ENGS 26 Final Report: Inverted Pendulum

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Introduction

A stick is a long, thin piece of wood, that hates being balanced. We will attempt to balance a stick in its upright position using a car and a control compensator. Successful completion of this project requires an accurate derivation of the transfer function of the system (the dynamics of the car and stick) and appropriate controller design and implementation to balance the stick in its upright position. We also need to carry out our wiring perfectly and remind our stick that she is capable of incredible things. We derive the open-loop transfer function by characterizing the dynamics of three systems: (1) the force of the wheels response to a voltage input, (2) the angle of the stick response to the force of the wheels, and (3) the voltage of the sensor response to the angle of the stick. The open-loop transfer functions (1) and (3) are determined experimentally, and transfer function (2) is derived through free-body diagram analysis. We use these transfer functions to characterize our plant, use the plant model to analytically design a compensator, and finally study the closed-loop response of the system both experimentally and analytically. The figure below shows the block diagram of the closed-loop system.

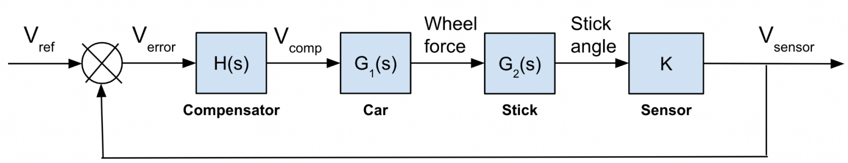
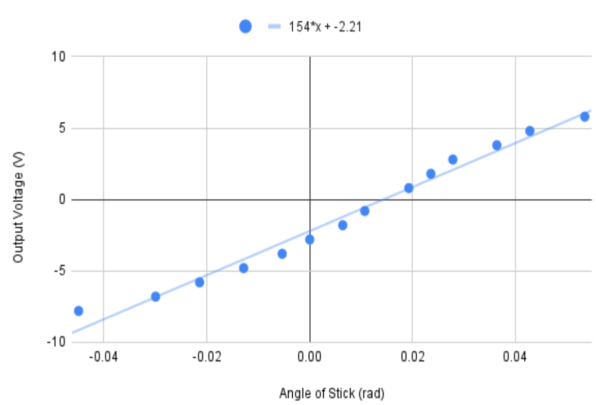


Figure 1: Block diagram of closed-loop inverted pendulum system.

We choose a PID controller for this project because we want to, above all, minimize the peak overshoot of our system response. The PID controller also provides us with both minimal steady-state error and rise times. If everything goes well, we expect the stick to rise to its equilibrium position quickly and stay at steady-state with a minimal amplitude of oscillation.

Sensor Characterization

We varied the angle of the stick and measured the output voltage in the range of - 7.8 V to 5.8 V. The sensor became saturated before the stick could reach either extreme of oscillation–therefore the range of voltages does not trace out the entire path of Sticky's potential motion. We plotted the linear portion of angle vs. voltage, where we converted the displacement from zero of the stick to a corresponding angle using



trigonometry. We characterized the sensor using the slope of this linear plot which was 154 V/rad.

Figure 2: Plot of voltage vs. angle of stick.

Motor/Car Characterization

We characterized our car and motor by supplying a 3 V step input directly to the motor and recording a video of our car's position response. Using Open Source Physics "Tracker Video Analysis and Modeling Tool" software, we created a position vs. time graph by tracing one point on the car. We converted this position vs. time plot to a velocity vs. time plot using MATLAB. Both plots are shown below.

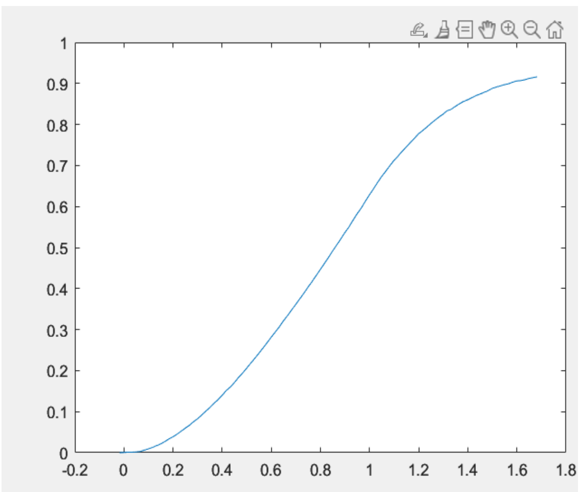


Figure 3: Plot of car position (m) vs. time (s).

