# MATH 367 - Week 1 Notes

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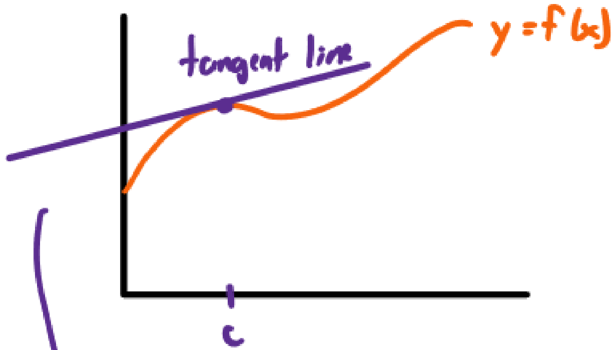
# Introduction

What is a derivative?

- rate of change of $y$ with respect to $x$.
- linearization




contains (c, f(c)) with slope f'(c)

We write the linearization as $y - f(c) = f'(c) \cdot (x - c)$. With $y = L(x)$, we then get

$$L(x) = f(c) - f'(c) \cdot (x - a) \ .$$

$L(x)$ is the best linear approximation to $f(x)$ near $x = c$. That is, $f'(c)$ is the best minimizing quantity for

$$|f(x) - (f(c) + m(x - c))\,|$$

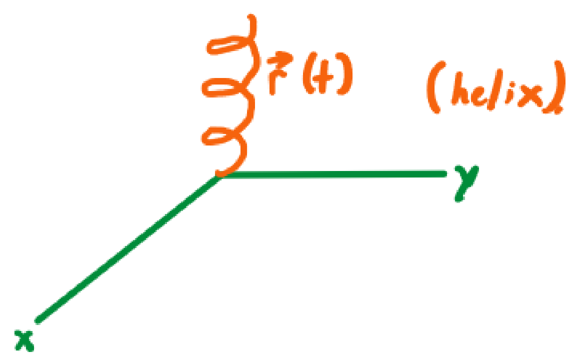
for $x$ close to $c$.

In this course, we will look at functions with many input variables and many output variables. Consider the following examples.

(i) $f(x, y, z) = xy + z^2$ (scalar-valued function)

(ii) $\vec{r}(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \\ t \end{pmatrix}$ (vector-valued function)

Note that the graph of the range of a vector-valued function is called a **<u>curve</u>**.




(iii) $\vec{f}(x, y) = \begin{pmatrix} -y \\ x \end{pmatrix}$ (vector field in 2D)

$\vec{f}(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}} \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix}$ (vector field in 3D).

(# of inputs = # of outputs)

(iv) Suppose $A$ is a an $m \times n$ matrix and define a function

$$\vec{f}(\vec{x}) = A\vec{x} + \vec{b} \in \mathbb{R}^m \ ,$$

where $\vec{b} \in \mathbb{R}^m$, and $\vec{x} \in \mathbb{R}^n$ is our input. This is a function from $\mathbb{R}^n \to \mathbb{R}^m$, called an **affine** function. (Note that this is a **linear transformation when** $\vec{b} = \vec{0}$).

What should be the derivative of $\vec{f}$? We want it to be $A$. Note that the equation above is similar to the slope equation

$$y = mx + b \ ,$$

whose derivative evaluates to $m$.

# The Basic Topology of $\mathbb{R}^n$

**Definition (Open Ball)**

The open ball centered at $\vec{x}_0$ with radius $r > 0$ is the set

$$B_r(\vec{x}_0) = \{\vec{x} \in \mathbb{R}^n : ||\vec{x} - \vec{x}_0|| < r\} \subseteq \mathbb{R}^n ,$$

