10.007: Modeling the Systems World

Lecture 1

Introduction to Mathematical Modelling and Optimization

Term 3, 2017

Singapore University of
Technology & Design

Topics and outcomes

Weeks 1–6: Dr. Selin Ahipasaoglu

- Optimization Paradigm
- Unconstrained Optimization
- Equality Constrained Optimization
- Inequality Constrained Optimization
  (Only Linear Optimization)
- Numerical solutions of (some) optimization problems.

Weeks 8–13: Dr. Sergey Kushnarev

- First- and higher-order ordinary differential equations.
- Laplace Transform.
- Systems of ODEs.

Schedule & exams & Grading

There are ten homework sets in total. (%15 of the final grade)

Quiz: February 9 (week 3), March 30 (week 10). (%10 each)
Exam: March 1 (week 6), April 27 (week 14). (%25 each)

There will be a make-up class for the CNY holiday on January 27th. Check your calendars.

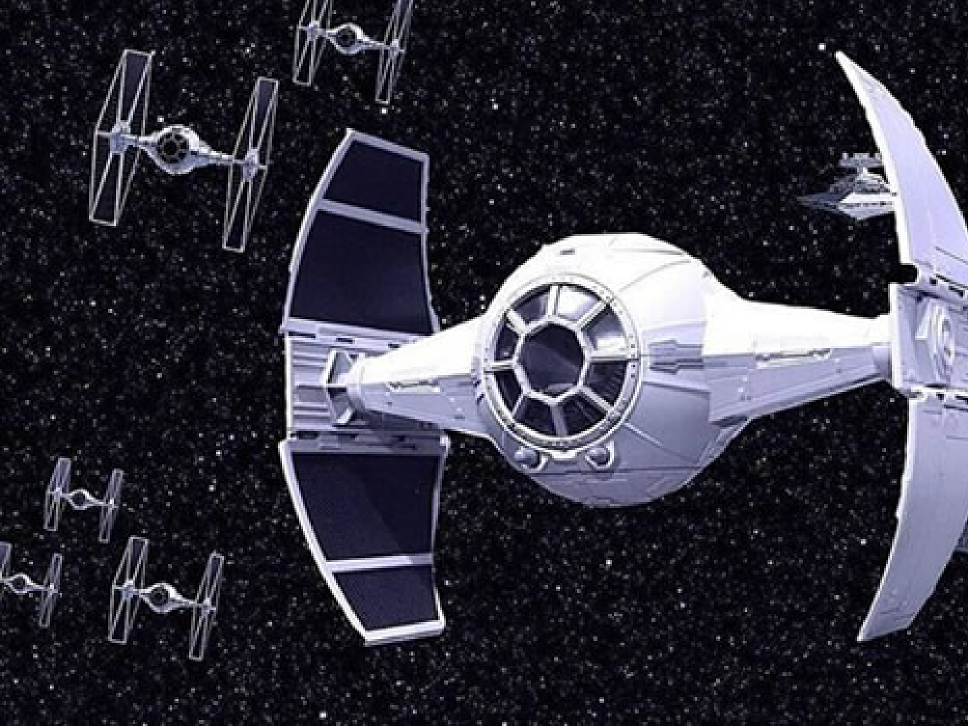
The 2D Design Project is 10% of the final grade.

Textbook: until recess week, mostly your Calculus book and the material given to you; some parts will be taken from *Applied Mathematical Programming* (available online).

Agenda & Reading

1. Motivation
2. Intro to Mathematical Optimization Paradigm
3. Modelling
4. Algebraic (abstract) formulations

TO READ: Please REVISE your Math 2 knowledge by reading Simmons 9.1-9.6, your MATH II notes, and AMP A.1-A.5.



Our very first optimization problem

Too much work, too little time!

Rebels are in the middle of building the new Rebel Fleet ... but they are running out of time: there are only 600 robot-hours left before the Starkiller Base is ready to fire on D'Qar...

- 4 MC80 star cruisers and 12 X-wing fighters to construct.
- Each MC80 star cruiser requires 100 robot-hours, each X-wing fighter requires 50 robot-hours.
- They estimate to lose 500 rebels for each unfinished MC80 or 300 rebels for each unfinished X-wing.
- At least two MC80's must be built to be able to command the counter-attacks.

