Problems in 02612 Constrained Optimization 2021
Exercise 05

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1 Problem 1 - Equality Constrained Quadratic Optimization

This problems illustrates how solution of the equality constrained convex quadratic program scales with problem size and factorization method applied.

Consider the convex quadratic optimization problem

$$\min_{u} \quad \frac{1}{2}\sum_{i=1}^{n+1}(u_i - \bar{u})^2 \tag{1.1a}$$

$$s.t. \quad -u_1 + u_n = -d_0 \tag{1.1b}$$

$$u_i - u_{i+1} = 0 \qquad i = 1, 2, \dots, n-2 \tag{1.1c}$$

$$u_{n-1} - u_n - u_{n+1} = 0 \tag{1.1d}$$

$\bar{u}$ and d_0 are parameters of the problem. The problem size can be adjusted selecting $n \geq 3$. Let $\bar{u} = 0.2$ and $d_0 = 1$. The constraints models a recycle system as depicted by the directed graph in figure 1.

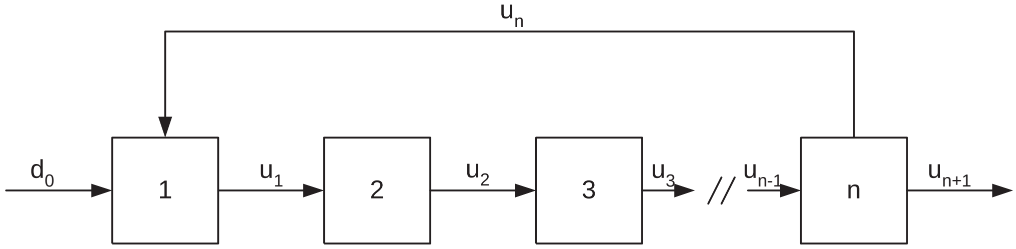

Figure 1: Directed graph representation of the constraints in (1.1). This graph represents a recycle system.

1. Express the problem in matrix form, i.e. in the form

$$\min_{x \in \mathbb{R}^n} \quad \phi = \frac{1}{2}x'Hx + g'x \tag{1.2a}$$

$$s.t. \quad A'x = b \tag{1.2b}$$

   Let $n = 10$. What is x, H, g, A, and b.

2. What is the Lagrangian function and the first order optimality conditions for the problem? Explain why the optimality conditions are both necessary and sufficient for this problem.

3. Make a Matlab function that constructs H, g, A, and b as function of n, $\bar{u}$, and d_0.

4. Make a Matlab function that constructs the KKT-matrix as function of n, $\bar{u}$, and d_0.

5. Make a Matlab function that solves (1.1) using an LU factorization.

6. Make a Matlab function that solves (1.1) using an LDL factorization.

7. Make a Matlab function that solves (1.1) using the Null-Space procedure based on QR-factorizations.

8. Make a Matlab function that solves (1.1) using the Range-Space procedure.

9. Evaluate the performance of your QP-solvers based on LU, LDL, Null-Space, and Range-Space factorizations by plotting the cputime as function of problem size (say in the range n=10-1000). Comment on the results.

10. Let $n = 100$ and plot the sparsity pattern of the KKT-matrix.

11. Make a function that treats the system as a sparse system (see sparse) using an LU-factorization. Evaluate the performance (cputime) of this solver as function of problem size (n=10-1000). Comment on the results.

12. Make a function that treats the system as a sparse system (see sparse) using an LDL-factorization. Evaluate the performance (cputime) of this solver as function of problem size (n=10-1000). Comment on the results.

13. Discuss the performance of the sparse solvers compared to the dense solvers.

2 Problem 2 - Inequality Constrained Quadratic Programming

Consider the QP in Example 16.4 (p.475) in Nocedal and Wright.

1. Make a contour plot of the problem.
2. Write the KKT-conditions for this problem.
3. Argue that the KKT-conditions are both necessary and sufficient optimality conditions.
4. Make a function for solution of convex equality constrained QPs (see Problem 1 and Problem 2).
5. Apply a conceptual active set algorithm to the problem. Use the iteration sequence in Figure 16.3 of Nocedal and Wright. Plot the iterations sequence in your contour plot. For each iteration (guess of working set) you should list the working set, the solution, x, and the Lagrange multipliers, λ.
6. Comment on the Lagrange multipliers at each iteration.
7. Explain the active-set method for convex QPs listed on p. 472 in N&W.
8. Use `linprog` to compute a feasible point to a QP. Apply and test this procedure to the problem in Example 16.4.
9. Implement the algorithm in p. 472 and test it it for the problem in Example 16.4. Print information for every iteration of the algorithm (i.e. the point x_k, the working set $\mathcal{W}_k$, etc) and list that in your report.
10. Test your active set algorithm for the problem in Example 16.4 using (16.47) and (16.48) in N&W. Do the same using `quadprog`.

3 Problem 3 - Markowitz Portfolio Optimization

This exercise illustrates use of quadratic programming in a financial application. By diversifying an investment into several securities it may be possible to reduce risk without reducing return. Identification and construction of such portfolios is called hedging. The Markowitz Portofolio Optimization problem is very simple hedging problem for which Markowitz was awarded the Nobel Price in 1990.

Consider a financial market with 5 securities.

Security	Covariance					Return
1	2.30	0.93	0.62	0.74	-0.23	15.10
2	0.93	1.40	0.22	0.56	0.26	12.50
3	0.62	0.22	1.80	0.78	-0.27	14.70
4	0.74	0.56	0.78	3.40	-0.56	9.02
5	-0.23	0.26	-0.27	-0.56	2.60	17.68

1. For a given return, R, formulate Markowitz' Portfolio optimization problem as a quadratic program.
2. What is the minimal and maximal possible return in this financial market?
3. Use `quadprog` to find a portfolio with return, $R = 10.0$, and minimal risk. What is the optimal portfolio and what is the risk (variance)?
4. Compute the efficient frontier, i.e. the risk as function of the return. Plot the efficient frontier as well as the optimal portfolio as function of return.

In the following we add a risk free security to the financial market. It has return $r_f = 2.0$.

1. What is the new covariance matrix and return vector.
2. Compute the efficient frontier, plot it as well as the (return,risk) coordinates of all the securities. Comment on the effect of a risk free security. Plot the optimal portfolio as function of return.
3. What is the minimal risk and optimal portfolio giving a return of $R = 15.00$. Plot this point in your optimal portfolio as function of return as well as on the efficient frontier diagram.