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Research Article

On the Derivatives of Bernstein Polynomials: An Application for the Solution of High Even-Order Differential Equations

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A new formula expressing explicitly the derivatives of Bernstein polynomials of any degree and for any order in terms of Bernstein polynomials themselves is proved, and a formula expressing the Bernstein coefficients of the general-order derivative of a differentiable function in terms of its Bernstein coefficients is deduced. An application of how to use Bernstein polynomials for solving high even-order differential equations by Bernstein Galerkin and Bernstein Petrov-Galerkin methods is described. These two methods are then tested on examples and compared with other methods. It is shown that the presented methods yield better results.

1. Introduction

Bernstein polynomials [1] have many useful properties, such as, the positivity, the continuity, and unity partition of the basis set over the interval $[0,1]$. The Bernstein polynomial bases vanish except the first polynomial at $x = 0$, which is equal to 1 and the last polynomial at $x = 1$, which is also equal to 1 over the interval $[0,1]$. This provides greater flexibility in imposing boundary conditions at the end points of the interval. The moments x^m is nothing but Bernstein polynomial itself. With the advent of computer graphics, Bernstein polynomial restricted to the interval $x \in [0,1]$ becomes important in the form of Bezier curves [2]. Many properties of the Bézier curves and surfaces come from the properties of the Bernstein polynomials. Moreover, Bernstein polynomials have been recently used for the solution of differential equations, (see, e.g., [3]).

The Bernstein polynomials are not orthogonal; so their uses in the least square approximations are limited. To overcome this difficulty, two approaches are used. The first approach is the basis transformation, for the transformation matrix between Bernstein polynomial basis and Legendre polynomial basis [4], between Bernstein polynomial basis and Chebyshev polynomial basis [5], and between Bernstein polynomial basis and Jacobi polynomial basis [6]. The second approach is the dual basis functions for Bernstein polynomials (see Jüttler [7]). Jüttler [7] derived an explicit formula for the dual basis function of Bernstein polynomials. The construction of the dual basis must be repeated at each time the approximation polynomial increased.

For spectral methods [8, 9], explicit formulae for the expansion coefficients of a general-order derivative of an infinitely differentiable function in terms of those of the original expansion coefficients of the function itself are needed. Such formulae are available for expansions in Chebyshev [10], Legendre [11], ultraspherical [12], Hermite [13], Jacobi [14], and Laguerre [15] polynomials. These polynomials have been used in both the solution of boundary value problems [16–19] and in computational fluid dynamics [8]. In most of these applications, use is made of formulae relating the expansion coefficients of derivatives appearing in the differential equation to those of the function itself, (see, e.g., [16–19]). This process results in an algebraic system or a system of differential equations for the expansion coefficients of the solution which then must be solved.

Due to the increasing interest on Bernstein polynomials, the question arises of how to describe their properties in terms of their coefficients when they are given in the Bernstein basis. Up to now, and to the best of our Knowledge, many formulae corresponding to those mentioned previously are unknown and are traceless in the literature for Bernstein polynomials. This partially motivates our interest in such polynomials.

Another motivation is concerned with the direct solution techniques for solving high even-order differential equations, using the Bernstein Galerkin approximation. Also, we use Bernstein Petrov-Galerkin approximation; we choose the trial functions to satisfy the underlying boundary conditions of the differential equations, and the test functions to be dual Bernstein polynomials which satisfy the orthogonality condition. The method leads to linear systems which are sparse for problems with constant coefficients. Numerical results are presented in which the usual exponential convergence behavior of spectral approximations is exhibited.

The remainder of this paper is organized as follows. In Section 2, we give an overview of Bernstein polynomials and the relevant properties needed in the sequel, and in Section 3, we prove the main results of the paper which are: (i) an explicit expression for the derivatives of Bernstein polynomials of any degree and for any order in terms of the Bernstein polynomials themselves and (ii) an explicit formula for the expansion coefficient of the derivatives of an infinitely differentiable function in terms of those of the original expansion coefficients of the functions itself. In Section 4, we discuss separately Bernstein Galerkin and Bernstein Petrov-Galerkin methods and describe how they are used to solve high even-order differential equations. Finally, Section 5 gives some numerical results exhibiting the accuracy and efficiency of our proposed numerical algorithms.

2. Relevant Properties of Bernstein Polynomials

The Bernstein polynomials of nth degree form a complete basis over $[0,1]$, and they are defined by

$$B_{i,n}(x) = \binom{n}{i} x^i (1-x)^{n-i}, \quad 0 \leq i \leq n, \tag{2.1}$$

where the binomial coefficients are given by $\binom{n}{i} = n!/i!(n-i)!$.

