# 10.007 Systems World
# Term 3, 2017

Homework Set 1

Due date: Tuesday, 31 January, 2017

1. (An optimal breakfast) John is taking a diet program and has only three types of food to choose for breakfast: corn, milk and bread. Each food's nutritional characteristic is described in the table below and each serving is 100 grams. John needs to find the optimal composition (i.e., amount of each food) of a breakfast having minimum cost, while the amount of vitamin is between 1200 IU and 2000 IU, sugar level is no larger than 400 grams, number of calories is between 600 kJ and 800 kJ and maximum number of servings for each type of food is 5. (You don't need to solve the problem, only provide the formulation.)

| Food (per 100g) | Cost (in \$) | Vitamin(in IU) | Sugar (in gr.) | Calories (in kJ) |
|---|---|---|---|---|
| Corn | 0.50 | 105 | 45 | 70 |
| Milk | 1.25 | 250 | 30 | 60 |
| Bread | 0.25 | 0 | 60 | 165 |

   (a) Identify the parameters.

   (b) Identify the decision variables.

   (c) Identify the objective function.

   (d) Identify the constraints.

   (e) Write an optimization model that can be used to find the optimal diet for John.

   (f) Replace the parameters of the model with symbolic parameters, so that your model becomes independent of the actual numerical values.

   **Solution:**

   (a) The parameters are: the values given in the table (such as the cost of each unit of corn, milk and bread, and the associated amount of vitamin, sugar, calories contained in each unit of corn, milk and bread), the required nutrition levels (such as the range for the amount of vitamin, the maximum sugar level, and the range for the number of calories), and the maximum number of serving for each type of food.

   (b) The decision variables are the amount of food chosen for the breakfast: the amount of corns, denoted by $x_c$, the amount of milk, denoted by $x_m$, and the amount of bread, denoted by $x_b$.

(c) The objective function is the total cost of the breakfast, which could be expressed as $0.5x_c + 1.25x_m + 0.25x_b$. Our objective is to minimize the cost.

(d) The constraints are related to the required nutrition level for the breakfast: the breakfast should contain vitamin between 1200 and 2000 ($1200 \leq 105x_c + 250x_m \leq 2000$), sugar no larger than 400 ($45x_c + 30x_m + 60x_b \leq 400$, and calories between 600 and 800 ($600 \leq 70x_c + 60x_m + 165x_b \leq 800$). Moreover, the maximum number of servings for each type of food is 5 ($0 \leq x_c, x_m, x_b \leq 5$)

(e) The resulting optimization model is the following:

$$\begin{aligned}
\min \quad & 0.5x_c + 1.25x_m + 0.25x_b \\
s.t. \quad & 105x_c + 250x_m \geq 1200 \\
& 105x_c + 250x_m \leq 2000 \\
& 45x_c + 30x_m + 60x_b \leq 400 \\
& 70x_c + 60x_m + 165x_b \geq 600 \\
& 70x_c + 60x_m + 165x_b \leq 800 \\
& x_c, x_m, x_b \geq 0 \\
& x_c, x_m, x_b \leq 5
\end{aligned}$$

(f) We need to replace the numerical parameters with symbols. The costs and nutrition characteristics are denoted in the following table:

| Food (per 100g) | Cost | Vitamin | Sugar | Calories |
|---|---|---|---|---|
| Corn | $t_c$ | $v_c$ | $s_c$ | $c_c$ |
| Milk | $t_m$ | $v_m$ | $s_m$ | $c_m$ |
| Bread | $t_b$ | $v_b$ | $s_b$ | $c_b$ |

Moreover, the required minimal and maximum amounts for vitamin are denoted respectively as $V_l$ and $V_h$, the maximum sugar level is denoted as $S_h$, the required minimal and maximum amounts for calories are denoted respectively as $C_l$ and $C_h$. And the maximum numbers of servings for corn, milk and bread are denoted respectively as $a_h$.

