MATH 367 - Week 6-7 Notes

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Line Integrals of Scalar Functions

- We are used to integrating $f(x,y)$ and $f(x,y,z)$ on 2D or 3D regions, respectively. We intepret these as volume/hypervolume.

- Today, we integrate $f(x,y)$ or $f(x,y,z)$ along a 2D or 3D curve C.

- The integrals are of the form:

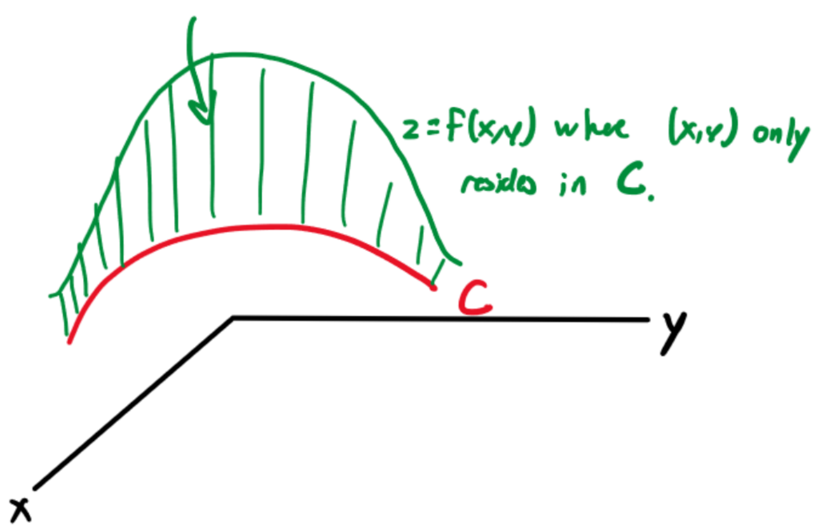
$$\int_C f(x,y)\ ds \tag{2D}$$

$$\int_C f(x,y,z)\ ds \tag{3D}$$

  where both are integrated with respect to the "arc-length element" ds.

- Recall that $\frac{ds}{dt} = v(t)$ is the speed. (Note that this implies that there is some parameterization for C.) So $ds = v(t)\ dt$. In terms of approximation, this says $\Delta s \approx v(t)\Delta t$.

- In 2D, $\int_C f(x,y)\ ds$ gives "bended area".



- Evaluating f along the curve C with the parameterization $\vec{r}(t) = \big[x(t), y(t), z(t)\big]^{\mathsf{T}}$, where $a \le t \le b$, gives

$$\int_C f(x,y)\ ds = \int_a^b f(x,y)\ ds = \int_a^b f(x(t),y(t))\ v(t)\ dt \tag{2D}$$

$$\int_C f(x,y,z)\ ds = \int_a^b f(x,y,z)\ ds = \int_a^b f(x(t),y(t),z(t))\ v(t)\ dt \tag{3D}$$

These are referred to as line integrals of scalar functions. These do not depend on what parameterization $\vec{r}(t)$ is used.

- Physical interpretation:
  - If C is a wire with linear density $\rho(x,y,z)$ with units kg/m, then $\int_C \rho(x,y,z)\ ds$ gives the mass of the wire segment in kg.

> The standard parameterization for the line segment connecting two points P_0 and P_1 is
>
> $$\vec{r}(t) = (1-t)P_0 + tP_1\ ,$$
>
> where $0 \le t \le 1$.

1. Suppose a straight wire connecting $(2,0,3)$ and $(1,2,1)$ has linear density $\rho(x,y) = x^2y^2$. Compute the mass.

Let $P_0 = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$ and $P_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. We use the parameterization

$$\begin{aligned} \vec{r}(t) &= (1-t)P_0 + tP_1 \\ &= (1-t)\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + t\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 2-2t \\ 0 \\ 3-3t \end{bmatrix} + \begin{bmatrix} t \\ 2t \\ t \end{bmatrix} \\ &= \begin{bmatrix} 2-t \\ 2t \\ 3-2t \end{bmatrix} \\ &= \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} , \end{aligned}$$

where $0 \leq t \leq 1$. So, the velocity is

$$\vec{v}(t) = \begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} ,$$

which means that the speed is

$$\begin{aligned} v(t) &= ||\vec{v}(t)|| \\ &= \sqrt{(-1)^2 + (2)^2 + (-2)^2} \\ &= \sqrt{1+4+4} \\ &= \sqrt{9} \\ &= 3 , \end{aligned}$$

where $v(t) > 0$ since speed is a positive scalar quantity. Noting that $ds = v(t)\ dt$, the line integral is

$$\begin{aligned} \int_C \rho(x,y)\ ds &= \int_{t=0}^{t=1} \rho(x(t),y(t))\ v(t)\ dt \\ &= \int_{t=0}^{t=1} x^2y^2 \cdot 3\ dt \\ &= \int_{t=0}^{t=1} 3x^2y^2\ dt \end{aligned}$$

$$
\begin{aligned}
&= \int_{t=0}^{t=1} 3 \cdot x(t)^2 \cdot y(t)^2 \ dt \\
&= 3 \int_{t=0}^{t=1} x(t)^2 \cdot y(t)^2 \ dt \\
&= 3 \int_{t=0}^{t=1} (2-t)^2 \cdot (2t)^2 \ dt \\
&= 3 \int_{t=0}^{t=1} (4 - 4t + t^2) \cdot 4t^2 \ dt \\
&= 3 \int_{t=0}^{t=1} (16t^2 - 16t^3 + 4t^4) \ dt \\
&= 3 \left[\frac{16}{3} t^3 - \frac{16}{4} t^4 + \frac{4}{5} t^5 \right]_{t=0}^{t=1} \\
&= 3 \left[\frac{16}{3} t^3 - 4t^4 + \frac{4}{5} t^5 \right]_{t=0}^{t=1} \\
&= 3 \left[\frac{16}{3} - 4 + \frac{4}{5} \right] \ .
\end{aligned}
$$

2. Let C be the intersection between $z = 2 - x^2 - 2y^2$ and $z = x^2$ in the **first octant** (i.e. $x \geq 0$, $y \geq 0$, $z \geq 0$). Find $\displaystyle\int_C xy\ ds$.

We have that

$$\begin{aligned}
2 - x^2 - 2y^2 &= z \\
2 - x^2 - 2y^2 &= x^2 \\
2 &= x^2 + 2y^2 + x^2 \\
2 &= 2x^2 + 2y^2 \\
1 &= x^2 + y^2 \ .
\end{aligned}$$

Now, try

$$\begin{aligned}
x &= \cos(t) \\
y &= \sin(t) \\
z &= x^2 = \cos^2(t) \ ,
\end{aligned}$$

where $0 \leq t \leq \frac{\pi}{2}$. Then the parameterization is given by

$$\vec{r}(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \\ \cos^2(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} \ .$$

So, the velocity is

$$\vec{v}(t) = \begin{bmatrix} -\sin(t) \\ \cos(t) \\ -2\cos(t)\sin(t) \end{bmatrix}$$

and the speed

$$\begin{aligned}
v(t) &= \sqrt{(-\sin(t))^2 + (\cos(t))^2 + (-2\cos(t)\sin(t))^2} \\
&= \sqrt{\sin^2(t) + \cos^2(t) + 4\cos^2(t)\sin^2(t)} \\
&= \sqrt{1 + 2\cos(t)\sin(t) \cdot 2\cos(t)\sin(t)} \\
&= \sqrt{1 + \sin(2t) \cdot \sin(2t)} \\
&= \sqrt{1 + \sin^2(2t)} \ .
\end{aligned}$$

Then

$$\begin{aligned}
\int_C xy\ ds &= \int_{t=0}^{t=\pi/2} x(t) \cdot y(t)\ ds \\
&= \int_{t=0}^{t=\pi/2} x(t) \cdot y(t) \cdot v(t)\ dt
\end{aligned}$$

$$= \int_{t=0}^{t=\pi/2} \cos(t) \cdot \sin(t) \cdot \sqrt{1 + \sin^2(t)} \ dt \ .$$