The derivatives of the nth degree Bernstein polynomials are polynomials of degree $n-1$ and are given by

$$DB_{i,n}(x) = n\left(B_{i-1,n-1}(x) - B_{i,n-1}(x)\right), \quad D \equiv \frac{d}{dx}. \tag{2.2}$$

The multiplication of two Bernstein basis is

$$B_{i,j}(x)B_{k,m}(x) = \frac{\binom{j}{i}\binom{m}{k}}{\binom{j+m}{i+k}} B_{i+k,j+m}(x), \tag{2.3}$$

and the moments of Bernstein basis are

$$x^m B_{i,n}(x) = \frac{\binom{n}{i}}{\binom{n+m}{i+m}} B_{i+m,n+m}(x). \tag{2.4}$$

Like any basis of the space Π_n, the Bernstein polynomials have a unique dual basis $(D_{0,n}, D_{1,n}, \ldots, D_{n,n})$ (also called the inverse or reciprocal basis) which consists of the $n+1$ dual basis functions

$$D_{i,n}(x) = \sum_{j=0}^{n} c_{i,j} B_{j,n}(x), \quad (j = 0, 1, \ldots, n), \tag{2.5}$$

where

$$c_{i,j} = \frac{(-1)^{i+j}}{\binom{n}{i}\binom{n}{j}} \sum_{k=0}^{\min(i,j)} (2k+1) \binom{n+k+1}{n-i}\binom{n-k}{n-i}\binom{n+k+1}{n-j}\binom{n-k}{n-j}, \quad (i,j = 0, 1, \ldots, n). \tag{2.6}$$

Jüttler [7] represented the dual basis function with respect to the Bernstein basis. The dual basis functions must satisfy the relation of duality

$$\int_0^1 B_{i,n}(x) D_{k,n}(x) dx = \delta_{i,k}. \tag{2.7}$$

Indefinite integral of Bernstein basis is given by

$$\int B_{i,n}(x)dx = \frac{1}{n+1}\sum_{j=i+1}^{n+1} B_{j,n+1}(x), \tag{2.8}$$

and all Bernstein basis function of the same order have the same definite integral over the interval $[0,1]$, namely,

$$\int_0^1 B_{i,n}(x)dx = \frac{1}{n+1}. \tag{2.9}$$

3. Derivatives of Bernstein Polynomials

The main objective of this section is to prove the following two theorems for the derivatives of $B_{i,n}(x)$ and Bernstein coefficients of the qth derivative of $f(x)$.

Theorem 3.1.

$$D^p B_{i,n}(x) = \frac{n!}{(n-p)!}\sum_{k=\max(0,i+p-n)}^{\min(i,p)} (-1)^{k+p}\binom{p}{k} B_{i-k,n-p}(x). \tag{3.1}$$

Proof. For $p = 1$, (3.1) leads us to go back to (2.2).

If we apply induction on p, assuming that (3.1) holds, we want to show that

$$D^{p+1} B_{i,n}(x) = \frac{n!}{(n-p-1)!}\sum_{k=\max(0,i+p+1-n)}^{\min(i,p+1)} (-1)^{k+p+1}\binom{p+1}{k} B_{i-k,n-p-1}(x). \tag{3.2}$$

If we differentiate (3.1), then we have (with application of relation (2.2))

$$\begin{aligned} D^{p+1} B_{i,n}(x) &= D\left(D^p B_{i,n}(x)\right) \\ &= D\left(\frac{n!}{(n-p)!}\sum_{k=\max(0,i+p-n)}^{\min(i,p)} (-1)^{k+p}\binom{p}{k} B_{i-k,n-p}(x)\right) \\ &= \frac{n!}{(n-p)!}\sum_{k=\max(0,i+p-n)}^{\min(i,p)} (-1)^{k+p}\binom{p}{k} D\left(B_{i-k,n-p}(x)\right) \\ &= \frac{n!(n-p)}{(n-p)!}\sum_{k=\max(0,i+p-n)}^{\min(i,p)} (-1)^{k+p}\binom{p}{k} \left(B_{i-k-1,n-p-1}(x) - B_{i-k,n-p-1}(x)\right), \end{aligned} \tag{3.3}$$

which can be written as

$$D^{p+1}B_{i,n}(x) = \frac{n!}{(n-p-1)!}\sum_{k=\max(0,i+p-n)}^{\min(i,p)}(-1)^{k+p}\binom{p}{k}B_{i-k-1,n-p-1}(x)$$
$$-\frac{n!}{(n-p-1)!}\sum_{k=\max(0,i+p+1-n)}^{\min(i,p)}(-1)^{k+p}\binom{p}{k}B_{i-k,n-p-1}(x). \tag{3.4}$$

Set $k = k-1$ in the first term of the right-hand side of relation (3.4) to get

$$D^{p+1}B_{i,n}(x) = \frac{n!}{(n-p-1)!}\sum_{k=\max(1,i+p+1-n)}^{\min(i,p+1)}(-1)^{k+p+1}\binom{p}{k-1}B_{i-k,n-p-1}(x)$$
$$+\frac{n!}{(n-p-1)!}\sum_{k=\max(0,i+p+1-n)}^{\min(i,p)}(-1)^{k+p+1}\binom{p}{k}B_{i-k,n-p-1}(x). \tag{3.5}$$