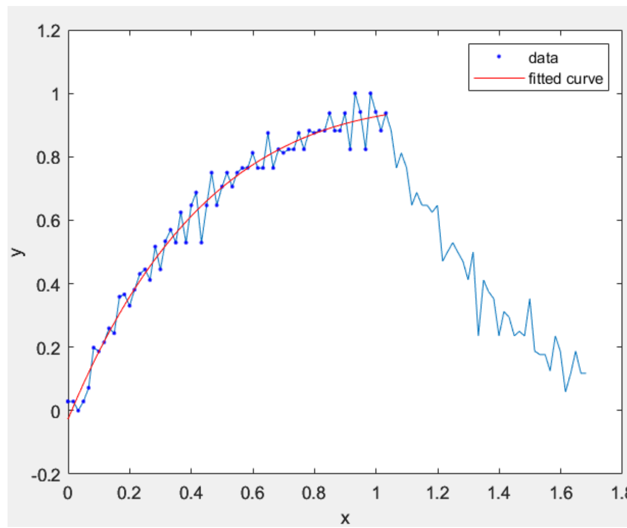


Figure 4: Plot of car velocity (m/s) vs. time (s).

While our data was noisy, the overall shape of the plot indicated a first order model. Using curve fitting tools in MATLAB, we found the equation of best fit for a first order model based on our data, shown on the graph below. The equation describing velocity-time is $v(t)=a-be^{-ct}$ where a=1.04 , b=1.075, and c=2.307. We used this model to determine the transfer function for velocity output from voltage input and then converted this velocity transfer function to force.

general form: $\dot{x}(t) = a - be^{-ct}$

$$\dot{X}(s) = L(\dot{x}(t)) = \frac{a}{s} - \frac{b}{s+c} = \frac{a(s+c) - bs}{s(s+c)}$$

Now, we divide both sides by $V(s) = \frac{3}{s}$

$$\frac{\dot{X}(s)}{V(s)} = \frac{s}{3}\,\frac{a(s+c) - bs}{s(s+c)} = \frac{a(s+c) - bs}{3(s+c)}$$

We multiply by s to take the derivative in the transform domain and find the transfer function relating acceleration to input voltage.

$$\frac{\ddot{X}(s)}{V(s)} = \frac{s(a(s+c) - bs)}{3(s+c)}$$

We multiply by mass to find the transfer function relating force output to input voltage.

$$\frac{F(s)}{V(s)} = (m+M)\frac{s(a(s+c) - bs)}{3(s+c)}$$

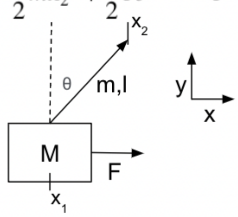
where a=1.04 , b=1.075, and c=2.307

Force to Stick Angle Characterization

To characterize the relationship between force on the car and angle of the stick, we performed the derivation below. We began with free body diagrams of each object, determined equations of motion using a lagrangian, linearized each equation around where the stick was vertical, made a small angle approximation, and finally, used Laplace Transform to find the transfer function relating input force and output angle.