Hence, we obtain the following model:

$$\begin{aligned}
\min \quad & t_c x_c + t_m x_m + t_b x_b \\
s.t. \quad & v_c x_c + v_m x_m + v_b x_b \geq V_l \\
& v_c x_c + v_m x_m + v_b x_b \leq V_h \\
& s_c x_c + s_m x_m + s_b x_b \leq S_h \\
& c_c x_c + c_m x_m + c_b x_b \geq C_l \\
& c_c x_c + c_m x_m + c_b x_b \leq C_h \\
& x_c, x_m, x_b \geq 0 \\
& x_c, x_m, x_b \leq a_h
\end{aligned}$$

2. (Geometry) Formulate the optimization models corresponding to the following problems. For each problem, identify parameters, decision variables, objective function and constraints. (You don't need to solve the problem, just the formulation is enough.)

**(a)** Given a line $ax + by = c$, find the point $P$ on the line that minimizes the distance from a given point $C = (x_C, y_C)$.

**(b)** Given a circle with centre $(x_c,y_c)$ and radius $r$, find the point on the circle that maximizes the value of the linear function $ax + by$.

**Solution:**

(a) The parameters are: the coordinates of the point $C = (x_C, y_C)$ and the parameters of the line ($a$, $b$ and $c$).

We need to find the coordinates of the point $P = (x_P, y_P)$ with minimum distance from $C$. We thus have two decision variables, $x_P$ and $y_P$.

The distance between $C$ and $P$ is given by the expression $\sqrt{(x_C - x_P)^2 + (y_C - y_P)^2}$, which we want to minimize.

We must account that $P$ belongs to the line $ax + by = c$, namely $ax_P + by_P = c$.

The resulting model is the following:

$$\begin{aligned} \min \quad & \sqrt{(x_C - x_P)^2 + (y_C - y_P)^2} \\ \text{s.t.} \quad & ax_P + by_P = c \end{aligned}$$

(b) The parameters are: the radius and centre of the circumference, $r$ and $(x_c,y_c)$, and the coefficients $a$ and $b$ characterizing the linear function $ax + by$.

The problem requires to find one specific line in the set of parallel lines $ax + by = z$. Since the value of $a$ and $b$ is given, we need to find the value of $z$ characterizing the line we are interested in. Furthermore, the problem requires finding the point $P$ (with coordinates $(x_P,y_P)$) in which the circumference and the line are tangent. We thus have three decision variables: $(x_P,y_P)$ and $z$.

The objective is to maximize the value of $z$.

Finally, we must introduce two constraints to account for the relation between $(x_P,y_P)$ and $z$. The first constraint determines the fact that the point $P$ belongs to the circumference, namely $\sqrt{(x_c - x_P)^2 + (y_c - y_P)^2} = r$, while the other constraints determines the fact that $P$ belongs to the line, namely $ax_P + by_P = z$.

The resulting model is the following:

$$\begin{aligned} \max \quad & z \\ \text{s.t.} \quad & ax_P + by_P = z \\ & \sqrt{(x_c - x_P)^2 + (y_c - y_P)^2} = r \end{aligned}$$

You can also wirte the same model simpy as:

$$\begin{aligned} \max \quad & ax_P + by_P \\ \text{s.t.} \quad & \sqrt{(x_c - x_P)^2 + (y_c - y_P)^2} = r \end{aligned}$$

3. The maximum function $\max(a, b)$ over $\mathbb{R}^2$ returns the largest value of the two arguments, a and b. More formally, it is defined as

$$\max\{a, b\} = \begin{cases} a, & \text{if} \quad a \geq b, \\ b, & \text{o.w.} \end{cases}$$

Consider the following (trivial) minimization problem:

$$\left. \begin{array}{rr} \min & (\max\{x/2, -2x\}) \\ & x \in \mathbb{R}. \end{array} \right\} \tag{1}$$

(a) Identify the objective function and the feasible region precisely. Plot the objective function as a function of the decision variable $x$ and find its minimum using basic arguments.

(b) Is the objective function linear? (Provide brief and precise arguments.)

(c) Introduce an auxiliary variable $y$. Consider the problem:

$$\left. \begin{array}{lrcl} \min & y & & \\ \text{s.t.:} & x/2 & \leq & y \\ & -2x & \leq & y \\ & x, y & \in & \mathbb{R}. \end{array} \right\} \tag{2}$$

Draw the feasible region of problem (2).

