec{a}||}{v^3} \\
&= \frac{\sqrt{(0)^2 + (0)^2 + (f''(t))^2}}{\left(\sqrt{1 + f(t)^2}\,\right)^3} \\
&= \frac{f''(t)}{\left(\sqrt{1 + f(t)^2}\,\right)^3} \\
&= \dots
\end{aligned}$$

3. Calculate $\lim_{x\to\infty} \kappa$ for $f(x) = e^x$.



We have that

$$\kappa = \frac{|e^x|}{(1+e^{2x})^{\frac{3}{2}}} = \frac{e^x}{(1+e^{2x})^{\frac{3}{2}}} \ .$$

Then

$$\begin{aligned}
\lim_{x\to\infty} \kappa &= \lim_{x\to\infty} \frac{e^x}{(1+e^{2x})^{\frac{3}{2}}} \\
&= \lim_{x\to\infty} \frac{e^x}{(e^{2x})^{\frac{3}{2}}} \\
&= \lim_{x\to\infty} \frac{e^x}{e^{3x}} \\
&= \lim_{x\to\infty} e^{x-3x} \\
&= \lim_{x\to\infty} e^{-2x} \\
&= 0 \ .
\end{aligned}$$

where the 1 in the denominator is negligible. Note that

$$\lim_{x\to-\infty} \frac{e^x}{(1+e^{2x})^{\frac{3}{2}}} = \frac{0}{(1+0)} = 0 \ .$$

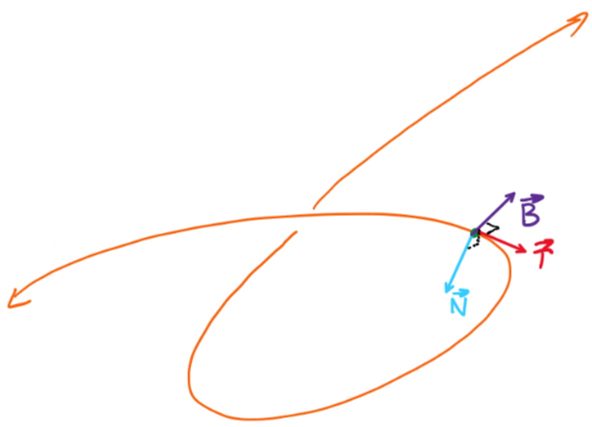
(a) Where is κ maximized for $f(x) = e^x$?

TNB Frames

For this section, all parametric curves are in $\mathbb{R}^3$.

Definition (The Unit Binormal)

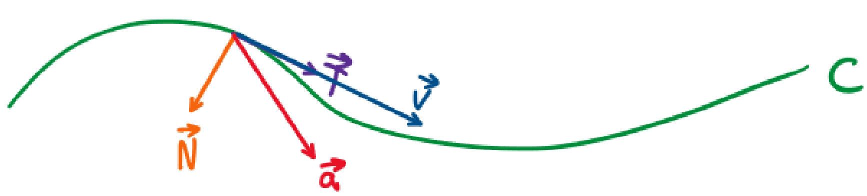
Let $\vec{B} = \vec{T} \times \vec{N}$. Then B is a unit vector perpendicular to both $\vec{T}$ and $\vec{N}$ and gives the direction of "twist".



A curve is uniquely determined by $\vec{T}, \vec{N}, \vec{B}$. We call the set $\{\vec{T}, \vec{N}, \vec{B}\}$ a **frame**. Note that the frame is an orthonormal basis for $\mathbb{R}^3$.

Remark: $\vec{N} = \dfrac{\vec{T}'}{||\vec{T}'||}$ is typically hard to calculate.

Observation: The plane spanned by $\vec{T}$ and $\vec{N}$ is the same as the plane spanned by $\vec{v}$ and $\vec{a}$.



- So, $\vec{v} \times \vec{a}$ is parallel (and in the same direction as $\vec{T} \times \vec{N}$. Indeed, this can be verified with the right-hand rule.

- Since $\vec{B}$ is a unit vector and $\vec{B} = \vec{T} \times \vec{N}$ is parallel to $\vec{v} \times \vec{a}$, this gives

$$\vec{B} = \frac{\vec{v} \times \vec{a}}{||\vec{v} \times \vec{a}||} \ .$$

Then this means that

$$\vec{N} = \vec{B} \times \vec{T} \ .$$

Proposition

(a) $\dfrac{d\vec{T}}{ds} = \kappa \vec{N}$, where s is the arc-length.

