MATH 367 - Week 8-9 Notes

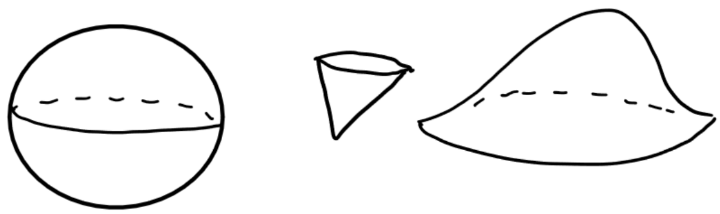
Jasraj Sandhu

October 2023

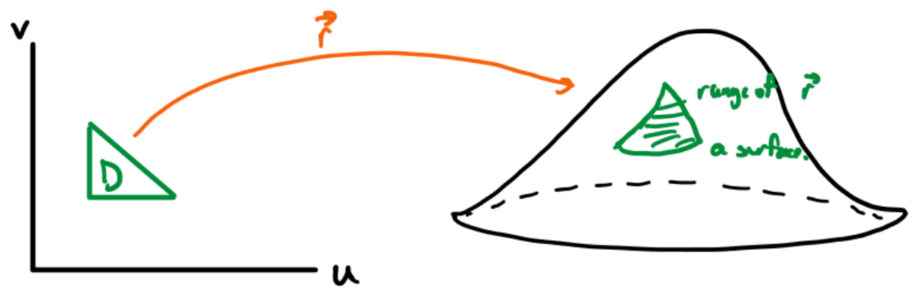
Parametric Surfaces

Definition (Parametric Surface)

The **range** of a function $\vec{r} : D \to \mathbb{R}^3$, where D is a region in $\mathbb{R}^2$ (as a subset of $\mathbb{R}^3$) is called a **parametric surface**.



- Usually, we use variables u, v for $\vec{r}$.



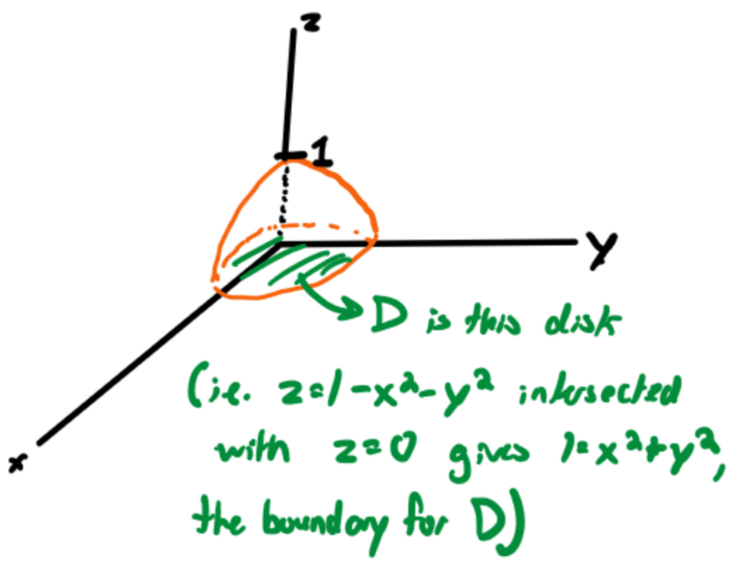
1. ("Function-type") Suppose $\mathscr{S}$ is a piece of the function $z = f(x, y, z)$, where $(x, y) \subseteq D$. Then

$$\vec{r}(x,y) = \begin{bmatrix} x \\ y \\ f(x,y) \end{bmatrix},$$

where $(x, y) \in D$, parameterizes $\mathscr{S}$. Note that from the definition of a parametric surface, the domain of $\vec{r}$, which we call D, is a region in $\mathbb{R}^2$. Similarly, for $y = g(x, z)$ and $x = h(y, z)$,

$$\vec{r}(x,z) = \begin{bmatrix} x \\ g(x,z) \\ z \end{bmatrix} \quad \text{and} \quad \vec{r}(y,z) = \begin{bmatrix} h(y,z) \\ y \\ z \end{bmatrix},$$

respectively. Consider the following example: $z = 1 - x^2 - y^2$ with $z \geq 0$ (a piece of a parabaloid).



Here,

$$\vec{r}(x,y) = \begin{bmatrix} x \\ y \\ 1 - x^2 - y^2 \end{bmatrix}$$

with (x, y) that satisfy $x^2 + y^2 \leq 1$ (since $z \geq 0$).

We can also use **cylindrical coordinates**. Recall that for cylindrical coordinates,

$$\begin{aligned} r &= \sqrt{x^2+y^2}\ , \\ \theta &= \tan^{-1}\left(\frac{y}{x}\right)\ , \\ x &= r\cos\theta\ , \\ y &= r\sin\theta\ . \end{aligned}$$

So, we have that

$$\begin{aligned} z &= 1-x^2-y^2 \\ &= 1-(x^2+y^2) \\ &= 1-\left(\sqrt{x^2+y^2}\ \right)^2 \\ &= 1-r^2\ . \end{aligned}$$

Now, we know that D is given by $x^2+y^2 \leq 1$. That is, $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$. So, we have

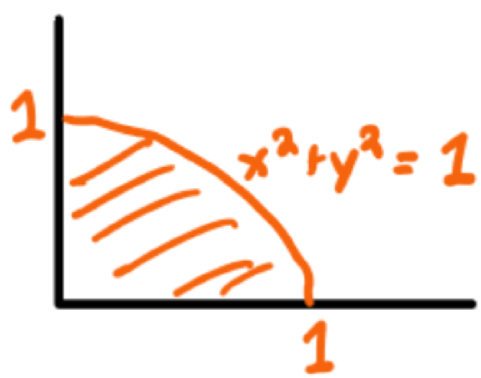
$$\vec{r}(r,\theta) = \begin{bmatrix} x \\ y \\ 1-x^2-y^2 \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ 1-r^2 \end{bmatrix}$$

2. (Implicit Equations) Consider $x^2 + y^2 + z^2 = R^2$, which is a sphere with radius R. Find a parameterization for the unit sphere $x^2 + y^2 + z^2 = 1$ in the first octant $(x \geq 0,\ y \geq 0,\ z \geq 0)$.

We can do this as a "function-type":

$$z = +\sqrt{1 - x^2 - y^2}\ ,$$

since $z \geq 0$.



So, we have that

$$\vec{r}(x, y) = \begin{bmatrix} x \\ y \\ \sqrt{1 - x^2 - y^2} \end{bmatrix}$$

where

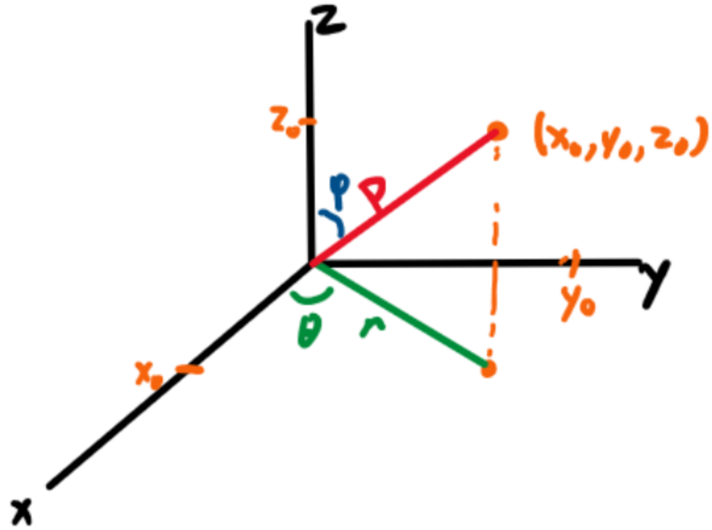
$$0 \leq x \leq 1$$

$$0 \leq y \leq \sqrt{1-x^2} \ .$$

In cylindrical coordinates, this is

$$\vec{r}(r,\theta) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ \sqrt{1-r^2} \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ \sqrt{1-r^2} \end{bmatrix} \ , \quad 0 \leq r \leq 1, \ 0 \leq \theta \leq \frac{\pi}{2}$$

As another alternative, we can use spherical coordinates!



For spherical coordinates, we have that

$$\begin{aligned}
\rho^2 &= x^2 + y^2 + z^2 \\
\theta &= \tan^{-1}\left(\frac{y}{x}\right) && \text{(same as polar)} \\
\varphi &= \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) \\
x &= \rho\sin\varphi\cos\theta \\
y &= \rho\sin\varphi\sin\theta \\
z &= \rho\cos\varphi \ ,
\end{aligned}$$

where $\rho \geq 0$, $0 \leq \theta \leq 2\pi$ and $0 \leq \varphi \leq \pi$.

