MATH 367 - Week 4 Notes

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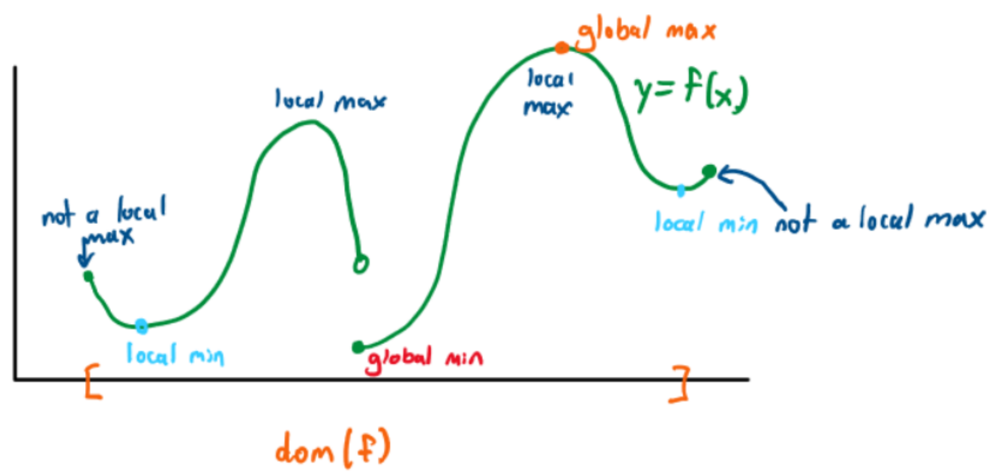
Optimization

- Goal: find maxima/minima for multivariable **scalar** functions subject to some constraint.

> **Definitions**
>
> Let $f(x_1, \ldots, x_n)$ be a scalar function.
>
> (i) An **absolute/global maximum** for f is a value $f(a_1, \ldots, a_n)$ such that $f(a_1, \ldots, a_n) \geq f(x_1, \ldots, x_n)$ for all $(x_1, \ldots, x_n)$ in $\mathrm{dom}(f)$.
>
> (ii) An **absolute/global minimum** for f is a value $f(a_1, \ldots, a_n)$ such that $f(a_1, \ldots, a_n) \leq f(x_1, \ldots, x_n)$ for all $(x_1, \ldots, x_n)$ in $\mathrm{dom}(f)$.
>
> (iii) A **local maximum** is a value $f(a_1, \ldots, a_n)$ such that there is an open ball B for which $f(a_1, \ldots, a_n) \geq f(x_1, \ldots, x_n)$ for all $(x_1, \ldots, x_n)$ in B.
>
> (iv) A **local minimum** is a value $f(a_1, \ldots, a_n)$ such that there is an open ball B for which $f(a_1, \ldots, a_n) \leq f(x_1, \ldots, x_n)$ for all $(x_1, \ldots, x_n)$ in B.



- **<u>Remark:</u>** In general, we only think of local extrema as located at interior points. Absolute/global extrema can be interior on the boundary.

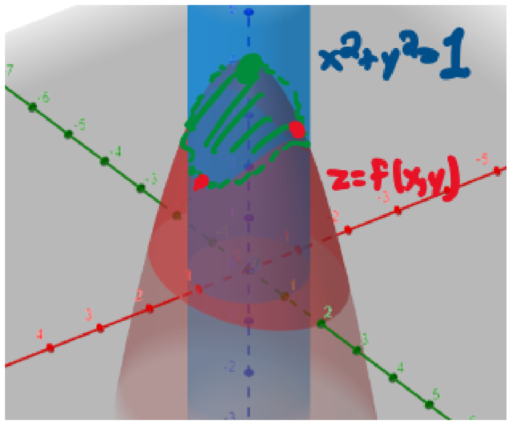
- **<u>Fact:</u>** If f is continuous on a closed and bounded domain, then it achieves **both** global maxima and minima. Morover, these extrema must occur at a

  (i) **critical point**, i.e. where $\nabla f = \vec{0}$.

  (ii) **singular point**, i.e. ∇f does not exist.

  (iii) **boundary point**

1. Find the max/min for $f(x,y) = 4 - 2x^2 - y^2$ on the region $x^2 + y^2 \leq 1$.



We have the following:

(i) The critical points of f occur at points where $\nabla f = \vec{0}$. This can be thought of as the direction of steepest ascent being zero, as that point is already at the peak of the "hill", so to speak. Now,

$$\begin{aligned}
\nabla f &= \begin{bmatrix} f_x \\ f_y \end{bmatrix} \\
&= \begin{bmatrix} (4 - 2x^2 - y^2)_x \\ (4 - 2x^2 - y^2)_y \end{bmatrix} \\
&= \begin{bmatrix} (0 - 4x - 0) \\ (0 - 0 - 2y) \end{bmatrix} \\
&= \begin{bmatrix} -4x \\ -2y \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \end{bmatrix} .
\end{aligned}$$

So, we have that $-4x = 0$ and $-2y = 0$, which means that the critical points of f occur when $x = 0$ and $y = 0$; that is when $(x,y) = (0,0)$.

(ii) The singular points of f are points where ∇f does not exist. In our example, we have that ∇f always exists, so there are no singular points.

(iii) The boundary points are those points literally on the boundary $x^2 + y^2 = 1$.