(b) $\dfrac{d\vec{B}}{ds}$ is parallel to $\vec{N}$.

(a) Note that

$$\vec{N} = \frac{d\vec{T}/dt}{||d\vec{T}/dt||} \ ,$$
$$\kappa = \left|\left|\frac{d\vec{T}}{ds}\right|\right| \ .$$

By the chain rule, we have that

$$\frac{d\vec{T}}{ds} = \frac{d\vec{T}}{dt} \cdot \frac{dt}{ds} \ ,$$

where t is a function of s. So,

$$\begin{aligned}
\frac{d\vec{T}}{ds} &= \frac{d\vec{T}}{dt} \cdot \frac{dt}{ds} \\
&= \vec{N} \left|\left|\frac{d\vec{T}}{dt}\right|\right| \cdot \frac{dt}{ds} \\
&= \vec{N} \left|\left|\frac{d\vec{T}}{ds}\right|\right| \cdot \left|\frac{ds}{dt}\right| \cdot \frac{dt}{ds} && \text{(chain rule)} \\
&= \vec{N} \left|\left|\frac{d\vec{T}}{ds}\right|\right| \cdot 1 \\
&= \vec{N} \cdot \kappa \\
&= \kappa \vec{N} \ .
\end{aligned}$$

(b) Any vector $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ can be decomposed with respect to $\{\vec{T}, \vec{N}, \vec{B}\}$ as

$$\vec{u} = u_1\vec{T} + u_2\vec{N} + u_3\vec{B} = (\vec{u} \cdot \vec{T})\vec{T} + (\vec{u} \cdot \vec{N})\vec{N} + (\vec{u} \cdot \vec{B})\vec{B} \ .$$

We will show that $\dfrac{d\vec{B}}{ds}$ is perpendicular to $\vec{T}$ and $\vec{B}$ (and therefore parallel to $\vec{N}$). We have that

$$s1 = ||\vec{B}||^2 = \left(\sqrt{\vec{B} \cdot \vec{B}}\ \right)^2 = \vec{B} \cdot \vec{B} \ .$$

Then differentiating both sides with respect to s gives us

$$\begin{aligned} 0 &= \frac{d}{ds}\left[\vec{B} \cdot \vec{B}\right] \\ 0 &= \frac{d\vec{B}}{ds} \cdot \vec{B} + \vec{B} \cdot \frac{d\vec{B}}{ds} \\ 0 &= \vec{B} \cdot \frac{d\vec{B}}{ds} + \vec{B} \cdot \frac{d\vec{B}}{ds} \\ 0 &= 2\vec{B} \cdot \frac{d\vec{B}}{ds} \ . \end{aligned}$$

From this we get that $\vec{B}$ is perpendicular to $\dfrac{d\vec{B}}{ds}$ since

$$\begin{aligned} 0 &= 2\vec{B} \cdot \frac{d\vec{B}}{ds} \\ 0 &= \vec{B} \cdot \frac{d\vec{B}}{ds} \ . \end{aligned}$$

Finally, since $\vec{T}$ and $\vec{B}$ are perpendicular,

$$0 = \vec{T} \cdot \vec{B}$$

and so differentiating both sides with respect to s gives us

$$\begin{aligned} 0 &= \frac{d}{ds}\left[\vec{T} \cdot \vec{B}\right] \\ 0 &= \frac{d\vec{T}}{ds} \cdot \vec{B} + \vec{T} \cdot \frac{d\vec{B}}{ds} \\ 0 &= 0 + \vec{T} \cdot \frac{d\vec{B}}{ds} \\ 0 &= \vec{T} \cdot \frac{d\vec{B}}{ds} \ , \end{aligned}$$

Thus, $\vec{T} \perp \dfrac{d\vec{B}}{ds}$.

Definition (Torsion)

Let τ be the constant so that

$$\frac{d\vec{B}}{ds} = -\tau\vec{N} \ .$$

We call τ the **torsion**. Note that τ is guaranteed to exist because of proposition (b).

