10.007: Modeling the Systems World

Week 01 - Cohort 1

Fundamentals of Mathematical Modelling

Term 3, 2017

Singapore University of Technology & Design

Learning objectives

We will recall and put to use the fundamentals of the optimization paradigm.

- Modelling many real-world problems
- Identifying decision variables, objective function, constraints, and parameters.
- Understanding definitions: feasible solution, feasible region, and optimal solution.

Activity 1 (15 minutes): formulation

You have a lot of homework and you have only 5 hours left!

- 6 Systems World and 10 Physical World problems are left.
- In 1 hour you can solve 2 SW problems or 3 PW problems.
- Each problem is worth 10 marks (SW) or 6 marks (PW).

Your goal is to obtain the highest sum of marks in total.

(a) Write down a mathematical model by identifying all the elements of your model.

(b) Draw the feasible region, identify three feasible solutions.

(c) What can you say about the objective function and the feasible region?

Activity 1a (solution)

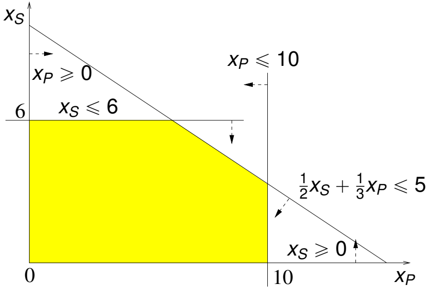
We must decide how many problems of each type should be solved. Introduce two decision variables: number of SysW problems (x_S), number of PhyW problems (x_P).

$$\left.\begin{array}{rrcl}
\text{max} & 10x_S + 6x_P & & \\
\text{subject to:} & & & \\
\text{(Time limit)} & \frac{1}{2}x_S + \frac{1}{3}x_P & \leqslant & 5 \\
\text{(SysW problems)} & x_S & \leqslant & 6 \\
\text{(PhyW problems)} & x_P & \leqslant & 10 \\
\text{(Nonnegativity)} & x_S & \geqslant & 0 \\
\text{(Nonnegativity)} & x_P & \geqslant & 0.
\end{array}\right\}$$

Activity 1b (solution)

Any point that satisfies the constraints is a feasible solution.

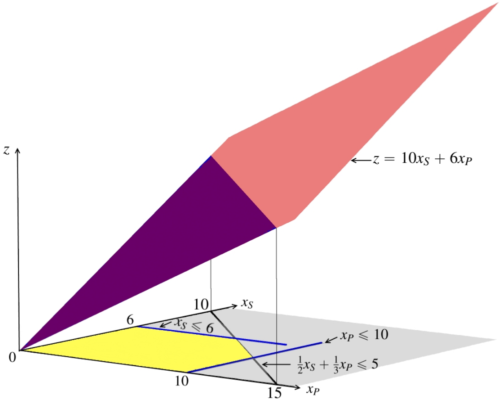
Some feasible points: $(0, 0)$, $(10, 0)$, $(2, 3)$.



The set of all points in the yellow area of the figure is the feasible region: these points satisfy all the constraints.

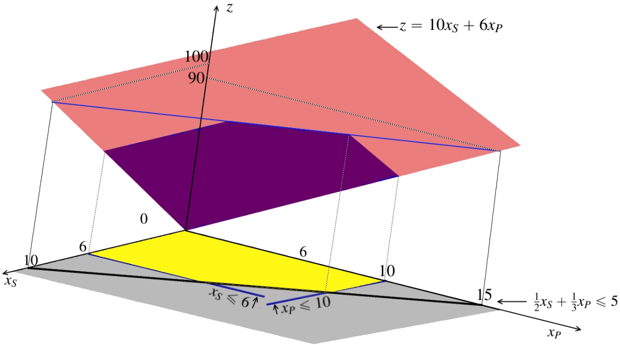
Activity 1b (solution cont'd)

The picture in 3D: on the horizontal plane we have the feasible region, on the z axis we have the value of the objective function $z = 10x_S + 6x_P$.



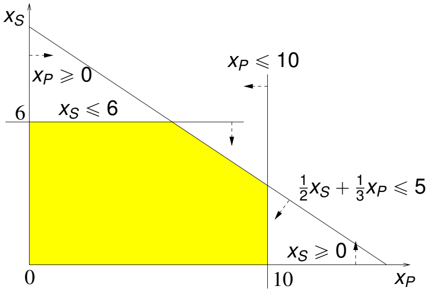
Activity 1b (solution cont'd)

Another point of view on the same picture: now it is easier to see how the feasible region "projects" onto the plane representing the objective function.



Activity 1c (solution)

The objective function is a linear function of the decision variables.



Each constraint is a halfspace, therefore, the feasible region is an intersection of halfspaces.