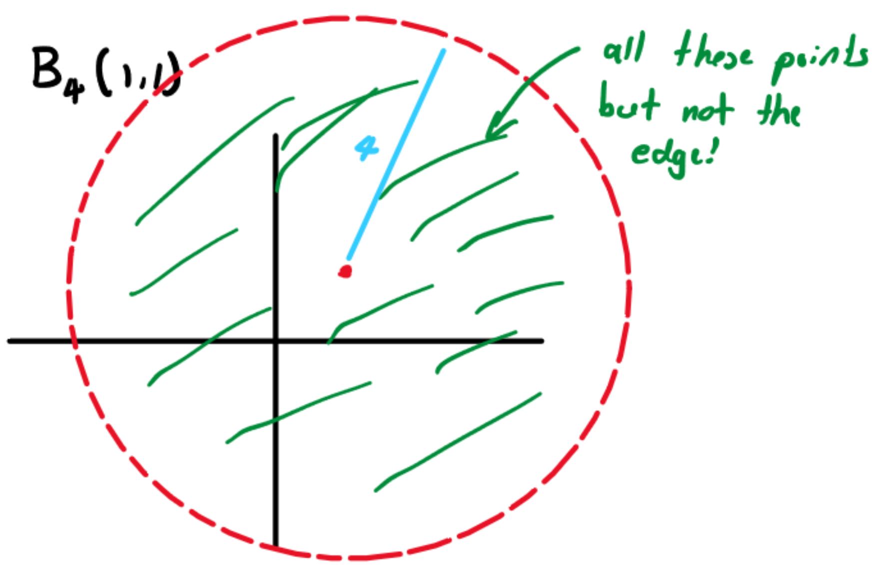
where $||\vec{x} - \vec{x}_0||$ is the Euclidean distance between $\vec{x}$ and $\vec{x}_0$.

For example, let $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$. Then the euclidean distance between $\vec{v}$ and $\vec{w}$ is given by

$$||\vec{v} - \vec{w}|| = \sqrt{\sum_{j=1}^{n} |v_j - w_j|^2} .$$

Essentially, an open ball is the set of all vectors $< r$ distance away from the vector $\vec{x}_0$, where $\vec{x}_0$ is the center of the ball whose radius is $r$.

Consider $B_4(1,1)$ in $\mathbb{R}^2$. This is the ball of radius 4 centered at $(1,1)$.



**Definition (Closed Ball)**

The closed ball of radius $r$ centered at $\vec{x}_0$ is the set

$$\overline{B_r}(\vec{x}_0) = \{\vec{x} \in \mathbb{R}^n : ||\vec{x} - \vec{x}_0|| \leq r\}$$

If we consider $\overline{B_4}(1,1)$, which is the closed ball of radius 4 centered at $(1,1)$, the image is the same as the one for the open ball $B_4(1,1)$, but with the edge included.

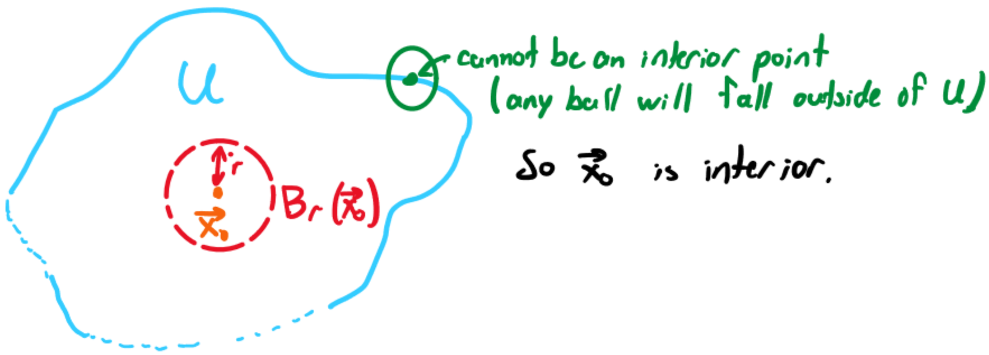
In one variable:

$$\begin{aligned} B_r(x_0) &= (x_0 - r,\ x_0 + r) \\ \overline{B_r} &= [x_0 - r,\ x_0 + r] \end{aligned}$$

Note that these are intervals, not matrices.

> **Definition (Interior Point)**
>
> Suppose $U \subseteq \mathbb{R}^n$. We say that $\vec{x}_0$ in $U$ is an **interior point** for $U$ if there is some $r > 0$ with $B_r(\vec{x}_0) \subseteq U$.