To find a formula:

$$\begin{aligned}
\frac{d\vec{B}}{ds} \cdot \vec{N} &= -\tau\vec{N} \cdot \vec{N} \\
&= -\tau \left(\vec{N} \cdot \vec{N}\right) \\
&= -\tau||\vec{N}||^2 \\
&= -\tau \cdot 1 \\
&= -\tau \ ,
\end{aligned}$$

where $||\vec{N}||^2 = 1$ since $\vec{N}$ is a unit vector. Then after choosing a parameterization and performing a bunch of algebra, we get that

$$\boxed{\tau = \frac{(\vec{v} \times \vec{a}) \cdot \vec{a}'}{||\vec{v} \times \vec{a}||^2}}$$

- When $\tau > 0$, motion is twisting **counter-clockwise**.
- When $\tau < 0$, motion is twisting **clockwise**.

Remark: $\vec{T}$, $\vec{N}$, $\vec{B}$, κ and τ do not depend on which parameterization $\vec{r}(t)$ you use for a curve C. (At a particular point, ensure that the correct value of t has been used for the $\vec{r}(t)$ you select.)

1. Let $\vec{r}(t) = \begin{bmatrix} \cos(\ln(t)) \\ \sin(\ln(t)) \\ t \end{bmatrix}$, where $t > 0$. Compute $\vec{T}$, $\vec{N}$, $\vec{B}$, κ, and τ.

Pure suffering.

Tangential and Normal Components of Acceleration

- Let $\vec{r}$ be a parameterization for a curve C in $\mathbb{R}^3$.

- Any vector $\vec{u}$ admits the decomposition:

$$\begin{aligned}\vec{u} &= u_1\vec{T} + u_2\vec{N} + u_3\vec{B} \\ &= (\vec{u}\cdot\vec{T})\vec{T} + (\vec{u}\cdot\vec{N})\vec{N} + (\vec{u}\cdot\vec{B})\vec{B}\ .\end{aligned}$$

- In particular,

$$\begin{aligned}\vec{a} &= a_1\vec{T} + a_2\vec{N} + a_3\vec{B} \\ &= (\vec{a}\cdot\vec{T})\vec{T} + (\vec{a}\cdot\vec{N})\vec{N} + (\vec{a}\cdot\vec{B})\vec{B}\ .\end{aligned}$$

- Since $\vec{B}$ is parallel to $\vec{v}\times\vec{a}$, it means that $\vec{B}$ is perpendicular to both $\vec{v}$ and $\vec{a}$. Since $\vec{a}$ and $\vec{B}$ are perpendicular, we get that $\vec{a}\cdot\vec{B} = 0$.

- So acceleration only has components, in the direction of $\vec{T}$ and $\vec{N}$.

- Recall that $\vec{r}(t) = \begin{bmatrix} x(t) & y(t) & z(t)\end{bmatrix}^\intercal$, $\vec{v}(t) = \begin{bmatrix} x'(t) & y'(t) & z'(t)\end{bmatrix}^\intercal$, and $\vec{a}(t) = \begin{bmatrix} x''(t) & y''(t) & z''(t)\end{bmatrix}^\intercal$. Then the component of acceleration in the direction of $\vec{T}$ (i.e. the tangential acceleration) is

$$\begin{aligned}a_T &= \vec{a}\cdot\vec{T} \\ &= \vec{a}\cdot\frac{\vec{v}}{||\vec{v}||} \\ &= \vec{a}\cdot\left(\frac{1}{||\vec{v}||}\cdot\vec{v}\right) \\ &= \frac{1}{||\vec{v}||}\cdot(\vec{a}\cdot\vec{v}) \\ &= \frac{1}{\sqrt{(x')^2+(y')^2+(z')^2}}\cdot\left(\begin{bmatrix} x'' \\ y'' \\ z''\end{bmatrix}\cdot\begin{bmatrix} x' \\ y' \\ z'\end{bmatrix}\right) \\ &= \frac{x''\cdot x' + y''\cdot y' + z''\cdot z'}{\sqrt{(x')^2+(y')^2+(z')^2}} \\ &= \dots \\ &= \sqrt{(x')^2+(y')^2+(z')^2} \\ &= v'\ .\end{aligned}$$

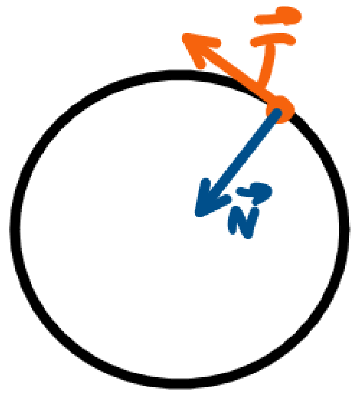
We call a_T the tangential component of acceleration.