Beginning with the equations of motion for the system:

$T = \frac{1}{2}M\dot{x}_1{}^2 + \frac{1}{2}m\dot{x}_2{}^2 + \frac{1}{2}I\theta'^2 \qquad I = \frac{1}{3}ml^2 \qquad \ddot{x}_2 = \frac{l}{2}\theta' \qquad m\dot{x}_2{}^2 = \int_0^l \rho A\dot{x}_2\,\mathrm{dr}$ $\qquad \theta'$ is the derivative of θ



$$x_2 = \langle x_1 + \mathrm{r}\sin\theta, \mathrm{r}\cos\theta\rangle$$
$$\dot{x}_2 = \langle \dot{x}_1 + \mathrm{r}\cos\theta\cdot\theta', -\mathrm{r}\sin\theta\cdot\theta'\rangle$$
$$\dot{x}_2{}^2 = \dot{x}_1{}^2 + 2r\dot{x}_1\theta'\cos\theta + r^2\theta'^2\cos\theta + r^2\theta'^2\sin\theta$$
$$\dot{x}_2{}^2 = \dot{x}_1{}^2 + 2r\dot{x}_1\theta'\cos\theta + r^2\theta'^2$$

$$T = \frac{1}{2}M\dot{x}_1{}^2 + \frac{1}{2}\int_0^l \rho A\left(\dot{x}_1{}^2 + 2r\dot{x}_1\theta'\cos\theta + r^2\theta'^2\right)\mathrm{dr}$$
$$= \frac{1}{2}M\dot{x}_1{}^2 + \frac{1}{2}\rho A\left(\dot{x}_1{}^2 l + l^2\dot{x}_1\theta'\cos\theta + \frac{l^3}{3}\theta'^2\right)$$ $\dot{x}_1 = \dot{x}$, now the only x variable of interest
$$= \frac{1}{2}M\dot{x}_1{}^2 + \frac{1}{2}m\dot{x}_1{}^2 + \frac{1}{2}\mathrm{ml}\dot{x}_1\theta'\cos\theta + \frac{ml^2}{6}\theta'^2$$

$$V = \int_0^l \rho\mathrm{Agrsin}\theta\mathrm{dr} - \mathrm{Fx} = \frac{\mathrm{mglcos}\theta}{2} - \mathrm{Fx}$$

$$L = T - V = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}\mathrm{ml}\dot{x}\theta'\cos\theta + \frac{ml^2}{6}\theta'^2 - \frac{\mathrm{mglcos}\theta}{2} + \mathrm{Fx}$$

$$\frac{d}{\mathrm{dt}}\frac{\delta L}{\delta\dot{x}} - \frac{\delta L}{\delta x} = \frac{d}{\mathrm{dt}}\left((M+m)\dot{x} + \frac{1}{2}\mathrm{ml}\theta'\cos\theta\right) - F = (M+m)\ddot{x} + \frac{1}{2}\mathrm{ml}\theta''\cos\theta - \frac{1}{2}\mathrm{ml}\theta'^2\sin\theta - F = 0$$
$$\frac{d}{\mathrm{dt}}\frac{\delta L}{\delta\theta'} - \frac{\delta L}{\delta\theta} = \frac{d}{\mathrm{dt}}\left(\frac{1}{2}\mathrm{ml}\theta'\cos\theta + \frac{ml^2}{6}\theta'^2\right) - \left(-\frac{1}{2}\mathrm{ml}\dot{x}\theta'\sin\theta + \frac{\mathrm{mglsin}\theta}{2}\right)$$
$$\frac{1}{2}\mathrm{ml}\ddot{x}\cos\theta - \frac{1}{2}\mathrm{ml}\dot{x}\theta'\sin\theta + \frac{ml^2}{6}\theta'^2 + \frac{1}{2}\mathrm{ml}\dot{x}\theta'\sin\theta - \frac{\mathrm{mgl}}{2}\sin\theta = 0$$

Equations:

$$\frac{1}{2}\mathrm{ml}\ddot{x}\cos\theta + \frac{ml^2}{6}\theta'' - \frac{\mathrm{mgl}}{2}\sin\theta = 0,\ \ddot{x}\cos\theta + \frac{l}{3}\theta'' - \mathrm{g}\sin\theta = 0$$
$$(M+m)\ddot{x} + \frac{1}{2}\mathrm{ml}\theta''\cos\theta - \frac{1}{2}\mathrm{ml}\theta'^2\sin\theta - F = 0$$