So, we have the following:

- For the full sphere of radius R, $x^2 + y^2 + z^2 = R^2$ becomes $\rho = R$. This is because

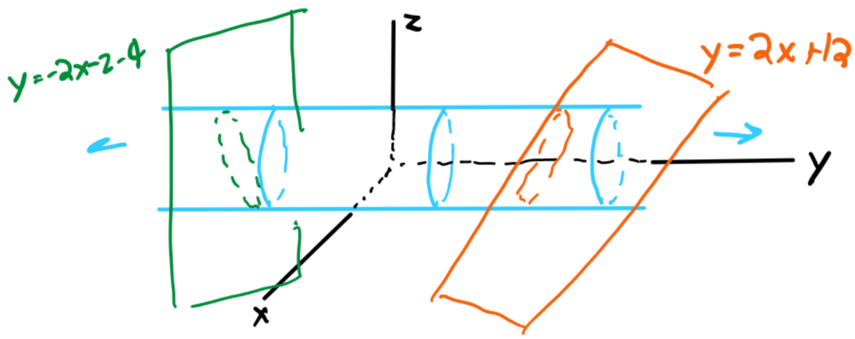
$$\begin{aligned} \rho^2 &= x^2 + y^2 + z^2 = R^2 \\ \rho &= R \ . \end{aligned}$$

- The first octant portion becomes $\rho = R$ ($\rho = 1$ for the unit sphere), $0 \leq \theta \leq \frac{\pi}{2}$, and $0 \leq \varphi \leq \frac{\pi}{2}$. Hence, a parameterization for the unit sphere in spherical coordinates can be given by

$$\vec{r}(\theta, \varphi) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \sin\varphi\cos\theta \\ \sin\varphi\sin\theta \\ \cos\varphi \end{bmatrix} ,$$

  where $0 \leq \theta \leq \frac{\pi}{2}$ and $0 \leq \varphi \leq \frac{\pi}{2}$.

3. Parameterize the surface $\mathscr{S}$ given by the section of $x^2+z^2=1$ bounded by the planes $y=-2x-z-4$ and $y=2x+12$. Note that $x^2+z^2=1$ is a cylinder of radius 1 that runs parallel to the y-axis.



Note that this shape is hollow, not a solid. Now, since $x^2+z^2=1$, we might try

$$x=\cos\theta \quad \text{and} \quad z=\sin\theta\ .$$

y is free to vary between $-2x-z-4$ and $2x+12$. So, try

$$\vec{r}(\theta,y)=\begin{bmatrix}\cos\theta\\ y\\ \sin\theta\end{bmatrix}\ ,$$

where $0\le\theta\le 2\pi$, $-2x-z-4\le y\le 2x+12$. Since $x=\cos\theta$ and $z=\sin\theta$, we get that

$$-2\cos\theta-\sin\theta-4\le y\le 2\cos\theta+12\ .$$

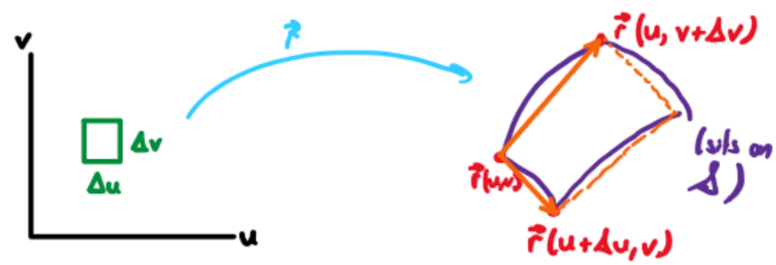
Surface Integrals of Scalar Functions

- Let $\mathscr{S}$ be a parametric surface given by $\vec{r}(u, v)$.

- The vector $\vec{r}_u(u_0, v_0)$ and $\vec{r}_v(u_0, v_0)$ span the tangent plane. The normal to this plane is given by

$$\vec{r}_u(u_0, v_0) \times \vec{r}_v(u_0, v_0) \ .$$

- The **area element** dS represents an infinitesimal piece of area on $\mathscr{S}$. Let's calculate this!



- **Key observation:** The area of the piece of $\mathscr{S}$ is approximately the area of the parallelogram. That is,

$$\mathscr{S} \approx ||(\vec{r}(u, v + \Delta v) - \vec{r}(u, v)) \times (\vec{r}(u + \Delta u, v) - \vec{r}(u, v))|| \ .$$

Now, recall from differentials in Calculus I that

$$\boxed{\begin{aligned} &\frac{dy}{dx} = f'(x) \\ \Longrightarrow\ & dy = f'(x)\ dx \\ \Longrightarrow\ & dy \approx \Delta y = f(x + \Delta x) - f(x) \end{aligned}}$$

Then using this fact, it follows that

$$\begin{aligned}
\mathscr{S} &\approx ||(\vec{r}(u, v + \Delta v) - \vec{r}(u, v)) \times (\vec{r}(u + \Delta u, v) - \vec{r}(u, v))|| \\
&\approx ||(\vec{r}_v(u, v)\Delta v) \times (\vec{r}_u(u, v)\Delta u)|| \\
&= ||\vec{r}_v \times \vec{r}_u||\Delta v \Delta u \\
&= ||\vec{n}||\Delta u \Delta v \ ,
\end{aligned}$$

where $\vec{n}$ is normal to the surface of the plane that $\vec{r}_v$ and $\vec{r}_u$ reside on.

- Let $\Delta u, \Delta v \to 0$. This gives us the area element to be

$$dS = ||\vec{n}|| \ du \ dv = ||\vec{n}|| \ dA \ ,$$

where $du \ dv = dA$.

- Given a scalar function (always 3 variables), we define

$$\boxed{\iint_{\mathscr{S}} f \ dS \stackrel{\text{def}}{=} \iint_D f(\vec{r}(u, v)) \cdot ||\vec{n}|| \ du \ dv}$$

where D is the set of (u, v) coordinates for $\mathscr{S}$. Note that the RHS of this equation is a double integral in (u, v).

- **Physical Interpretation:**
  - If $\rho(x, y, z)$ gives area density along the surface $\mathscr{S}$, then

$$\text{mass}(\mathscr{S}) = \iint_{\mathscr{S}} \rho \ dS \ .$$

  - If $\rho \equiv 1$, this returns the surface area of $\mathscr{S}$.

1. Confirm that the surface area of $x^2+y^2+z^2=R^2$ (a sphere of radius R) is $4\pi R^2$.

We can use spherical coordinates. Recall that for spherical coordinates,

$$\begin{aligned}
\rho^2 &= x^2+y^2+z^2 \\
\theta &= \tan^{-1}\left(\frac{y}{x}\right) \\
\varphi &= \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) = \cos^{-1}\left(\frac{z}{\rho}\right) \\
x &= \rho\sin\varphi\cos\theta \\
y &= \rho\sin\varphi\sin\theta \\
z &= \rho\cos\varphi \ ,
\end{aligned}$$

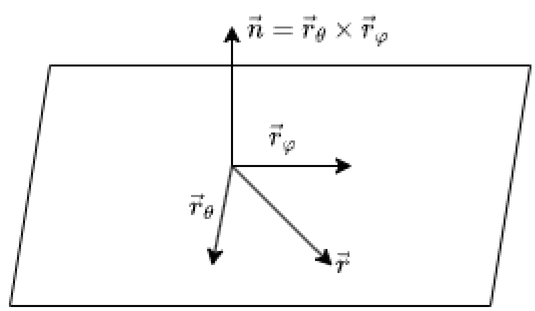
where $0 \le \theta \le 2\pi$ and $0 \le \varphi \le \pi$. Here ρ is the distance from the origin to the point of interest. This ρ is not to be confused with the ρ used for linear density. So,

$$\vec{r}(\theta,\varphi) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \rho\sin\varphi\cos\theta \\ \rho\sin\varphi\sin\theta \\ \rho\cos\varphi \end{bmatrix} = \begin{bmatrix} \rho\cos\theta\sin\varphi \\ \rho\sin\theta\sin\varphi \\ \rho\cos\varphi \end{bmatrix}$$

where $0 \le \theta \le 2\pi$ and $0 \le \varphi \le \pi$. Now, for a sphere we know that $\rho = R$. That is, the distance from the origin to any point on the sphere is simply the radius of the sphere. So, we get that

$$\vec{r}(\theta,\varphi) = \begin{bmatrix} \rho\cos\theta\sin\varphi \\ \rho\sin\theta\sin\varphi \\ \rho\cos\varphi \end{bmatrix} = \begin{bmatrix} R\cos\theta\sin\varphi \\ R\sin\theta\sin\varphi \\ R\cos\varphi \end{bmatrix} \ .$$

Now, we need $||\vec{n}||$. Recall that $\vec{n}$ is the vector normal to the plane tangent to $\vec{r}(\theta,\varphi)$. Indeed, on this plane live the vectors $\vec{r}_\theta$ and $\vec{r}_\varphi$. Of course, this can be visualized:



Observe that $\vec{r} = \vec{r}_\theta + \vec{r}_\varphi$. So, we have that

$$\vec{r}_\theta = \begin{bmatrix} (R\cos\theta\sin\varphi)_\theta \\ (R\sin\theta\sin\varphi)_\theta \\ (R\cos\varphi)_\theta \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{\partial}{\partial\theta}\,(R\cos\theta\sin\varphi) \\ \dfrac{\partial}{\partial\theta}\,(R\sin\theta\sin\varphi) \\ \dfrac{\partial}{\partial\theta}\,(R\cos\varphi) \end{bmatrix}$$

$$= \begin{bmatrix} R\sin\varphi\cdot\dfrac{\partial}{\partial\theta}\,(\cos\theta) \\ R\sin\varphi\cdot\dfrac{\partial}{\partial\theta}\,(\sin\theta) \\ R\cos\varphi\cdot\dfrac{\partial}{\partial\theta}\,(1) \end{bmatrix}$$

$$= \begin{bmatrix} R\sin\varphi\cdot(-\sin\theta) \\ R\sin\varphi\cdot(\cos\theta) \\ R\cos\varphi\cdot 0 \end{bmatrix}$$

$$= \begin{bmatrix} -R\sin\theta\sin\varphi \\ R\cos\theta\sin\varphi \\ 0 \end{bmatrix}$$

and

$$\vec{r}_\varphi = \begin{bmatrix} (R\cos\theta\sin\varphi)_\varphi \\ (R\sin\theta\sin\varphi)_\varphi \\ (R\cos\varphi)_\varphi \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{\partial}{\partial\varphi}\,(R\cos\theta\sin\varphi) \\ \dfrac{\partial}{\partial\varphi}\,(R\sin\theta\sin\varphi) \\ \dfrac{\partial}{\partial\varphi}\,(R\cos\varphi) \end{bmatrix}$$

$$= \begin{bmatrix} R\cos\theta \cdot \dfrac{\partial}{\partial\varphi}(\sin\varphi) \\ R\sin\theta \cdot \dfrac{\partial}{\partial\varphi}(\sin\varphi) \\ R \cdot \dfrac{\partial}{\partial\varphi}(\cos\varphi) \end{bmatrix}$$

$$= \begin{bmatrix} R\cos\theta \cdot \cos\varphi \\ R\sin\theta \cdot \cos\varphi \\ R \cdot (-\sin\varphi) \end{bmatrix}$$

$$= \begin{bmatrix} R\cos\theta\cos\varphi \\ R\sin\theta\cos\varphi \\ -R\sin\varphi \end{bmatrix} .$$

Then

$$\begin{aligned} \vec{n} &= \vec{r}_\theta \times \vec{r}_\varphi \\ &= \dots \end{aligned}$$

and so

$$\begin{aligned} ||\vec{n}|| &= R^2\sqrt{\cos^2\theta\sin^4\varphi + \sin^2\theta\sin^4\varphi + \sin^2\varphi\cos^2\varphi} \\ &= R^2\sqrt{\sin^4\varphi(\cos^2\theta + \sin^2\theta) + \sin^2\varphi\cos^2\varphi} \\ &= R^2\sqrt{\sin^4\varphi \cdot 1 + \sin^2\varphi\cos^2\varphi} \\ &= R^2\sqrt{\sin^4\varphi + \sin^2\varphi\cos^2\varphi} \\ &= R^2\sqrt{\sin^2\varphi(\sin^2\varphi + \cos^2\varphi)} \\ &= R^2\sqrt{\sin^2\varphi \cdot 1} \\ &= R^2\sqrt{\sin^2\varphi} \\ &= R^2\sin\varphi\ , \end{aligned}$$

where $\sin\varphi \geq 0$ since $0 \leq \varphi \leq \pi$. Thus, the surface area of a sphere $x^2 + y^2 + z^2 = R^2$ is

$$\begin{aligned} \iint_{\mathscr{S}} 1\ dS &= \int_0^{2\pi}\int_0^{\pi} 1 \cdot ||\vec{n}||\ d\varphi\ d\theta \\ &= \int_0^{2\pi}\int_0^{\pi} ||\vec{n}||\ d\varphi\ d\theta \\ &= \int_0^{2\pi}\int_0^{\pi} R^2\sin\varphi\ d\varphi\ d\theta \end{aligned}$$

$$
\begin{aligned}
&= R^2 \int_0^{2\pi} \int_0^{\pi} \sin\varphi \; d\varphi \; d\theta \\
&= R^2 \int_0^{2\pi} \left(\int_0^{\pi} \sin\varphi \; d\varphi \right) \; d\theta \\
&= R^2 \int_0^{2\pi} \left(- \int_0^{\pi} - \sin\varphi \; d\varphi \right) \; d\theta \\
&= -R^2 \int_0^{2\pi} \left(\int_0^{\pi} - \sin\varphi \; d\varphi \right) \; d\theta \\
&= -R^2 \int_0^{2\pi} \Big[\cos\varphi \Big]_{\varphi=0}^{\pi} \; d\theta \\
&= -R^2 \int_0^{2\pi} (\cos(\pi) - \cos(0)) \; d\theta \\
&= -R^2 \int_0^{2\pi} (-1 - 1) \; d\theta \\
&= -R^2 \int_0^{2\pi} -2 \; d\theta \\
&= 2R^2 \int_0^{2\pi} d\theta \\
&= 2R^2 \int_0^{2\pi} \theta^0 \; d\theta \\
&= 2R^2 \Big[\theta \Big]_{\theta=0}^{2\pi} \\
&= 2R^2 (2\pi - 0) \\
&= 2R^2 \cdot 2\pi \\
&= 4\pi R^2 \; .
\end{aligned}
$$

2. Find $\displaystyle\iint_{\mathscr{S}} z\ dS$, where $\mathscr{S}$ is given by $z = \sqrt{2xy}$ for $0 \le x \le 5$ and $0 \le y \le 2$.

Note that $z = \sqrt{2xy}$ is a "function type". So, try using x and y as parameters.

$$\vec{r}(x,y) = \begin{bmatrix} x \\ y \\ \sqrt{2xy} \end{bmatrix},$$

where $0 \le x \le 5$ and $0 \le y \le 2$. Now, recall that from earlier in the course (see page 11 of Week 2 Notes) that the normal for a function $z = f(x,y)$

$$\boxed{\vec{n} = \begin{bmatrix} f_x \\ f_y \\ -1 \end{bmatrix} \quad \text{or} \quad \vec{n} = \begin{bmatrix} -f_x \\ -f_y \\ 1 \end{bmatrix}}$$

Similarly, for a function type $y = g(x,z)$,

$$\vec{n} = \begin{bmatrix} g_x \\ -1 \\ g_z \end{bmatrix} \quad \text{or} \quad \vec{n} = \begin{bmatrix} -g_x \\ 1 \\ -g_z \end{bmatrix}$$

and for a function type $x = h(y,z)$,

$$\vec{n} = \begin{bmatrix} -1 \\ h_y \\ h_z \end{bmatrix} \quad \text{or} \quad \vec{n} = \begin{bmatrix} 1 \\ -h_y \\ -h_z \end{bmatrix}.$$

Then, for our function type $z = \sqrt{2xy}$, we will use

$$\vec{n} = \begin{bmatrix} \dfrac{\partial z}{\partial x} \\ \dfrac{\partial z}{\partial y} \\ -1 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial}{\partial x}\left(\sqrt{2xy}\,\right) \\ \dfrac{\partial}{\partial y}\left(\sqrt{2xy}\,\right) \\ -1 \end{bmatrix} = \begin{bmatrix} \dfrac{y}{\sqrt{2xy}} \\ \dfrac{x}{\sqrt{2xy}} \\ -1 \end{bmatrix}.$$

So,

$$\begin{aligned} ||\vec{n}|| &= \sqrt{\left(\frac{y}{\sqrt{2xy}}\right)^2 + \left(\frac{x}{\sqrt{2xy}}\right)^2 + (-1)^2} \\ &= \sqrt{\frac{y^2}{2xy} + \frac{x^2}{2xy} + 1} \\ &= \sqrt{\frac{y^2}{2xy} + \frac{x^2}{2xy} + \frac{2xy}{2xy}} \end{aligned}$$

$$
\begin{aligned}
&= \sqrt{\frac{x^2+y^2+2xy}{2xy}} \\
&= \sqrt{\frac{1}{2xy}\left(x^2+y^2+2xy\right)} \\
&= \sqrt{\frac{1}{2xy}} \cdot \sqrt{x^2+y^2+2xy} \\
&= \frac{\sqrt{1}}{\sqrt{2xy}} \cdot \sqrt{x^2+y^2+2xy} \\
&= \frac{1}{\sqrt{2xy}} \cdot \sqrt{x^2+y^2+2xy} \\
&= \frac{1}{\sqrt{2xy}} \cdot \sqrt{x^2+2xy+y^2} \\
&= \frac{1}{\sqrt{2xy}} \cdot \sqrt{(x+y)^2} \\
&= \frac{1}{\sqrt{2xy}} \cdot (x+y) \\
&= \frac{x+y}{\sqrt{2xy}} \,.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\int\int_S z\ dS &= \int_0^5 \int_0^2 z \cdot ||\vec{n}||\ dy\ dx \\
&= \int_0^5 \int_0^2 \sqrt{2xy} \cdot \frac{x+y}{\sqrt{2xy}}\ dy\ dx \\
&= \int_0^5 \int_0^2 (x+y)\ dy\ dx \\
&= \int_0^5 \left(\int_0^2 (x+y)\ dy\right)\ dx \\
&= \int_0^5 \left(\int_0^2 x\ dy + \int_0^2 y\ dy\right)\ dx \\
&= \int_0^5 \left(x\int_0^2 dy + \int_0^2 y^1\ dy\right)\ dx \\
&= \int_0^5 \left(x\Big[y\Big]_0^2 + \left[\frac{y^2}{2}\right]_0^2 dy\right)\ dx \\
&= \int_0^5 \left(x(2-0) + \left(\frac{2^2}{2} - \frac{0^2}{2}\right)\ dy\right)\ dx \\
&= \int_0^5 (2x+2)\ dx
\end{aligned}
$$