We need to substitute the boundary into the function $x^2 + y^2 = 1$. Rearranging this gives us $y^2 = 1 - x^2$. So, we get that

$$\begin{aligned} f(x, y) &= 4 - 2x^2 - y^2 \\ &= 4 - 2x^2 - (1 - x^2) \\ &= 4 - 2x^2 - 1 + x^2 \\ &= 3 - x^2 \ , \end{aligned}$$

where (x, y) belongs to $x^2+y^2 = 1$ and $-1 \leq x \leq 1$ (since the cross-section of the cylinder is a circle of radius 1). We can define.

$$g(x) := f(x, y) = 3 - x^2 \ .$$

Then optimizing g on the closed interval $[-1, 1]$ using the closed interval method gives us

$$g'(x) = -2x = 0 \quad \text{when} \quad x = 0$$

That is, $x = 0$ is a critical point for $g(x)$ since $g'(x) = 0$ at $x = 0$ and $x = 0$ is defined for $g(x)$. We call g the "boundary function" for this optimization. Then evaluating g at the critical point and the endpoints gives us

$$\begin{aligned} g(0) &= 3 - (0)^2 = 3 \\ g(-1) &= 3 - (-1)^2 = 3 - 1 = 2 \\ g(1) &= 3 - (1)^2 = 3 - 1 = 2 \ . \end{aligned}$$

Then since $g(x) := f(x, y) = 4 - 2x^2 - y^2$, we get that

$$f(x, y) = 4 - 2x^2 - y^2 = 3$$

when $x = 0$ and $y = \pm 1$, and

$$f(x, y) = 4 - 2x^2 - y^2 = 2$$

when $x = \pm 1$ and $y = 0$. Now, we compare f on the critical points and boundary points. We have that f evaluated at the critical point $(x, y) = (0, 0)$ gives

$$f(0, 0) = 4 \ .$$

For the boundary, the above analysis says we need only look at $f(0, \pm 1)$ and $f(\pm 1, 0)$. So,

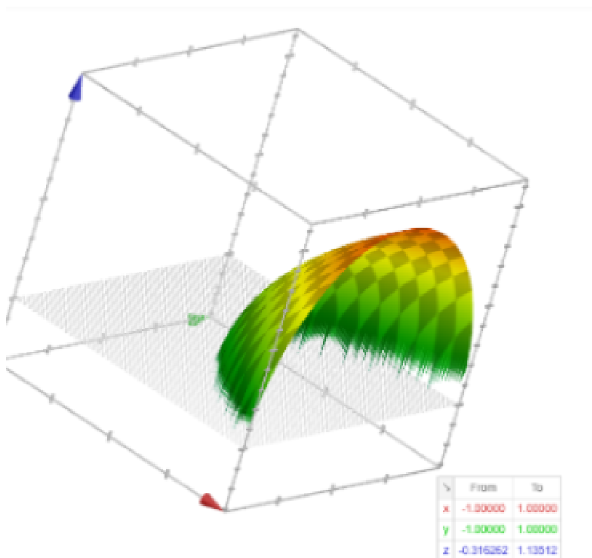
$$f(0, \pm 1) = 3$$

and

$$f(\pm 1, 0) = 2 \ .$$

Thus, $f(0, 0) = 4$ is the maximum and $f(\pm 1, 0)$ is the minimum. That is, the maximum point is $(0, 0, 4)$, and the minimum points are $(-1, 0, 2)$ and $(1, 0, 2)$.

2. Find the max/min for $f(x,y) = \sqrt{x - y^2}$ on $(x-4)^2 + y^2 = 1$ (circle of radius 1 centered at $4, 0$.



We should note that

$$\begin{aligned} \mathrm{dom}(f) &= \{(x,y) : x - y^2 \geq 0\} \\ &= \{(x,y) : x \geq y^2\} \ . \end{aligned}$$

We have the following:

(i) The critical points of f occur at points where $\nabla f = \vec{0}$. For the partials, we have that

$$\begin{aligned} f_x &= \frac{\partial f}{\partial x} \\ &= \frac{\partial}{\partial x} \left[f(x,y) \right] \\ &= \frac{\partial}{\partial x} \left[\sqrt{x - y^2} \right] \\ &= \frac{\partial \left(\sqrt{x - y^2} \right)}{\partial (x - y^2)} \cdot \frac{\partial (x - y^2)}{\partial x} \\ &= \frac{1}{2\sqrt{x - y^2}} \cdot \left(\frac{\partial}{\partial x} [x] - \frac{\partial}{\partial x} \left[y^2 \right] \right) \\ &= \frac{1}{2\sqrt{x - y^2}} \cdot (1 - 0) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\sqrt{x-y^2}} \cdot 1 \\
&= \frac{1}{2\sqrt{x-y^2}}
\end{aligned}
$$

and

$$
\begin{aligned}
f_y &= \frac{\partial f}{\partial y} \\
&= \frac{\partial}{\partial y}\left[f(x,y)\right] \\
&= \frac{\partial}{\partial y}\left[\sqrt{x-y^2}\right] \\
&= \frac{\partial\left(\sqrt{x-y^2}\right)}{\partial(x-y^2)} \cdot \frac{\partial(x-y^2)}{\partial y} \\
&= \frac{1}{2\sqrt{x-y^2}} \cdot \left(\frac{\partial}{\partial y}[x] - \frac{\partial}{\partial y}\left[y^2\right]\right) \\
&= \frac{1}{2\sqrt{x-y^2}} \cdot (0-2y) \\
&= -\frac{2y}{2\sqrt{x-y^2}} \ .
\end{aligned}
$$

However, we get that

$$
\begin{aligned}
\nabla f &= \begin{bmatrix} f_x \\ f_y \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2\sqrt{x-y^2}} \\ -\frac{2y}{2\sqrt{x-y^2}} \end{bmatrix} \\
&\neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \ ,
\end{aligned}
$$

which means that f has no critical points.