We can see that this is a difficult integral to solve. So, let's try a different parameterization. We can see that $z = x^2$ is a "function type", and so we can let $x = t$. Then $z = x^2 = t^2$ and

$$\begin{aligned}
2 - x^2 - 2y^2 &= z \\
2 - x^2 - 2y^2 &= x^2 \\
2 - 2y^2 &= 2x^2 \\
1 - y^2 &= x^2 \\
1 - x^2 &= y^2 \\
y^2 &= 1 - x^2 \\
y &= \sqrt{1 - x^2} \\
y &= \sqrt{1 - t^2} \ .
\end{aligned}$$

So, we get the paramaterization

$$\vec{r}(t) = \begin{bmatrix} t \\ \sqrt{1-t^2} \\ t^2 \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} \ ,$$

where $0 \leq t \leq 1$ for x, y, and z to be in the first octant. Note that there is no need to consider $y = -\sqrt{1-t^2}$ since we are only concerned with the first octant (i.e. $y \geq 0$). Then the velocity is

$$\vec{v}(t) = \begin{bmatrix} 1 \\ -\dfrac{2t}{2\sqrt{1-t^2}} \\ 2t \end{bmatrix}$$

and the speed is

$$\begin{aligned}
v(t) &= \sqrt{(1)^2 + \left(-\frac{2t}{2\sqrt{1-t^2}}\right)^2 + (2t)^2} \\
&= \sqrt{1 + \frac{4t^2}{4(1-t^2)} + 4t^2} \\
&= \sqrt{1 + \frac{t^2}{1-t^2} + 4t^2} \ .
\end{aligned}$$

So,

$$\int_C xy \ ds = \int_{t=0}^{t=1} x(t) \cdot y(t) \cdot v(t) \ dt$$

$$
\begin{aligned}
&= \int_{t=0}^{t=1} t \cdot \sqrt{1-t^2} \cdot \sqrt{1 + \frac{t^2}{1-t^2} + 4t^2} \ dt \\
&= \int_{t=0}^{t=1} t \cdot \cdot \sqrt{(1-t^2)\left(1 + \frac{t^2}{1-t^2} + 4t^2\right)} \ dt \\
&= \int_{t=0}^{t=1} t \cdot \sqrt{(1-t^2) + t^2 + 4t^2(1-t^2)} \ dt \\
&= \int_{t=0}^{t=1} t \cdot \sqrt{1 - t^2 + t^2 + 4t^2 - 4t^4} \ dt \\
&= \int_{t=0}^{t=1} t \cdot \sqrt{1 + 4t^2 - 4t^4} \ dt \ .
\end{aligned}
$$

Now, since only powers of t appear let $u = t^2$. Then

$$
\begin{aligned}
\frac{du}{dt} &= \frac{d}{dt}\left[t^2\right] \\
\frac{du}{dt} &= 2t \\
du &= 2t \ dt \ ,
\end{aligned}
$$

and the upper and lower endpoints become

$$
\begin{aligned}
u(1) &= (1)^2 = 1 \\
u(0) &= (0)^2 = 0 \ .
\end{aligned}
$$

So,

$$
\begin{aligned}
\int_C xy \ ds &= \int_{t=0}^{t=1} t \cdot \sqrt{1 + 4t^2 - 4t^4} \ dt \\
&= \frac{1}{2} \int_{t=0}^{t=1} 2t \cdot \sqrt{1 + 4t^2 - 4t^4} \ dt \\
&= \frac{1}{2} \int_{t=0}^{t=1} \sqrt{1 + 4t^2 - 4t^4} \cdot 2t \ dt \\
&= \frac{1}{2} \int_{u=0}^{u=1} \sqrt{1 + 4u - 4u^2} \ du \ .
\end{aligned}
$$

We can solve this integral by completing the square of the radical term.

$$
\begin{aligned}
1 + 4u - 4u^2 &= -4u^2 + 4u + 1 \\
&= -4\left(u^2 - u - \frac{1}{4}\right) \\
&= -4\left(\left(u^2 - u + \frac{1}{4}\right) - \frac{1}{4} - \frac{1}{4}\right) \\
&= -4\left(\left(u^2 - u + \frac{1}{4}\right) - \frac{2}{4}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= -4\left(\left(u-\frac{1}{2}\right)^2-\frac{1}{2}\right)\\
&= -4\left(u-\frac{1}{2}\right)^2+2\\
&= 2-4\left(u-\frac{1}{2}\right)^2\\
&= 2\left(1-2\left(u-\frac{1}{2}\right)^2\right)\ .
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\int_C xy\ ds &= \frac{1}{2}\int_{u=0}^{u=1}\sqrt{1+4u-4u^2}\ du\\
&= \frac{1}{2}\int_{u=0}^{u=1}\sqrt{2\left(1-2\left(u-\frac{1}{2}\right)^2\right)}\ du\\
&= \frac{1}{2}\int_{u=0}^{u=1}\sqrt{2}\cdot\sqrt{\left(1-2\left(u-\frac{1}{2}\right)^2\right)}\ du\\
&= \frac{\sqrt{2}}{2}\int_{u=0}^{u=1}\sqrt{\left(1-2\left(u-\frac{1}{2}\right)^2\right)}\ du\\
&= \dots
\end{aligned}
$$

(To be continued)

Vector Fields, Conservative Fields, and Potentials

Definitions

Recall that a **vector field** is a function $\vec{F} : U \to \mathbb{R}^n$, where $U \subseteq \mathbb{R}^n$ is typically open. For us, we restrict to $n = 2$ or $n = 3$. If we write

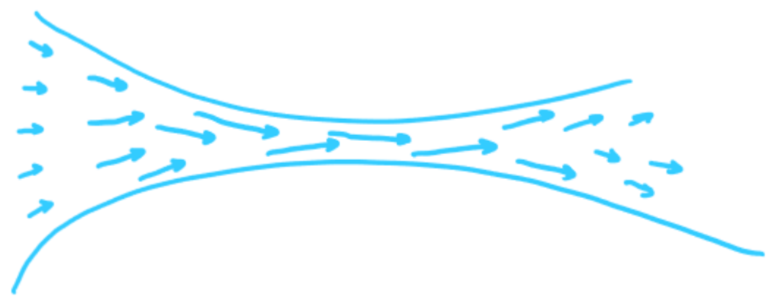
$$\vec{F} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \quad \text{or} \quad \vec{F} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} ,$$

we typically require the F_i to be twice differentiable, with the second order partials also continuous. If Φ is a scalar function, then $\nabla\Phi$ is a vector field. We call a vector field $\vec{F}$ for which there is a Φ with $\nabla\Phi = \vec{F}$ a **conservative field**. In this case, Φ is the **potential** for $\vec{F}$.

Remark: We can visualize vector fields by plotting a sample of $\vec{F}(x_i, y_i, z_i)$ at some choice for (x_i, y_i, z_i). We do so by drawing the vector $\vec{F}(x_i, y_i, z_i)$ with point of origination at (x_i, y_i, z_i).

1. **(Velocity Field for a Fluid)**

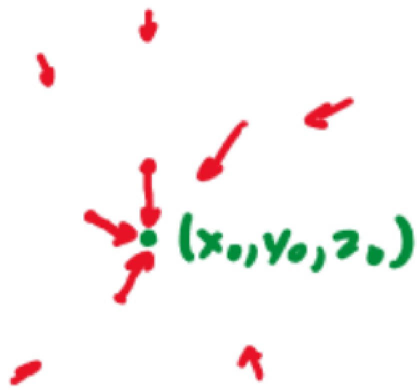
   Consider fluid flow through a fixed passage. Let $\vec{v}(x, y)$ denote the velocity of the fluid at (x, y).



2. The gravitational field of a point-mass with mass m at (x_0, y_0, z_0) is given by

$$\vec{F}(x,y,z) = \frac{-km}{\left[(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2\right]^{3/2}} \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} \ ,$$

where k is a fixed constant. Note that $\vec{F}$ always points towards the point-mass.



$\vec{F}$ is conservative, and a potential is given by

$$\Phi(x,y,z) = \frac{km}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}} \ .$$

The electrostatic force is the same thing, but opposite in sign and some different constant k.

3. **Rigid Body Rotation**

Suppose we rotate the entire xy-plane counter-clockwise about the z-axis with angular velocity Ω rad/s, where $\Omega > 0$.