How should the rebels allocate the remaining time to minimize the number of casualities?

Our very first optimization problem

We must decide how many ships of each type should be built.

Introduce two decision variables: number of MC80s (x_C), number of X-wings (x_W).

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$$
\left.
\begin{array}{rrcl}
\text{min} & 500(4 - x_C) + 300(12 - x_W) & & \\
\text{subject to:} & & & \\
\text{(Robot-hours)} & 100x_C + 50x_W & \leqslant & 600 \\
\text{(\# MC80s)} & x_C & \leqslant & 4 \\
\text{(\# X-wings)} & x_W & \leqslant & 12 \\
\text{(min. \# MC80s)} & x_C & \geqslant & 2 \\
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$$

(Optimal) Solution: $x_C = 2, x_W = 8$.

(Optimal) Objective function value:
$500(4 - 2) + 300(12 - 8) = 2200$.

Our very first optimization problem

Note that we have worked with a different objective function in class. Instead of minimizing the lives that are lost, our goal was maximizing the number of lives that are saved.

$$
\left.
\begin{array}{rrcl}
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\end{array}
\right\}
$$

The two formulations are equal. (Can you see why?)

The optimization paradigm

From Wikipedia: mathematical optimization is the selection of a best element (with regard to some criteria) from a given set of available alternatives.

An optimization problem is defined by the following elements:

1. Decision variables
2. A single objective function
3. Constraints
4. Parameters

Decision variables

$$
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\begin{array}{rrcl}
\text{min} & 500(4 - x_C) + 300(12 - x_W) & & \\
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The decision variables are the elements that are under the control of the decision maker.

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\end{array}\right\}$$

The decision variables are the elements that are under the control of the decision maker.

- The number of ships of each type that we build.
- The level of investments in a portfolio.
- The number of produced items of a certain type.

Objective function

$$
\left.
\begin{array}{rrcl}
\text{min} & 500(4 - x_C) + 300(12 - x_W) & & \\
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The objective function is a function of the decision variables. It corresponds to a measure of performance that we want to maximize or minimize.

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The objective function is a function of the decision variables. It corresponds to a measure of performance that we want to maximize or minimize.

- Minimize the casualities in the upcoming fight.
- Maximize the expected return of the portfolio.
- Minimize the total production cost.

Constraints

$$
\left.\begin{array}{rrcl}
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\text{subject to:} & & & \\
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The constraints are restrictions that define which values of the decision variables are allowed. Typically they are expressed as equalities or inequalities.

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The constraints are restrictions that define which values of the decision variables are allowed. Typically they are expressed as equalities or inequalities.

- We cannot exceed the number of robot-hours available.
- We cannot exceed the total investment capital.
- We must meet the minimum production level.

Parameters

$$\left.\begin{array}{rrcl}
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\text{subject to:} & & & \\
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The parameters are the data of the problem. If the parameters change, we obtain a different problem of the same class – that is, with a similar structure.

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The parameters are the data of the problem. If the parameters change, we obtain a different problem of the same class – that is, with a similar structure.

- The number of robot-hours required by each type of ship.
- The expected return rate of each asset in the portfolio.
- The quantity of resources necessary to manufacture each item.

Your turn now!

Optimization is everywhere. Can you name a few examples?

Your turn now!

Your turn now!

Your turn now!

Your turn now!

Your turn now!

Your turn now!

A production planning problem

We want to plan production of Chairs and Tables.

Each Chair takes 2 hours of Cutting and 1 hour of Assembly.
Each Table takes 1 hour of Cutting and 3 hours of Assembly.

The daily production capacity is 20 hours in the Cutting department and 30 hours in the Assembly department. Chairs are sold for $15 each and Tables are sold for $20 each.

Write a model to determine the production plan that maximizes profit.

Production planning problem: solution

Decision variables: number of chairs (x_C), number of tables (x_T).

$$\left.\begin{array}{rrcl}
\max & 15x_C + 20x_T & & \\
\text{subject to:} & & & \\
\text{(Cutting dep.:)} & 2x_C + 1x_T & \leqslant & 20 \\
\text{(Assembly dep.:)} & 1x_C + 3x_T & \leqslant & 30 \\
 & x_C & \geqslant & 0 \\
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Production planning problem: solution

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& x_C & \geqslant & 0 \\
& x_T & \geqslant & 0.
\end{array}\right\}$$

(Optimal) Solution: $x_C = 6, x_T = 8$.