Linearizing around $\theta = 0$:

$$\ddot{x}\cos\theta + \frac{l}{3}\theta'' = \mathrm{g}\sin\theta \rightarrow \ddot{x} + \frac{l}{3}\theta'' = g\theta$$
$$(M+m)\ddot{x} + \frac{1}{2}\mathrm{ml}\theta''\cos\theta - \frac{1}{2}\mathrm{ml}\theta'^2\sin\theta = F \rightarrow (M+m)\ddot{x} + \frac{1}{2}\mathrm{ml}\theta'' - \frac{1}{2}\mathrm{ml}\theta'^2\theta = F$$

We ignore the nonlinear term, resulting in:

$$\ddot{x} + \frac{l}{3}\theta'' = g\theta$$

$$(M+m)\ddot{x} + \frac{1}{2}\mathrm{ml}\theta'' = F$$

Then we take the Laplace transform:

$$s^2X + \frac{s^2 l}{3}\Theta = g\Theta$$

$$(M+m)s^2X + \frac{1}{2}\mathrm{ml}s^2\Theta = F$$

Finally we rearrage algebraically and use substitution to find the transfer function from Θ to F

$$X = \frac{3g - s^2 l}{3s^2}$$

$$(M+m)s^2\left(\frac{3g - s^2 l}{3s^2}\right) + \frac{1}{2}\mathrm{ml}s^2\Theta = F$$

$$\left((M+m)g - \frac{\mathrm{Ml}}{3}s^2 + \frac{\mathrm{ml}}{6}s^2\right)\Theta = F$$

$$\frac{\Theta}{F} = \frac{6}{6(M+m)g + (m-2M)ls^2}$$

Final Plant Transfer Function

From this modeling, we arrived at a transfer function for our plant.

$$\frac{V_{\mathrm{sensor}}}{V_{\mathrm{ref}}} = \frac{F}{V_{\mathrm{ref}}} \cdot \frac{\Theta}{F} \cdot \frac{V_{\mathrm{sensor}}}{\Theta}$$

$$\frac{V_{\mathrm{sensor}}}{V_{\mathrm{ref}}} = \frac{103.4s^2 - 7088s}{16.236s^3 + 37.456s^2 - 563.961 - 1301.059}$$

Analysis of Uncompensated System

We evaluated the stability of the plant system by using root locus to find where the roots of the system lie. The root locus plot of the plant system is on the next page.

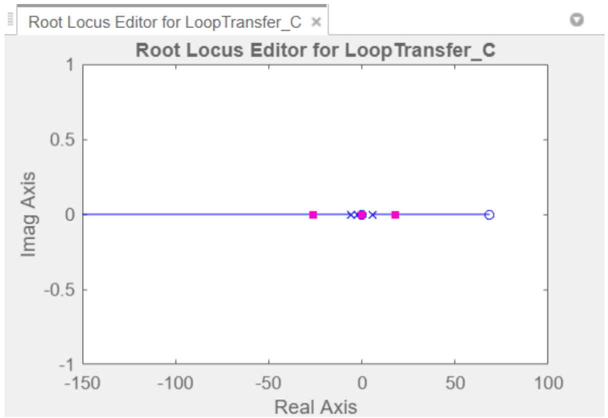


Figure 5: Root Locus plot of plant system.

The root locus plot of the plant system has a pole in the right-half plane, which indicates the plant system will be unstable without a controller. This indicates that the plant step response will go to infinity in the time-domain. We chose a preferred controller design below.

Design Criterion and Controller Design

We worked in Matlab to design a PID controller that created a stable closed-loop step response for our plant. We implemented a PID controller in Matlab's control system designer, using the SISOTOOL command and changed our controller parameters (poles, zeros, and gain) until the system met our specifications. Our specifications were system stability, less than 50% peak overshoot, and settling time under 1 sec– all of which our system met (with peak overshoot 39% and settling time of 0.65 sec). A main aim of our design was to respond well to disturbances, so we also made sure that our system had a gain margin of over 6dB and a phase margin of greater than 30 degrees. Our system meet these requirements as well. Below are our calculations used to determine whether the controller met the specifications above.