Let $\vec{v}(x, y)$ denote the velocity vector for a point located at (x, y), where

$$\vec{v}(x, y) = \begin{bmatrix} -\Omega y \\ \Omega x \end{bmatrix} .$$

Is $\vec{v}$ conservative? Suppose it was conservative. Then there exists Φ such that

$$\nabla \Phi = \begin{bmatrix} \Phi_x \\ \Phi_y \end{bmatrix} = \begin{bmatrix} -\Omega y \\ \Omega x \end{bmatrix} = \vec{v} .$$

We can integrate $\Phi_x = -\Omega y$ with respect to x, which gives us

$$\begin{aligned} \Phi &= \int \Phi_x \, dx \\ &= \int -\Omega y \, dx \\ &= -\Omega y \int dx \\ &= -\Omega y x + f(y) \\ &= -\Omega x y + f(y) , \end{aligned}$$

where $f(y)$ is some arbitrary function in y. Similarly, we can integrate $\Phi_y = \Omega_x$ with respect to y, which gives us

$$\begin{aligned} \Phi &= \int \Phi_y \, dy \\ &= \int \Omega x \, dy \end{aligned}$$

$$
\begin{aligned}
&= \Omega x \int dy \\
&= \Omega xy + g(x) \ ,
\end{aligned}
$$

where $g(x)$ is some arbitrary function in x. Then since $\Phi = -\Omega xy + f(y)$ and $\Phi = \Omega xy + g(x)$, we get that

$$
\begin{aligned}
\Omega xy + g(x) &= -\Omega xy + f(y) \\
2\Omega xy &= f(y) - g(x) \qquad (*)
\end{aligned}
$$

Now,

- If $x = 0$, then

  $$
  \begin{aligned}
  0 = \Omega \cdot 0 \cdot y &= f(y) - g(0) \\
  \implies f(y) &= g(0)
  \end{aligned}
  $$

  for all y. So, f is constant.

- If $y = 0$, then

  $$
  \begin{aligned}
  0 = \Omega \cdot x \cdot 0 &= f(0) - g(x) \\
  \implies g(x) &= f(0)
  \end{aligned}
  $$

  for all x. So, g is constant.

Then by (*), Ωxy is constant. This is a contradiction, since $\Omega > 0$.

Fact

For a twice continuously differentiable scalar function Φ, we have **equality of second order mixed partials**. That is,

$$\Phi_{xy} = \Phi_{yx} \ , \quad \Phi_{xz} = \Phi_{zx} \ , \quad \Phi_{yz} = \Phi_{zy} \ .$$

Necessary Condition for the Existence of a Potential

If a vector field $\vec{F} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$ is conservative, then

$$F_{1,y} = F_{2,x} \ .$$

If $\vec{F} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$ is conservative, then

$$\begin{aligned} F_{2,x} &= F_{1,y} \\ F_{3,x} &= F_{1,z} \\ F_{3,y} &= F_{2,z} \end{aligned} \tag{**}$$

Typical usage: If any of these fail, then we know $\vec{F}$ is NOT conservative. Essentially, if the contrapositive does not hold, then $\vec{F}$ can not be conservative. For example, in the case of rigid body rotation, we had

$$\vec{v}(x, y) = \begin{bmatrix} -\Omega_y \\ \Omega_x \end{bmatrix} \ .$$

However,

$$(-\Omega y)_y = -\Omega \neq \Omega = (\Omega x)_x \ .$$

Why is the necessary condition the way it is? If $\vec{F} = \nabla\Phi = \begin{bmatrix} \Phi_x \\ \Phi_y \\ \Phi_z \end{bmatrix}$, then (**) becomes

$$\begin{aligned} \Phi_{yx} &= \Phi_{xy} \\ \Phi_{zx} &= \Phi_{xz} \\ \Phi_{yz} &= \Phi_{zy} \end{aligned}$$

1. Let $\vec{F}(x,y) = \dfrac{1}{x^2+y^2}\begin{bmatrix} -y \\ x \end{bmatrix}$, where $(x,y) \neq (0,0)$. Is $\vec{F}$ conservative? If it is, find a potential.

We have that

$$\vec{F}(x,y) = \frac{1}{x^2+y^2}\begin{bmatrix} -y \\ x \end{bmatrix} = \begin{bmatrix} \dfrac{-y}{x^2+y^2} \\ \dfrac{x}{x^2+y^2} \end{bmatrix} .$$

Using the test, we get that

$$\left(\frac{-y}{x^2+y^2}\right)_y = \frac{-(x^2+y^2)+y\cdot 2y}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

and

$$\left(\frac{x}{x^2+y^2}\right)_x = \frac{x^2+y^2-x\cdot 2x}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2} .$$

So, we have that

$$\left(\frac{-y}{x^2+y^2}\right)_y = \frac{-x^2+y^2}{(x^2+y^2)^2} = \left(\frac{x}{x^2+y^2}\right)_x .$$

So, $\vec{F}$ could be conservative. Now, suppose there exists a scalar-valued function Φ such that

$$\nabla\Phi(x,y) = \begin{bmatrix} \Phi_x \\ \Phi_y \end{bmatrix} = \begin{bmatrix} \dfrac{-y}{x^2+y^2} \\ \dfrac{x}{x^2+y^2} \end{bmatrix} = \vec{F}(x,y) .$$

If we integrate $\Phi_x = \dfrac{-y}{x^2+y^2}$ with respect to x, we get

$$\begin{aligned} \int \Phi_x \; dx &= \int \frac{-y}{x^2+y^2} \; dx \\ \Phi &= \int \frac{-y}{x^2+y^2} \; dx \\ &\vdots \\ -\Phi &= \tan^{-1}\left(\frac{x}{y}\right) + f(y) \end{aligned}$$

and if we integrate $\Phi_y = \dfrac{x}{x^2+y^2}$ with respect to y, we get

$$\int \Phi_y \; dy = \int \frac{x}{x^2+y^2} \; dy$$

$$
\begin{aligned}
\Phi &= \int \frac{x^2}{y^2}\, dy \\
&\vdots \\
\Phi &= \tan^{-1}\left(\frac{y}{x}\right) + g(x)\ .
\end{aligned}
$$

Φ is never defined when $x = 0$, so it cannot be a potential for $\vec{F}$. Hence, $\vec{F}$ is not conservative. (More later)

2. Determine if the following fields are conservative. If they are, find a potential.

   (a) $\vec{F}(x,y) = \dfrac{1}{x^2-y^2}\begin{bmatrix} x \\ -y \end{bmatrix}$, where $x \neq \pm y$.

   We have that

$$\vec{F}(x,y) = \frac{1}{x^2-y^2}\begin{bmatrix} x \\ -y \end{bmatrix} = \begin{bmatrix} \dfrac{x}{x^2-y^2} \\ \dfrac{-y}{x^2+y^2} \end{bmatrix} .$$

   Using the test, we have that

$$\left(\frac{x}{x^2-y^2}\right)_y = \frac{2xy}{(x^2-y^2)^2}$$

   and

$$\left(\frac{-y}{x^2+y^2}\right)_x = \frac{2xy}{(x^2+y^2)} .$$

   So, we get that

$$\left(\frac{x}{x^2-y^2}\right)_y = \frac{2xy}{(x^2-y^2)^2} = \left(\frac{-y}{x^2+y^2}\right)_x ,$$

   which means that $\vec{F}$ could be conservative. Now, suppose there exists a scalar-valued function $\Phi(x,y)$ such that

$$\nabla\Phi(x,y) = \begin{bmatrix} \Phi_x \\ \Phi_y \end{bmatrix} = \begin{bmatrix} \dfrac{x}{x^2-y^2} \\ \dfrac{-y}{x^2+y^2} \end{bmatrix} = \vec{F}(x,y) .$$