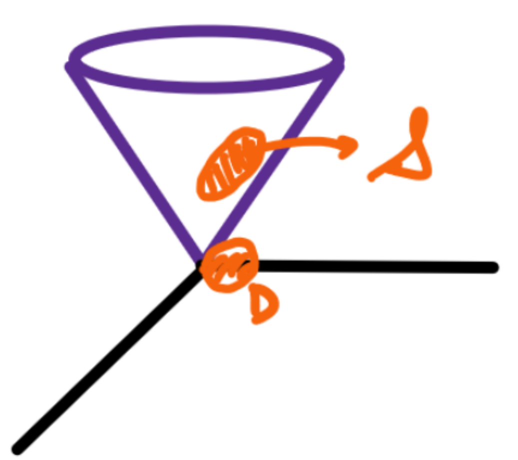
$$
\begin{aligned}
&= \int_0^5 2x \; dx + \int_0^5 2 \; dx \\
&= 2\int_0^5 x \; dx + 2\int_0^5 dx \\
&= 2\left[\frac{x^2}{2}\right]_0^5 + 2\Big[x\Big]_0^5 \\
&= 2\left(\frac{5^2}{2} - \frac{0^2}{2}\right) + 2(5-0) \\
&= 2\left(\frac{25}{2}\right) + 2(5) \\
&= 25 + 10 \\
&= 35 \; .
\end{aligned}
$$

3. Compute $\displaystyle\int_{\mathscr{S}} f \; dS$ (typo in notes?) for the following.

(a) $f \equiv 1$, and $\mathscr{S}$ is the piece of the cone $z = \sqrt{x^2 + y^2}$ within $x^2 + y^2 = 2ay$ where $a > 0$.

Since $f \equiv 1$, this means that that integral $\displaystyle\int_{\mathscr{S}} = f \; dS$ returns the surface area of $\mathscr{S}$. (See `https://www.geogebra.org/calculator/yw4abzhc`)

From the link, we see that $\mathscr{S}$ is the region of intersection between the cone and the cylinder (this is best seen from a top-down view). Note that D is the circle centered at $(0, a)$ (i.e. lies on the xy-plane).



We have that

$$\begin{aligned} x^2 + y^2 &= 2ay \\ x^2 + y^2 - 2ay &= 0 \\ x^2 + (y^2 - 2ay) &= 0 \\ x^2 + ((y^2 - 2ay + a^2) - a^2) &= 0 \\ x^2 + ((y-a)^2 - a^2) &= 0 \\ x^2 + (y-a)^2 - a^2 &= 0 \\ x^2 + (y-a)^2 &= a^2 \; . \end{aligned}$$

In cartesian coordinates,

$$\vec{r}(x, y) = \begin{bmatrix} x \\ y \\ \sqrt{x^2 + y^2} \end{bmatrix} \; ,$$

where $x^2+(y-a)^2 \leq a^2$ (this condition gives the interior of the circle of intersection between the cylinder and cone). So, we want to find

$$\begin{aligned}\iint_{\mathscr{S}} f\ dS - \iint_{\mathscr{S}} 1\ dS \\ - \iint_{x^2+(y-a)^2\leq a^2} 1 \cdot ||\vec{n}||\ dA\ .\end{aligned}$$

We have that

$$\begin{aligned}\vec{n} &= \begin{bmatrix} f_x \\ f_y \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} \left(\sqrt{x^2+y^2}\right)_x \\ \left(\sqrt{x^2+y^2}\right)_y \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{2x}{2\sqrt{x^2+y^2}} \\ \dfrac{2y}{2\sqrt{x^2+y^2}} \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{x}{\sqrt{x^2+y^2}} \\ \dfrac{y}{\sqrt{x^2+y^2}} \\ -1 \end{bmatrix}\ ,\end{aligned}$$

and so

$$\begin{aligned}||\vec{n}|| &= \sqrt{\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2 + (-1)^2} \\ &= \sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + 1} \\ &= \sqrt{\frac{x^2+y^2}{x^2+y^2} + 1} \\ &= \sqrt{1+1} \\ &= \sqrt{2}\ .\end{aligned}$$

Thus,

$$\iint_{\mathscr{S}} f\ dS = \iint_{\mathscr{S}} 1\ dS$$

$$
\begin{aligned}
&= \int\int_{x^2+(y-a)^2 \le a^2} 1 \cdot ||\vec{n}||\ dA \\
&= \int\int_{x^2+(y-a)^2 \le a^2} 1 \cdot \sqrt{2}\ dA \\
&= \int\int_{x^2+(y-a)^2 \le a^2} \sqrt{2}\ dA \\
&= \sqrt{2} \cdot \text{Area (circle of radius } a) \\
&= \sqrt{2}\pi a^2\ .
\end{aligned}
$$

Alternatively, we convert to shifted polar coordinates before finding $\vec{n}$. Recall that for polar coordinates

$$
\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta \\
r &= \sqrt{x^2+y^2}
\end{aligned}
$$

So for shifted polar coordinates, since the cylinder is centered at $(0, a)$, we have that

$$
\begin{aligned}
x &= ar\cos\theta \\
y &= ar\sin\theta + a
\end{aligned}
$$

Then

$$
\vec{r}(r,\theta) = \begin{bmatrix} ar\cos\theta \\ ar\sin\theta + a \\ \sqrt{x^2+y^2} \end{bmatrix} = \begin{bmatrix} ar\cos\theta \\ ar\sin\theta + a \\ \sqrt{x^2+y^2} \end{bmatrix}
$$

for $0 \le r \le 1$ (since the sphere has a radius of 1) and $0 \le \theta \le 2\pi$ (full circle). For the last component $z = \sqrt{x^2+y^2}$, we have that

$$
\begin{aligned}
z &= \sqrt{x^2+y^2} \\
&= \sqrt{(ar\cos\theta)^2 + (ar\sin\theta + a)^2} \\
&= \sqrt{a^2r^2\cos^2\theta + a^2r^2\sin^2\theta + 2a^2r\sin\theta + a^2} \\
&= \sqrt{a^2r^2(\cos^2\theta + \sin^2\theta) + 2a^2r\sin\theta + a^2} \\
&= \sqrt{a^2r^2(1) + 2a^2r\sin\theta + a^2} \\
&= \sqrt{a^2r^2 + 2a^2r\sin\theta + a^2} \\
&= \sqrt{a^2(r^2 + 2r\sin\theta + 1)} \\
&= \sqrt{a^2} \cdot \sqrt{r^2 + 2r\sin\theta + 1} \\
&= a\sqrt{r^2 + 2r\sin\theta + 1}\ .
\end{aligned}
$$

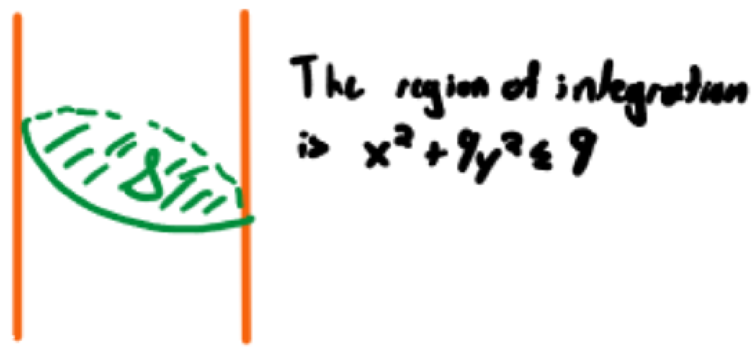
So,

$$\vec{r}(r,\theta) = \begin{bmatrix} ar\cos\theta \\ ar\sin\theta + a \\ \sqrt{x^2+y^2} \end{bmatrix} = \begin{bmatrix} ar\cos\theta \\ ar\sin\theta + a \\ a\sqrt{r^2+2r\sin\theta+1} \end{bmatrix} .$$

Then we compute $\vec{n}$, which gives us $||\vec{n}||$. Finally, computing the surface integral should give us

$$\begin{aligned} \iint_{\mathscr{S}} f \; dS &= \iint_D 1 \cdot ||\vec{n}|| \; d\theta \; dr \\ &= \int_0^1 \int_0^{2\pi} ||\vec{n}|| \; d\theta \; dr \\ &= \sqrt{2}\pi a^2 \; . \end{aligned}$$

(b) Find $\displaystyle\iint_{\mathscr{S}} z^2 \, dS$, where $\mathscr{S}$ is the piece of $x + 2y + z = 1$ inside of the elliptic cylinder $x^2 + 9y^2 = 9$.