- The component of acceleration in the direction of $\vec{N}$ is

$$\begin{aligned} a_N &= \vec{a} \cdot \vec{N} \\ &= \vec{a} \cdot \frac{\vec{T}\,'}{||\vec{T}\,'||} \\ &= \text{(some algebra happens)} \\ &= v^2\kappa \ . \end{aligned}$$

We call a_N the normal component of acceleration.

1. Let $\vec{r}(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$. Find a_T and a_N. Note that $\vec{r}(t)$ is the paramaterization of the unit circle.



We have that the velocity is

$$\vec{v}(t) = \vec{r}\,'(t) = \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix} \ ,$$

the speed is

$$\begin{aligned} v(t) &= \sqrt{(-\sin(t))^2 + (\cos(t))^2} \\ &= \sqrt{\sin^2(t) + \cos^2(t)} \\ &= \sqrt{1} \\ &= 1 \ , && \text{(since } v(t) > 0\text{)} \end{aligned}$$

and the acceleration is

$$\vec{a}(t) = \vec{v}\,'(t) = \begin{bmatrix} -\cos(t) \\ -\sin(t) \end{bmatrix} \ .$$

Now, since $v(t) = 1$, this means that $v'(t) = \frac{d}{dt}[v(t)] = \frac{d}{dt}[1] = 0$ (speed is constant along the curve). Hence, $a_T = v'(t) = 0$ and

$$\begin{aligned} a_N &= v^2\kappa \\ &= v(t)^2 \cdot \frac{||\vec{v}(t) \times \vec{a}(t)||}{v(t)^3} \\ &= (1)^2 \cdot \frac{||\vec{v}(t) \times \vec{a}(t)||}{(1)^3} \end{aligned}$$

$$
\begin{aligned}
&= ||\vec{v}(t) \times \vec{a}(t)|| \\
&= \left|\left|\det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin(t) & \cos(t) & 0 \\ -\cos(t) & -\sin(t) & 0 \end{bmatrix}\right|\right| \\
&= \left|\left|(\sin^2(t) + \cos^2(t))\hat{k}\right|\right| \\
&= \left|\left|\begin{bmatrix} 0 \\ 0 \\ \sin^2(t) + \cos^2(t) \end{bmatrix}\right|\right| \\
&= \left|\left|\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right|\right| \\
&= \sqrt{0^2 + 0^2 + 1^2} \\
&= \sqrt{1} \\
&= 1\ .
\end{aligned}
$$

In fact, a is actually parallel (and equal) to $\vec{N}$.

2. Let $\vec{r}(t) = \begin{bmatrix} \cos(t^2) \\ \sin(t^2) \end{bmatrix}$. Find a_T and a_N. Note that $\vec{r}(t)$ can be thought of as a particle moving around the unit circle faster than in the previous 1.

We have that the velocity is

$$\vec{v}(t) = \vec{r}\,'(t) = \begin{bmatrix} -2t\sin(t^2) \\ 2t\cos(t^2) \end{bmatrix} ,$$

the speed is

$$\begin{aligned} v(t) &= \sqrt{(-2t\sin(t^2))^2 + (2t\cos(t^2))^2} \\ &= \sqrt{4t^2\sin^2(t^2) + 4t^2\cos^2(t^2)} \\ &= \sqrt{4t^2\left(\sin^2(t^2) + \cos^2(t^2)\right)} \\ &= \sqrt{4t^2 \cdot 1} \\ &= \sqrt{4t^2} \\ &= 2t \ , \end{aligned}$$

and the acceleration is

$$\begin{aligned} \vec{a}(t) &= \vec{v}\,'(t) \\ &= \begin{bmatrix} \dfrac{d}{dt}\left[-2t\sin(t^2)\right] \\ \dfrac{d}{dt}\left[2t\cos(t^2)\right] \end{bmatrix} \\ &= \begin{bmatrix} -2 \cdot \dfrac{d}{dt}\left[t\sin(t^2)\right] \\ 2 \cdot \dfrac{d}{dt}\left[t\cos(t^2)\right] \end{bmatrix} \\ &= \begin{bmatrix} -2 \cdot \left(\sin(t^2) + 2t^2\cos(t^2)\right) \\ 2 \cdot \left(\cos(t^2) - 2t^2\sin(t^2)\right) \end{bmatrix} \\ &= \begin{bmatrix} -2\sin(t^2) - 4t^2\cos(t^2) \\ 2\cos(t^2) - 4t^2\sin(t^2) \end{bmatrix} . \end{aligned}$$

