10.007 Systems World
Term 3, 2017

Homework Set 2

Due date: Monday, 6 February, 2017

1. Given $f(x) = e^{-x^2}$

   a) Compute $\hat{\hat{f}}_1(x)$, the second order Taylor approximation of $f(x)$ at $\bar{x} = 1$.

   b) Verify that this is a good approximation, i.e.

   $$\hat{\hat{f}}_1(1) = f(1), \quad \hat{\hat{f}}_1'(1) = f'(x), \quad \text{and} \quad \hat{\hat{f}}_1''(1) = f''(1).$$

   Solution:
   a) We have $f'(x) = -2xe^{-x^2}$ and $f''(x) = -2e^{-x^2} + 4x^2e^{-x^2}$.

   So

   $$\begin{aligned} \hat{\hat{f}}_1(x) &= f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2 \\ &= e^{-1} - 2e^{-1}(x-1) + \frac{1}{2}(-2e^{-1} + 4e^{-1})(x-1)^2 \\ &= e^{-1} - 2e^{-1}x + 2e^{-1} + e^{-1}(x^2 - 2x + 1) \\ &= 4e^{-1} - 4xe^{-1} + e^{-1}x^2 \\ &= e^{-1}x^2 - 4e^{-1}x + 4e^{-1} \end{aligned}$$

   b)

   - $\hat{\hat{f}}_1(1) = e^{-1}(1)^2 - 4e^{-1}(1) + 4e^{-1} = e^{-1} = e^{-1^2} = f(1)$.
   - $\hat{\hat{f}}_1'(1) = 2e^{-1} - 4e^{-1} = -2e^{-1} = -2e^{-1^2} = f'(1)$.
   - $\hat{\hat{f}}_1''(1) = 2e^{-1} = -2e^{-1^2} + 4e^{-1^2} = f''(1)$.

2. Compute the second order Taylor approximation of $f(x, y) = \sqrt{1 + x^2 + y^2}$ at $(\bar{x}, \bar{y}) = (0, 0)$.

Solution: We have

$$f_x(x,y) = \frac{x}{\sqrt{1+x^2+y^2}},$$

$$f_y(x,y) = \frac{y}{\sqrt{1+x^2+y^2}},$$

$$f_{xx}(x,y) = \frac{\sqrt{1+x^2+y^2} - x^2(1+x^2+y^2)^{-1/2}}{1+x^2+y^2},$$

$$f_{yy}(x,y) = \frac{\sqrt{1+x^2+y^2} - y^2(1+x^2+y^2)^{-1/2}}{1+x^2+y^2},$$

$$f_{xy}(x,y) = -xy(1+x^2+y^2)^{-3/2},$$

$$f_{yx}(x,y) = -xy(1+x^2+y^2)^{-3/2}.$$

So

$$f(0,0) = 1, \quad f_x(0,0) = 0, \quad f_y(0,0) = 0,$$

$$f_{xx}(0,0) = 1, \quad f_{xy}(0,0) = 0, \quad f_{yx}(0,0) = 0, \quad f_{yy}(0,0) = 1.$$

Then

$$\begin{aligned}
\hat{\hat{f}}_{(0,0)}(x,y) &= f(0,0) + \nabla f(0,0)^T(x-0,y-0) + \frac{1}{2}(x-0,y-0)^T H_f(0,0)(x-0,y-0) \\
&= 1 + \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \frac{1}{2} \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
&= 1 + \frac{1}{2}(x^2+y^2).
\end{aligned}$$

3. Is the function $f(x,y) = x^3 - 3xy^2$ convex, concave or neither in $\mathbb{R}^2$? Justify your answer.

Solution: We compute the Hessian $H_f(x,y)$:

$$f_x(x,y) = 3x^2 - 3y^2, \quad f_y(x,y) = -6xy,$$

$$f_{xx}(x,y) = 6x, \quad f_{xy}(x,y) = -6y, \quad f_{yx}(x,y) = -6y, \quad f_{yy}(x,y) = -6x.$$

So

$$H_f(x,y) = \begin{bmatrix} 6x & -6y \\ -6y & -6x \end{bmatrix}.$$

Taking, for instance, $(x,y) = (1,0)$ we can see that $H_f(1,0) = \begin{bmatrix} 6 & 0 \\ 0 & -6 \end{bmatrix}$ has positive and negative eigenvalues and therefore is indefinite. Hence f is neither convex or concave in $\mathbb{R}^2$.