(Also see `https://www.geogebra.org/calculator/crtzgw4j`)

$\mathscr{S}$ is a plane, and its normal is given by

$$\vec{n} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} ,$$

where the components of $\vec{n}$ are the coefficients of the piece $x^2 + 2y + z = 1$ inside of the elliptical cylinder. Note that we could also have used

$$\vec{n} = \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix} .$$

So, we have that

$$\begin{aligned} ||\vec{n}|| &= \sqrt{(1)^2 + (2)^2 + (1)^2} \\ &= \sqrt{1 + 4 + 1} \\ &= \sqrt{6} \ . \end{aligned}$$

Then, noting that $z = 1 - x - 2y$ after rearranging for z, the surface integral is

$$\begin{aligned} \iint_{\mathscr{S}} z^2 \, dS &= \iint_D z^2 \cdot ||\vec{n}|| \, dA \\ &= \iint_{x^2+9y^2 \le 9} (1 - x - 2y)^2 \cdot \sqrt{6} \, dA \ . \end{aligned}$$

Now, note that the elliptic cylinder $x^2 + 9y^2 = 9$ has an x-radius of 3. This makes sense, since

$$x^2 + 9y^2 = 9$$

$$\begin{aligned}
\frac{x^2}{9} + y^2 &= 1 \\
\frac{x^2}{3} + \frac{y^2}{1} &= 1 \\
\frac{(x-0)^2}{3^2} + \frac{(y-0)^2}{1^2} &= 1
\end{aligned}$$

which means that $0 \leq x \leq 3$ is the restriction on x. Also, from the equation of the given elliptic cylinder,

$$\begin{aligned}
x^2 + 9y^2 &= 9 \\
9y^2 &= 9 - x^2 \\
y^2 &= \frac{9-x^2}{9} \\
y &= \sqrt{\frac{9-x^2}{9}} \ ,
\end{aligned}$$

which means that the restriction on y is

$$-\sqrt{\frac{9-x^2}{9}} \leq y \leq \sqrt{\frac{9-x^2}{9}} \ .$$

So, plugging these restrictions in for the surface integral gives

$$\begin{aligned}
\iint_{\mathscr{S}} z^2 \ dS &= \iint_{x^2+9y^2 \leq 9} (1-x-2y)^2 \cdot \sqrt{6} \ dA \\
&= \int_{-3}^{3} \int_{-\sqrt{\frac{9-x^2}{9}}}^{-\sqrt{\frac{9-x^2}{9}}} (1-x-2y)^2 \cdot \sqrt{6} \ dy \ dx \ ,
\end{aligned}$$

which is a difficult integral to solve. We can instead use elliptical coordinates. (To be continued.)

(c) Find $\displaystyle\iint_{\mathscr{S}} (x+y)\ dS$, where $\mathscr{S}$ is the section of $y = x^2 + z^2 - 2$ to the left of $x + y = 1$.

Note that $y = x^2 + z^2 - 2$ is a paraboloid which can be rewritten as

$$\begin{aligned} x^2 + z^2 - 2 &= y \\ x^2 + z^2 &= y + 2 \\ \frac{x^2}{1} + \frac{z^2}{1} &= \frac{y}{1} + 2 \ . \end{aligned}$$

Here we have that since the coefficient in the denominator of y is positive, the paraboloid opens up in the positive y-direction. Also, since the coefficients in the denominators of x and z are the same, this means that the cross section of the parabola is circular. Lastly, the $+2$ on the RHS of the equation tells us that the apex of the parabola is at $y = -2$ (i.e. the point $(0, -2, 0)$). For the plane, we have that it has intercepts at $x = 1$ and $y = 1$. So, the parameterization of the surface $\mathscr{S}$ is given by

$$\vec{r}(x, z) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ x^2 + z^2 - 2 \\ z \end{bmatrix}$$

for $-2 \le x^2 + z^2 - 2 \le 1 - x$. Let's instead use polar coordinates. Then

$$\begin{aligned} r &= \sqrt{x^2 + z^2} \\ x &= r\cos\theta \\ z &= r\sin\theta \ . \end{aligned}$$

So,

$$\vec{r}(r, \theta) = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$
\begin{aligned}
&= \begin{bmatrix} r\cos\theta \\ x^2+z^2-2 \\ r\sin\theta \end{bmatrix} \\
&= \begin{bmatrix} r\cos\theta \\ (r\cos\theta)^2+(r\sin\theta)^2-2 \\ r\sin\theta \end{bmatrix} \\
&= \begin{bmatrix} r\cos\theta \\ r^2\cos^2\theta+r^2\sin^2\theta-2 \\ r\sin\theta \end{bmatrix} \\
&= \begin{bmatrix} r\cos\theta \\ r^2(\cos^2\theta+\sin^2\theta)-2 \\ r\sin\theta \end{bmatrix} \\
&= \begin{bmatrix} r\cos\theta \\ r^2\cdot 1-2 \\ r\sin\theta \end{bmatrix} \\
&= \begin{bmatrix} r\cos\theta \\ r^2-2 \\ r\sin\theta \end{bmatrix} .
\end{aligned}
$$

Since $y = r^2 - 2$, this means that

$$
\begin{aligned}
&-2 \le y \le 1 - x \\
&-2 \le r^2 - 2 \le 1 - r\cos\theta .
\end{aligned}
$$

The maximum value for r is when

$$
\begin{aligned}
r^2 - 2 &= 1 - r\cos\theta \\
r^2 + r\cos\theta - 3 &= 0 \\
1r^2 + \cos(\theta)r - 3 &= 0 .
\end{aligned}
$$

Note that we put it in this form as we want to express r not in terms of itself. Here, we can use the quadratic formula and take the positive solution (since we are looking for $r \ge 0$). Then we get that

$$
\begin{aligned}
r &= \frac{-(\cos\theta) + \sqrt{(\cos\theta)^2 - 4(1)(-3)}}{2(1)} \\
&= \frac{-\cos\theta + \sqrt{\cos^2\theta + 12}}{2} .
\end{aligned}
$$

So, we get that the restriction on r is

$$
0 \le r \le \frac{-\cos\theta + \sqrt{\cos^2\theta + 12}}{2} .
$$

Of course, the restriction on θ is $0 \leq \theta \leq 2\pi$ as usual. Now, the normal is given by

$$
\begin{aligned}
\vec{n} &= \vec{r}_r \times \vec{r}_\theta \\
&= \begin{bmatrix} (r\cos\theta)_r \\ (r^2-2)_r \\ (r\sin\theta)_r \end{bmatrix} \times \begin{bmatrix} (r\cos\theta)_\theta \\ (r^2-2)_\theta \\ (r\sin\theta)_\theta \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta \\ 2r \\ \sin\theta \end{bmatrix} \times \begin{bmatrix} -r\sin\theta \\ 0 \\ r\cos\theta \end{bmatrix} \\
&= \det\left(\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & 2r & \sin\theta \\ -r\sin\theta & 0 & r\cos\theta \end{bmatrix} \right) \\
&= \left(2r^2\cos\theta\right)\hat{i} - \left(r\cos^2\theta + r\sin^2\theta\right)\hat{j} + \left(2r^2\sin\theta\right)\hat{k} \\
&= \left(2r^2\cos\theta\right)\hat{i} - \left(r(\cos^2\theta + \sin^2\theta)\right)\hat{j} + \left(2r^2\sin\theta\right)\hat{k} \\
&= \left(2r^2\cos\theta\right)\hat{i} - (r\cdot 1)\,\hat{j} + \left(2r^2\sin\theta\right)\hat{k} \\
&= \left(2r^2\cos\theta\right)\hat{i} - (r)\,\hat{j} + \left(2r^2\sin\theta\right)\hat{k} \\
&= \left(2r^2\cos\theta\right)\hat{i} + (-r)\,\hat{j} + \left(2r^2\sin\theta\right)\hat{k} \\
&= \begin{bmatrix} 2r^2\cos\theta \\ -r \\ 2r^2\sin\theta \end{bmatrix} ,
\end{aligned}
$$

and so

$$
\begin{aligned}
||\vec{n}|| &= \sqrt{\left(2r^2\cos\theta\right)^2 + (-r)^2 + \left(2r^2\sin\theta\right)^2} \\
&= \sqrt{4r^4\cos^2\theta + r^2 + 4r^4\sin^2\theta} \\
&= \sqrt{4r^4\cos^2\theta + 4r^4\sin^2\theta + r^2} \\
&= \sqrt{4r^4(\cos^2\theta + \sin^2\theta) + r^2} \\
&= \sqrt{4r^4\cdot 1 + r^2} \\
&= \sqrt{4r^4 + r^2} \\
&= \sqrt{r^2(4r^2+1)} \\
&= \sqrt{r^2}\cdot\sqrt{4r^2+1} \\
&= r\sqrt{4r^2+1} \ .
\end{aligned}
$$

Thus, the surface integral is

$$\begin{aligned}
&\iint_{\mathscr{S}} (x+y)\ dS \\
&= \int_0^{\frac{-\cos\theta+\sqrt{\cos^2\theta+12}}{2}} \int_0^{2\pi} (r\cos\theta + (r^2-2)) \cdot ||\vec{n}||\ d\theta\ dr \\
&= \int_0^{\frac{-\cos\theta+\sqrt{\cos^2\theta+12}}{2}} \int_0^{2\pi} (r\cos\theta + r^2-2) \cdot r\sqrt{4r^2+1}\ d\theta\ dr \\
&= \int_0^{\frac{-\cos\theta+\sqrt{\cos^2\theta+12}}{2}} \int_0^{2\pi} (r^2 + r\cos\theta-2) \cdot r\sqrt{4r^2+1}\ d\theta\ dr\ .
\end{aligned}$$