(d) Are the feasible regions of (1) and (2) the same? Explain why (1) and (2) yield the same optimal value for the $x$ variable. (While we do not require a formal proof of this statement, be as precise as possible.)

(e) Apply the ideas discussed above to write a new problem with linear constraints that yields the same optimum value for the $x_1, x_2$ variables as the following problem:

$$\left. \begin{array}{lrcl} \min & |x_1| + |x_2| & & \\ \text{s.t.:} & x_1 + 2x_2 & \geq & 4 \\ & x_1 & \leq & 5 \\ & x_2 & \leq & 3 \\ & x_1, x_2 & \in & \mathbb{R}. \end{array} \right\}$$

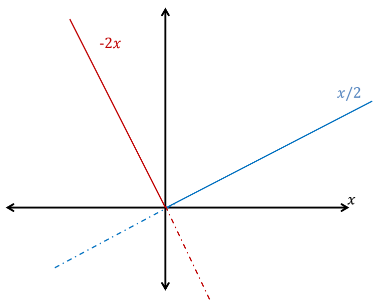
(Hint: Observe that $|a| = \max\{a, -a\}$ for any $a \in \mathbb{R}$.)

**Solution:**

(a) The objective function is

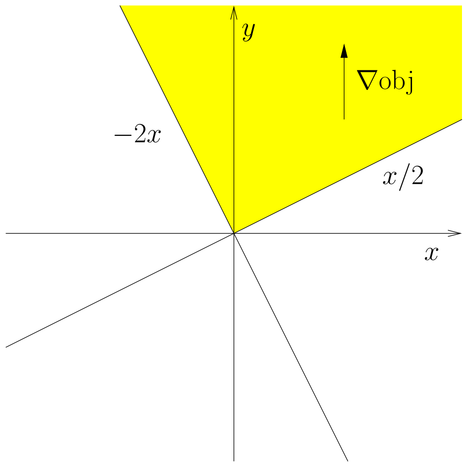
$$\max\{x/2, -2x\} = \begin{cases} x/2, & x \geq 0, \\ -2x, & x < 0, \end{cases}$$

and the feasible set is the whole real line $\mathbb{R}$. We plot the function below, which has "two pieces". This type of functions are referred to as "piecewise linear functions". The minimum objective function value is attained at the intersection of the two lines, i.e., at $x^* = 0$. It is easy to see that this must be the minimum as the function is always nonnegative.




(b) It is not linear as we cannot write it in the form $ax+b$ for some $a$ and $b$ in $\mathbb{R}$.

(c) The feasible region of the new problem is represented below. The gradient of the objective function points upwards (remember we are minimizing so we go in the opposite direction), and the minimum value 0 is attained at $x=0, y=0$.




Note that this formulation has an objective function which is differentiable, therefore we can work with its gradient. This was not true for the original formulation!

(d) The feasible regions are *not* the same: (1) has only one decision variable so its feasible region lives in $\mathbb{R}$, whereas (2) has two decision variables so its feasible region lives in $\mathbb{R}^2$. However the optimal solutions for the $x$ variable coincide for the following reason. The constraints $x/2 \leq y, -2x \leq y$ ensure that $y \geq \max\{x/2, -2x\}$. Because we are minimizing $y$ and $y$ is not involved in other constraints, for any value of $x$ the smallest possible value of $y$ is exactly $y = \max\{x/2, -2x\}$. Finally, notice that we are free to choose $x$, so (2) attains a minimum when $x$ attains the minimum of $\max\{x/2, -2x\}$.

(e) Introducing auxiliary variables $z_1 = |x_1|$ and $z_2 = |x_2|$, we obtain the following formulation:

$$\left.\begin{array}{lrcl} \min & z_1 + z_2 & & \\ \text{s.t.:} & x_1 + 2x_2 & \geq & 4 \\ & x_1 & \leq & 5 \\ & x_2 & \leq & 3 \\ & z_1 & \geq & x_1 \\ & z_1 & \geq & -x_1 \\ & z_2 & \geq & x_2 \\ & z_2 & \geq & -x_2 \\ & x_1, x_2, z_1, z_2 & \in & \mathbb{R}. \end{array}\right\}$$

4. (Maxima and Minima). Given the following single variable and multivariable functions identify all of the local/global maxima/minima if they exist. Justify your answers using the optimality tests you have learned in Math 2. (It is helpful to sketch a graph. You may use a graphing calculator, matlab or other software to aid yourself in sketching.)