It can be easily shown that

$$\begin{aligned}
D^{p+1}B_{i,n}(x) &= \frac{n!}{(n-p-1)!}\sum_{k=\max(0,i+p+1-n)}^{\min(i,p+1)}(-1)^{k+p+1}\left(\binom{p}{k-1}+\binom{p}{k}\right)B_{i-k,n-p-1}(x)\\
&= \frac{n!}{(n-p-1)!}\sum_{k=\max(0,i+p+1-n)}^{\min(i,p+1)}(-1)^{k+p+1}\binom{p+1}{k}B_{i-k,n-p-1}(x)\\
&= \frac{n!}{(n-p-1)!}\sum_{k=\max(0,i+p+1-n)}^{\min(i,p+1)}(-1)^{k+p+1}\binom{p+1}{k}B_{i-k,n-p-1}(x),
\end{aligned} \tag{3.6}$$

which completes the induction and proves the theorem. □

We can express the Bernstein polynomial of any degree $B_{k,n}(x)$ in terms of any higher degree basis $B_{k,n+p}(x)$ using the following lemma.

Lemma 3.2.

$$B_{k,n}(x) = \sum_{j=k}^{k+p}\frac{\binom{n}{k}\binom{p}{j-k}}{\binom{n+p}{j}}B_{j,n+p}(x). \tag{3.7}$$

For proof, see, Farouki and Rajan [20].

Let $f(x)$ be a differentiable function of degree n defined on the interval $[0,1]$, then we can write

$$f(x)=\sum_{i=0}^{n} a_{i,n} B_{i,n}(x). \tag{3.8}$$

Further, let $a_{i,n}^{(q)}$ denote the Bernstein coefficients of the qth derivative of $f(x)$, that is,

$$f^{(q)}(x)=\frac{d^q f(x)}{dx^q}=\sum_{i=0}^{n} a_{i,n}^{(q)} B_{i,n}(x), \quad a_{i,n}^{(0)}=a_{i,n}. \tag{3.9}$$

Then, we can state and prove the following theorem.

Theorem 3.3.

$$a_{i,n}^{(q)}=\sum_{k=-q}^{q} C_k(i,n,q)\, a_{i-k,n}, \tag{3.10}$$

where

$$C_k(i,n,q)=q!\sum_{m=0}^{q}(-1)^{m+q}\binom{q}{m}\binom{i}{m+k}\binom{n-i}{q-m-k}. \tag{3.11}$$

Proof. Since

$$\begin{gathered} f(x)=\sum_{i=0}^{n} a_{i,n} B_{i,n}(x), \\ f^{(q)}(x)=\frac{d^q f(x)}{dx^q}=\sum_{i=0}^{n} a_{i,n} D^{(q)} B_{i,n}(x), \end{gathered} \tag{3.12}$$

then making use of Theorem 3.1 (formula (3.1)) immediately yields

$$\begin{aligned} f^{(q)}(x) &= \sum_{i=0}^{n} a_{i,n}\frac{n!}{(n-q)!}\sum_{k=\max(0,i+q-n)}^{\min(i,q)}(-1)^{k+q}\binom{q}{k} B_{i-k,n-q}(x) \\ &= \sum_{i=0}^{n} a_{i,n}\frac{n!}{(n-q)!}\sum_{k=0}^{q}(-1)^{k+q}\binom{q}{k} B_{i-k,n-q}(x). \end{aligned} \tag{3.13}$$

If we change the degree of Bernstein polynomials using (3.7), then we can write

$$\begin{aligned}
f^{(q)}(x) &= \sum_{i=0}^{n} a_{i,n} \frac{n!}{(n-q)!} \sum_{k=0}^{q} (-1)^{k+q} \binom{q}{k} \sum_{m=0}^{q} \frac{\binom{n-q}{i-k}\binom{q}{m}}{\binom{n}{i-k+m}} B_{i+m-k,n}(x) \\
&= \sum_{i=0}^{n} a_{i,n} \frac{n!}{(n-q)!} \left(\sum_{k=0}^{q} (-1)^{k+q} \binom{q}{k} \sum_{m=0}^{q} \frac{\binom{n-q}{i-k}\binom{q}{m}}{\binom{n}{i-k+m}} \right) B_{i+m-k,n}(x) \\
&= \frac{n!}{(n-q)!} \sum_{i=0}^{n} a_{i,n} \left[\sum_{k=0}^{q} (-1)^{k+q} \binom{q}{k} \binom{n-q}{i-k} \sum_{m=0}^{q} \frac{\binom{q}{m}}{\binom{n}{i-k+m}} B_{i+m-k,n}(x) \right].
\end{aligned} \tag{3.14}$$