The inner integral evaluates to

$$\begin{aligned}
&\int_0^{2\pi} (r^2 + r\cos\theta - 2) \cdot r\sqrt{4r^2+1}\ d\theta \\
&= \int_0^{2\pi} \left(r^3\sqrt{4r^2+1} + r^2\cos\theta\sqrt{4r^2+1} - 2r\sqrt{4r^2+1}\right)\ d\theta \\
&= \int_0^{2\pi} r^3\sqrt{4r^2+1}\ d\theta + \int_0^{2\pi} r^2\cos\theta\sqrt{4r^2+1}\ d\theta - \int_0^{2\pi} 2r\sqrt{4r^2+1}\ d\theta \\
&= r^3\sqrt{4r^2+1}\int_0^{2\pi} d\theta + r^2\sqrt{4r^2+1}\int_0^{2\pi}\cos\theta\ d\theta - 2r\sqrt{4r^2+1}\int_0^{2\pi} d\theta \\
&= r^3\sqrt{4r^2+1}\ \Big[\theta\Big]_0^{2\pi} + r^2\sqrt{4r^2+1}\ \Big[\sin\theta\Big]_0^{2\pi} - 2r\sqrt{4r^2+1}\ \Big[\theta\Big]_0^{2\pi} \\
&= r^3\sqrt{4r^2+1}\cdot(2\pi-0) + r^2\sqrt{4r^2+1}\cdot(0-0) - 2r\sqrt{4r^2+1}\cdot(2\pi-0) \\
&= 2\pi r^3\sqrt{4r^2+1} + 0 - 4\pi r\sqrt{4r^2+1} \\
&= 2\pi r^3\sqrt{4r^2+1} - 4\pi r\sqrt{4r^2+1}\ .
\end{aligned}$$

To save space, let $k = \frac{-\cos\theta+\sqrt{\cos^2\theta+12}}{2}$. Then

$$\begin{aligned}
&\iint_{\mathscr{S}} (x+y)\ dS \\
&= \int_0^k \left(2\pi r^3\sqrt{4r^2+1} - 4\pi r\sqrt{4r^2+1}\right)\ dr \\
&= \int_0^k 2\pi r^3\sqrt{4r^2+1}\ dr - \int_0^k 4\pi r\sqrt{4r^2+1}\ dr \\
&= 2\pi\int_0^k r^3\sqrt{4r^2+1}\ dr - 4\pi\int_0^k r\sqrt{4r^2+1}\ dr\ .
\end{aligned}$$

Now, let $u = 4r^2+1$. Then $du = 8r\ dr$, and the new bounds of integration are

$$\begin{aligned}
u(k) &= u\left(\frac{-\cos\theta + \sqrt{\cos^2\theta + 12}}{2}\right) \\
&= 4\left(\frac{-\cos\theta + \sqrt{\cos^2\theta + 12}}{2}\right)^2 + 1 \\
&= 4 \cdot \frac{\left(-\cos\theta + \sqrt{\cos^2\theta + 12}\right)^2}{4} + 1 \\
&= \left(-\cos\theta + \sqrt{\cos^2\theta + 12}\right)^2 + 1 \\
&= \left(\cos^2\theta - 2\cos\theta\sqrt{\cos^2\theta + 12} + (\cos^2\theta + 12)\right) + 1 \\
&= \left(2\cos^2\theta - 2\cos\theta\sqrt{\cos^2\theta + 12} + 12\right) + 1 \\
&= 2\cos^2\theta - 2\cos\theta\sqrt{\cos^2\theta + 12} + 13 \\
&= 2\cos\theta\left(\cos\theta - \sqrt{\cos^2\theta + 12}\right) + 13
\end{aligned}$$

and

$$u(0) = 4(0)^2 + 1 = 0 + 1 = 1 \ .$$

Also, since $u = r^2 + 1$, we get that

$$r^2 = u - 1 \ .$$

To save space, let $m = 2\cos\theta\left(\cos\theta - \sqrt{\cos^2\theta + 12}\right) + 13$. Then

$$\begin{aligned}
&\iint_{\mathscr{S}} (x + y)\ dS \\
&= 2\pi \int_0^k r^3\sqrt{4r^2 + 1}\ dr - 4\pi \int_0^k r\sqrt{4r^2 + 1}\ dr \\
&= 2\pi \int_0^k \frac{1}{8} \cdot 8 \cdot r \cdot r^2\sqrt{4r^2 + 1}\ dr - 4\pi \int_0^k \frac{1}{8} \cdot 8r\sqrt{4r^2 + 1}\ dr \\
&= 2\pi \cdot \frac{1}{8} \int_0^k r^2\sqrt{4r^2 + 1} \cdot 8r\ dr - 4\pi \cdot \frac{1}{8} \int_0^k \sqrt{4r^2 + 1} \cdot 8r\ dr \\
&= \frac{\pi}{4} \int_0^k r^2\sqrt{4r^2 + 1} \cdot 8r\ dr - \frac{\pi}{2} \int_0^k \sqrt{4r^2 + 1} \cdot 8r\ dr \\
&= \frac{\pi}{4} \int_1^m (u - 1)\sqrt{u}\ du - \frac{\pi}{2} \int_1^m \sqrt{u}\ du \\
&= \frac{\pi}{4} \int_1^m (u - 1)u^{1/2}\ du - \frac{\pi}{2} \int_1^m u^{1/2}\ du \\
&= \frac{\pi}{4} \int_1^m \left(u^{3/2} - u^{1/2}\right)\ du - \frac{\pi}{2} \left[\frac{u^{3/2}}{3/2}\right]_1^m
\end{aligned}$$

$$= \frac{\pi}{4}\left(\int_1^m u^{3/2}\ du - \int_1^m u^{1/2}\ du\right) - \frac{\pi}{2}\left[\frac{2u^{3/2}}{3}\right]_1^m$$
$$= \frac{\pi}{4}\left(\left[\frac{u^{5/2}}{5/2}\right]_1^m - \left[\frac{u^{3/2}}{3/2}\right]_1^m\right) - \frac{\pi}{2}\left(\frac{2(m)^{3/2}}{3} - \frac{2(1)^{3/2}}{3}\right)$$
$$= \frac{\pi}{4}\left(\left[\frac{2u^{5/2}}{5}\right]_1^m - \left[\frac{2u^{3/2}}{3}\right]_1^m\right) - \frac{\pi}{2}\left(\frac{2(m)^{3/2}}{3} - \frac{2}{3}\right)$$
$$= \frac{\pi}{4}\left(\left(\frac{2(m)^{5/2}}{5} - \frac{2(1)^{5/2}}{5}\right) - \left(\frac{2(m)^{3/2}}{3} - \frac{2(1)^{3/2}}{3}\right)\right) - \frac{\pi}{2}\left(\frac{2(m)^{3/2} - 2}{3}\right)$$
$$= \frac{\pi}{4}\left(\left(\frac{2(m)^{5/2}}{5} - \frac{2}{5}\right) - \left(\frac{2(m)^{3/2}}{3} - \frac{2}{3}\right)\right) - \frac{\pi\left(2(m)^{3/2} - 2\right)}{6}$$
$$= \frac{\pi}{4}\left(\frac{2(m)^{5/2}}{5} - \frac{2}{5} - \frac{2(m)^{3/2}}{3} + \frac{2}{3}\right) - \frac{2\pi(m)^{3/2} - 2\pi}{6}$$
$$= \frac{\pi}{4}\left(\frac{2(m)^{5/2} - 2}{5} - \frac{2(m)^{3/2} + 2}{3}\right) - \frac{2\left(\pi(m)^{3/2} - \pi\right)}{6}$$
$$= \frac{\pi\left(2(m)^{5/2} - 2\right)}{20} - \frac{\pi\left(2(m)^{3/2} + 2\right)}{12} - \frac{\pi(m)^{3/2} - \pi}{3}$$
$$= \frac{2\pi(m)^{5/2} - 2\pi}{20} - \frac{2\pi(m)^{3/2} + 2\pi}{12} - \frac{\pi(m)^{3/2} - \pi}{3}$$
$$= \frac{2\left(\pi(m)^{5/2} - \pi\right)}{20} - \frac{2\left(\pi(m)^{3/2} + \pi\right)}{12} - \frac{\pi(m)^{3/2} - \pi}{3}$$
$$= \frac{\pi(m)^{5/2} - \pi}{10} - \frac{\pi(m)^{3/2} + \pi}{6} - \frac{\pi(m)^{3/2} - \pi}{3}$$
$$= \frac{\pi(m)^{5/2} - \pi}{10} - \frac{\pi(m)^{3/2} + \pi}{6} - \frac{\pi(m)^{3/2} - \pi}{3}$$

Surface Integrals of Vector Fields (Flux)

Definition (Orientable)

A surface $\mathscr{S}$ is **orientable** if its normal vector $\vec{n}$ defines a smooth vector field on $\mathscr{S}$.

Note: An orientable surface has two **sides**.

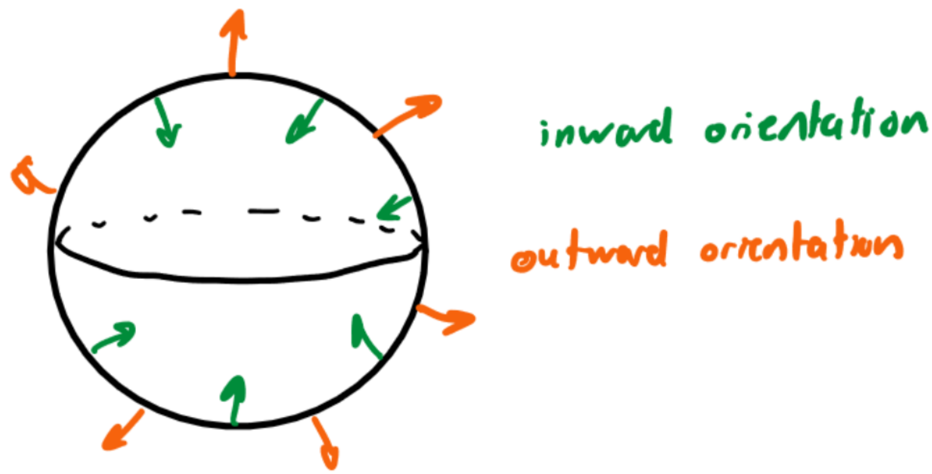
Definition (Orientation)

The side that is chosen for the orientable surface gives the **orientation** of the surface.

