10.007 Systems World
Term 3, 2017

Homework Set 1

Due date: Tuesday, 31 January, 2017

1. (An optimal breakfast) John is taking a diet program and has only three types of food to choose for breakfast: corn, milk and bread. Each food's nutritional characteristic is described in the table below and each serving is 100 grams. John needs to find the optimal composition (i.e., amount of each food) of a breakfast having minimum cost, while the amount of vitamin is between 1200 IU and 2000 IU, sugar level is no larger than 400 grams, number of calories is between 600 kJ and 800 kJ and maximum number of servings for each type of food is 5. (You don't need to solve the problem, only provide the formulation.)

Food (per 100g)	Cost (in $)	Vitamin(in IU)	Sugar (in gr.)	Calories (in kJ)
Corn	0.50	105	45	70
Milk	1.25	250	30	60
Bread	0.25	0	60	165

   (a) Identify the parameters.
   (b) Identify the decision variables.
   (c) Identify the objective function.
   (d) Identify the constraints.
   (e) Write an optimization model that can be used to find the optimal diet for John.
   (f) Replace the parameters of the model with symbolic parameters, so that your model becomes independent of the actual numerical values.

2. (Geometry) Formulate the optimization models corresponding to the following problems. For each problem, identify parameters, decision variables, objective function and constraints. (You don't need to solve the problem, just the formulation is enough.)

   (a) Given a line $ax + by = c$, find the point P on the line that minimizes the distance from a given point $C = (x_C, y_C)$.

   (b) Given a circle with centre (x_c,y_c) and radius r, find the point on the circle that maximizes the value of the linear function $ax + by$.

3. The maximum function $\max(a, b)$ over $\mathbb{R}^2$ returns the largest value of the two arguments, a and b,. More formally, it is defined as

$$\max\{a, b\} = \begin{cases} a, & \text{if} \quad a \geq b, \\ b, & \text{o.w.} \end{cases}$$

Consider the following (trivial) minimization problem:

$$\left.\begin{array}{rr} \min & (\max\{x/2, -2x\}) \\ & x \in \mathbb{R}. \end{array}\right\} \tag{1}$$

(a) Identify the objective function and the feasible region precisely. Plot the objective function as a function of the decision variable x and find its minimum using basic arguments.

(b) Is the objective function linear? (Provide brief and precise arguments.)

(c) Introduce an auxiliary variable y. Consider the problem:

$$\left.\begin{array}{lrcl} \min & y & & \\ \text{s.t.:} & x/2 & \leq & y \\ & -2x & \leq & y \\ & x, y & \in & \mathbb{R}. \end{array}\right\} \tag{2}$$

Draw the feasible region of problem (2).

(d) Are the feasible regions of (1) and (2) the same? Explain why (1) and (2) yield the same optimal value for the x variable. (While we do not require a formal proof of this statement, be as precise as possible.)

(e) Apply the ideas discussed above to write a new problem with linear constraints that yields the same optimum value for the x_1, x_2 variables as the following problem:

$$\left.\begin{array}{lrcl} \min & |x_1| + |x_2| & & \\ \text{s.t.:} & x_1 + 2x_2 & \geq & 4 \\ & x_1 & \leq & 5 \\ & x_2 & \leq & 3 \\ & x_1, x_2 & \in & \mathbb{R}. \end{array}\right\}$$

(Hint: Observe that $|a| = \max\{a, -a\}$ for any $a \in \mathbb{R}$.)

4. (Maxima and Minima). Given the following single variable and multivariable functions identify all of the local/global maxima/minima if they exist. Justify your answers using the optimality tests you have learned in Math 2. (It is helpful to sketch a graph. You may use a graphing calculator, matlab or other software to aid yourself in sketching.)

(a) $f(x) = x^2 - 6x + 8$

(b) $f(x) = \frac{x}{x^2+1}$

(c) $f(x, y) = (x - 1)^2 - y^2$

(d) $f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$