10.007: Modeling the Systems World

Lecture 3

Unconstrained Optimization (Convex Objective Function)

Term 3, 2017

Singapore University of
Technology & Design

Overview

- Convexity and global optimality.
- “Easy” problems.

An example

Find the global minimum of the following problem:

$$\left.\begin{array}{ll} \min & x^2 + 3x + y^2 - 2y \\ \text{s.t.:} & \\ & \begin{array}{rcl} x^2 + y^2 & \leqslant & 4 \\ x - y & \leqslant & 0 \end{array} \end{array}\right\}$$

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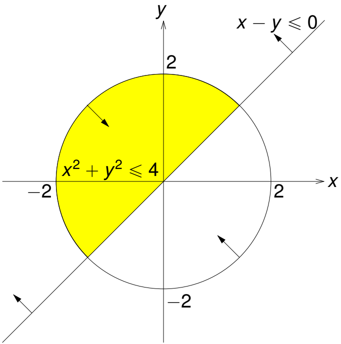

Relaxation of the problem

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The relaxed (unconstrained) problem must have an objective function value at least as good as the constrained problem! (Can you see why?)

Solving the relaxation

Let $f(x, y)$ be the objective function, and set $\nabla f = 0$:

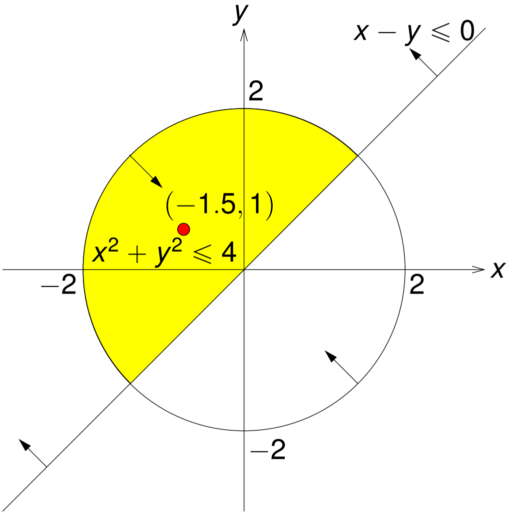
$$\begin{aligned} \frac{\partial f}{\partial x} &= 2x + 3 = 0 \\ \frac{\partial f}{\partial y} &= 2y - 2 = 0. \end{aligned}$$

Solution: $x = -3/2, y = 1$. Compute the Hessian at $(-3/2, 1)$:

$$\det \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 4 > 0.$$

Local minimum at $(-3/2, 1)$, it is inside the feasible region.
Therefore, it is a local minimum to the constrained problem too!
Is this a global minimum?

An example



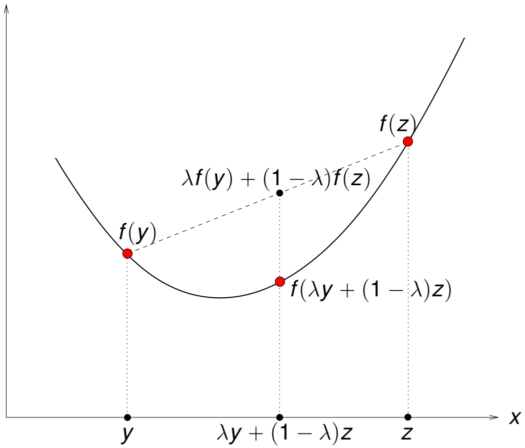
Go global!

Solving an optimization problem typically requires finding a global optimum.

However, it is very difficult to write algorithms that find global optima. In most cases, we can only guarantee local optimality.

There are problems for which global optimality follows from local optimality: these problems are considered “easy”. This is the “ideal” situation.

Convexity: intuition



A function is convex if the line between $(y, f(y)), (z, f(z))$ lies above the graph of the function over the segment $\overline{yz}$.

Convexity

Definition

A function $f(x) : \mathbb{R}^n \to \mathbb{R}$ is called convex if for every $y, z \in \mathbb{R}^n$ and every scalar $0 \leqslant \lambda \leqslant 1$ we have:

$$f(\lambda y + (1-\lambda)z) \leqslant \lambda f(y) + (1-\lambda)f(z).$$

It is called strictly convex if for every $y \neq z \in \mathbb{R}^n$ and every scalar $0 < \lambda < 1$ we have:

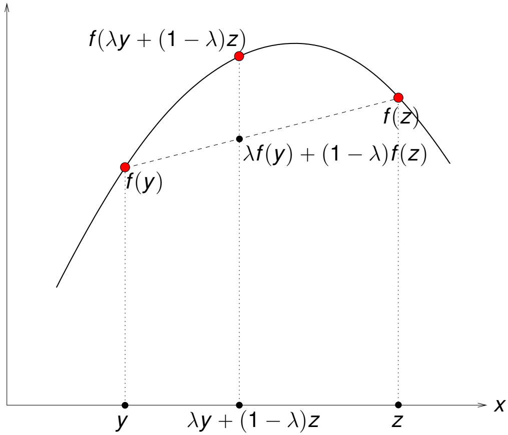
$$f(\lambda y + (1-\lambda)z) < \lambda f(y) + (1-\lambda)f(z).$$

The expression $\lambda y + (1-\lambda)z$ is simply the line segment between y and z when λ ranges from 0 to 1.

Concavity

Definition

A function f is concave if $-f$ is convex.



Necessary and Sufficient Conditions for Minimizing Convex Functions

Theorem

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function, and assume that f is continuously differentiable at point x^*. Then x^* is a global minimum of f if and only if $\nabla f(x^*) = \vec{0}$.

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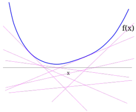
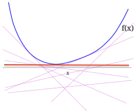
Sketch of Proof: Think about how a single variable convex function looks, the tangent line is always 'below' the function. When the tangent line is horizontal at a point $(x^*, f(x^*))$, all the function values are greater than or equal to $f(x^*)$.





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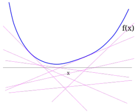

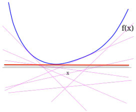

This means that we have a very simple procedure to find the global minimum of a convex and differentiable function.

Optimality conditions for convex functions

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a twice continuously differentiable convex function. We want to solve: $\min_{x \in \mathbb{R}^n} f(x)$.

Because the function is convex, the approach becomes simpler:

1. Find all x^* such that $\nabla f(x^*) = \vec{0}$. These are global minima.
2. (Optional unless requested) Check if $H_f(x^*)$ is pd. If yes, x^* is a strict global minimum.

If we are not interested in checking if global minima are strict, solving Step 1 is sufficient. As before, Step 1 is often very hard even for the convex case.

Back to the example

Recall our initial problem:

$$\left.\begin{array}{lrcl} \min & x^2+3x+y^2-2y & & \\ \text{s.t.:} & & & \\ & x^2+y^2 & \leqslant & 4 \\ & x-y & \leqslant & 0 \end{array}\right\}$$

The objective function is convex (sum of convex functions – see previous cohort). By the Theorem, a local minimum is a global minimum.

The point $(-3/2, 1)$ is the global optimum of the problem unconstrained problem. It is also feasible for the constrained problem, therefore, it is a global minimum of that as well. We are done!

Summary

- Local/global optima.
- Convex functions .
- Implications of convexity on global optimality.