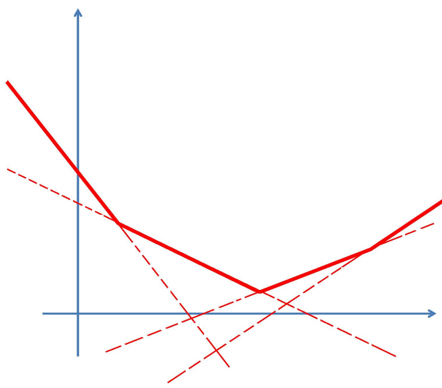

Figure 1: Maximum of 4 linear functions

4. Given some positive integer n and some real numbers a_i, b_i for $i = 1, \ldots, n$, prove that $f(x) = \max_{i=1,\ldots,n}\{a_i x + b_i\}$ is a convex function using the definition of a convex function. Illustrate with a small sketch what this function looks like.

   Solution: Let $\lambda \in [0, 1]$. For any x and y, we have

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &= \max_i\{a_i(\lambda x + (1-\lambda)y) + b_i\} \\ &= \max_i\{\lambda(a_i x + b_i) + (1-\lambda)(a_i y + b_i)\} \\ &\leq \max_i\{\lambda(a_i x + b_i)\} + \max_i\{(1-\lambda)(a_i y + b_i)\} \\ &= \lambda \max_i\{a_i x + b_i\} + (1-\lambda)\max_i\{a_i y + b_i\} \\ &= \lambda f(x) + (1-\lambda) f(y). \end{aligned}$$

   Therefore $f(x)$ is a convex function. For the inequality, we used the fact that the maximum of the sum of two functions is always smaller than or equal to the sum of the maximum of the two functions (that is, each one maximized individually). See Figure 1 for a simple illustration with $n = 4$.

   You can show that the function is convex easily by applying the rules of "Operations that preserve convexity" from Cohort 2.2. It is stated that $\max\{f(x), g(x)\}$ is convex if f, g are convex. Therefore, if we take the first two linear functions $a_1 x + b_1, a_2 x + b_2$, we have that $\max\{a_1 x + b_1, a_2 x + b_2\}$ is convex. Then we can apply the same argument and say that $\max\{\max\{a_1 x + b_1, a_2 x + b_2\}, a_3 x + b_3\}$ is convex. Iterating this argument up to n proves the result.

5. For $f(x, y) = 3(x+1)^2 + (y-2)^2 + 1$

   a) Find all maxima/minima.

b) Justify whether the maxima/minima found in part (a) are local or global. (Hint: You can find a very simple argument to justify your answer.)

Solution:
a) We find the points such that $\nabla f(x, y) = (0, 0)$:

$$\nabla f(x, y) = \begin{bmatrix} 6(x+1) \\ 2(y-2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

has unique solution $x = -1$ and $y = 2$. Then $(x, y) = (-1, 2)$ is the only critical point.

Since $H_f(x, y) = \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix}$ is positive definite (Its eigenvalues are positive.), $(-1, 2)$ is a local minimum.

There are no other maxima or minima.

b) It is easy to see that $f(x) \geq 1$ for any x, y. (Square terms must be at least 0!) Since $f(-1, 2) = 1$, (-1,2) must be a global minimum.

We can also use the following argument (which we have learned in Week 3!): $f(x, y)$ is convex since $H_f(x, y)$ is positive definite (its eigenvalues are both positive) <u>everywhere</u>. Therefore $(-1, 2)$ is a global minimum.