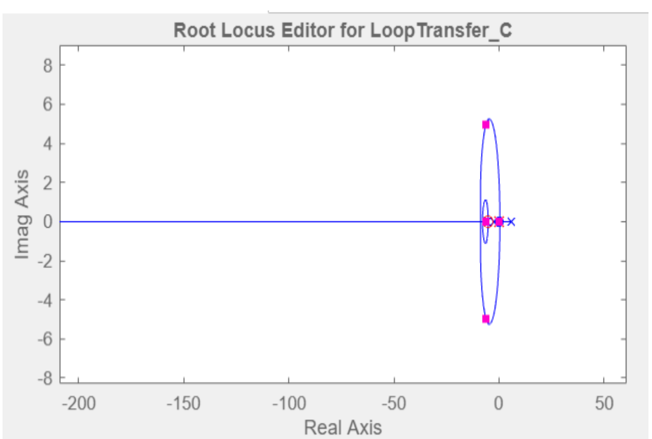


Figure 6: Root Locus plot of closed-loop system.

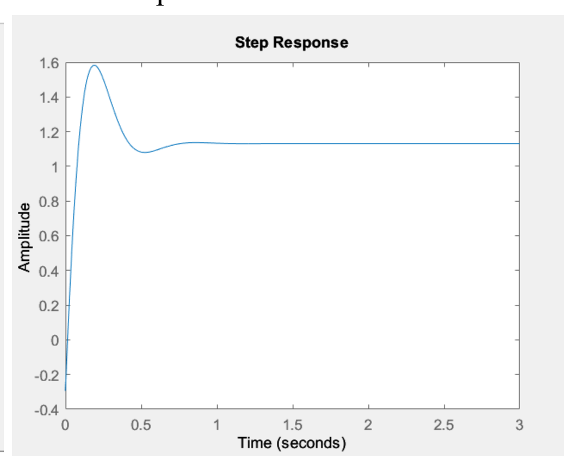


Figure 7: Step-response of closed-loop system.

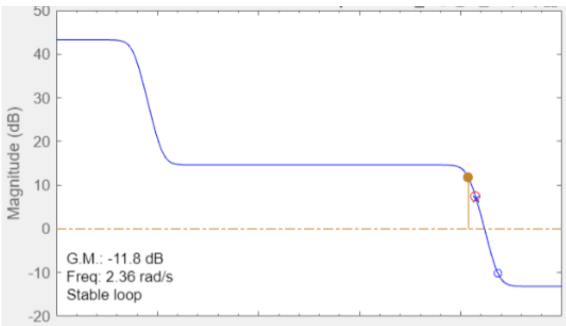


Figure 8: Gain Margn of Closed-loop system.

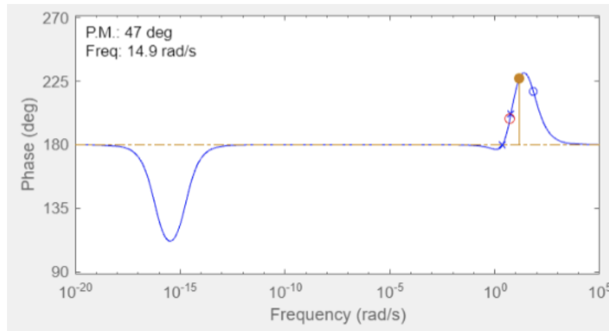


Figure 9: Phase Margn of Closed-loop system.

Controller Transfer Function:

$$-0.9023 \times \frac{(1 + 0.19s)\,(1 + 0.19s)}{s}$$

One may notice that we have not included zero steady-state error as one of our specifications. Unfortunately, we were unable to derive a controller transfer function that resulted in zero steady-state error. We found this surprising because we expected any PID controller to eliminate steady-state error. The steady-state error for the controller we chose was 13.06%. We were able to solve this problem by setting our actual Vref to be equal to (desired Vref/1.1306). This means that our Vout at steady state is equal to (1.1306)(desired Vref/1.1306)=desired Vref. For our car, we found our desired reference input to be -2.8V (where the stick was at equilibrium). Then, we set our actual reference voltage to be =2.8/1.1306=-2.4766 V. This way, we completely eliminated the issue of having a steady-state error.