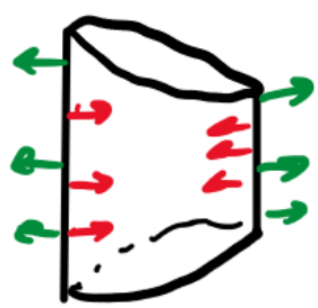
Definition (Closed Surface)

An orientable surface with no edge nor hole is called **closed**.

1. A sphere is orientable and closed.



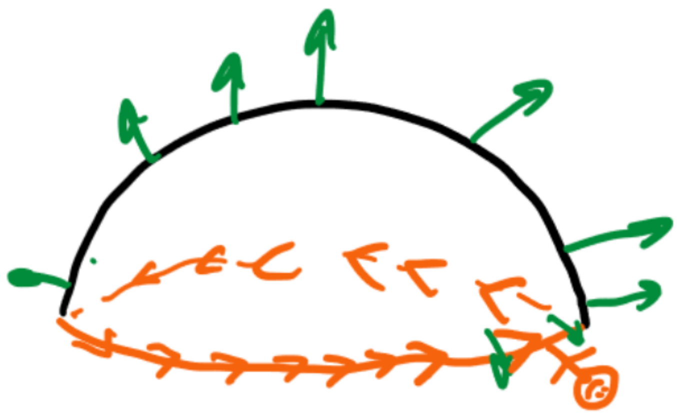
2. A cylindrical section given by $x^2 + y^2 = 1$ with z bounded by 2 planes is **orientable** but **not closed** (the ends are missing).



3. The **Möbius band** is non-orientable (it has only one side). The normal vector is necessarily given by two distinct vectors at a given point, so $\vec{n}$ does not define a function.

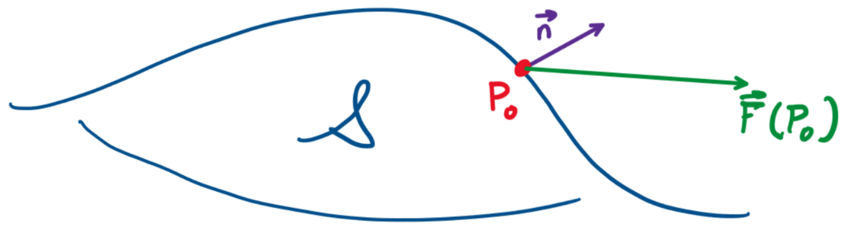
4. For a non-closed surface, its boundary (or edges) are given by curves. For example, the upper hemisphere $z = \sqrt{1 - x^2 - y^2}$ has a boundary curve $x^2 + y^2 = 1$ at $z = 0$. In other words, the boundary of a hemisphere is a circle.

- The chosen orientation of $\mathscr{S}$ produces an orientation (i.e. direction of travel) on each **boundary curve**.
  - This orientation is said to be **positive** if when travelling along the curve with our heads pointing in the same direction as the normal vector $\vec{n}$, the surface $\mathscr{S}$ is to our **left**.
  - For example, on the hemisphere with an outward normal,



Flux Integrals

- How do we integrate a vector field on $\mathscr{S}$?



- We are interested in flow across the parametric surface $\mathscr{S}$. We will integrate the component of $\vec{F}(P_0)$ that is parallel to $\vec{n}$ across all of $\mathscr{S}$.

- This parallel component is the quantity that is the coefficient of the projection of $\vec{F}(P_0)$ onto the unit normal $\vec{n}$

$$\text{proj}_{\vec{n}}\vec{F}(P_0) = \frac{\vec{F}(P_0) \cdot \vec{n}}{||\vec{n}||} \cdot \frac{\vec{n}}{||\vec{n}||} \ .$$

  That is, the quantity of interest is

$$\frac{\vec{F}(P_0) \cdot \vec{n}}{||\vec{n}||} \ .$$

- The area element is still

$$dS = ||\vec{n}|| \ dA \ .$$

- So, the flux integral (the surface integral of a vector field) is given by

$$\boxed{\iint_{\mathscr{S}} \vec{F} \cdot d\vec{S} := \iint_D \vec{F} \cdot \frac{\vec{n}}{||\vec{n}||} \cdot ||\vec{n}|| \ dA = \iint_D \vec{F} \cdot \vec{n} \ dA}$$

  where $d\vec{S} = \vec{n} \ dA$.

1. Compute $\iint_{\mathscr{S}} \vec{F} \cdot d\vec{S}$ for the following.

(a) $\vec{F}(x, y, z) = (-3xy,\ -z,\ 2y)$ and $\mathscr{S}$ is the plane $x + y + z = 4$ in the first octant whose normal is pointing away from $(0, 0, 0)$.

From the equation of the plane, we get that the x-intercept occurs when

$$\begin{aligned} x + 0 + 0 &= 4 \\ x &= 4\ , \end{aligned}$$

the y-intercept occurs when

$$\begin{aligned} 0 + y + 0 &= 4 \\ y &= 4\ , \end{aligned}$$

and the z-intercept occurs when

$$\begin{aligned} 0 + 0 + z &= 4 \\ z &= 4\ . \end{aligned}$$

Expressing this as function type in terms of z gives us the parmaterization

$$\vec{r}(x, y) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 4 - x - y \end{bmatrix}$$

Since we are only looking at the portion of the surface $\mathscr{S}$ in the first octant, the restriction on x is

$$0 \le x \le 4\ .$$

Note that since $z = 4 - x - y$, we get that

$$\begin{aligned} z &\ge 0 \\ 4 - x - y &\ge 0 \\ 4 - x &\ge y \\ y &\le 4 - x\ . \end{aligned}$$

So, the restriction on y is

$$0 \le y \le 4 - x\ .$$

Now, the normal to this plane that faces away from the $(0, 0, 0)$ is given by $(1, 1, 1)$. We want to evaluate the flux integral

$$\iint_{\mathscr{S}} \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(x, y)) \cdot \vec{n}\ dA\ .$$

Recall that

$$\vec{F} = \begin{bmatrix} -3xy \\ -z \\ 2y \end{bmatrix} .$$

So, we have that

$$\begin{aligned}
\vec{F}(\vec{r}(x,y)) \cdot \vec{n} &= \vec{F}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) \cdot \vec{n} \\
&= \vec{F}\left(\begin{bmatrix} x \\ y \\ 4-x-y \end{bmatrix}\right) \cdot \vec{n} \\
&= \begin{bmatrix} -3(x)y \\ -(4-x-y) \\ 2(y) \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -3xy \\ x+y-4 \\ 2y \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\
&= -3xy + (x+y-4) + 2y \\
&= -3xy + x + y - 4 + 2y \\
&= -3xy + x + 3y - 4 \ .
\end{aligned}$$

Thus,

$$\begin{aligned}
&\iint_{\mathscr{S}} \vec{F} \cdot d\vec{S} \\
&= \iint_D \vec{F}(\vec{r}(x,y)) \cdot \vec{n} \ dA \\
&= \int_0^{4-x} \int_0^4 \vec{F}(\vec{r}(x,y)) \cdot \vec{n} \ dx \ dy \\
&= \int_0^{4-x} \int_0^4 (-3xy + x + 3y - 4) \ dx \ dy \\
&= \int_0^{4-x} \left(\int_0^4 (-3xy + x + 3y - 4) \ dx\right) \ dy \\
&= \int_0^{4-x} \left(\int_0^4 (-3xy + x + 3y - 4) \ dx\right) \ dy \\
&= \int_0^{4-x} \left(\int_0^4 -3xy \ dx + \int_0^4 x \ dx + \int_0^4 3y \ dx - \int_0^4 4 \ dx\right) \ dy \\
&= \int_0^{4-x} \left(-3y\int_0^4 x \ dx + \int_0^4 x \ dx + 3y\int_0^4 dx - 4\int_0^4 dx\right) \ dy \\
&= \int_0^{4-x} \left(-3y\left[\frac{x^2}{2}\right]_0^4 + \left[\frac{x^2}{2}\right]_0^4 + 3y\Big[x\Big]_0^4 - 4\Big[x\Big]_0^4\right) \ dy
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{4-x} (-24y + 8 + 12y - 16) \; dy \\
&= \int_0^{4-x} (-12y - 8) \; dy \\
&= \int_0^{4-x} -12y \; dy - \int_0^{4-x} 8 \; dy \\
&= -12 \int_0^{4-x} y \; dy - 8 \int_0^{4-x} dy \\
&= -12 \left[\frac{y^2}{2} \right]_{y=0}^{4-x} - 8 \Big[y \Big]_{y=0}^{4-x} \\
&= -12 \left(\frac{(4-x)^2}{2} \right) - 8(4-x) \\
&= -6(16 - 8x + x^2) - 32 + 8x \\
&= -96 + 48x - 6x^2 - 32 + 8x \\
&= -6x^2 + 56x - 128 \; .
\end{aligned}
$$

(b) $\mathscr{S}$ is the piece of $x = 4 - y^2 - z^2$ in front of $x = -2$ whose normal is pointing towards the x-axis, and $\vec{F} = (x - 2,\ y^2,\ 2y)$.