Then

$$a_T = v'(t) = \frac{d}{dt}[v(t)] = \frac{d}{dt}[2t] = 2 .$$

Now, instead of solving for a_N the usual way, we can use a shortcut. We know that the acceleration consists of the normal and tangential components, and so we can express $\vec{a}$ as the following linear combination:

$$\vec{a} = a_T\vec{T} + a_N\vec{N} \ ,$$

where $a_N \geq 0$. Since $\vec{T} \perp \vec{N}$, we use the Pythagarous law which gives us

$$\begin{aligned} ||\vec{a}||^2 &= ||a_T \vec{T}||^2 + ||a_N \vec{N}||^2 \\ &= a_T^2 \cdot ||\vec{T}||^2 + a_N^2 \cdot ||\vec{N}||^2 \\ &= a_T^2 \cdot 1 + a_N^2 \cdot 1 \\ &= a_T^2 + a_N^2 \ , \end{aligned}$$

which means that

$$\begin{aligned} a_N &= \sqrt{||\vec{a}||^2 + a_N^2} \\ &= \sqrt{||\vec{a}||^2 + 2^2} \\ &= \sqrt{||\vec{a}||^2 + 4} \ . \end{aligned}$$

3. Let $\vec{r}(t) = \begin{bmatrix} \frac{1}{3}t^3 & \frac{4}{5}t^{5/2} & t^2 \end{bmatrix}^{\mathsf{T}}$, where $t \geq 0$. Compute $\vec{T}$, $\vec{N}$, $\vec{B}$, a_T, a_N, κ, τ.

We have that the velocity is

$$\vec{v}(t) = \vec{r}\,'(t) = \begin{bmatrix} t^2 \\ 2t^{3/2} \\ 2t \end{bmatrix} \ ,$$

the speed is

$$\begin{aligned} v(t) &= ||\vec{v}(t)|| \\ &= \sqrt{(t^2)^2 + (2t^{3/2})^2 + (2t)^2} \\ &= \sqrt{t^4 + 4t^3 + 4t^2} \\ &= \sqrt{t^2(t^2 + 4t + 4)} \\ &= \sqrt{t^2} \cdot \sqrt{t^2 + 4t + 4} \\ &= t\sqrt{t^2 + 4t + 4} \\ &= t\sqrt{(t+2)^2} \\ &= t(t+2) \\ &= t^2 + 2t \ , \end{aligned}$$

and the acceleration is

$$\vec{a}(t) = \vec{v}\,'(t) = \begin{bmatrix} 2t \\ 3t^{1/2} \\ 2 \end{bmatrix} \ .$$

Then the tangent vector is

$$\begin{aligned} \vec{T} &= \frac{\vec{v}}{||\vec{v}||} \\ &= \frac{1}{||\vec{v}||} \cdot \vec{v} \\ &= \frac{1}{v} \cdot \vec{v} \\ &= \frac{1}{t^2 + 2t} \begin{bmatrix} t^2 \\ 2t^{3/2} \\ 2t \end{bmatrix} \\ &= \frac{1}{t^2 + 2t} \cdot t \begin{bmatrix} t \\ 2t^{1/2} \\ 2 \end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
&= \frac{t}{t^2 + 2t} \begin{bmatrix} t \\ 2t^{1/2} \\ 2 \end{bmatrix} \\
&= \frac{t}{t(t+2)} \begin{bmatrix} t \\ 2t^{1/2} \\ 2 \end{bmatrix} \\
&= \frac{1}{t+2} \begin{bmatrix} t \\ 2t^{1/2} \\ 2 \end{bmatrix} \ .
\end{aligned}
$$

We have that

$$\vec{v} \times \vec{a} = t^{3/2} \begin{bmatrix} -2 \\ 2t^{1/2} \\ -t \end{bmatrix} \ ,$$

and so

$$
\begin{aligned}
||\vec{v} \times \vec{a}|| &= t^{3/2} \cdot \sqrt{(-2)^2 + (2t^{1/2})^2 + (-t)^2} \\
&= t^{3/2} \cdot \sqrt{4 + 2t + t^2} \\
&= t^{3/2} \cdot \sqrt{(t+2)^2} \\
&= t^{3/2} \cdot (t+2) \ .
\end{aligned}
$$