Expanding the two summation $\sum_{k=0}^{q}\sum_{m=0}^{q}$ and rearranging the coefficients of $B_{i+k,n}$ from $-q \leq k \leq q$, we get

$$\begin{aligned}
f^{(q)}(x) &= \frac{n!}{(n-q)!} \sum_{i=0}^{n} a_{i,n} \left[\sum_{k=-q}^{q} \frac{1}{\binom{n}{i+k}} B_{i+k,n}(x) \sum_{m=0}^{q} (-1)^{m+q} \binom{q}{m} \binom{n-q}{i-m} \binom{q}{m+k} \right] \\
&= \frac{n!}{(n-q)!} \sum_{i=k}^{n+k} a_{i-k,n} \sum_{k=-q}^{q} \frac{1}{\binom{n}{i}} B_{i,n}(x) \sum_{m=0}^{q} (-1)^{m+q} \binom{q}{m} \binom{n-q}{i-k-m} \binom{q}{m+k} \\
&= \frac{n!}{(n-q)!} \sum_{i=0}^{n} a_{i-k,n} \sum_{k=-q}^{q} \frac{1}{\binom{n}{i}} B_{i,n}(x) \sum_{m=0}^{q} (-1)^{m+q} \binom{q}{m} \binom{n-q}{i-k-m} \binom{q}{m+k} \\
&= \sum_{i=0}^{n} \left[\frac{n!}{(n-q)!} \sum_{k=-q}^{q} \frac{1}{\binom{n}{i}} \sum_{m=0}^{q} (-1)^{m+q} \binom{q}{m} \binom{n-q}{i-k-m} \binom{q}{m+k} a_{i-k,n} \right] B_{i,n}(x) \\
&= \sum_{i=0}^{n} \left[q! \sum_{k=-q}^{q} \sum_{m=0}^{q} (-1)^{m+q} \binom{q}{m} \binom{i}{m+k} \binom{n-i}{q-m-k} a_{i-k,n} \right] B_{i,n}(x) \\
&= \sum_{i=0}^{n} a_{i,n}^{(q)} B_{i,n}(x),
\end{aligned} \tag{3.15}$$

and this completes the proof of Theorem 3.3. □

The following two corollaries will be of fundamental importance in what follows.

Corollary 3.4.

$$\int_0^1 B_{i,n}^{(p)}(x)B_{j,n}(x)dx = \frac{n!\binom{n}{j}}{(2n-p+1)(n-p)!}\sum_{k=\max(0,i+p-n)}^{\min(i,p)}(-1)^{k+p}\frac{\binom{p}{k}\binom{n-p}{i-k}}{\binom{2n-p}{i+j-k}}. \tag{3.16}$$

Proof. We can express explicitly the pth derivatives of Bernstein polynomials from Theorem 3.1 to obtain

$$\begin{aligned}\int_0^1 B_{i,n}^{(p)}(x)B_{j,n}(x)dx &= \int_0^1 \frac{n!}{(n-p)!}\sum_{k=\max(0,i+p-n)}^{\min(i,p)}(-1)^{k+p}\binom{p}{k}B_{i-k,n-p}(x)B_{j,n}(x)dx \\ &= \frac{n!}{(n-p)!}\sum_{k=\max(0,i+p-n)}^{\min(i,p)}(-1)^{k+p}\binom{p}{k}\int_0^1 B_{i-k,n-p}(x)B_{j,n}(x)dx.\end{aligned} \tag{3.17}$$

Now, (3.16) can be easily derived by using (2.3). Thanks to (2.9), we have

$$\begin{aligned}\int_0^1 B_{i,n}^{(p)}(x)B_{j,n}(x)dx &= \frac{n!}{(n-p)!}\sum_{k=\max(0,i+p-n)}^{\min(i,p)}(-1)^{k+p}\binom{p}{k}\int_0^1 \frac{\binom{n-p}{i-k}\binom{n}{j}}{\binom{2n-p}{i+j-k}}B_{i+j-k,2n-p}(x)dx \\ &= \frac{n!}{(n-p)!}\sum_{k=\max(0,i+p-n)}^{\min(i,p)}(-1)^{k+p}\binom{p}{k}\frac{\binom{n-p}{i-k}\binom{n}{j}}{\binom{2n-p}{i+j-k}}\int_0^1 B_{i+j-k,2n-p}(x)dx \\ &= \frac{n!}{(n-p)!}\sum_{k=\max(0,i+p-n)}^{\min(i,p)}(-1)^{k+p}\binom{p}{k}\frac{\binom{n-p}{i-k}\binom{n}{j}}{\binom{2n-p}{i+j-k}}\frac{1}{2n-p+1}.\end{aligned} \tag{3.18}$$

□

Corollary 3.5.

