MATHEMATICAL BACKGROUND 4

Differential equations

A **differential equation** is a relation between a function and its derivatives, as in

$$a\frac{d^2f}{dx^2}+b\frac{df}{dx}+cf=0 \qquad \text{(MB4.1)}$$

where f is a function of the variable x and the factors a, b, c may be either constants or functions of x. If the unknown function depends on only one variable, as in this example, the equation is called an **ordinary differential equation**; if it depends on more than one variable, as in

$$a\frac{\partial^2f}{\partial x^2}+b\frac{\partial^2f}{\partial y^2}+cf=0 \qquad \text{(MB4.2)}$$

it is called a **partial differential equation**. Here, f is a function of x and y, and the factors a, b, c may be either constants or functions of both variables. Note the change in symbol from d to ∂ to signify a *partial derivative* (see *Mathematical background 1*).

MB4.1 The structure of differential equations

The **order** of the differential equation is the order of the highest derivative that occurs in it: both examples above are second-order equations. Only rarely in science is a differential equation of order higher than 2 encountered.

A **linear differential equation** is one for which, if f is a solution, then so is constant $\times f$. Both examples above are linear. If the 0 on the right were replaced by a different number or a function other than f, then they would cease to be linear.

Solving a differential equation means something different from solving an algebraic equation. In the latter case, the solution is a value of the variable x (as in the solution $x=2$ of the quadratic equation $x^2-4=0$). The solution of a differential equation is the entire function that satisfies the equation, as in

$$\frac{d^2f}{dx^2}+f=0 \quad \text{has the solution} \quad f=A\sin x+B\cos x \qquad \text{(MB4.3)}$$

with A and B constants. The process of finding a solution of a differential equation is called **integrating** the equation. The solution in eqn MB4.3 is an example of a **general solution** of a differential equation, that is, it is the most general solution of the equation and is expressed in terms of a number of constants (A and B in this case). When the constants are chosen to accord with certain specified **initial conditions** (if one variable is the time) or certain **boundary conditions** (to fulfil certain spatial restrictions on the solutions), we obtain the **particular solution** of the equation. The particular solution of a first-order differential equation requires one such condition; a second-order differential equation requires two.

• A brief illustration

If we are informed that $f(0)=0$, then, because from eqn MB4.3 it follows that $f(0)=B$, we can conclude that $B=0$. That still leaves A undetermined. If we are also told that $df/dx=2$ at $x=0$ (that is, $f'(0)=2$, where the prime denotes a first derivative), then, because the general solution (but with $B=0$) implies that $f'(x)=A\cos x$, we know that $f'(0)=A$, and therefore $A=2$. The particular solution is therefore $f(x)=2\sin x$. Figure MB4.1 shows a series of particular solutions corresponding to different boundary conditions. •

MB4.2 The solution of ordinary differential equations

The first-order linear differential equation

$$\frac{df}{dx}+af=0 \qquad \text{(MB4.4a)}$$

with a a function of x or a constant can be solved by direct integration. To proceed, we use the fact that the quantities df and dx (called *differentials*) can be treated algebraically like any quantity and rearrange the equation into

$$\frac{df}{f}=-adx \qquad \text{(MB4.4b)}$$

and integrate both sides. For the left-hand side, we use the familiar result $\int dy/y=\ln y+\text{constant}$. After pooling all the constants into a single constant A, we obtain:

$$\ln f=-\int adx+A \qquad \text{(MB4.4c)}$$

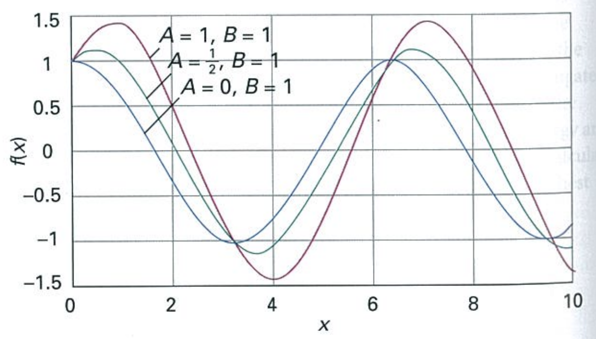

Fig. MB4.1 The solution of the differential equation in eqn MB4.3 with three different boundary conditions (as indicated by the resulting values of the constants A and B).