Controller Implementation

Following the specifications of our Matlab simulation, we built a controller for our car. The physical design can be seen in the schematic below.

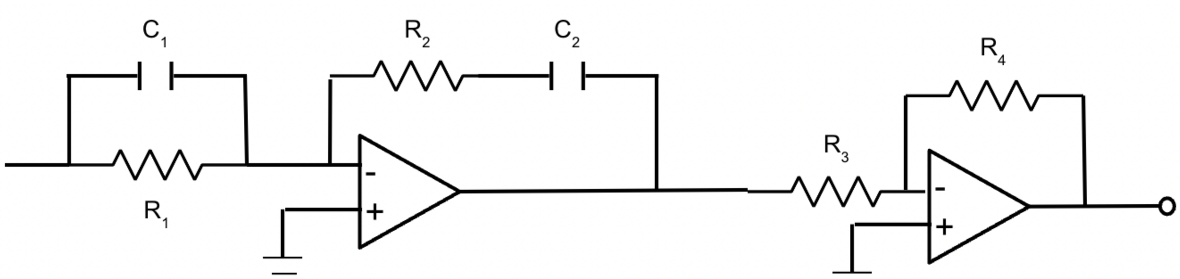


Figure 10: Op-amp representation of chosen PID controller.

We iterated our controller, meaning it differs slightly from the transfer function from our Matlab simulation due to limitations in our materials (fixed values of resistors, capacitors, etc) and iterations based on our assessment of the controller's performance.

The final transfer function for our controller is:

$$H(s) = \frac{R_4}{R_3} \cdot \frac{R_2}{R_1} \cdot \frac{1}{R_2 C_2} \cdot \frac{(C_1 R_1 s + 1)(C_2 R_2 s + 1)}{s}$$

$$H(s) = 1.32 \frac{(1 + 0.19s)^2}{s}$$

R_1=10 kΩ, R_2=10 kΩ, R_3=40 kΩ, R_4=10 kΩ, C_1=19 μF, C_2= 19 μF

Evaluating Our Design

Experimental Performance

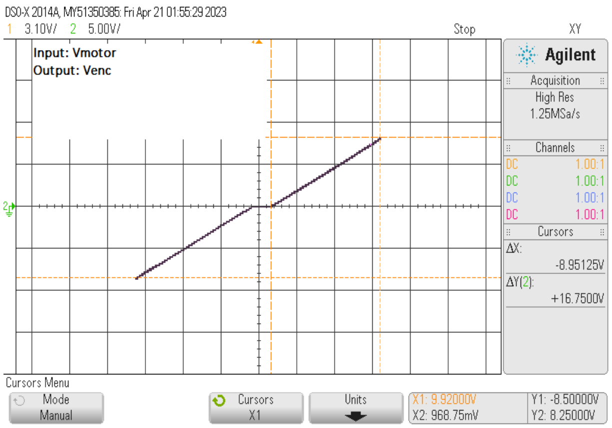
We found that the experimental performance of the car behaved similarly to the theoretical model when V_{out} was disturbed in the negative direction – the car recovered to equilibrium almost immediately. However, the response of the car to a positive V_{out} disturbance did not match the predicted response. The car traveled a significant distance before dealing with the disturbance, indicating that the car had a higher peak overshoot and rise time compared to the theoretical model. We found this issue with positive disturbances was universal for all controller components values tested, indicating that the issue was not with system design but rather with the system itself. We investigated this peculiarity with the following analysis.

Nonlinear System Qualities

The system demonstrates non-linearity due to two key factors: an asymmetrical torque characteristic of the motor and an asymmetrical output behavior of the operational amplifier (op-amp).

Firstly, the motor used in the system may exhibit different torque characteristics depending on the direction of rotation. This means that the torque generated when the motor rotates in one direction may differ from the torque produced when it rotates in the opposite direction. Consequently, the system's response will not be linear, as the relationship between the applied voltage and the resulting torque will not be uniform across all operating conditions.