$$\int_0^1 B_{i,n}^{(p)}(x)D_{j,n}(x)dx = \frac{n!}{(n-p)!}\sum_{k=\max(0,i+p-n)}^{\min(i,p)}(-1)^{k+p}\frac{\binom{p}{k}\binom{n-p}{i-k}\binom{p}{j-i+k}}{\binom{n}{j}}. \tag{3.19}$$

Proof. Using Theorem 3.1, we get

$$\int_0^1 B_{i,n}^{(p)}(x)D_{j,n}(x)dx = \int_0^1 \frac{n!}{(n-p)!}\sum_{k=\max(0,i+p-n)}^{\min(i,p)}(-1)^{k+p}\binom{p}{k}B_{i-k,n-p}(x)D_{j,n}(x)dx. \tag{3.20}$$

It follows immediately from (3.7) and (2.7) that

$$\begin{aligned}
\int_0^1 B_{i,n}^{(p)}(x) D_{j,n}(x)dx &= \frac{n!}{(n-p)!}\sum_{k=\max(0,i+p-n)}^{\min(i,p)}(-1)^{k+p}\binom{p}{k}\sum_{q=i-k}^{i-k+p}\frac{\binom{n-p}{i-k}\binom{p}{q-i+k}}{\binom{n}{q}} \\
&\quad\times\int_0^1 B_{q,n}(x)D_{j,n}(x)dx \\
&= \frac{n!}{(n-p)!}\sum_{k=\max(0,i+p-n)}^{\min(i,p)}(-1)^{k+p}\binom{p}{k}\sum_{q=i-k}^{i-k+p}\frac{\binom{n-p}{i-k}\binom{p}{q-i+k}}{\binom{n}{q}}\delta_{q,j} \\
&= \frac{n!}{(n-p)!}\sum_{k=\max(0,i+p-n)}^{\min(i,p)}(-1)^{k+p}\binom{p}{k}\frac{\binom{n-p}{i-k}\binom{p}{j-i+k}}{\binom{n}{j}}.
\end{aligned} \tag{3.21}$$

□

4. An Application for the Solution of High Even-Order Differential Equations

4.1. Bernstein Galerkin Method

Consider the solution of the differential equation

$$u^{(2m)} + \sum_{i=1}^{2m-1}\gamma_i u^{(i)} + \gamma_0 u = f(x), \quad x \in [0,1], \tag{4.1}$$

subject to the following boundary conditions

$$u^{(q)}(0) = 0, \quad u^{(q)}(1) = 0, \quad 0 \le q \le m-1. \tag{4.2}$$

Let us first introduce some basic notation which will be used in the sequel. We set

$$\begin{aligned}
S_N &= \{B_{0,N}(x), B_{1,N}(x), \dots, B_{N,N}(x)\}, \\
W_N &= \left\{v \in S_N\colon v^{(q)}(0) = v^{(q)}(1) = 0;\ 0 \le q \le m-1\right\},
\end{aligned} \tag{4.3}$$

then the Bernstein-Galerkin approximation to (4.1) is to find $u_N \in W_N$ such that

$$\left(u_N^{(2m)}, v_N\right) + \sum_{i=1}^{2m-1}\gamma_i\left(u_N^{(i)}, v_N\right) + \gamma_0(u_N, v_N) = (f, v_N), \quad \forall v_N \in W_N, \tag{4.4}$$

where $(u,v) = \int_I u(x)v(x)dx$ is the inner product in $L^2(I)$, and its norm will be denoted by $\|\cdot\|$.

It is of fundamental importance to note here that the crucial task in applying the Galerkin-spectral Bernstein approximations is how to choose an appropriate basis for W_N such that the linear system resulting from the Bernstein-Galerkin approximation to (4.4) is as simple as possible.

We can choose the basis functions $\phi_k(x)$ to be of the form

$$\phi_k(x) = B_{k,N}(x), \tag{4.5}$$

where $\phi_k(x) \in W_N$ for all $k = m, m+1, \dots, N-m$. The $2m$ boundary conditions lead to the first m and the last m expansion coefficients to be zero.

Therefore, for $N \geq 2m$, we have

$$W_N = \operatorname{span}\{\phi_m(x), \phi_{m+1}(x), \dots, \phi_{N-m}(x)\}. \tag{4.6}$$

It is now clear that (4.4) is equivalent to

$$\left(u_N^{(2m)}, \phi_k(x)\right) + \sum_{i=1}^{2m-1} \gamma_i \left(u_N^{(i)}, \phi_k(x)\right) + \gamma_0 (u_N, \phi_k(x)) = (f, \phi_k(x)), \quad \forall k = m, m+1, \dots, N-m. \tag{4.7}$$

