02612 Constrained Optimization
Lecture 2 Exercises

John Bagterp Jørgensen

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1 Problem 1 - Quadratic Optimization

Consider the problem

$$\begin{aligned}
\min_x \quad & f(x) = 3x_1^2 + 2x_1x_2 + x_1x_3 + 2.5x_2^2 + 2x_2x_3 + 2x_3^2 - 8x_1 - 3x_2 - 3x_3 && \text{(1.1a)} \\
s.t. \quad & x_1 + x_3 = 3 && \text{(1.1b)} \\
& x_2 + x_3 = 0 && \text{(1.1c)}
\end{aligned}$$

1. Write the constraints in the form $c_i(x) = 0$.
2. Write the Lagrangian of this problem.
3. Write the first order optimality conditions for this problem.
4. Find the optimal solution

Write the problem in the form

$$\begin{aligned}
\min_x \quad & f(x) = \frac{1}{2}x'Hx + g'x && \text{(1.2a)} \\
s.t. \quad & A'x = b && \text{(1.2b)}
\end{aligned}$$

1. What are H, g, A, b
2. Write the Lagrangian for this problem
3. Use the first order optimality conditions for (1.2).
4. Find the minimizer using these optimality conditions.
5. Use sufficient conditions to proove that the point found is a minimizer.

2 Problem 2 - Linear Optimization

Consider the problem

$$\min_{x \in \mathbb{R}^2} \quad f(x) = g'x \tag{2.1a}$$

$$s.t. \quad c(x) = A'x - b \geq 0 \tag{2.1b}$$

with

$$g = \begin{bmatrix} 1 & -2 \end{bmatrix}'$$

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 & -5 \\ 0 & 1 & -1 & -5 & 1 \end{bmatrix}$$

$$b = \begin{bmatrix} 0 & 0 & -2 & -20 & -15 \end{bmatrix}'$$

1. Draw a contour plot of (2.1).
2. Locate the minimizer using the contour plot.
3. Write the Lagrange function of (2.1).
4. Write the first-order optimality conditions of (2.1).
5. Write the equations to solve using the active set method. Use the contour plot to determine the active set.
6. Determine the solution numerically using the active set method.

3 Problem 3 - Nonlinear Optimization

Consider the equality constrained problem

$$\min_{x \in \mathbb{R}^2} \quad f(x) = x_1^2 + x_2^2 + 3x_2 \tag{3.1a}$$

$$s.t. \quad c(x) = x_1^2 + (x_2 + 1)^2 - 1 = 0 \tag{3.1b}$$

1. Make a contour plot of (3.1)
2. Plot the objective functions in the feasible region as function of x_2.

Show that

1. $x = 0$ is a KKT point
2. The Lagrange multiplier at 0 is positive
3. $d^2f/dx^2(0)$ is positive definite
4. 0 is *not* a local minimizer of the problem

Find the minimizer of the problem and use a sufficient condition to prove that it is a minimizer.

4 Problem 4 - Nonlinear Optimization

Consider the constrained optimization problem

$$\min_{x,y} \quad f(x,y) = (x^2 + y - 11)^2 + (x + y^2 - 7)^2 \tag{4.1a}$$

$$s.t. \quad c_1(x,y) = (x+2)^2 - y \geq 0 \tag{4.1b}$$

$$c_2(x,y) = -4x + 10y \geq 0 \tag{4.1c}$$

1. Make a contour plot of the problem.
2. Locate all minimizers of (4.1) using the contour plot.
3. Write the Lagrange function of (4.1).
4. Locate all stationary points of the Lagrangian using the contour plot.
5. Write the first order optimality conditions for this problem.
6. Use the first order optimality conditions and `fsolve` to locate the minimizers numerically.
7. Write the second order sufficient optimality conditions for (4.1) and use these to prove that the stationary points are minimizers.