   **(a)** $f(x) = x^2 - 6x + 8$

   **(b)** $f(x) = \frac{x}{x^2+1}$

   **(c)** $f(x, y) = (x-1)^2 - y^2$

   **(d)** $f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

   **Solution:** Note that we will answer this question using Hessian for some of the parts. Nevertheless, you were expected to use the second-order derivative tests from Math II in your homework (since you have not learned the Hessian tests until Week 2 formally. Obviously, for one or two variable functions, these methods are equivalent.).

(a) The first-order derivative is $f'(x) = 2x - 6$, which is equal to 0 in $x = 3$. The second-order derivative is $f''(x) = 2 > 0$, so the point (3,-1) must be a local minimum. By looking at the sketch, we can conclude that the point (3,-1) is also a global minimum.

(b) The first-order derivative is

$$f'(x) = \frac{1 \cdot (x^2+1) - x \cdot 2x}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$

The denominator is always positive, so we just have to identify the cases for which the numerator is 0. This happens for $x = \pm 1$, so the critical points are (1,1/2) and (-1,-1/2). To determine whether these points are local/global maxima/minima, we can calculate the second-order derivative or study the behaviour of $f'(x)$ in proximity of the two critical points. Let's try with the second option: the denominator of $f'(x)$ is always positive, so by studying the numerator we can determine when $f'(x)$ is positive or negative. There are three cases:

- For $x < -1$, $f'(x) < 0$;
- For $-1 < x < 1$, $f'(x) > 0$;

- For $x > 1$, $f'(x) < 0$.

This implies that (-1,-1/2) and (1,1/2) are a local minimum and maximum, respectively. This finding is confirmed by the sketch, from which we can also say that the two points are a global minimum and maximum.

(c) The gradient of $f$ is $\nabla f(x, y) = (2(x-1), -2y)$, so the point (1,0) is the only stationary point. The Hessian matrix in (1,0) is

$$H_f(1,0) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \tag{3}$$

The determinant of $H_f(1,0)$ is $-4 < 0$, so the point (1,0) cannot be a local minimum. It is a saddle, as shown in the sketch.

Since the only stationary point is a sandle point, this function has neither a maximum nor a minimum. In fact, we can show that this function is unbounded below and above, but considering the following two rays as $\alpha$ goes to infinity: $(\alpha, 0)$ and $(0, -\alpha)$.

(d) The gradient of $f$ is $\nabla f(x, y) = \big(4x^3 - 4x + 4y, 4y^3 + 4x - 4y\big)$, so the stationary points are (0,0), $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$. The Hessian matrix is

$$H_f(x, y) = \begin{pmatrix} 12x^2 - 4 & 4 \\ 4 & 12y^2 - 4 \end{pmatrix} \tag{4}$$

In the points $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$, the Hessian is

$$H_f(\sqrt{2}, -\sqrt{2}) = H_f(-\sqrt{2}, \sqrt{2}) = \begin{pmatrix} 20 & 4 \\ 4 & 20 \end{pmatrix} \tag{5}$$

which is positive definite, so both points are local minima.

In the point (0,0) the Hessian is

$$H_f(0,0) = \begin{pmatrix} -4 & 4 \\ 4 & -4 \end{pmatrix} \tag{6}$$

The matrix is negative semidefinite but not negative definite, so we need to check further. Let's analyse the behaviour of $f(x, y)$ in the proximity of (0,0). If we consider the points belonging to the $x$-axis, namely all points ($x$,0), we see that the difference $f(x, y) - f(0, 0) = x^4 - 2x^2$ takes negative values in a neighbourhood of $x = 0$. If we consider the points $(x, x)$, we notice that the difference $f(x, y) - f(0, 0) = 4x^4$ takes positive values in a neighbourhood of $x = 0$. We can conclude that (0,0) is a saddle, since there are points in $(x, y)$ such that $f(x, y) > f(0, 0)$ and $f(x, y) < f(0, 0)$. This is confirmed by the sketch, from which we also deduce that the points $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$ are global minima.