• A brief illustration

Suppose that in eqn MB4.4a the factor $a = 2x$; then the general solution, eqn MB4.4c, is

$$\ln f = -2\int x\mathrm{d}x + A = -x^2 + A$$

(We have absorbed the constant of integration into the constant A.) Therefore

$$f = \mathrm{e}^{A}\mathrm{e}^{-x^2}$$

If we are told that $f(0) = 1$, then we can infer that $A = 0$ and therefore that $f = \mathrm{e}^{-x^2}$. •

The solution even of first-order differential equations quickly becomes more complicated. A nonlinear first-order equation of the form

$$\frac{\mathrm{d}f}{\mathrm{d}x} + af = b \qquad \text{(MB4.5a)}$$

with a and b functions of x (or constants) has a solution of the form

$$f\mathrm{e}^{\int a\mathrm{d}x} = \int \mathrm{e}^{\int a\mathrm{d}x}\, b\mathrm{d}x + A \qquad \text{(MB4.5b)}$$

as may be verified by differentiation. Mathematical software packages can often perform the required integrations.

Second-order differential equations are in general much more difficult to solve than first-order equations. One powerful approach commonly used to lay siege to second-order differential equations is to express the solution as a power series:

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \qquad \text{(MB4.6)}$$

and then to use the differential equation to find a relation between the coefficients. This approach results, for instance, in the Hermite polynomials that form part of the solution of the Schrödinger equation for the harmonic oscillator (Section 8.5). Many of the second-order differential equations that occur in this text are tabulated in compilations of solutions or can be solved with mathematical software, and the specialized techniques that are needed to establish the form of the solutions may be found in mathematical texts.

MB4.3 The solution of partial differential equations

The only partial differential equations that we need to solve are those that can be separated into two or more ordinary differential equations by the technique known as **separation of variables**. To discover if the differential equation in eqn MB4.2 can be solved by this method we suppose that the full solution can be factored into functions that depend only on x or only on y, and write $f(x,y) = X(x)Y(y)$. At this stage there is no guarantee that the solution can be written in this way. Substituting this trial solution into the equation and recognizing that

$$\frac{\partial^2 XY}{\partial x^2} = Y\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} \qquad \frac{\partial^2 XY}{\partial y^2} = X\frac{\mathrm{d}^2 Y}{\mathrm{d}y^2}$$

we obtain

$$aY\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} + bX\frac{\mathrm{d}^2 Y}{\mathrm{d}y^2} + cXY = 0$$

We are using d instead of ∂ at this stage to denote differentials because each of the functions X and Y depends on one variable, x and y, respectively. Division through by XY turns this equation into

$$\frac{a}{X}\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} + \frac{b}{Y}\frac{\mathrm{d}^2 Y}{\mathrm{d}y^2} + c = 0$$

Now suppose that a is a function only of x, b a function of y, and c a constant. (There are various other possibilities that permit the argument to continue.) Then the first term depends only on x and the second only on y. If x is varied, only the first term can change. But, as the other two terms do not change and the sum of the three terms is a constant (0), even that first term must be a constant. The same is true of the second term. Therefore because each term is equal to a constant, we can write

$$\frac{a}{X}\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} = c_1 \qquad \frac{b}{Y}\frac{\mathrm{d}^2 Y}{\mathrm{d}y^2} = c_2 \qquad \text{with} \qquad c_1 + c_2 = -c$$

We now have two ordinary differential equations to solve by the techniques described in Section MB4.2. An example of this procedure is given in Section 8.2, for a particle in a two-dimensional region.