Let us denote

$$\begin{gathered} f_k = (f, \phi_k(x)), \qquad \mathbf{f} = (f_m, f_{m+1}, \dots, f_{N-m})^T, \\ u_N(x) = \sum_{k=m}^{N-m} a_k \phi_k(x), \qquad \mathbf{a} = (a_m, a_{m+1}, \dots, a_{N-m})^T, \\ A = (a_{kj}), \quad B_i = \left(b_{kj}^i\right), \quad m \leq k, j \leq N-m. \end{gathered} \tag{4.8}$$

Then, (4.7) is equivalent to the following matrix equation

$$\left(A + \sum_{i=1}^{2m-1} \gamma_i B_i + \gamma_0 B_0\right)\mathbf{a} = \mathbf{f}, \tag{4.9}$$

where the elements of the matrices A, B_i, and B_0, $i = 1, 2, \dots, 2m-1$ are given explicitly using Corollary 3.4, as follows:

$$\begin{aligned} a_{kj} &= \left(B_{j,N}^{(2m)}, B_{k,N}\right) = \int_0^1 B_{j,N}^{(2m)}(x) B_{k,N}(x)dx \\ &= \frac{N!\binom{N}{k}}{(2N-2m+1)(N-2m)!} \sum_{r=\max(0,j-N+2m)}^{\min(j,2m)} (-1)^r \frac{\binom{2m}{r}\binom{N-2m}{j-r}}{\binom{2N-2m}{j+k-r}}, \end{aligned}$$

$$
\begin{aligned}
b_{kj}^{i} &= \left(B_{j,N}^{(i)}, B_{k,N}\right) = \int_0^1 B_{j,N}^{(i)}(x) B_{k,N}(x)dx \\
&= \frac{N!\binom{N}{k}}{(2N-i+1)(N-i)!} \sum_{r=\max(0,j-N+i)}^{\min(j,i)} (-1)^r \frac{\binom{i}{r}\binom{N-i}{j-r}}{\binom{2N-i}{j+k-r}}, \\
b_{kj}^{0} &= \left(B_{j,N}, B_{k,N}\right) = \int_0^1 B_{j,N}(x) B_{k,N}(x)dx \\
&= \frac{1}{2N+1} \frac{\binom{N}{j}\binom{N}{k}}{\binom{2N}{j+k}}.
\end{aligned}
\tag{4.10}
$$

4.2. Bernstein Petrov-Galerkin Method

The Petrov-Galerkin method generates a sequence of approximate solutions that satisfy a weak form of the original differential equation as tested against polynomials in a dual space. To describe this method and the full discretization more precisely, we introduce some basic notation. We set

$$
\begin{gathered}
W_N = \left\{ v \in S_N \colon u^{(q)}(0) = u^{(q)}(1) = 0,\ 0 \le q \le m-1 \right\}, \\
W_N^* = \left\{ v \in S_N^* \right\}.
\end{gathered}
\tag{4.11}
$$

Denoting by S_N and S_N^* the spaces of Bernstein polynomials of degree $\le N$ and dual Bernstein of degree $\le N$, then the Bernstein Petrov-Galerkin approximation to (4.1) is, to find $u_N \in W_N$ such that

$$
\left(u_N^{(2m)}, v_N\right) + \sum_{i=1}^{2m-1} \gamma_i \left(u_N^{(i)}, v_N\right) + \gamma_0 (u_N, v_N) = (f, v_N), \quad \forall v_N \in W_N^*. \tag{4.12}
$$

We choose the trial Bernstein functions to satisfy the underlying boundary conditions of the differential equation, and we choose the test dual Bernstein functions to satisfy the orthogonality condition. Consider the test and trial functions of expansion $\phi_k(x)$ and $\psi_k(x)$ to be of the form

$$
\begin{aligned}
\phi_k(x) &= B_{k,N}(x), \\
\psi_k(x) &= D_{k,N}(x),
\end{aligned}
\tag{4.13}
$$

where $\phi_k(x) \in W_N$ and $\psi_k(x) \in W_N^*$, for all $k = m, m+1, \ldots, N-m$. The $2m$ boundary conditions lead to the first m and the last m expansion coefficients to be zero.

Therefore, for $N \geq 2m$, we have

$$W_N = \operatorname{span}\{\phi_m(x), \phi_{m+1}(x), \dots, \phi_{N-m}(x)\},$$
$$W_N^* = \operatorname{span}\{\psi_m(x), \psi_{m+1}(x), \dots, \psi_{N-m}(x)\}, \tag{4.14}$$

and, accordingly, (4.12) is equivalent to

$$\left(u_N^{(2m)}, \psi_k(x)\right) + \sum_{i=1}^{2m-1} \gamma_i \left(u_N^{(i)}, \psi_k(x)\right) + \gamma_0 (u_N, \psi_k(x)) = (f, \psi_k(x)), \quad \forall k = m, m+1, \dots, N-m. \tag{4.15}$$