(ii) f has no singular points.

(iii) f has boundary points at $(x-4)^2 + y^2 = 1$.

Now, we need to subsitute the boundary into the function $(x-4)^2+y^2 = 1$. Rearranging the boundary equation for y^2 gives us $y^2 = 1-(x-4)^2$. So, we get that

$$
f(x,y) = \sqrt{x-y^2}
$$

$$
\begin{aligned}
&= \sqrt{x - (1 - (x-4)^2)} \\
&= \sqrt{x - 1 + (x-4)^2}\ ,
\end{aligned}
$$

where (x, y) is a point belonging to $(x-4)^2 + y^2 = 1$ and $3 \leq x \leq 5$ (since this is the circle of radius 1 centered at $(4, 0)$). Define $g(x) := \sqrt{x - 1 + (x-4)^2}$ to be the boundary function for this optimization. Then optimizing g on the closed interval $[3, 5]$ gives us

$$
\begin{aligned}
g'(x) &= \frac{d}{dx}[g(x)] \\
&= \frac{d}{dx}\left[\sqrt{x - 1 + (x-4)^2}\right] \\
&= \frac{d\left(\sqrt{x - 1 + (x-4)^2}\right)}{d(x - 1 + (x-4)^2)} \cdot \frac{d(x - 1 + (x-4)^2)}{dx} \\
&= \frac{1}{2\sqrt{x - 1 + (x-4)^2}} \cdot \left(\frac{d}{dx}[x] - \frac{d}{dx}[1] + \frac{d}{dx}\left[(x = 4)^2\right]\right) \\
&= \frac{1}{2\sqrt{x - 1 + (x-4)^2}} \cdot \left(1 - 0 + \left(\frac{d((x-4)^2)}{d(x-4)} \cdot \frac{d(x-4)}{dx}\right)\right) \\
&= \frac{1}{2\sqrt{x - 1 + (x-4)^2}} \cdot (1 + (2(x-4) \cdot 1)) \\
&= \frac{1}{2\sqrt{x - 1 + (x-4)^2}} \cdot (1 + 2(x-4)) \\
&= \frac{1 + 2(x-4)}{2\sqrt{x - 1 + (x-4)^2}}\ .
\end{aligned}
$$

Now, we know that critical points for g exist when $g'(x) = 0$ or $g'(x)$ is undefined. However, g is continuous everywhere, so we don't need to consider when $g(x)$ is undefined. So, g has a critical point at

$$
\begin{aligned}
g'(x) &= 0 \\
\frac{1 + 2(x-4)}{2\sqrt{x - 1 + (x-4)^2}} &= 0 \\
1 + 2(x-4) &= 0 \\
1 + 2x - 8 &= 0 \\
2x - 7 &= 0 \\
2x &= 7 \\
x &= \frac{7}{2}\ .
\end{aligned}
$$

Now, evaluating g at the critical point and the endpoints of the interval $[3, 5]$, we have that

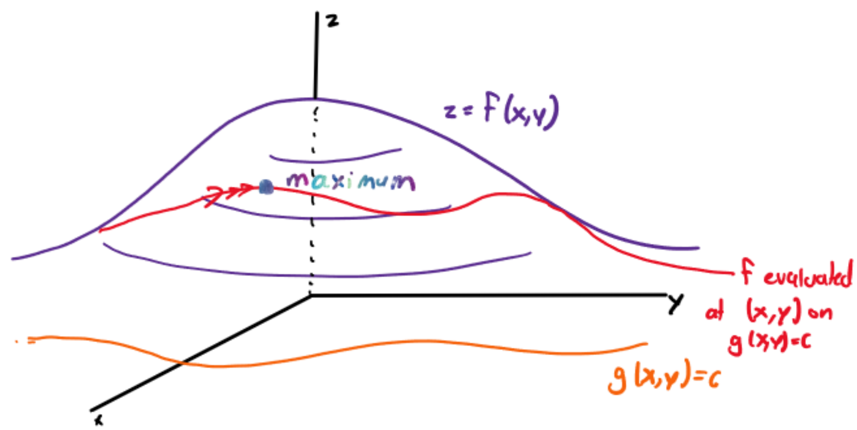
$$
g\left(\frac{7}{2}\right) = \sqrt{2.75}
$$

$$g(3) = \sqrt{3}$$
$$g(5) = \sqrt{5}\ .$$

Thus, $\sqrt{2.75}$ is a minumum point and $\sqrt{5}$ is a maximum point.

Method of Lagrange Multipliers

- Type of problem to address: Find the max/min for $f(x, y)$ (or $f(x, y, z)$ or $f(x_1, x_2, \ldots, x_n)$) subject to a constraint $g(x, y) = c$ (we are only looking at boundary points).



Move everything to 2D using a contour plot

plotting a bunch of $f(x,y)=d$ for various d.

- **<u>Key observation:</u>** The gradients of f and g are **parallel** at the location of the maximum.

- **Method of Lagrange Multipliers**: Find **all** places where the gradients are parallel. Evaluate f at all of them. We write this as

$$\nabla f(x,y) = \lambda \nabla g(x,y)$$
$$g(x,y) = c \ ,$$

where λ is called the Lagrange multiplier. This is a system of equations that can be solved!

