02612 Constrained Optimization 2022
Exam Assignment
Hand-in deadline: May 30, 2022, 13:30

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April 5, 2022

1 Equality Constrained Convex QP

In this problem, we consider the equality constrained convex QP

$$\min_{x} \quad \phi = \frac{1}{2}x'Hx + g'x \tag{1.1a}$$

$$s.t. \quad A'x = b \tag{1.1b}$$

with $H \succ 0$.

1. What is the Lagrangian function for this problem?

2. What is the first order necessary optimality conditions for this problem? Are they also sufficient and why?

3. Implement solvers for solution of the problem (1.1) that are based on an LU-factorization (dense), LU-factorization (sparse), LDL-factorization (dense), LDL-factorization (sparse), a range-space factorization, and a null-space factorization. You must provide pseudo-code and source code for your implementation.
   The solvers for the individual factorizatons must have the interface `[x,lambda]=EqualityQPSolverXX(H,g,A,b)` where `XX` can be e.g. `LUdense`, `LUsparse`, etc. You must make a system that can switch between the different solvers as well.
   It should have an interface like `[x,lambda]=EqualityQPSolver(H,g,A,b,solver)`, where `solver` is a flag used to switch between the different factorizations.

4. Test your algorithms on the test problem with the data

```
H =
5.0000    1.8600    1.2400    1.4800   -0.4600
1.8600    3.0000    0.4400    1.1200    0.5200
1.2400    0.4400    3.8000    1.5600   -0.5400
1.4800    1.1200    1.5600    7.2000   -1.1200
-0.4600    0.5200   -0.5400   -1.1200    7.8000

g =
-16.1000
-8.5000
-15.7000
-10.0200
-18.6800

A =
16.1000    1.0000
8.5000    1.0000
15.7000    1.0000
10.0200    1.0000
18.6800    1.0000

b =
15
1
```

Compute the solution for different values of `b(1)` in the range `[8.5 18.68]`.

5. Test your implementation on a size dependent problem structure and report the results. You are free to chose the problems that you want to use for testing your algorithm.

2 Quadratic Program (QP)

We consider the quadratic program (QP) in the form (assume that A has full column rank)

$$\min_x \quad \phi = \frac{1}{2}x'Hx + g'x \tag{2.1a}$$
$$s.t. \quad A'x = b \tag{2.1b}$$
$$l \leq x \leq u \tag{2.1c}$$

1. What is the Lagrangian function for this problem (2.1)?
2. Write the nesessary and sufficient optimality conditions for this problem (2.1).
3. Write pseudo-code for a primal-dual interior-point algorithm for solution of this problem (2.1). Explain each major step in your algorithm.
4. Implement the primal-dual interior-point algorithm for (2.1) and test it. You must provide commented code as well as driver files to test your code, documentation that it works, and performance statistics.
5. Compare the performance of your primal-dual interior-point algorithm and `quadprog` from Matlab (or equivalent QP library functions). Provide scripts that demonstrate how you compare the software and comment on the tests and the results.
6. Consider a QP in the form (2.1). Use H, g, A and b from problem 1. Let `l=zeros(5,1)` and `u = ones(5,1)`. Compute the solution for different values of `b(1)` in the range `[8.5 18.68]`. Test the primal-dual interior-point QP algorithm and the library QP algorithm e.g. `quadprog`, MOSEK, Gurobi, and `cvx` for this problem. Plot the solution as well as solution statistics (number of iterations, cpu time, etc).

3 Linear Program (LP)

In this problem we consider a linear program in the form (assume that A has full column rank)

$$
\begin{aligned}
\min_{x} \quad & \phi = g'x && (3.1a)\\
s.t. \quad & A'x = b && (3.1b)\\
& l \leq x \leq u && (3.1c)
\end{aligned}
$$

1. What is the Lagrangian function for this problem (3.1)?
2. Write the nesessary and sufficient optimality conditions for this problem (3.1).
3. Write pseudo-code for a primal-dual interior-point algorithm for solution of this problem (3.1). Explain each major step in your algorithm.
4. Implement the primal-dual interior-point algorithm and test it. You must provide commented code as well as driver files to test your code, documentation that it works, and performance statistics.
5. Test your LP algorithm on the following test problem with the data

```
g =
-16.1000
-8.5000
-15.7000
-10.0200
-18.6800

A =
1.0000
1.0000
1.0000
1.0000
1.0000

b =
1
```

   `l=zeros(5,1)` and `u=ones(5,1)`.

6. Compare the performance of your primal-dual interior-point algorithm and `linprog` from Matlab (or equivalent LP library functions). Provide scripts that demonstrate how you compare the software and comment on the tests and the results. You should solve the problem using your primal-dual interior-point algorithm and a library algorithm e.g. `linprog`, MOSEK, Gurobi, and `cvx`.