(Optimal) Objective function value: $15 \cdot 6 + 20 \cdot 8 = 250$.

Abstracting: from numbers to parameters

Call Chairs and Tables items 1 and 2.

Call $p_1 = 15$ and $p_2 = 20$ the selling price of items 1 and 2.

Call Cutting and Assembly departments *a* and *b*.

Call h_{a1} the number of hours in department *a* necessary for manufacturing item 1, h_{a2} for item 2. H_a is the total number of hours available in dept. *a*. Similarly for dept. *b*.

Rewrite the production planning model using these symbols.

First level of abstraction

Initial model:

$$\left.\begin{array}{rrcl} \max & 15x_C + 20x_T & & \\ \text{s.t.:} & & & \\ & 2x_C + 1x_T & \leqslant & 20 \\ & 1x_C + 3x_T & \leqslant & 30 \\ & x_C & \geqslant & 0 \\ & x_T & \geqslant & 0. \end{array}\right\}$$

New (abstract) model:

$$\left.\begin{array}{rrcl} \max & p_1x_1 + p_2x_2 & & \\ \text{s.t.:} & & & \\ & h_{a1}x_1 + h_{a2}x_2 & \leqslant & H_a \\ & h_{b1}x_1 + h_{b2}x_2 & \leqslant & H_b \\ & x_1 & \geqslant & 0 \\ & x_2 & \geqslant & 0. \end{array}\right\}$$

One step further

Now there are n items $\{1, \ldots, n\}$ and m departments $\{1, \ldots, m\}$.

Data:

- p_j = selling price item $j = 1, \ldots, n$.
- H_i = number of hours available dept. $i = 1, \ldots, m$.
- h_{ij} = number of hours required by item j in dept. i.

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Can you write down a model for this problem? The model will contain symbols only.

The algebraic formulation

Decision variables: x_j = # items of type $j = 1, \ldots, n$.

$$
\left.
\begin{array}{rrrcl}
 & \max & \sum_{j=1}^{n} p_j x_j & & \\
 & \text{s.t.:} & & & \\
\text{(Availability dept.:)} & \forall i = 1, \ldots, m & \sum_{j=1}^{n} h_{ij} x_j & \leqslant & H_i \\
 & \forall j = 1, \ldots, n & x_j & \geqslant & 0.
\end{array}
\right\}
$$