Activity 2 (20 minutes): supply/demand

We buy fresh oranges (O) and sell two products: fresh oranges (F) and orange juice (J). Denote unit prices and quantities by:

P_o = buying price O
Q_o = quantity O

P_f = selling price F
Q_f = quantity F

P_j = selling price J
Q_j = quantity J.

$$\begin{aligned} P_o &= 3 + 0.0005Q_o \\ P_f &= 4 - 0.001Q_f - 0.0002Q_j \\ P_j &= 7.5 - 0.0002Q_f - 0.005Q_j. \end{aligned}$$

Assume that we can make 1 unit of juice from 2 fresh oranges. Our goal is to maximize profit.

(a) Write down a mathematical model by identifying all the elements of your model.

(b) What can you say about the objective function and the feasible region? (See next cohort!)

Activity 2a (solution)

Decision variables: quantities Q_o, Q_f, Q_j respectively of oranges bought, oranges sold unprocessed, units of juice.

Profit = revenue − cost, so:

$$\text{Profit} = P_f Q_f + P_j Q_j - P_o Q_o.$$

We cannot sell more oranges than we buy:

$$Q_f + 2Q_j \leqslant Q_o.$$

$$\left.\begin{array}{rrcl} \max & P_f Q_f + P_j Q_j - P_o Q_o & & \\ \text{s.t.:} & & & \\ \text{(Availability)} & Q_f + 2Q_j - Q_o & \leqslant & 0 \\ & Q_f, Q_j, Q_o & \geqslant & 0. \end{array}\right\}$$

Activity 2a (solution)

Rework the objective function:

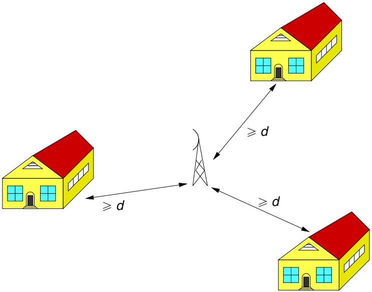
$$
\begin{aligned}
P_f Q_f + P_j Q_j - P_o Q_o &= (4 - 0.001 Q_f - 0.0002 Q_j) Q_f \\
&\quad + (7.5 - 0.0002 Q_f - 0.005 Q_j) Q_j \\
&\quad - (3 + 0.0005 Q_o) Q_o \\
&= 4Q_f + 7.5Q_j - 3Q_o - 0.0004 Q_j Q_f \\
&\quad - 0.001 Q_f^2 - 0.005 Q_j^2 - 0.0005 Q_o^2.
\end{aligned}
$$

$$
\left.
\begin{array}{rrcl}
\max & \multicolumn{3}{l}{4Q_f + 7.5Q_j - 3Q_o - 0.0004 Q_j Q_f} \\
 & \multicolumn{3}{l}{-0.001 Q_f^2 - 0.005 Q_j^2 - 0.0005 Q_o^2} \\
\text{s.t.:} & & & \\
\text{(Availability)} & Q_f + 2Q_J - Q_o & \leqslant & 0 \\
 & Q_f, Q_j, Q_o & \geqslant & 0.
\end{array}
\right\}
$$

Activity 3 (15 minutes): antenna placement

We want to cover m customers with a single antenna. The location of each customer is denoted by $(x_i, y_i), i = 1, \ldots, m$.

It must be at least d meters away from each customer. Transmission power is affected only by distance, and is inversely proportional with constant κ to the square of the distance.



Your goal is to maximize the sum of the transmission power to all customers, while satisfying regulations.

(a) Write down a model by identifying all the elements of your model.

(b) What can you say about the objective function and the feasible region?

Activity 3 (solution)

Decision variables: x, y coordinates of the antenna.

$$\left.\begin{array}{rrcl} \max & \sum_{i=1}^{m} \left(\frac{\kappa}{(x-x_i)^2+(y-y_i)^2}\right) & & \\ \text{s.t.:} & & & \\ \forall i = 1, \ldots, m, & (x-x_i)^2 + (y-y_i)^2 & \geqslant & d^2. \end{array}\right\}$$

Do you think that this problem is easier, more difficult or similar in difficulty to the “homework optimization” problem? What about the “orange juice” problem?

We will see that this problem is probably harder than both, because it does not belong to the class of convex problems: more on this later in the course.

Summary

- Modeling, modeling and more modeling!
- Definitions of feasible solution, feasible region, and infeasible problem.
- Algebraic formulations to get used to “abstract” modeling.

A note on Feasibility and optimality

Among all the feasible solutions, we are interested in those solutions that maximize or minimize the objective function.

A feasible solution that maximizes (minimizes) the objective function is called an optimal solution, or sometimes simply an optimum. There can be multiple optima, all of which have the same objective function value.

A problem with no feasible solution is called infeasible. Such a problem does not have an optimal solution.