1. Find the max/min for $f(x, y, z) = x + 2y + z^2$ on $x^2 + y^2 + z^2 \leq 1$.

   Note that there are no singular points in $x^2 + y^2 + z^2 < 1$ (interior) since ∇f exists at every point. Also, since

$$\begin{aligned}
\nabla f &= \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} \\
&= \begin{bmatrix} (x + 2y + z^2)_x \\ (x + 2y + z^2)_y \\ (2 + 2y + z^2)_z \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 2 \\ 2z \end{bmatrix} \\
&\neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad ,
\end{aligned}$$

   f does not have any critical points. So, we only consider boundary points. We will use Lagrange for the boundary. Let $g(x, y, z) = x^2 + y^2 + z^2$ be the boundary function. Then we find all the places where the gradient of f and g are parallel, which is given by

$$\begin{aligned}
\nabla f(x, y, z) &= \lambda \nabla g(x, y, z) \\
\begin{bmatrix} 1 \\ 2 \\ 2z \end{bmatrix} &= \lambda \begin{bmatrix} g_x \\ g_y \\ g_z \end{bmatrix} \\
\begin{bmatrix} 1 \\ 2 \\ 2z \end{bmatrix} &= \lambda \begin{bmatrix} (x^2 + y^2 + z^2)_x \\ (x^2 + y^2 + z^2)_y \\ (x^2 + y^2 + z^2)_z \end{bmatrix} \\
\begin{bmatrix} 1 \\ 2 \\ 2z \end{bmatrix} &= \lambda \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix} \\
\begin{bmatrix} 1 \\ 2 \\ 2z \end{bmatrix} &= \begin{bmatrix} 2\lambda x \\ 2\lambda y \\ 2\lambda z \end{bmatrix} \quad .
\end{aligned}$$

   From this, we get the following system of equations

$$1 = 2\lambda x \tag{1}$$
$$2 = 2\lambda y \tag{2}$$
$$2z = 2\lambda z \tag{3}$$
$$x^2 + y^2 + z^2 = 1 \tag{4}$$

Note that it is not necessary to find λ. Only x, y, z are of interest to us. Now, from observation, we see that x, y, λ are non-zero. They must be non-zero because equations (1), (2), and (3) will not hold if x, y, or λ were zero. So, if we divide equation (2) by equation (1), we get that

$$\begin{aligned} \frac{2}{1} &= \frac{2\lambda y}{2\lambda x} \\ 2 &= \frac{y}{x} \\ y &= 2x \ . \end{aligned}$$

Now, we must consider the following cases.

- (Case 1: $z = 0$). If $z = 0$, then equation (3) still holds, and equation (4) becomes

$$\begin{aligned} x^2 + y^2 + z^2 &= 1 \\ x^2 + y^2 + 0^2 &= 1 \\ x^2 + y^2 &= 1 \ . \end{aligned}$$

  Then using the fact that $y = 2x$, which we derived earlier, it follows that

$$\begin{aligned} x^2 + y^2 &= 1 \\ x^2 + (2x)^2 &= 1 \\ x^2 + 4x^2 &= 1 \\ 5x^2 &= 1 \\ x^2 &= \frac{1}{5} \\ x &= \pm\frac{1}{\sqrt{5}} \ . \end{aligned}$$

  Then plugging this result back into $y = 2x$ gives us that

$$y = 2x = 2 \cdot \left(\pm\frac{1}{\sqrt{5}}\right) = \pm\frac{2}{\sqrt{5}} \ .$$

  Hence, this case gives the solutions:

$$\begin{aligned} &\left(\frac{1}{\sqrt{5}},\ \frac{2}{\sqrt{5}},\ 0\right) \\ &\left(\frac{1}{\sqrt{5}},\ -\frac{2}{\sqrt{5}},\ 0\right) \\ &\left(-\frac{1}{\sqrt{5}},\ \frac{2}{\sqrt{5}},\ 0\right) \\ &\left(-\frac{1}{\sqrt{5}},\ -\frac{2}{\sqrt{5}},\ 0\right) \ . \end{aligned}$$

- (Case 2: $z \neq 0$). If $z \neq 0$, then equation (3) gives us

  $$\begin{aligned} 2z &= 2\lambda z \\ 1 &= \lambda \ . \end{aligned}$$

  Here we see that we just so happened to find λ, which makes our lives a little easier (even though knowledge of λ isn't really required for these problems). Nevertheless, since $\lambda = 1$, equation (1) gives us

  $$\begin{aligned} 1 &= 2\lambda x \\ 1 &= 2x \\ x &= \frac{1}{2} \ , \end{aligned}$$

  and equation (2) gives us

  $$\begin{aligned} 2 &= 2\lambda y \\ 2 &= 2y \\ y &= 1 \ . \end{aligned}$$

  So, plugging these into equation (4) gives us

  $$\begin{aligned} x^2 + y^2 + z^2 &= 1 \\ \left(\frac{1}{2}\right)^2 + (1)^2 + z^2 &= 1 \\ \frac{1}{4} + 1 + z^2 &= 1 \\ \frac{1}{4} + z^2 &= 0 \\ z^2 &= -\frac{1}{4} \ , \end{aligned}$$

  which means that the system of equations has no solution for the case that $z =\neq 0$, since we cannnot solve for z.