We have that paraboloid $x = 4 - y^2 - z^2$ can be written as

$$\begin{aligned} 4 - y^2 - z^2 &= x \\ -x + 4 &= y^2 + z^2 \\ \frac{x}{-1} + 4 &= \frac{y^2}{1} + \frac{z^2}{1} \ . \end{aligned}$$

Since the term in the denominator of x is negative, this means that the paraboloid opens along the negative x-direction. Also, since the coefficient in the denominators of y and z are the same, this means that the cross section of the paraboloid is a circle. Lastly, the $+4$ implies that the apex of the parabola is at $x = 4$ (i.e. at the point $(4, 0, 0)$). Since this surface is a paraboloid, we can use polar coordinates. In the case of this surface,

$$\begin{aligned} r &= \sqrt{y^2 + z^2} \\ y &= r\cos\theta \\ z &= r\sin\theta \ . \end{aligned}$$

So, we have that

$$\begin{aligned} \vec{r}(r,\theta) &= \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ &= \begin{bmatrix} 4 - y^2 - z^2 \\ r\cos\theta \\ r\sin\theta \end{bmatrix} \\ &= \begin{bmatrix} 4 - (y^2 + z^2) \\ r\cos\theta \\ r\sin\theta \end{bmatrix} \\ &= \begin{bmatrix} 4 - r^2 \\ r\cos\theta \\ r\sin\theta \end{bmatrix} \ . \end{aligned}$$

Now, since $x = -2$ and $x = 4 - y^2 - z^2$,

$$\begin{aligned} -2 &= 4 - y^2 - z^2 \\ -6 &= -y^2 - z^2 \\ y^2 + z^2 &= 6 \\ \sqrt{y^2 + z^2} &= \sqrt{6} \\ r &= \sqrt{6} \ , \end{aligned}$$

which means that the restriction on r is $0 \le r \le \sqrt{6}$. Of course, the restriction on θ is $0 \le \theta \le 2\pi$. Then the normal to this surface is

$$\begin{aligned}
\vec{n} &= \vec{r}_r \times \vec{r}_\theta \\
&= \begin{bmatrix} \left(4 - r^2\right)_r \\ (r\cos\theta)_r \\ (r\sin\theta)_r \end{bmatrix} \times \begin{bmatrix} \left(4 - r^2\right)_\theta \\ (r\cos\theta)_\theta \\ (r\sin\theta)_\theta \end{bmatrix} \\
&= \begin{bmatrix} -2r \\ \cos\theta \\ \sin\theta \end{bmatrix} \times \begin{bmatrix} 0 \\ -r\sin\theta \\ r\cos\theta \end{bmatrix} \\
&= \det\left(\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -2r & \cos\theta & \sin\theta \\ 0 & -r\sin\theta & r\cos\theta \end{bmatrix}\right) \\
&= \left(r\cos^2\theta + r\sin^2\theta\right)\hat{\imath} - \left(-2r^2\cos\theta\right)\hat{\jmath} + \left(2r^2\sin\theta\right)\hat{k} \\
&= \left(r(\cos^2\theta + \sin^2\theta)\right)\hat{\imath} + \left(2r^2\cos\theta\right)\hat{\jmath} + \left(2r^2\sin\theta\right)\hat{k} \\
&= (r \cdot 1)\,\hat{\imath} + \left(2r^2\cos\theta\right)\hat{\jmath} + \left(2r^2\sin\theta\right)\hat{k} \\
&= (r)\,\hat{\imath} + \left(2r^2\cos\theta\right)\hat{\jmath} + \left(2r^2\sin\theta\right)\hat{k} \\
&= \begin{bmatrix} r \\ 2r^2\cos\theta \\ 2r^2\sin\theta \end{bmatrix} .
\end{aligned}$$

Now, we want to evaluate

$$\iint_{\mathscr{S}} \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(y,z)) \cdot \vec{n}\; dA .$$

Note that evaluating this requires us to parameterize the surface in cartesian coordinates. We can do this since $x = 4 - y^2 - z^2$ is a function type. So, we have that

$$\vec{r}(y,z) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 - y^2 - z^2 \\ y \\ z \end{bmatrix} ,$$

for $y^2 + z^2 \le 6$. Also note that we will need to use a normal in terms of cartesian coordinates. Recall that for a function type $x = g(y,z)$,

$$\vec{n} = \begin{bmatrix} -1 \\ g_y \\ g_z \end{bmatrix} .$$

Then it follows that

$$\vec{n} = \begin{bmatrix} -1 \\ \left(4 - y^2 - z^2\right)_y \\ \left(4 - y^2 - z^2\right)_z \end{bmatrix} = \begin{bmatrix} -1 \\ -2y \\ -2z \end{bmatrix} .$$

Since we are interested in a normal that points towards the x-axis, this normal will suffice. Now, recall that

$$\vec{F} = \begin{bmatrix} x - z \\ y^2 \\ 2y \end{bmatrix} .$$

Then

$$\begin{aligned}
\vec{F}(\vec{r}(y,z)) \cdot \vec{n} &= \vec{F}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) \cdot \vec{n} \\
&= \vec{F}\left(\begin{bmatrix} 4 - y^2 - z^2 \\ y \\ z \end{bmatrix}\right) \cdot \vec{n} \\
&= \begin{bmatrix} (4 - y^2 - z^2) - z \\ (y)^2 \\ 2(y) \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -2y \\ -2z \end{bmatrix} \\
&= \begin{bmatrix} 4 - y^2 - z^2 - z \\ y^2 \\ 2y \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -2y \\ -2z \end{bmatrix} \\
&= (-4 + y^2 + z^2 + z) - 2y^3 - 4yz \\
&= -4 + y^2 + z^2 + z - 2y^3 - 4yz \ .
\end{aligned}$$

Thus, noting that $dA = r\ dr\ d\theta$ in polar coordinates,

$$\begin{aligned}
&\iint \mathscr{S}\vec{F} \cdot d\vec{S} \\
&= \iint_D \vec{F}(\vec{r}(y,z)) \cdot \vec{n}\ dA \\
&= \iint_{y^2+z^2\le 6} (-4 + y^2 + z^2 + z - 2y^3 - 4yz)\ dA \\
&= \int_0^{2\pi} \int_0^{\sqrt{6}} (-4 + r^2 + r\sin\theta - 2(r\cos\theta)^3 - 4(r\cos\theta)(r\sin\theta)) \cdot r\ dr\ d\theta \\
&= \int_0^{2\pi} \int_0^{\sqrt{6}} (-4 + r^2 + r\sin\theta - 2r^3\cos^3\theta - 4r^2\cos\theta\sin\theta) \cdot r\ dr\ d\theta \\
&= \int_0^{2\pi} \int_0^{\sqrt{6}} (-4 + r^2 + 0 - 0 - 0) \cdot r\ dr\ d\theta \\
&= \int_0^{2\pi} \int_0^{\sqrt{6}} (-4 + r^2) \cdot r\ dr\ d\theta \\
&= \int_0^{2\pi} \int_0^{\sqrt{6}} (-4r + r^3)\ dr\ d\theta
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{2\pi} \left(\int_0^{\sqrt{6}} (-4r + r^3)\ dr \right)\ d\theta \\
&= \int_0^{2\pi} \left(\int_0^{\sqrt{6}} -4r\ dr + \int_0^{\sqrt{6}} r^3\ dr \right)\ d\theta \\
&= \int_0^{2\pi} \left(-4 \int_0^{\sqrt{6}} r\ dr + \int_0^{\sqrt{6}} r^3\ dr \right)\ d\theta \\
&= \int_0^{2\pi} \left(-4 \left[\frac{r^2}{2} \right]_0^{\sqrt{6}} + \left[\frac{r^4}{4} \right]_0^{\sqrt{6}} \right)\ d\theta \\
&= \int_0^{2\pi} \left(-4 \left(\frac{\left(\sqrt{6}\,\right)^2}{2} - \frac{0^2}{2} \right) + \left(\frac{\left(\sqrt{6}\,\right)^4}{4} - \frac{0^4}{4} \right) \right)\ d\theta \\
&= \int_0^{2\pi} \left(-4 \left(\frac{6}{2} \right) + \left(\frac{36}{4} - 0 \right) \right)\ d\theta \\
&= \int_0^{2\pi} (-12 + 9)\ d\theta \\
&= \int_0^{2\pi} -3\ d\theta \\
&= -3 \int_0^{2\pi} d\theta \\
&= -3 \Big[\theta \Big]_0^{2\pi} \\
&= -3(2\pi - 0) \\
&= -6\pi\ .
\end{aligned}
$$

(c) $\mathscr{S}$ is the piece of $y = 4z + x^3 + 6$ with (x, z) coordinates given by the region bounded by $z = x^3$, $z = 1$, and $x = 0$. The vector field is $\vec{F} = (1,\ 4z,\ z - y)$ and $\vec{n}$ is pointing away from the origin.