   If we integrate $\Phi_x = \dfrac{x}{x^2-y^2}$ with respect to x, then

$$\begin{aligned}
\Phi &= \int \Phi_x \; dx \\
&= \int \frac{x}{x^2-y^2} \; dx \\
&= \int \frac{1}{x^2-y^2} \cdot x \; dx \\
&= \frac{1}{2}\int \frac{1}{x^2-y^2} \cdot 2x \; dx \\
&= \frac{1}{2}\int \frac{1}{u} \; du
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2} \cdot \ln |u| \\
&= \frac{1}{2} \cdot \ln |x^2 - y^2| + f(y)
\end{aligned}
$$

and if we integrate $\Phi_y = \dfrac{-y}{x^2 - y^2}$ with respect to y, then

$$
\begin{aligned}
\Phi &= \int \Phi_y \ dy \\
&= \int \frac{-y}{x^2 - y^2} \ dy \\
&= \int \frac{1}{x^2 - y^2} \cdot (-y) \ dy \\
&= \frac{1}{2} \int \frac{1}{x^2 - y^2} \cdot (-2y) \ dy \\
&= \frac{1}{2} \int \frac{1}{w} \ dw \\
&= \frac{1}{2} \cdot \ln |w| \\
&= \frac{1}{2} \cdot \ln |x^2 - y^2| + g(x) \ .
\end{aligned}
$$

So, we get that

$$
\frac{1}{2} \cdot \ln |x^2 - y^2| + f(y) = \Phi = \frac{1}{2} \cdot \ln |x^2 - y^2| + g(x) \ .
$$

Then by taking $f(y) = g(x) = 0$, we get

$$
\Phi(x, y) = \frac{1}{2} \ln |x^2 - y^2| \ ,
$$

where $x \neq \pm y$ (same domain as $\vec{F}$).

(b) $\vec{F}(x,y,z) = \begin{bmatrix} y\cos(xy)+3 \\ x\cos(xy)-1 \\ -\sin(2z) \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$.

By the test, we check if $(F_1)_y = (F_2)_x$, $(F_3)_x = (F_1)_z$, and $(F_3)_y = (F_2)_z$.

(i) Check if $(F_1)_y = (F_2)_x$:

$$\begin{aligned}
(F_1)_y &= (y\cos(xy)+3)_y \\
&= \frac{\partial}{\partial y}\left[y\cos(xy)+3\right] \\
&= \frac{\partial}{\partial y}\left[y\cos(xy)\right] + \frac{\partial}{\partial y}[3] \\
&= \left(\frac{\partial}{\partial y}[y]\cdot\cos(xy) + y\cdot\frac{\partial}{\partial y}[\cos(xy)]\right) + 0 \\
&= 1\cdot\cos(xy) + y\cdot(-\sin(xy)\cdot x) \\
&= \cos(xy) - xy\sin(xy) \\
&= 1\cdot\cos(xy) + x\cdot(-\sin(xy)\cdot y) \\
&= \left(\frac{\partial}{\partial x}[x]\cdot\cos(xy) + x\cdot\frac{\partial}{\partial x}[\cos(xy)]\right) - 0 \\
&= \frac{\partial}{\partial x}[x\cos(xy)] - \frac{\partial}{\partial x}[1] \\
&= \frac{\partial}{\partial x}[x\cos(xy)-1] \\
&= (x\cos(xy)-1)_x \\
&= (F_2)_x\ .
\end{aligned}$$

So $(F_1)_y = (F_2)_x$.

(ii) Check if $(F_3)_x = (F_1)_z$:

$$\begin{aligned}
(F_3)_x &= (-\sin(2z))_x \\
&= \frac{\partial}{\partial x}[-\sin(2z)] \\
&= 0 \\
&= \frac{\partial}{\partial z}[y\cos(xy)+3] \\
&= (y\cos(xy)+3)_z \\
&= (F_1)_x\ .
\end{aligned}$$

So $(F_3)_x = (F_1)_z$.

(iii) Check if $(F_3)_y = (F_2)_z$:

$$\begin{aligned}(F_3)_y &= (-\sin(2z))_y \\ &= \frac{\partial}{\partial y}[-\sin(2z)] \\ &= 0 \\ &= \frac{\partial}{\partial z}[x\cos(xy) - 1] \\ &= (x\cos(xy) - 1)_z \\ &= (F_2)_z \ .\end{aligned}$$

So $(F_3)_y = (F_2)_x$.

So, we have that $\vec{F}$ could be a potential. Now, suppose there exists a scalar-valued function $\Phi(x, y, z)$ such that

$$\nabla\Phi(x,y,z) = \begin{bmatrix}\Phi_x \\ \Phi_y \\ \Phi_z\end{bmatrix} = \begin{bmatrix} y\cos(xy) + 3 \\ x\cos(xy) - 1 \\ -\sin(2z)\end{bmatrix} = \vec{F}(x,y,z) \ .$$

Now, we integrate each of Φ_x, Φ_y, and Φ_z with respect to their respective variables.

- Integrating $\Phi_x = y\cos(xy) + 3$ with respect to x gives us

$$\begin{aligned}\Phi &= \int \Phi_x \ dx \\ &= \int (y\cos(xy) + 3) \ dx \\ &= \int y\cos(xy) \ dx + \int 3 \ dx \\ &= \int \cos(xy) \cdot y \ dx + 3\int dx \\ &= \int \cos(u) \ du + 3\int dx \\ &= \sin(u) + 3x + f_1(y,z) \\ &= \sin(xy) + 3x + f_1(y,z) \ .\end{aligned}$$

- Integrating $\Phi_y = x\cos(xy) - 1$ with respect to y gives us

$$\begin{aligned}
\Phi &= \int \Phi_y \ dy \\
&= \int (x\cos(xy) - 1) \ dy \\
&= \int x\cos(xy) \ dy - \int 1 \ dy \\
&= \int \cos(xy) \cdot x \ dy - \int y^0 \ dy \\
&= \int \cos(w) \ dw - \int y^0 \ dy \\
&= \sin(w) - y + f_2(x,z) \\
&= \sin(xy) - y + f_2(x,z) \ .
\end{aligned}$$

- Integrating $\Phi_z = -\sin(2x)$ with respect to z gives us

$$\begin{aligned}
\Phi &= \int \Phi_z \ dz \\
&= \int -\sin(2x) \ dz \\
&= \frac{1}{2}\int -\sin(2z) \cdot 2 \ dz \\
&= \frac{1}{2}\int -\sin(v) \ dv \\
&= \frac{1}{2}\cos(v) + f_3(x,y) \\
&= \frac{1}{2}\cos(2z) + f_3(x,y) \ .
\end{aligned}$$

So, we get that Φ must be all three of the above. That is, it must hold that $\Phi = \sin(xy) + 3x + f_1(y,z)$, $\Phi = \sin(xy) - y + f_2(x,z)$, and $\Phi = \frac{1}{2}\cos(2z) + f_3(x,y)$. So, we get that

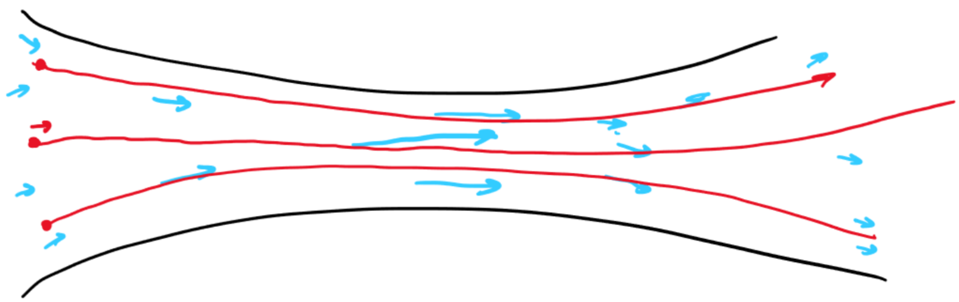
$$\begin{aligned}
3\Phi(x,y,z) &= \Phi(x,y,z) + \Phi(x,y,z) + \Phi(x,y,z) \\
&= 2\sin(xy) + \frac{1}{2}\cos(2z) + 3x - y + f_1(y,z) + f_2(x,z) + f_3(x,y) \\
&= \ldots
\end{aligned}$$

Field Lines

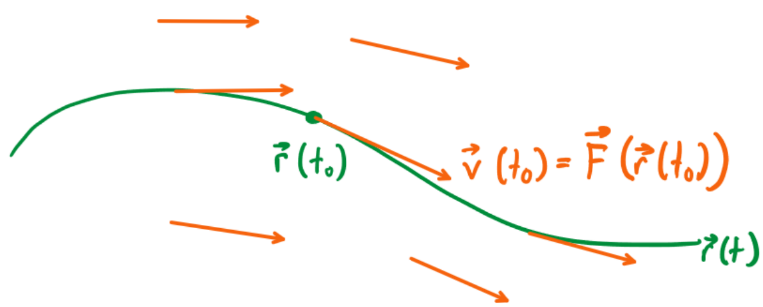
Definition (Field Line)

A field line for a vector field $\vec{F}$ is a parametric curve $\vec{r}(t)$ with the property that the velocity vector for $\vec{r}(t)$ is $\vec{F}(\vec{r}(t))$. That is,

$$\vec{v}(t) = \vec{F}(\vec{r}(t)) \ .$$



A particle dropped in will follow a field line.