Then the unit binormal is

$$
\begin{aligned}
\vec{B} &= \frac{\vec{v} \times \vec{a}}{||\vec{v} \times \vec{a}||} \\
&= \frac{1}{||\vec{v} \times \vec{a}||} \cdot (\vec{v} \times \vec{a}) \\
&= \frac{1}{t^{3/2} \cdot (t+2)} \cdot t^{3/2} \begin{bmatrix} -2 \\ 2t^{1/2} \\ -t \end{bmatrix} \\
&= \frac{1}{t+2} \begin{bmatrix} -2 \\ 2t^{1/2} \\ -t \end{bmatrix}
\end{aligned}
$$

The curvature is

$$
\begin{aligned}
\kappa &= \frac{||\vec{v} \times \vec{a}||}{v^3} \\
&= \frac{t^{3/2} \cdot (t+2)}{(t^2 + 2t)^3} \\
&= \frac{t^{3/2} \cdot (t+2)}{(t(t+2))^3}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{t^{3/2} \cdot (t+2)}{t^3(t+2)^3} \\
&= \frac{t^{3/2}}{t^3} \cdot \frac{t+2}{(t+2)^3} \\
&= t^{-3/2} \cdot \frac{1}{(t+2)^2} \\
&= \frac{1}{t^{3/2}} \cdot \frac{1}{(t+2)^2} \\
&= \frac{1}{t^{3/2}(t+2)^2} \ .
\end{aligned}
$$

Now, the tangential component of acceleration is

$$a_T = v'(t) = \frac{d}{dt}[v(t)] = \frac{d}{dt}(t^2 + 2t) = 2t + 2 \ ,$$

and the normal component of acceleration is

$$
\begin{aligned}
a_N &= v^2 \kappa \\
&= (t^2+2t)^2 \cdot \frac{1}{t^{3/2}(t+2)^2} \\
&= (t^2+2t)^2 \cdot \frac{t^{3/2}(t+2)}{(t^2+2t)^3} \\
&= \frac{t^{3/2}(t+2)}{t^2+2t} \\
&= \frac{t^{3/2}(t+2)}{t(t+2)} \\
&= \frac{t^{3/2}}{t} \cdot \frac{t+2}{t+2} \\
&= t^{1/2} \cdot 1 \\
&= t^{1/2} \\
&= \sqrt{t} \ .
\end{aligned}
$$

Since

$$\vec{a}'(t) = \begin{bmatrix} 2 \\ \frac{3}{2\sqrt{t}} \\ 0 \end{bmatrix} \ ,$$

the torsion is

$$\tau = \frac{(\vec{v} \times \vec{a}) \cdot \vec{a}'}{||\vec{v} \times \vec{a}||^2}$$

$$
\begin{aligned}
&= \frac{1}{||\vec{v}\times\vec{a}||^2}\cdot((\vec{v}\times\vec{a})\bullet\vec{a}\,') \\
&= \frac{1}{(t^{3/2}(t+2))^2}\cdot((\vec{v}\times\vec{a})\bullet\vec{a}\,') \\
&= \frac{1}{t^3(t+2)^2}\cdot((\vec{v}\times\vec{a})\bullet\vec{a}\,') \\
&= \dots \\
&= -\frac{1}{t^{3/2}(2+t)^2}\ .
\end{aligned}
$$

Finally, the normal vector is

$$
\begin{aligned}
\vec{N} &= \vec{B}\times\vec{T} \\
&= \dots \\
&= \frac{1}{(t+2)^2}\begin{bmatrix} 4t^{1/2}+2t^{3/2} \\ 4-t^2 \\ -4t^{1/2}-2t^{3/2} \end{bmatrix}\ .
\end{aligned}
$$

Note that $\vec{N}$ could also be found via

$$
\begin{aligned}
\vec{a} &= a_T\vec{T}+a_N\vec{N} \\
\vec{a}-a_T\vec{T} &= a_N\vec{N} \\
\vec{N} &= \frac{1}{a_N}\cdot(\vec{a}-a_T\vec{T})\ .
\end{aligned}
$$