So, we only care about case 1. Now, we evaluate f at the candidate points we found from the only valid case, which was case 1. So, we get that

$$\begin{aligned} f\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right) &= \frac{1}{\sqrt{5}} + 2\left(\frac{2}{\sqrt{5}}\right) + 0^2 \\ &= \frac{1}{\sqrt{5}} + \frac{4}{\sqrt{5}} \\ &= \frac{5}{\sqrt{5}} \\ &= \frac{5\sqrt{5}}{5} \end{aligned}$$

$$= \sqrt{5}\ ,$$

$$\begin{aligned} f\left(\frac{1}{\sqrt{5}},\ -\frac{2}{\sqrt{5}},\ 0\right) &= \frac{1}{\sqrt{5}} + 2\left(-\frac{2}{\sqrt{5}}\right) + 0^2 \\ &= \frac{1}{\sqrt{5}} - \frac{4}{\sqrt{5}} \\ &= -\frac{3}{\sqrt{5}}\ , \end{aligned}$$

$$\begin{aligned} f\left(-\frac{1}{\sqrt{5}},\ \frac{2}{\sqrt{5}},\ 0\right) &= -\frac{1}{\sqrt{5}} + 2\left(\frac{2}{\sqrt{5}}\right) + 0^2 \\ &= -\frac{1}{\sqrt{5}} + \frac{4}{\sqrt{5}} \\ &= \frac{3}{\sqrt{5}}\ , \end{aligned}$$

and

$$\begin{aligned} f\left(-\frac{1}{\sqrt{5}},\ -\frac{2}{\sqrt{5}},\ 0\right) &= -\frac{1}{\sqrt{5}} + 2\left(-\frac{2}{\sqrt{5}}\right) + 0^2 \\ &= -\frac{1}{\sqrt{5}} - \frac{4}{\sqrt{5}} \\ &= -\frac{5}{\sqrt{5}} \\ &= -\frac{5\sqrt{5}}{5} \\ &= -\sqrt{5}\ . \end{aligned}$$

Thus, the maximum is $\sqrt{5}$ which occurs at $\left(\frac{1}{\sqrt{5}},\ \frac{2}{\sqrt{5}},\ 0\right)$, and the minimum is $-\sqrt{5}$ which occurs at $\left(-\frac{1}{\sqrt{5}},\ -\frac{2}{\sqrt{5}},\ 0\right)$.

2. Find the max/min for $f(x,y,z) = xyz$ subject to the constraint $x^2+2y^2+z^2 = 4$.

Let $g(x,y,z) = x^2 + 2y^2 + z^2$. Then

$$\begin{aligned}
\nabla f &= \lambda \nabla g \\
\begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} &= \lambda \begin{bmatrix} g_x \\ g_y \\ g_z \end{bmatrix} \\
\begin{bmatrix} (xyz)_x \\ (xyz)_y \\ (xyz)_z \end{bmatrix} &= \lambda \begin{bmatrix} (x^2+2y^2+z^2)_x \\ (x^2+2y^2+z^2)_y \\ (x^2+2y^2+z^2)_z \end{bmatrix} \\
\begin{bmatrix} yz \\ xz \\ xy \end{bmatrix} &= \lambda \begin{bmatrix} 2x \\ 4y \\ 2z \end{bmatrix} \\
\begin{bmatrix} yz \\ xz \\ xy \end{bmatrix} &= \begin{bmatrix} 2\lambda x \\ 4\lambda y \\ 2\lambda z \end{bmatrix}
\end{aligned}$$

So, we get the following system of equations:

$$\begin{aligned}
yz &= 2\lambda x && (2) \\
xz &= 4\lambda y && (3) \\
xy &= 2\lambda z && (4) \\
x^2 + 2y^2 + z^2 &= 4 && (1)
\end{aligned}$$

Observe that if any one of x, y, or z are 0, then we get that $f(x,y,z) = 0$. So, we assume that x, y, z are non-zero. Also, equation (1) would not hold if $x = y = z = 0$. Now, if we divide equation (3) by equation (2), we get that

$$\begin{aligned}
\frac{xz}{yz} &= \frac{4\lambda y}{2\lambda x} \\
\frac{x}{y} &= \frac{2y}{x} \\
x^2 &= 2y^2 \ .
\end{aligned}$$

If we divide equation (4) by equation (2), we get that

$$\begin{aligned}
\frac{xy}{yz} &= \frac{2\lambda z}{2\lambda x} \\
\frac{x}{z} &= \frac{z}{x} \\
x^2 &= z^2 \ .
\end{aligned}$$

Then plugging this information into equation (1) gives us

$$\begin{aligned} 4 &= x^2 + 2y^2 + z^2 \\ &= x^2 + x^2 + x^2 \\ &= 3x^2 \ . \end{aligned}$$

From this, we get that

$$\begin{aligned} 4 &= 3x^2 \\ \frac{4}{3} &= x^2 \\ x &= \pm\sqrt{\frac{4}{3}} \\ x &= \pm\frac{2}{\sqrt{3}} \ . \end{aligned}$$

Then since $x^2 = z^2$, this means that

$$z = \pm\frac{2}{\sqrt{3}} \ .$$

Also, since $x^2 = 2y^2$, this means that

$$\begin{aligned} 2y^2 &= \frac{4}{3} \\ y^2 &= \frac{4}{6} \\ y^2 &= \frac{2}{3} \\ y &= \pm\frac{\sqrt{2}}{\sqrt{3}} \ . \end{aligned}$$

This gives us a total of 8 points to check! Some of these points will evaluate to the same value due to the way the function f is defined, luckily.

$$\begin{aligned} f\left(\pm\frac{2}{\sqrt{3}},\ \pm\frac{\sqrt{2}}{\sqrt{3}},\ \frac{2}{\sqrt{3}}\right) &= \frac{4\sqrt{2}}{\left(\sqrt{3}\right)^3} \ , \\ f\left(\pm\frac{2}{\sqrt{3}},\ \frac{\sqrt{2}}{\sqrt{3}},\ \pm\frac{2}{\sqrt{3}}\right) &= \frac{4\sqrt{2}}{\left(\sqrt{3}\right)^3} \ , \\ f\left(\frac{2}{\sqrt{3}},\ \pm\frac{\sqrt{2}}{\sqrt{3}},\ \pm\frac{2}{\sqrt{3}}\right) &= \frac{4\sqrt{2}}{\left(\sqrt{3}\right)^3} \ , \\ f\left(\ldots\right) &= -\frac{4\sqrt{2}}{\left(\sqrt{3}\right)^3} \ . \end{aligned}$$