1. **(Solid Body Rotatation)** Let $\vec{F}(x,y) = \begin{bmatrix} -y \\ x \end{bmatrix}$. We know that

$$
\begin{aligned}
\vec{F}(\vec{r}(t)) &= \vec{F}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \\
&= \begin{bmatrix} -y(t) \\ x(t) \end{bmatrix} \\
&= \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} \\
&= \vec{v}(t) \ .
\end{aligned}
$$

So, we have that $x'(t) = \dfrac{dx}{dt} = -y(t)$ and $y'(t) = \dfrac{dy}{dt} = x(t)$. From this, we can try to write these in cartesian coordinates via the chain rule:

$$
\frac{dy}{dx} = \frac{\left(\dfrac{dy}{dt}\right)}{\left(\dfrac{dx}{dt}\right)} = \frac{y'(t)}{x'(t)} = \frac{x(t)}{-y(t)} = \frac{x}{-y} = -\frac{x}{y} \ .
$$

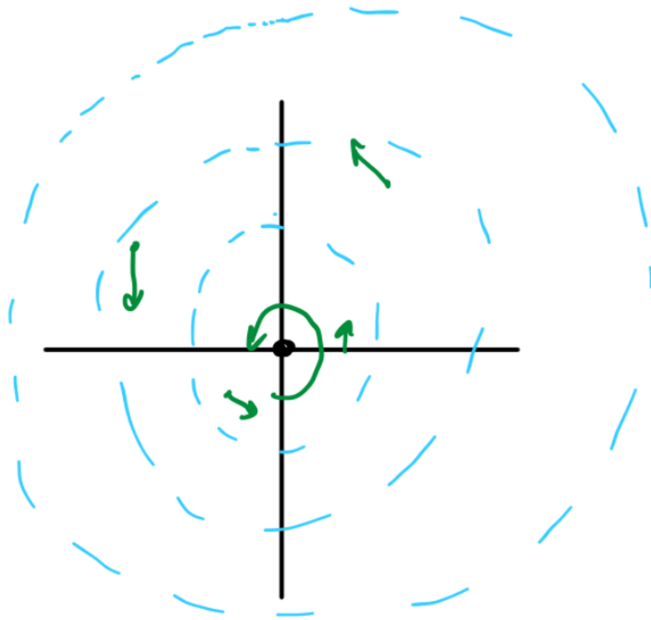
Then applying separation of variables gives

$$
y \ dy = -x \ dx \ .
$$

So, integrating both sides with respect to their appropriate variables gives

$$
\begin{aligned}
\int y \ dy &= \int -x \ dx \\
\int y \ dy &= -\int x \ dx \\
\frac{1}{2}y^2 + c_1 &= -\frac{1}{2}x^2 + c_2 \\
\frac{1}{2}y^2 &= -\frac{1}{2}x^2 + c_2 - c_1 \\
\frac{1}{2}y^2 &= -\frac{1}{2}x^2 + c \\
y^2 &= -x^2 + 2c \\
y^2 + x^2 &= 2c \ .
\end{aligned}
$$

These are circles! (See next page.)



2. $\vec{F}(x,y) = \dfrac{1}{x^2+y^2}\begin{bmatrix} -y \\ x \end{bmatrix}$.

We know that

$$\begin{aligned}\vec{F}(\vec{r}(t)) &= \vec{F}\left(x(t), y(t)\right) \\ &= \frac{1}{x(t)^2+y(t)^2}\begin{bmatrix} -y(t) \\ x(t) \end{bmatrix} \\ &= \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} \\ &= \vec{v}(t)\ .\end{aligned}$$

So, we get that

$$x'(t) = \frac{dx}{dt} = -\frac{y(t)}{x(t)^2+y(t)^2}$$

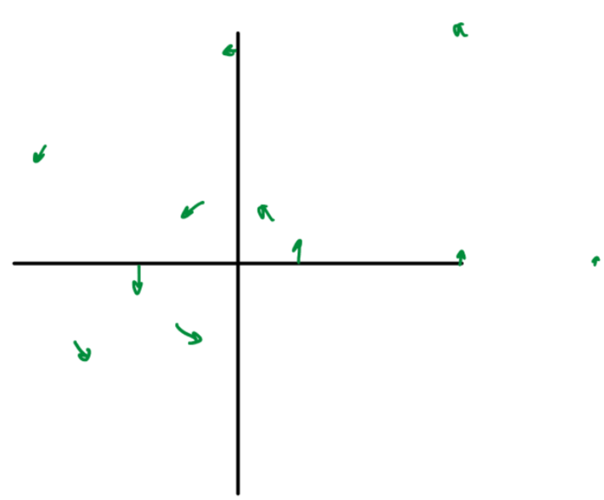
and

$$y'(t) = \frac{dy}{dt} = \frac{x(t)}{x(t)^2+y(t)^2}\ .$$

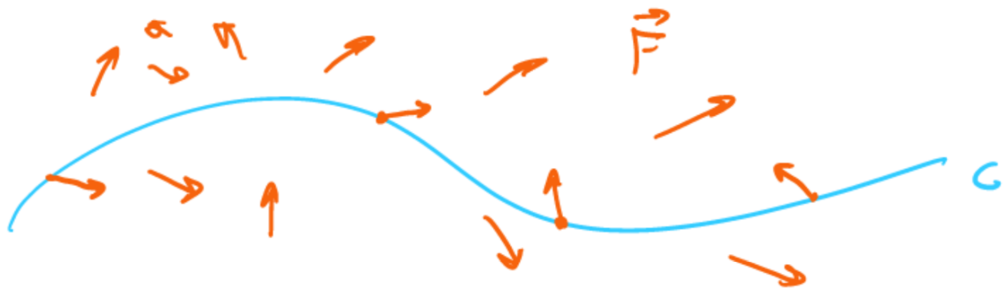
From this, we can try to write these in cartesian coordinates via the chain rule:

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{y'(t)}{x'(t)} = \frac{\frac{x(t)}{x(t)^2+y(t)^2}}{\frac{-y(t)}{x(t)^2+y(t)^2}} = -\frac{x(t)}{y(t)} = -\frac{x}{y}\ .$$

These are also circles (but slow speed along circles further out).



Line Integrals of Vector Fields

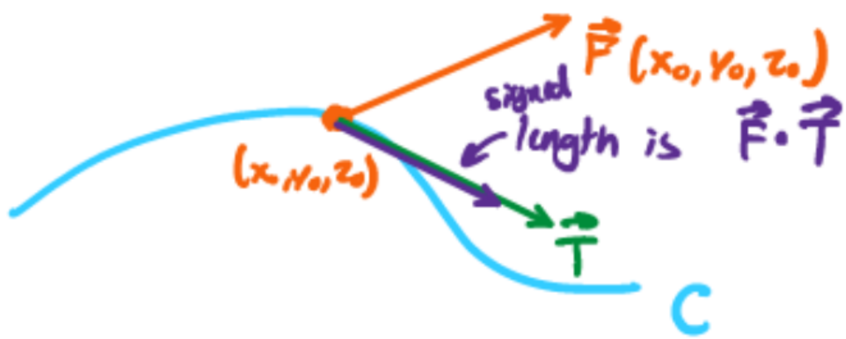


- We wish to define

$$\int_C \vec{F} \ ,$$

  where $\vec{F}$ is a vector field in 2D or 3D, and C is a parametric curve in $\mathbb{R}^2$ or $\mathbb{R}^3$.

- We need to integrate a <u>scalar</u> quantity along the curve associated with $\vec{F}$.