Let us denote

$$\hat{f}_k = (f, \psi_k(x)), \qquad \hat{\mathbf{f}} = \left(\hat{f}_m, \hat{f}_{m+1}, \dots, \hat{f}_{N-m}\right)^T,$$
$$u_N(x) = \sum_{n=m}^{N-m} v_n \phi_n(x), \qquad \mathbf{v} = (v_m, v_{m+1}, \dots, v_{N-m})^T. \tag{4.16}$$
$$\tilde{A} = (\hat{a}_{kj}), \quad \hat{B}_i = \left(\hat{b}_{kj}^i\right), \quad m \leq k, j \leq N-m, \; 0 \leq i \leq 2m-1.$$

Then, (4.15) is equivalent to the following matrix equation:

$$\left(\hat{A} + \sum_{i=1}^{2m-1} \gamma_i \hat{B}_i + \gamma_0 \hat{B}_0\right) \mathbf{v} = \hat{\mathbf{f}}. \tag{4.17}$$

If we take $\phi_k(x)$ and $\psi_k(x)$ as defined in (4.13) and if we denote $\hat{a}_{kj} = (\phi_j^{(2m)}(x), \psi_k(x))$ and $\hat{b}_{kj}^i = (\phi_j^{(i)}(x), \psi_k(x))$. Then, the elements $(\hat{a}_{kj})$, $(\hat{b}_{kj}^i)$, and $(\hat{b}_{kj}^0)$ for $m \leq k, j \leq N-m$, $i = 1, 2, \dots, 2m-1$ are given explicitly by using Corollary 3.5, as follows:

$$\begin{aligned}
\hat{a}_{kj} &= \left(B_{j,N}^{(2m)}, D_{k,N}\right) = \int_0^1 B_{j,N}^{(2m)}(x) D_{k,N}(x)\,dx \\
&= \frac{N!}{(N-2m)!\binom{N}{k}} \sum_{r=\max(0,j-N+2m)}^{\min(j,2m)} (-1)^r \binom{2m}{r} \binom{N-2m}{j-r} \binom{2m}{r-j+k}, \\
\hat{b}_{kj}^i &= \left(B_{j,N}^{(i)}, D_{k,N}\right) = \int_0^1 B_{j,N}^{(i)}(x) D_{k,N}(x)\,dx \\
&= \frac{N!}{(N-i)!\binom{N}{k}} \sum_{r=\max(0,j-N+i)}^{\min(j,i)} (-1)^r \binom{i}{r} \binom{N-i}{j-r} \binom{i}{r-j+k}, \\
\hat{b}_{kj}^0 &= \left(B_{j,N}, D_{k,N}\right) = \delta_{k,j}.
\end{aligned} \tag{4.18}$$

4.3. Using Coefficients of Differentiated Expansions

Here, we shall use Theorem 3.3 for the solution of the $2m$th-order differential (4.1)-(4.2). We approximate $u(x)$ by an expansion of Bernstein polynomials

$$u_N(x) = \sum_{i=0}^{N} a_{i,N} B_{i,N}(x). \tag{4.19}$$

We seek to determine $a_{i,N}$, $i = m, 1, \ldots, N - m$, using Petrov-Galerkin method. Note here that we set $a_i = a_{n-i} = 0$, $0 \leq i \leq m - 1$ to ensure that the boundary conditions (4.2) are satisfied. Since $u_N^{(2m)}(x)$ and $u_N^{(i)}(x)$ are polynomials of degree at most $N - 2m$ and $N - i$, respectively, we may write

$$u_N^{(s)}(x) = \sum_{i=m}^{N-m} a_{i,N}^{(s)} B_{i,N}(x), \tag{4.20}$$

where

$$a_{i,N}^{(s)} = s! \sum_{k=-s}^{s} \sum_{j=0}^{s} (-1)^{j+s} \binom{s}{j} \binom{i}{j+k} \binom{N-i}{s-j-k} a_{i-k,N}. \tag{4.21}$$

It is to be noted here that (4.21) is obtained by making use of relation (3.11). The coefficients $a_{i,N}$ are chosen so that $u_N(x)$ satisfies

$$u_N^{(2m)}(x) + \sum_{i=1}^{2m-1} \gamma_i u_N^{(i)}(x) + \gamma_0 u_N(x) = f(x). \tag{4.22}$$

Substituting (4.19) and (4.20) into (4.22), multiplying by $D_{m,N}$, and integrating over the interval $[0, 1]$ yield

$$a_{k,N}^{(2m)} + \sum_{i=1}^{2m-1} \gamma_i a_{k,N}^{(i)} + \gamma_0 a_{k,N} = f_k, \quad k = m, m+1, \ldots, N-m, \tag{4.23}$$

where

$$f_m = \int_0^1 f(x) D_{m,N} dx. \tag{4.24}$$

Thus, there are $(N-2m+1)$ equations for the $(N-2m+1)$ unknowns $a_{m,N}, a_{m+1,N}, \ldots, a_{N-m,N}$, in order to obtain a solution; it is only necessary to solve (4.23) with the help of (4.21) for the $(N - 2m + 1)$ unknowns coefficients $a_{i,N}$, $(m \leq i \leq N - m)$.