3. Find the max/min for $f(x,y,z) = x^2 - yz$ subject to the constraint $x^2 + y^2 + z^2 \leq 1$.

   - Step 1: Find criticial points and singular points in the interior $x^2 + y^2 + z^2 < 1$. We have that

     $$\nabla f = \begin{bmatrix} (x^2 - yz)_x \\ (x^2 - yz)_y \\ (x^2 - yz)_z \end{bmatrix} = \begin{bmatrix} 2x \\ -z \\ -y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

     when $x = y = z = 0$. So, we have one critical point which occurs at $f(0,0,0)$. Note that there are no singular points as ∇f exists for every point for f.

   - Step 2: Use Lagrange on the boundary $x^2 + y^2 + z^2 = 1$. Let $g(x) = x^2 + y^2 + z^2$. Then

     $$\begin{aligned} \nabla f &= \lambda \nabla g \\ \begin{bmatrix} 2x \\ -z \\ -y \end{bmatrix} &= \lambda \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix} \\ \begin{bmatrix} 2x \\ -z \\ -y \end{bmatrix} &= \begin{bmatrix} 2\lambda x \\ 2\lambda y \\ 2\lambda z \end{bmatrix} . \end{aligned}$$

     This gives us the following system of equations:

     $$x^2 + y^2 + z^2 = 1 \tag{1}$$
     $$2x = 2\lambda x \tag{2}$$
     $$-z = 2\lambda y \tag{3}$$
     $$-y = 2\lambda z \tag{4}$$

     Note that if $\lambda = 0$, we have that $\nabla f = 0$, which we already dealt with when we found the critical point.

Parametric Curves

Definition (Parametric Curve)

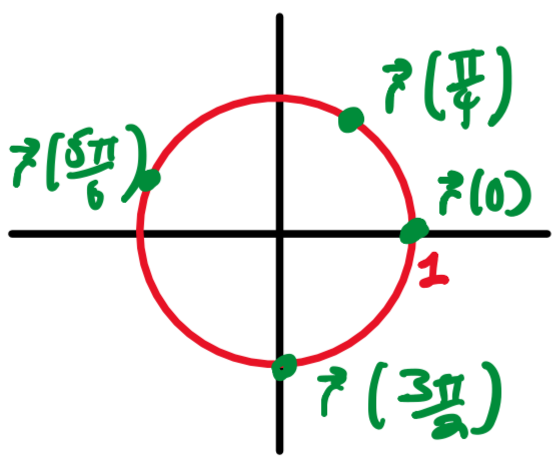
A parametric curve C is the range of a vector-valued function $\vec{r}(t)$ that takes on values in $\mathbb{R}^2 \to \mathbb{R}^3$, or generally $\mathbb{R}^n$. $\vec{r}$ is called a **parameterization of** C (parameterizations are generally **not unique**).

Remark: C is usually expressed as one or more several equations in cartesian coordinates ($x, y, z,$ **etc**).

1. Consider the **unit circle** given by $x^2 + y^2 = 1$. This equation has a standard parameterization

$$\vec{r}(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} ,$$

for $t \in [0, 2\pi]$.



Note that

$$\vec{r}(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$$

for $t \in [0, \pi]$ is NOT a parametization for the unit circle (but it is for the upper semicircle). Other choices for parameterizations are

$$\vec{r}(t) = \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix} \text{ for } 0 \leq t \leq 2\pi ,$$

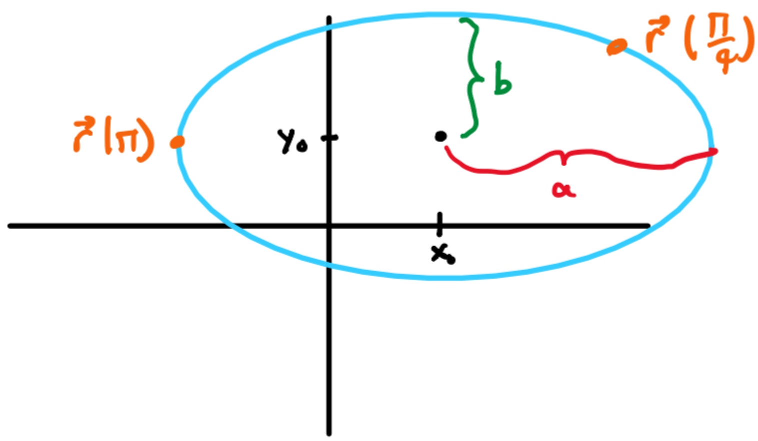
$$\vec{r}(t) = \begin{bmatrix} -\cos(t) \\ \sin(t) \end{bmatrix} \text{ for } 0 \le t \le 2\pi \ ,$$

$$\vec{r}(t) = \begin{bmatrix} \cos(2\pi t) \\ \sin(2\pi t) \end{bmatrix} \text{ for } 0 \le t \le 1 \ .$$

These all satisfy $x^2 + y^2 = 1$ and "hit" every such (x, y) on the curve.

2. Consider a **general ellipse**. The ellipse with an x-radius of a and a y-radius of b, centered at (x_0, y_0) is given by

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1 \ .$$



To get the standard parameterization, we can reverse engineer with the equation of an ellipse. We have that

$$\frac{(x-x_0)^2}{a^2} = \cos^2(t) \ ,$$
$$\frac{(y-y_0)^2}{b^2} = \sin^2(t) \ .$$

This gives us

$$x(t) = x_0 + a\cos(t)$$
$$y(t) = y_0 + b\sin(t) \ .$$

Thus, the standard parameterization of a general ellipse is given by

$$\vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x_0 + a\cos(t) \\ y_0 + b\sin(t) \end{bmatrix} \ ,$$

for $0 \le t \le 2\pi$.