- With respect to the $\vec{T}, \vec{N}, \vec{B}$ frame at (x_0, y_0, z_0), we can write

$$\vec{F}(x_0, y_0, z_0) = (\vec{F} \cdot \vec{T})\vec{T} + (\vec{F} \cdot \vec{N})\vec{N} + (\vec{F} \cdot \vec{B})\vec{B} \ .$$

  Here, $(\vec{F} \cdot \vec{T})$, $(\vec{F} \cdot \vec{N})$, and $(\vec{F} \cdot \vec{B})$ are all reasonable choices for scalar quantities we can integrate.

- $(\vec{F} \cdot \vec{T})$ is the tangential component of $\vec{F}$. It gives the contribution to $\vec{F}$ in the direction $\vec{T}$ (i.e. along the direction of the curve).

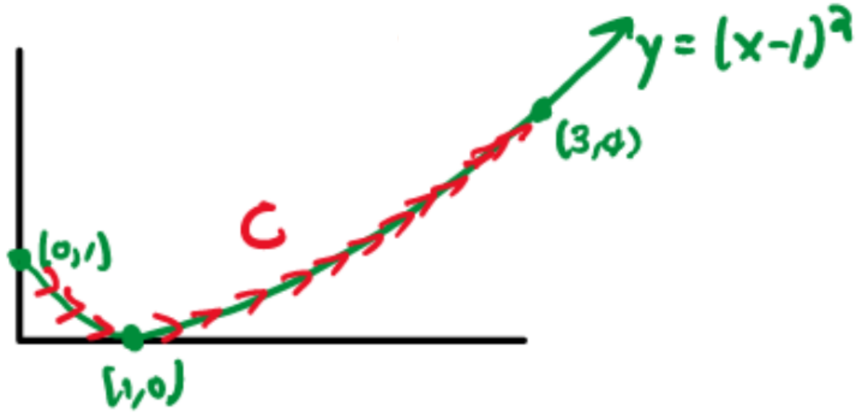
- We have that
$$\vec{F} \cdot \vec{T} = \vec{F} \cdot \frac{\vec{v}}{||\vec{v}||} ,$$
which can be thought of as the coefficient of the projection of $\vec{F}$ onto $\vec{v}$. Note that the projection of $\vec{F}$ onto $\vec{v}$ is
$$\text{proj}_v \vec{F} = \frac{\vec{F} \cdot \vec{v}}{||\vec{v}||^2} \cdot \vec{v} .$$

- So, this means that
$$\int_C \vec{F}$$
should be the integral of $\vec{F} \cdot \vec{T}$ (a scalar) over the curve C. Then the formula for the line integral of a scalar function gives
$$\begin{aligned} \int_C \left(\vec{F} \cdot \vec{T}\right) \, ds &= \int_C \left(\vec{F}(\vec{r}(t)) \cdot \frac{\vec{v}(t)}{||\vec{v}(t)||}\right) \, ds \\ &= \int_C \vec{F}(\vec{r}(t)) \cdot \frac{\vec{v}(t)}{v(t)} \cdot v(t) \, dt \\ &= \int_C \vec{F}(\vec{r}(t)) \cdot \vec{v}(t) \, dt \\ &= \int_C \left(\vec{F}(\vec{r}(t)) \cdot \vec{v}(t)\right) \, dt , \end{aligned}$$
where $\vec{r}(t)$ is a paramaterization of C.

- This gives the line integral of a vector field $\vec{F}$ over C as
$$\boxed{\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{v}(t) \, dt}$$
where $d\vec{r} = \vec{v}(t) \, dt$. Note that the left hand side is usually read as "the integral of $\vec{F}$ dot the velocity element".

- If $\vec{F}$ is a force, then we interpret this integral as the integral of the force field applied in the direction of a particle's motion along the curve. This is referred to as the **work**.

1. Let $\vec{F}(x,y) = \begin{bmatrix} y^2 \\ x^2 - 4 \end{bmatrix}$, and let C be the portion of the graph $y = (x-1)^2$ from $(0,1)$ to $(3,4)$.



Note that $y = (x-1)^2$ is a "function type". So, we can use the parameterization

$$\vec{r}(t) = \begin{bmatrix} t \\ (t-1)^2 \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} .$$

where $0 \leq t \leq 3$. Then the velocity is

$$\vec{v}(t) = \begin{bmatrix} 1 \\ 2(t-1) \end{bmatrix} .$$

Hence,

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \int_0^3 \vec{F}(\vec{r}(t)) \cdot \vec{v}(t) \ dt \\
&= \int_0^3 \vec{F}(x(t),\ y(t)) \cdot (1,\ 2(t-1))^\mathsf{T} \ dt \\
&= \int_0^3 (y(t)^2,\ x(t)^2 - 4)^\mathsf{T} \cdot (1,\ 2(t-1))^\mathsf{T} \ dt \\
&= \int_0^3 ((t-1)^4,\ t^2 - 4)^\mathsf{T} \cdot (1,\ 2(t-1))^\mathsf{T} \ dt \\
&= \int_0^3 ((t-1)^4 + 2t^2(t-1) - 8(t-1)) \ dt \\
&= \int_0^3 ((t-1)^4 + 2t^3 - 2t^2 - 8t + 8) \ dt \\
&= \dots
\end{aligned}$$

$$= \left[\frac{1}{5}(t-1)^5 + \frac{1}{2}t^4 - \frac{2}{3}t^3 - 4t^2 + 8t\right]_{t=0}^{3} .$$

Remarks

(i) The direction along C matters! If $\vec{r}_1(t)$ is a parameterization of C and $\vec{r}_2(t)$ is also a parameterization of C but travelling in the opposite direction, then

$$\int_C \vec{F}(\vec{r}_2(t)) \cdot \vec{v}_2 \ dt = -\int_C \vec{F}(\vec{r}_1(t)) \cdot d\vec{v}_1(t) \ .$$

We typically write this more simply as

$$\int_C \vec{F} \cdot d\vec{r} = -\int_{-C} \vec{F} \cdot dr \ .$$

- (ii) Two paramaterizations of C going in the same direction will give the <u>same</u> integral.

- (iii) If we write $\vec{F}(x,y,z) = \begin{bmatrix} F_1 & F_2 & F_3 \end{bmatrix}^{\mathsf{T}}$, many books write

$$\int_C \vec{F} \cdot dr = \int_C F_1 \ dx + \int_C F_2 \ dy + \int_C F_3 \ dz \ .$$

This is essentially the result of performing the dot product.

- For conservative fields, where $\vec{F} = \nabla \Phi$, we have

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{v}(t)\ dt \\
&= \int_a^b \nabla \Phi(\vec{r}(t)) \cdot \vec{r}\,'(t)\ dt \\
&= \int_a^b \frac{d}{dt} \left[\Phi(\vec{r}(t))\right]\ dt && \text{(chain rule)} \\
&= \Phi(\vec{r}(b)) - \Phi(\vec{r}(a))\ . && \text{(F.T.C. 2)}
\end{aligned}$$

Hence, we can conclude with the following:

Path Independence

If $\vec{F}$ is conservative, only the endpoints of the curve C contribute to $\displaystyle\int_C \vec{F} \cdot dr$. That is, if C_1 and C_2 both start and end at the same place, then

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}\ .$$

- C is called closed if its start equals its end.

i.e. $\vec{r}(a) = \vec{r}(b)$

Independence of Path Theorem

Suppose D is open and connected (like an open ball) and $\vec{F}$ is a vector field on D. Then the following are equivalent:

(i) $\displaystyle\oint_C \vec{F} \cdot d\vec{r} = 0$ for all closed curves in D.

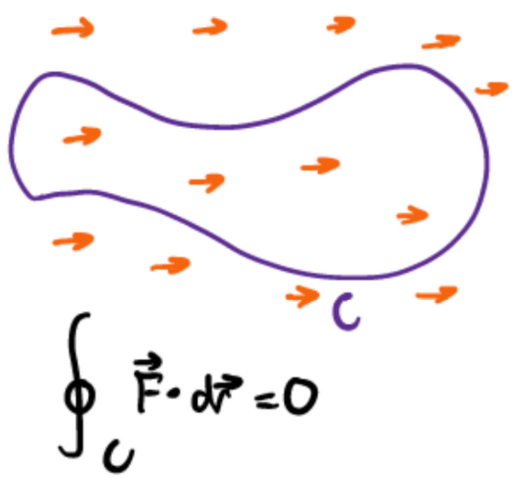
(ii) $\displaystyle\int_C \vec{F} \cdot d\vec{r}$ is independent of path.