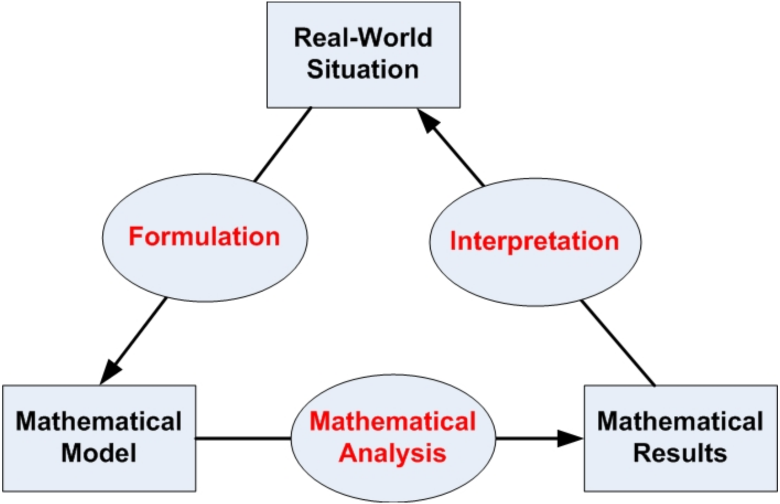
The algebraic formulation

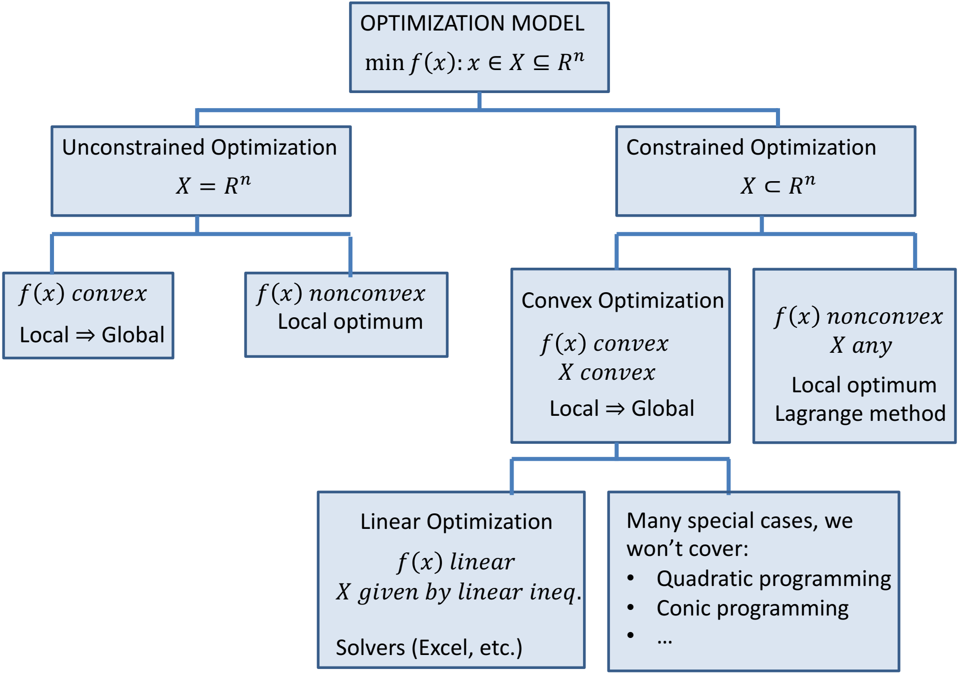
Decision variables: x_j = # items of type $j = 1, \dots, n$.

$$\left.\begin{array}{rrrcl} & \max & \sum_{j=1}^{n} p_j x_j & & \\ & \text{s.t.:} & & & \\ \text{(Availability dept.:)} & \forall i = 1, \dots, m & \sum_{j=1}^{n} h_{ij} x_j & \leqslant & H_i \\ & \forall j = 1, \dots, n & x_j & \geqslant & 0. \end{array}\right\}$$

- The model is independent of the data, and it depends on the problem only: reusability.
- It can be coded directly on a computer.
- This often leads to better understanding!

Process of mathematical modeling




OPTIMIZATION MODEL
$\min f(x): x \in X \subseteq R^n$
Unconstrained Optimization
$X = R^n$
Constrained Optimization
$X \subset R^n$
$f(x)$ convex
Local ⇒ Global
$f(x)$ nonconvex
Local optimum
Convex Optimization
$f(x)$ convex
X convex
Local ⇒ Global
$f(x)$ nonconvex
X any
Local optimum
Lagrange method
Linear Optimization
$f(x)$ linear
X given by linear ineq.
Solvers (Excel, etc.)
Many special cases, we won't cover:
• Quadratic programming
• Conic programming
• …

Summary

- Optimization is everywhere!
- Structure of an optimization problem: optimization paradigm.
- Abstraction and algebraic formulations.
- Modeling as a design process.

A note on terminology

Students tend to get confused with the optimization jargon. Here are some useful terms:

- Feasible solution: A solution which satisfies all the constraints of the problem.
- Infeasible solution: A solution which does NOT satisfiy all the constraints of the problem, i.e., it violates at least one of them.
- Feasible region: The set of all feasible solutions.

A note on terminology

More useful terms for a minimization problem:

- Optimal solution: A feasible solution whose objective function value is less than or equal to that of any other feasible solution. Sometimes, referred to as the optimizer or minimizer.
- Optimal Set: The set of all optimal solutions.
- Optimal (objective function) value: The objective function value of an optimal solution (if it exists).

A little friendly warning!

Failures with a D grade will have to attend the bootcamp, which will be held in late August and early September. The attendance of the bootcamp is compulsory, attending less than 80% of the bootcamp will result in an immediate failure of the bootcamp. Do not plan for any overseas trip during the bootcamp period if you are in the danger zone of failing the course based on your performance during the term (consult your instructors if you are not sure).

Failures with an F grade will have to retake the course in the next academic year.