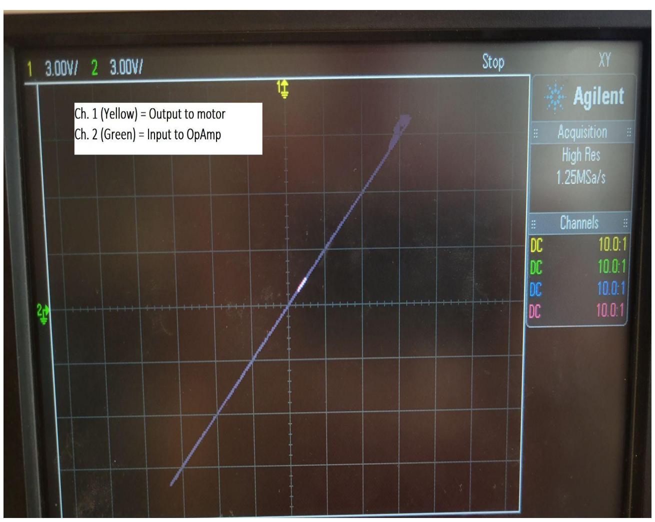
Figure 11: XY Plot of Motor input versus Motor Encoder



From fig. 11 on the left, the input is on the x-axis, and the output is on the y-axis. Knowing this, we can see that the motors used for the final project has a torque bias when the input is above 0V. The response when polarity is negative is great, but has some stiction for a couple volts past 0V. This indicates that our system will have a poor response when moving to the left (direction of positive polarity for input voltage).

Secondly, the op-amp, which plays a crucial role in signal amplification and feedback control, may introduce asymmetry in its output response. Op-amps typically have differential inputs, which means that the output behavior may differ depending on whether the input is positive or negative. This can result in varying amplification or attenuation characteristics, leading to non-linearities in the system's overall response. In this case, the op amps used for the final project (as specified in the datasheet) are less responsive for positive linearity, requiring 8/10 of a volt (for the non-inverting side/positive) and 2/10 of a volt (for the inverting side/negative).

Figure 12: XY Plot of Input to OpAmp to OpAmp Output



Looking at the fig. 12 to the left, beginning with an input of 8.5V on the x-axis from a wave generator, you can see phase shift and other non-linear properties. From an operational standpoint, this shows that the system can handle small deviations from system equilibrium (inverted pendulum) because the input to the operational amplifier will be small. But as we have sudden disturbance, like a push to the left (where positive polarity is defined), then we can see there will be a worse response. One fix would be to add a flyback diode.

The presence of these non-linearities poses challenges in accurately modeling and controlling the inverted pendulum system. It requires careful consideration and compensation techniques to account for the asymmetrical torque characteristics of the motor and the op-amp's non-ideal output behavior.

Conclusion

Our project was acceptable, with our system meeting the specifications we determined for it (stability, less than 50% peak overshoot, and less than one second settling time). We were also able to minimize our steady state error by changing our system's V_{ref} which allowed our car to work properly despite our system having 13% error. If we were to expand on this project in the future, we ultimately would have liked to try more controller designs like the lead-lag which offer more system tuning. We felt more comfortable with the PID and felt it was adequate, but after spending time understanding system dynamics, we realized that the PID may not have been the best controller. This is the result of PID controllers being designed for linear systems, and their performance can degrade when applied to nonlinear systems. Nonlinearities can lead to inaccurate control responses and instability, as PID controllers assume a linear relationship between the control input and the system output. In addition, PID controllers are not inherently robust to disturbances or noise in the system. Therefore, they may struggle to maintain stability and accurate control in the presence of external disturbances or noise. This was showcased in our demonstration when the system was able to stabilize the stick well, but once there was a big disturbance, it took more time for the car to return to it's low effort control. As a group, we learned a lot about applications for control theory and enjoyed getting to practice everything we learned this term and gain a deeper level of understanding and appreciation for the course concepts.