(iii) $\vec{F}$ is conservative.

- We use the notation

$$\oint_C \vec{F} \cdot d\vec{r}$$

for a line integral over a closed curve. This quantity is also called **circulation**, as it measures the tendency of the field to circulate about the curve C.



Large $\oint_C \vec{F} \cdot d\vec{r}$

1. Compute $\int_C \vec{F} \cdot d\vec{r}$ for $\vec{F}(x,y,z) = \begin{bmatrix} y^2 \\ 2xy - z \\ -y \end{bmatrix}$, where $\vec{r}(t) = \begin{bmatrix} \sqrt{t^2-1} \\ e^{t^2} \\ \cos^2(\pi t) \end{bmatrix}$ for $0 \le t \le 3$.

We have that

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \int_0^3 \vec{F}(\vec{r}(t)) \cdot \vec{v}(t)\ dt \\ &= \int_0^3 \vec{F}(x(t), y(t), z(t)) \cdot \vec{v}(t)\ dt\ . \end{aligned}$$

So,

$$\begin{aligned} \vec{F}(\vec{r}(t)) &= \vec{F}(x(t), y(t), z(t)) \\ &= \vec{F}\left(\begin{bmatrix} \sqrt{t^2-1} \\ e^{t^2} \\ \cos^2(\pi t) \end{bmatrix}\right) \\ &= \vec{F}\left(\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}\right) \\ &= \begin{bmatrix} y(t)^2 \\ 2 \cdot x(t) \cdot y(t) - z(t) \\ -y(t) \end{bmatrix} \\ &= \begin{bmatrix} \left(e^{t^2}\right)^2 \\ 2(\sqrt{t^2-1})(e^{t^2}) - \cos^2(\pi t) \\ -(e^{t^2}) \end{bmatrix} \\ &= \begin{bmatrix} e^{t^4} \\ 2\sqrt{t^2-1}e^{t^2} - \cos^2(\pi t) \\ -e^{t^2} \end{bmatrix} \end{aligned}$$

and

$$\vec{v}(t) = \begin{bmatrix} t/\sqrt{t^2-1} \\ 2te^{t^2} \\ 2\pi\cos(\pi t)(-\sin(\pi t)) \end{bmatrix}\ .$$

If we plug these back into the formula, we will get an integral that's probably impossible to solve. So, let's try a different approach. We have that $\vec{F}$ could be conservative. Recall that if $\vec{F} = \nabla\Phi$, then

$$\int_C \vec{F} \cdot d\vec{r} = \Phi(\vec{r}(b)) - \Phi(\vec{r}(a))\ ,$$

where a and b are the endpoints our interval. Note that we can assume equality of second order partials for $\vec{F}$. Let's try to find Φ. Since

$$\nabla\Phi(x,y,z) = \begin{bmatrix}\Phi_x\\ \Phi_y\\ \Phi_z\end{bmatrix} = \begin{bmatrix}y^2\\ 2xy-z\\ -y\end{bmatrix} = \vec{F}(x,y,z)\ ,$$

we get that

$$\begin{aligned}
&\Phi_x = y^2 \implies \Phi = xy^2 + f_1(y,z)\\
&\Phi_y = 2xy - z \implies \Phi = xy^2 - yz + f_2(x,z)\\
&\Phi_z = -y \implies \Phi = -yz + f_3(x,y)
\end{aligned}$$

Now, let $f_1(y,z) = -yz,\ f_3(x,y) = xy^2,\ f_2(x,z) = 0$. Then we solve for like so:

$$\begin{aligned}
\Phi &= xy^2 - yz\\
-\ \Phi &= xy^2 - yz + 0\\
+\ \Phi &= -yz + xy^2\\
\hline
\Phi &= xy^2 - yz
\end{aligned}$$

Then

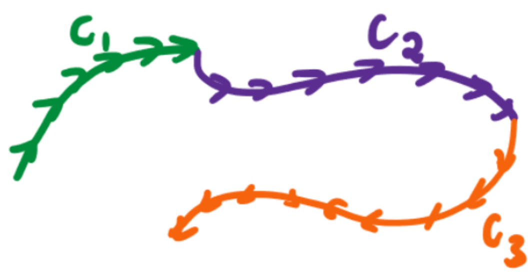
$$\Phi(x,y,z) = xy^2 - yz$$

is a potential. Hence, for conservative fields we have that

$$\begin{aligned}
\int_C \vec{F}\cdot d\vec{r} &= \Phi(\vec{r}(3)) - \Phi(\vec{r}(0))\\
&= \Phi\left(\begin{bmatrix}\sqrt{3^2-1}\\ e^{3^2}\\ \cos^2(3\pi)\end{bmatrix}\right) - \Phi\left(\begin{bmatrix}\sqrt{0^2-1}\\ e^{0^2}\\ \cos^2(0)\end{bmatrix}\right)\\
&= \Phi\left(\begin{bmatrix}\sqrt{9-1}\\ e^{9}\\ (-1)^2\end{bmatrix}\right) - \Phi\left(\begin{bmatrix}\sqrt{0-1}\\ e^{0^2}\\ \cos^2(0)\end{bmatrix}\right)\\
&= \dots
\end{aligned}$$

Curve Decomposition

If C can be decomposed into curves $C_1, C_2, \ldots, C_n$ with final point of C_j equal to the initial point in C_{j+1}.

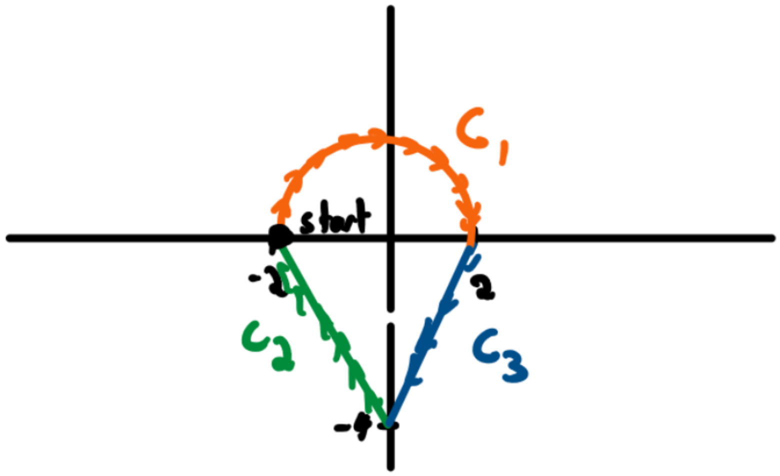


then we can define

$$\int_C \vec{F} \cdot d\vec{r} = \sum_{j=1}^{n} \int_{C_j} \vec{F} \cdot d\vec{r}_j \ .$$

(A consequence of this is we still integrate over continuous curves C that have a finite number of singular points.)

1. Integrate $\vec{F}(x,y) = \begin{bmatrix} -y^2 \\ x^2 \end{bmatrix}$ on C below.



- For C_1, it is a circle of radius 2 (at least, the top part of a circle). So, we have the parameterization

$$\vec{r}_1(t) = \begin{bmatrix} -2\cos(t) \\ 2\sin(t) \end{bmatrix}$$

  for $0 \leq t \leq \pi$.

- For C_2, it is a line from $(0,-4)$ to $(-2,0)$, which can be described by the parameterization

$$\begin{aligned} \vec{r}_2(t) &= (1-t)\begin{bmatrix} 0 \\ -4 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ -4+4t \end{bmatrix} + \begin{bmatrix} -2t \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -2t \\ -4+4t \end{bmatrix} \end{aligned}$$

  for $0 \leq t \leq 1$.

- For C_3, it is a line from $(2,0)$ to $(0,-4)$, which can be described by the parameterization

$$\begin{aligned} \vec{r}_3(t) &= (1-t)\begin{bmatrix} 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ -4 \end{bmatrix} \\ &= \begin{bmatrix} 2-2t \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -4t \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} 2-2t \\ -4t \end{bmatrix}$$

for $0 \le t \le 1$.