3. Consider a "function type" parametric curve. For $y = g(x)$, we can rewrite this as $\vec{r}(t) = \begin{bmatrix} t \\ g(t) \end{bmatrix}$ (so $t = x$).

   Perhaps we should consider an example. Let $g(x) = x^2 + 1$. Then this can be parameterized as

$$\vec{r}(t) = \begin{bmatrix} t \\ g(t) \end{bmatrix} = \begin{bmatrix} t \\ t^2 + 1 \end{bmatrix} .$$

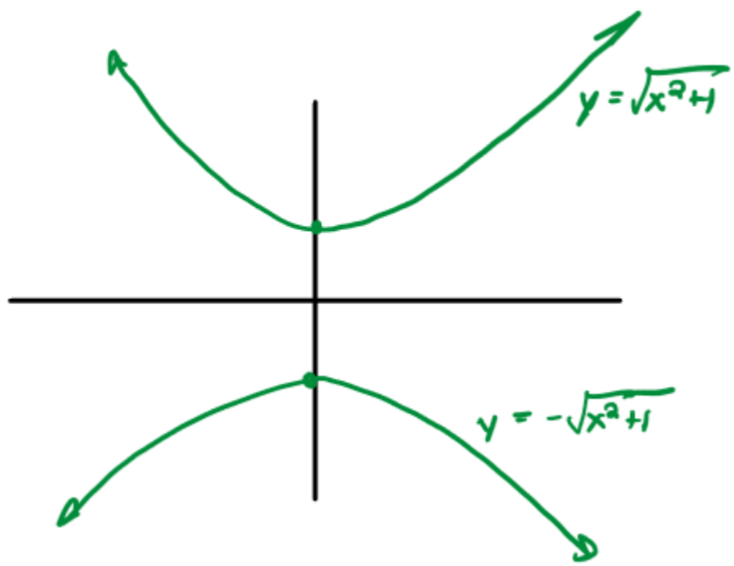
4. Consider the hyperbolic cosine and sine functions

$$\cosh(t) = \frac{e^t + e^{-t}}{2} \ ,$$
$$\sinh(t) = \frac{e^t - e^{-t}}{2} \ .$$

(Indeed, we can compare these to

$$\cos(t) = \frac{e^{it} + e^{-it}}{2} \ ,$$
$$\sin(t) = \frac{e^{it} - e^{-it}}{2} \ ,$$

which comes from $e^{it} = \cos(t) + \sin(t)$.) These naturally parameterize hyperbolas. E.g. $y^2 - x^2 = 1$

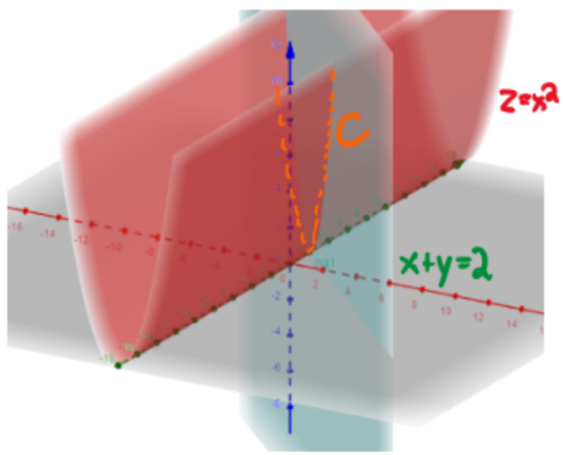


We can use the fact that $\cosh^2(t) - \sinh^2 t = 1$ (hyperbolic trig identity).

- For the upper half, one paramaterization is $\vec{r}(t) = \begin{bmatrix} \sinh(t) \\ \cosh(t) \end{bmatrix}$, $t \in \mathbb{R}$.
- For the upper half, one parameterization is $\vec{r}(t) = \begin{bmatrix} \sinh(t) \\ -\cosh(t) \end{bmatrix}$, $t \in \mathbb{R}$.

5. (Three-dimensions) Find a parametrization for the curve of intersection between the following pairs of surfaces.

   (a) $z = x^2$ and $x + y = 2$.



Note that $z = x^2$ is a parabolic cylinder (parallel to the y-axis) and $x + y = 2$ runs parallel to the z-axis. We want

$$\vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} ,$$

where each of $x(t)$, $y(t)$, and $z(t)$ must satisfy both the equations $z = x^2$ and $x + y = 2$. Try $x = t$ (for a "function type" $z = x^2$) so that $z = x^2 = t^2$. Then the second equation becomes

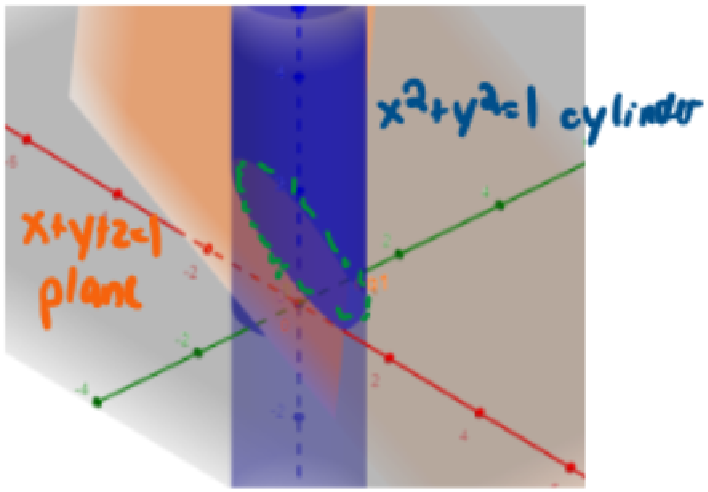
$$\begin{aligned} x + y &= 2 \\ y &= 2 - x \\ y &= 2 - t\ . \end{aligned}$$

Hence, a parameterization for the curve of intersection between $z = x^2$ and $x + y = 2$ is

$$\vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} t \\ 2 - t \\ t^2 \end{bmatrix} ,$$

for $t \in \mathbb{R}$.