For example,

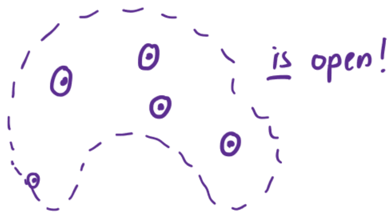


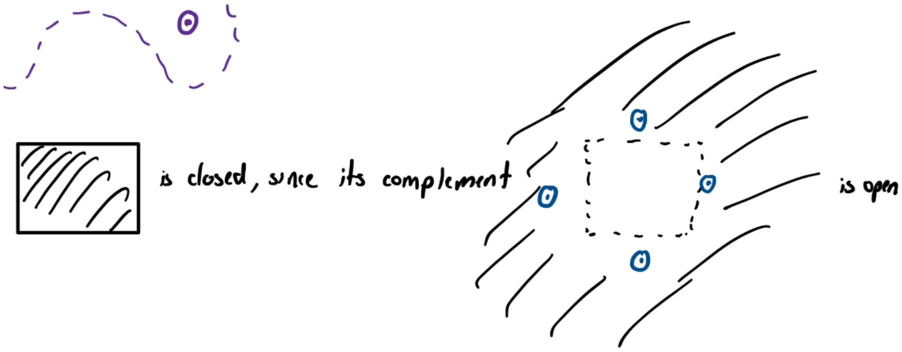

> **Definition (Interior)**
>
> The set of all interior points for $U$ is called the **interior** of $U$ (often written as $U^{\circ}$ or $\mathrm{int}(U)$).

> **Definition (Open Set)**
>
> $U$ is **open** if $U^{\circ} = U$ (that is, every point is interior).

> **Definition (Closed Set)**
>
> $U$ is **closed** if $\mathbb{R}^n \setminus U$ (the complement of $U$) is open.

Consider the following examples.



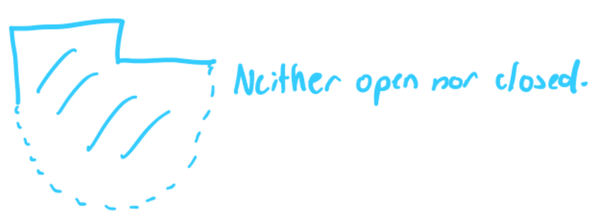

This makes sense.

- For the first example above, the set $U$ is the inside of the strange shape, and so since every point is an interior point, we say that $U$ is indeed an open set (by the definition of open set).
- For the second example above, $U$ is closed. If we consider the complement of the second image (the rectangle), which is the the third image, we see that the complement of $U$, which we'll call $U^{\mathsf{C}}$, is the outside of the rectangle. Since every point is indeed part of $U^{\mathsf{C}}$ (outside the rectangle), we say that $U$ is indeed a closed set.

**Remarks:**

(i) A set is open if and only if it contains **no** boundary points.

(ii) A set is closed if and only if it contains **all** boundary points.



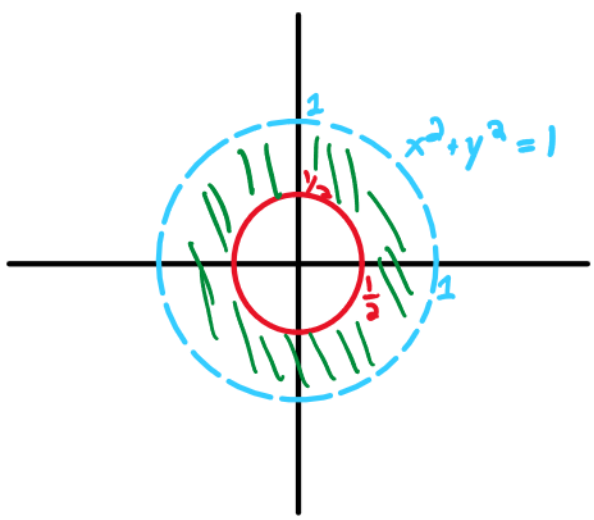
**Ex:** Find and sketch the domain of

$$\vec{f}(x,y) = \begin{pmatrix} \ln(1-x^2-y^2) \\ \sqrt{x^2+y^2-\frac{1}{4}} \end{pmatrix}.$$