4 Nonlinear Program (NLP)

We consider a nonlinear program in the form

$$\min_{x} \quad f(x) \tag{4.1a}$$

$$s.t. \quad g_l \leq g(x) \leq g_u \tag{4.1b}$$

$$x_l \leq x \leq x_u \tag{4.1c}$$

We assume that the involved functions are sufficiently smooth for the algorithms discussed in this course to work. Assume that $\nabla g(x)$ has full column rank.

1. What is the Lagrangian function for the nonlinear program (4.1)?

2. What is the necessary first order optimality conditions for the nonlinear program (4.1)?

3. What are the sufficient second order optimality conditions for the nonlinear program (4.1)?

4. Consider Himmelblau's test problem. Convert this problem into the form (4.1). Provide the contour plot of the problem and locate all stationary points.

5. Solve the test problem and other nonlinear optimization problems you choose using a library function for nonlinear programs, e.g. `fmincon` in Matlab, IPOPT, and `CasADi (IPOPT called from CasADi)`.

6. Explain, discuss and implement an SQP procedure with a damped BFGS approximation to the Hessian matrix for the problem (4.1). Make a table with the iteration sequence for different starting points. Plot the iteration sequence in a contour plot. Discuss the results.

7. Explain, discuss and implement the SQP procedure with a damped BFGS approximation to the Hessian matrix and line search for the problem (4.1). Make a table with the iteration sequence. Make a table with relevant statistics (function calls etc). Plot the iteration sequence in a contour plot. Discuss the results.

8. Explain, discuss, and implement a Trust Region based SQP algorithm for this problem (4.1). Make a table with the iteration sequence. Make a table with relevant statistics (function calls etc). Plot the iteration sequence in a contour plot. Discuss the results

9. Solve a number of test problems you choose using library optimization algorithms and your own nonlinear optimization algorithms. Use the test to document that your algorithm works and to document its efficiency compared to library functions.

5 Markowitz Portfolio Optimization

This exercise illustrates use of quadratic programming in a financial application. By diversifying an investment into several securities it may be possible to reduce risk without reducing return. Identification and construction of such portfolios is called hedging. The Markowitz Portofolio Optimization problem is very simple hedging problem for which Markowitz was awarded the Nobel Price in 1990.

Consider a financial market with 5 securities.

Security	Covariance					Return
1	2.50	0.93	0.62	0.74	-0.23	16.10
2	0.93	1.50	0.22	0.56	0.26	8.50
3	0.62	0.22	1.90	0.78	-0.27	15.70
4	0.74	0.56	0.78	3.60	-0.56	10.02
5	-0.23	0.26	-0.27	-0.56	3.90	18.68

Optimal solution as function of return

1. For a given return, R, formulate Markowitz' Portfolio optimization problem as a quadratic program.
2. What is the minimal and maximal possible return in this financial market?
3. Compute a portfolio with return, $R = 12.0$, and minimal risk. What is the optimal portfolio and what is the risk (variance)?
4. Compute the efficient frontier, i.e. the risk as function of the return. Plot the efficient frontier as well as the optimal portfolio as function of return.

Bi-criterion optimization

1. Formulate the optimization problem as a bi-criterion of the variance and the return, i.e. an objective function in the form $\phi = \alpha V\{\boldsymbol{R}\} - (1-\alpha)E\{\boldsymbol{R}\}$, where $\boldsymbol{R}$ is the stochastic portfolio return.
2. Compute the solution using your own algorithms (e.g. Problem 1-3) for different values of α in the interval $[0, 1]$. Plot the solutions and also report the solver statistics.
3. Compare the solution to the solution obtained using `quadprog`, MOSEK, Guroi, and `cvx`. Also compare the solver statistics to the solver statistics for your own algorithm (Problem 2).

Risk-free asset

In the following we add a risk free security to the financial market. It has return $r_f = 0.0$.

1. What is the new covariance matrix and return vector.

2. Compute the efficient frontier, plot it as well as the (return,risk) coordinates of all the securities. Comment on the effect of a risk free security. Plot the optimal portfolio as function of return.

3. What is the minimal risk and optimal portfolio giving a return of $R = 14.00$. Plot this point in your optimal portfolio as function of return as well as on the efficient frontier diagram.

4. Discuss the solution and an appropriate solver for the problem.

Report

You are allowed to work on the assignment in groups. You must hand in an individual report that you write yourself for the assignment. The following must be uploaded to Learn: 1) one pdf file of the report, 2) one zip-file containing all Matlab and Latex code etc used to prepare the report. In addition you must print the pdf file of your report and hand it in to may mail box in Building 303B Room 112.

Labels, fontsize, and visibility of all figures must be made in a professional manner. Include key matlab code in the report (use syntax high lighting - and print the report in color), and provide all matlab code in the appendix that you can refer to. The report should include a desciption and discussion of the mathematical methods and algorithms that you use, as well as a discussion of the results that you obtain. We want you to demonstrate that you can critically reflect on the methods used, their properties, and the results that you obtain.

The deadline for handing in the report is Monday May 30, 2022 at 13:30.