So, we get that

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \sum_{j=1}^{n} \int_{C_j} \vec{F} \cdot d\vec{r}_j \\
&= \sum_{j=1}^{3} \int_{C_j} \vec{F} \cdot d\vec{r}_j \\
&= \int_{C_1} \vec{F} \cdot d\vec{r}_1 + \int_{C_2} \vec{F} \cdot d\vec{r}_2 + \int_{C_3} \vec{F} \cdot d\vec{r}_3 \ .
\end{aligned}$$

Now, we evaluate each of these integrals.

- For C_1,

$$\int_{C_1} \vec{F} \cdot d\vec{r}_1 = \int_0^{\pi} \vec{F}(\vec{r}_1(t)) \cdot \vec{v}(t)\ dt \ .$$

So, we have that

$$\begin{aligned}
\vec{F}(\vec{r}_1(t)) &= \vec{F}\left(\begin{bmatrix} -2\cos(t) \\ 2\sin(t) \end{bmatrix}\right) \\
&= \vec{F}\left(\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}\right) \\
&= \begin{bmatrix} -y(t)^2 \\ x(t)^2 \end{bmatrix} \\
&= \begin{bmatrix} -(2\sin(t))^2 \\ (-2\cos(t))^2 \end{bmatrix} \\
&= \begin{bmatrix} -4\sin^2(t) \\ 4\cos^2(t) \end{bmatrix}
\end{aligned}$$

and

$$\vec{v}_1(t) = \begin{bmatrix} 2\sin(t) \\ 2\cos(t) \end{bmatrix} \ .$$

Then

$$\begin{aligned}
\vec{F}(\vec{r}_1(t)) \cdot \vec{v}_1\ dt &= \begin{bmatrix} -4\sin^2(t) \\ 4\cos^2(t) \end{bmatrix} \cdot \begin{bmatrix} 2\sin(t) \\ 2\cos(t) \end{bmatrix} \\
&= (-4\sin^2(t) \cdot 2\sin(t)) + (4\cos^2(t) \cdot 2\cos(t))
\end{aligned}$$

$$
\begin{aligned}
&= -8\sin^3(t) + 8\cos^3(t) \\
&= -8\sin(t)\sin^2(t) + 8\cos(t)\cos^2(t) \\
&= -8\sin(t)(1-\cos^2(t)) + 8\cos(t)(1-\sin^2(t)) \ .
\end{aligned}
$$

So,

$$
\begin{aligned}
&\int_{C_1} \vec{F} \cdot d\vec{r}_1 \\
&= \int_0^\pi \vec{F}(\vec{r}_1(t)) \cdot \vec{v}(t) \ dt \\
&= \int_0^\pi (-8\sin(t)(1-\cos^2(t)) + 8\cos(t)(1-\sin^2(t))) \ dt \\
&= \int_0^\pi -8\sin(t)(1-\cos^2(t)) \ dt + \int_0^\pi 8\cos(t)(1-\sin^2(t))) \ dt \\
&= -8\int_0^\pi \sin(t)(1-\cos^2(t)) \ dt + 8\int_0^\pi \cos(t)(1-\sin^2(t))) \ dt
\end{aligned}
$$

Now, let $u = \cos(t)$ and $w = \sin(t)$. Then

$$
\begin{aligned}
du &= -\sin(t) \ dt \ , \\
u(\pi) &= \cos(\pi) = -1 \ , \\
u(0) &= \cos(0) = 1
\end{aligned}
$$

and

$$
\begin{aligned}
dw &= \cos(t) \ dt \ , \\
w(\pi) &= \sin(\pi) = 0 \ , \\
w(0) &= \sin(0) = 0 \ .
\end{aligned}
$$

This gives

$$
\begin{aligned}
\int_0^\pi \sin(t)(1-\cos^2(t)) \ dt &= \int_0^\pi (1-\cos^2(t)) \cdot \sin(t) \ dt \\
&= -\int_0^\pi (1-\cos^2(t)) \cdot (-\sin(t)) \ dt \\
&= -\int_1^{-1} (1-u^2) \cdot du \\
&= -\left(-\int_{-1}^1 (1-u^2) \cdot du\right) \\
&= \int_{-1}^1 (1-u^2) \cdot du \\
&= \int_{-1}^1 1 \ du - \int_{-1}^1 u^2 \ du
\end{aligned}
$$

$$
\begin{aligned}
&= u\Big|_{-1}^{1} - \left[\frac{u^3}{3}\right]_{-1}^{1} \\
&= (1-(-1)) - \left(\frac{(1)^3}{3} - \frac{(-1)^3}{3}\right) \\
&= 2 - \left(\frac{1}{3} - \left(-\frac{1}{3}\right)\right) \\
&= 2 - \frac{2}{3} \\
&= \frac{6}{3} - \frac{2}{3} \\
&= \frac{4}{3}
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^\pi \cos(t)(1-\sin^2(t))\ dt &= \int_0^\pi (1-\sin^2(t)) \cdot \cos(t)\ dt \\
&= \int_0^0 (1-w^2)\ dw \\
&= 0\ .
\end{aligned}
$$

So,

$$
\begin{aligned}
&\int_{C_1} \vec{F} \cdot d\vec{r}_1 \\
&= -8\int_0^\pi \sin(t)(1-\cos^2(t))\ dt + 8\int_0^\pi \cos(t)(1-\sin^2(t)))\ dt \\
&= -8\left(\frac{4}{3}\right) + 8(0) \\
&= -\frac{32}{3} + 0 \\
&= -\frac{32}{3}\ .
\end{aligned}
$$

- For C_2,

2. (Almost conservative field) Let $\vec{F}(x,y) = \begin{bmatrix} e^x \sin(y) + 3y \\ e^x \cos(y) + 2x - 2y \end{bmatrix}$ on the curve C given by $4x^2 + 9y^2 = 16$ (ellipse). Find $\oint_C \vec{F} \cdot d\vec{r}$.

Direct evaluation gives a bad integral (difficult to solve). Idea: find a Φ and a "nice" $\vec{G}$ so that

$$\vec{F} = \nabla\Phi + \vec{G} \ .$$

Then using this potential, we can evaluate

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C \left(\nabla\Phi\vec{G}\right) \ d\vec{r} = \oint_C \nabla\Phi \ d\vec{r} + \oint_C \vec{G} \ d\vec{r} \ .$$

We can find ∇ (note that $\vec{F}$ is not actually conservative, but we're still finding a potential ∇ anyways). Let $\Phi(x,y)$ be a scalar-valued function such that

$$\nabla\Phi(x,y) = \begin{bmatrix} \Phi_x \\ \Phi_y \end{bmatrix} = \begin{bmatrix} e^x \sin(y) + 3y \\ e^x \cos(y) + 2x - 2y \end{bmatrix} = \vec{F}(x,y) \ .$$

Then integrating Φ_x gives

$$\begin{aligned} \Phi &= \int \Phi_x \ dx \\ &= \int e^x \sin(y) + 3y \ dx \\ &= \int e^x \sin(y) \ dx + \int 3y \ dx \\ &= \sin(y) \int e^x \ dx + 3y \int dx \\ &= e^x \sin(y) + 3xy + f(y) \end{aligned}$$

and integrating Φ_y with respect to y gives

$$\begin{aligned} \Phi &= \int \Phi_y \ dy \\ &= \int e^x \cos(y) + 2x - 2y \ dy \\ &= \int e^x \cos(y) \ dy + \int 2x \ dy - \int 2y \ dy \\ &= e^x \int \cos(y) \ dy + 2x \int dy - 2 \int y \ dy \\ &= e^x \sin(y) + 2xy - y^2 + g(x) \ . \end{aligned}$$

So, we have that

$$\begin{aligned} e^x \sin(y) + 3xy + f(y) &= e^x \sin(y) + 2xy - y^2 + g(x) \\ 3xy + f(y) &= 2xy - y^2 + g(x) \\ xy + f(y) &= -y^2 + g(x) \ . \end{aligned}$$

Here,

$$\Phi = e^x \sin(y) - y^2$$

gives something close to a potential.