**Answer:** In order for $(x,y) \in \text{dom}(\vec{f})$, it must be the case that $(x,y) \in \text{dom}(\ln(1-x^2-y^2)$ and $(x,y) \in \text{dom}\left(\sqrt{x^2+y^2-\frac{1}{4}}\right)$. So, using the fact that the domain of $\ln(x)$ is $(0,\infty)$ and the domain of $\sqrt{x}$ is $[0,\infty)$, we get that

$$\begin{aligned} \text{dom}(\vec{f}) &= \left\{(x,y) \in \mathbb{R}^2 : 1-x^2-y^2 > 0 \text{ and } x^2+y^2-\frac{1}{4} \geq 0\right\} \\ &= \left\{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1 \text{ and } x^2+y^2 \geq \frac{1}{4}\right\} \\ &= \left\{(x,y) \in \mathbb{R}^2 : \frac{1}{4} \leq x^2+y^2 < 1\right\} \end{aligned}$$

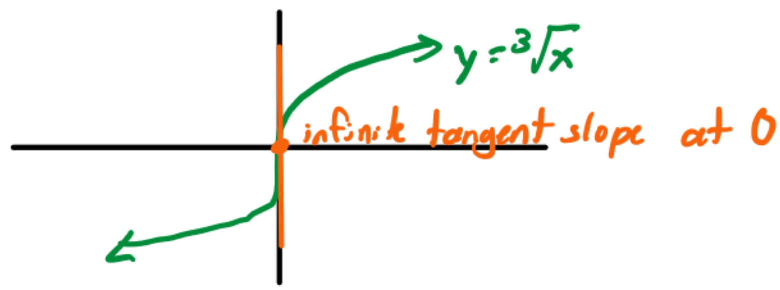
Recall that $x^2+y^2 = r^2$ is the circle of radius $r$ centered at $(0,0)$. The following is the sketch of $x^2+y^2=1$ and $x^2+y^2=\frac{1}{4}$, the circle centered at $(0,0)$ with radii 1 and $\frac{1}{2}$, respectively.



In single variable calculus, we generally define derivatives on the interior of the domain of $f$ (or a subset of the interior). For example,

$$f(x) = \sqrt{x}\ , \qquad \mathrm{dom}(f) = [0,\infty)$$
$$f'(x) = \frac{1}{2\sqrt{x}}\ , \quad \mathrm{dom}(f') = (0,\infty) = [0,\infty)^{\circ}$$

$$g(x) = \sqrt[3]{x}\ , \qquad \mathrm{dom}(g) = \mathbb{R}$$
$$g'(x) = \frac{1}{3\sqrt[3]{x^2}}\ , \quad \mathrm{dom}(g') = (-\infty,0) \cup (0,\infty)$$



## The Derivative

**Definition (Derivative)**

Suppose $\vec{f} : U \to \mathbb{R}^m$ where $U \subseteq \mathbb{R}^n$. We say that $f$ is differentiable at $\vec{x}_0 \in U^{\text{o}}$ if there is an $m \times n$ matrix $A$ such that

$$\lim_{\vec{x} \to \vec{x}_0} \frac{||\vec{f}(\vec{x}) - \vec{f}(\vec{x}_0) - A(\vec{x} - \vec{x}_0)||}{||\vec{x} - \vec{x}_0||} = 0 \ ,$$

where the numerator is the Euclidean distance in $\mathbb{R}^m$ and the denominator is the Euclidean distance in $\mathbb{R}^n$. When this is the case, $A$ must be unique. We call this matrix the derivative of $\vec{f}$ at $\vec{x}_0$ and write it as

$$D\vec{f}(\vec{x}_0) = A \ .$$

We can compare this definition to the single variable derivative definition:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \ .$$

We can get this limit definition of a derivative of a single variable in a form similar to that of our definition above.

$$\begin{aligned} f'(x_0) &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \\ 0 &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \\ 0 &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} - \frac{f'(x_0)(x - x_0)}{x - x_0} \\ 0 &= \lim_{x \to x_0} \left| \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{x - x_0} \right| \ . \end{aligned}$$