(b) $x^2 + y^2 = 1$ and $x + y + z = 1$.



Note that we previously found the parameterization for the unit circle $x^2 + y^2 = 1$:

$$x = \cos(t)$$
$$y = \sin(t)$$

for $0 \le t \le 2\pi$. Then the second equation gives

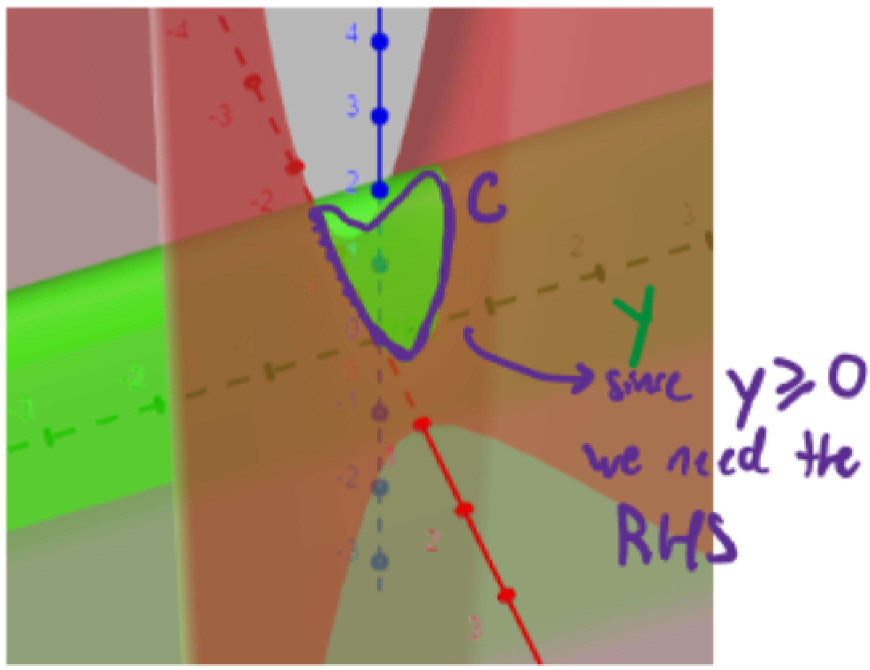
$$\begin{aligned} x + y + z &= 1 \\ z &= 1 - x - y \\ z &= 1 - \cos(t) - \sin(t) \ . \end{aligned}$$

Thus, a parameterization for the curve of intersection between $x^2 + y^2 = 1$ and $x + y + z = 1$ is

$$\vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} \cos(t) \\ \sin(t) \\ 1 - \cos(t) - \sin(t) \end{bmatrix} ,$$

for $0 \le t \le 2\pi$.

(c) $x^2 - 3y^2 + z = 1$ and $3x^2 + z = 2$, $y \geq 0$.



Note that $x^2 - 3y^2 + z = 1$ is a hyperboloid and $3x^2 + z = 2$ is a parabolic cylinder. There are multiple ways to approach these types of problems. Let's try and attack it from different angles (not literal angles, of course).

- <u>**Attempt 1:**</u> Let $x = t$. Then

$$\begin{aligned} 3x^2 + z &= 2 \\ z &= 2 - 3x^2 \\ z &= 2 - 3t^2 \ . \end{aligned}$$

  So, plugging x and z into the first equation to solve for y gives us

$$\begin{aligned} x^2 - 3y^2 + z &= 1 \\ 3y^2 &= x^2 + z - 1 \\ 3y^2 &= t^2 + (2 - 3t^2) - 1 \\ 3y^2 &= t^2 + 2 - 3t^2 - 1 \\ 3y^2 &= -2t^2 + 1 \\ 3y^2 &= 1 - 2t^2 \end{aligned}$$

$$
\begin{aligned}
y^2 &= \frac{1-2t^2}{3} \\
y &= \sqrt{\frac{1-2t^2}{3}}\ ,
\end{aligned}
$$

where we only care about the positive root since we are concerned with only $y \geq 0$. Note that $1 - 2t^2 \geq 0$, and so

$$
\begin{aligned}
&\frac{1}{2} \geq t^2 \\
\implies &-\frac{1}{\sqrt{2}} \leq t \leq \frac{1}{\sqrt{2}}\ .
\end{aligned}
$$

- **Attempt 2:** From the first equation, we have that

$$
\begin{aligned}
x^2 - 3y^2 + z &= 1 \\
x^2 &= 1 + 3y^2 - z\ ,
\end{aligned}
$$

and so solving the second equation for z gives us

$$
\begin{aligned}
3x^2 + z &= 2 \\
z &= 3x^2 - 2 \\
z &= 3(1 + 3y^2 - z) - 2 \\
z &= 3 + 9y^2 - 3z - 2 \\
z &= 1 + 9y^2 - 3z \\
&\vdots
\end{aligned}
$$

(To be continued ...)