Table 1: E_p and E_r for $N = 2, 4, \ldots, 18$.

N	BGM (E_p)	BPGM (E_p)	BGM (E_r)	BPGM (E_r)
2	3.639×10^{-2}	1.494×10^{-1}	4.052×10^{-1}	9.598×10^{-1}
4	3.830×10^{-4}	1.534×10^{-2}	8.845×10^{-3}	1.428×10^{-1}
6	1.217×10^{-6}	6.194×10^{-4}	4.213×10^{-5}	7.191×10^{-3}
8	1.827×10^{-9}	1.264×10^{-5}	7.901×10^{-8}	1.732×10^{-4}
10	1.594×10^{-12}	1.547×10^{-7}	7.483×10^{-11}	2.425×10^{-6}
12	1.110×10^{-15}	1.260×10^{-9}	4.042×10^{-14}	2.214×10^{-8}
14	3.331×10^{-16}	7.326×10^{-12}	1.356×10^{-14}	1.422×10^{-10}
16	2.220×10^{-16}	3.197×10^{-14}	3.165×10^{-15}	6.763×10^{-13}
18	2.220×10^{-16}	4.163×10^{-16}	7.299×10^{-15}	1.189×10^{-14}

5. Numerical Results

We solve in this section several numerical examples by using the algorithms presented in the previous section. Comparisons between Bernstein Galerkin method (BGM), Bernstein Petrov-Galerkin method (BPGM), and other methods proposed in [21–24] are made. We consider the following examples.

Example 5.1. Consider the boundary value problem (see, [22])

$$u^{(2)}(x) - u(x) = \left(4 - 2x^2\right) \sin x + 4x \cos x, \quad x \in [0, 1], \tag{5.1}$$

subject to the boundary conditions $u(0) = u(1) = 0$, with the exact solution $u(x) = (x^2 - 1) \sin(x)$.

Table 1 lists the maximum pointwise error (E_p) and maximum absolute relative error (E_r) of $u - u_N$ using the BGM and BPGM with various choices of N. Table 1 shows that our methods have better accuracy compared with the quintic nonpolynomial spline method developed in [22]; it is also shown that, in the case of solving linear system of order 14, we obtain a maximum absolute error of order 10^{-16}. It is worthy noting here that the method of [22] gives the maximum absolute error 6.5×10^{-14} but by solving a linear system of order 64 instead of order 14 in our case.

Example 5.2. We consider the fourth-order two point boundary value problem (see, [21])

$$\begin{gathered} u^{(4)}(x) - 3u(x) = -2e^x, \quad x \in [0, 1], \\ u(0) = 1, \qquad u(1) = e, \qquad u'(0) = 1, \qquad u'(1) = e, \end{gathered} \tag{5.2}$$

with the analytical solution $u(x) = e^x$.

Table 2 lists the maximum pointwise error and maximum absolute relative error of $u - u_N$ using the BGM and BPGM with various choices of N. In Table 3, a comparison between the error obtained by using BGM, BPGM, the sinc-Galerkin, and modified decomposition methods (see, [21]) is displayed. This definitely shows that our methods are more accurate.

Table 2: (E_p) and (E_r) for $N = 4, 6, \ldots, 18$.

N	BGM (E_p)	BPGM (E_p)	BGM (E_r)	BPGM (E_r)
4	1.259×10^{-4}	3.134×10^{-4}	9.394×10^{-5}	1.684×10^{-4}
6	1.575×10^{-7}	8.646×10^{-6}	1.089×10^{-7}	4.477×10^{-6}
8	1.256×10^{-10}	1.246×10^{-7}	8.392×10^{-11}	6.341×10^{-8}
10	6.817×10^{-14}	1.121×10^{-9}	4.463×10^{-14}	5.637×10^{-10}
12	1.332×10^{-15}	6.944×10^{-12}	6.563×10^{-16}	3.465×10^{-12}
14	1.332×10^{-15}	3.286×10^{-14}	6.498×10^{-16}	1.594×10^{-14}
16	1.332×10^{-15}	1.332×10^{-15}	6.609×10^{-16}	6.193×10^{-16}
18	1.776×10^{-15}	1.776×10^{-15}	6.849×10^{-16}	7.707×10^{-16}

Table 3: Comparison between different methods for Example 5.2.

Error	BGM	BPGM	Sinc-Galerkin in [21]	Decomposition in [21]
E_p	1.8×10^{-15}	1.8×10^{-15}	3.7×10^{-9}	2.5×10^{-8}

Table 4: (E_p) and (E_r) for $N = 6, 8, \ldots, 18$.

N	BGM (E_p)	BPGM (E_p)	BGM (E_r)	BPGM (E_r)
6	4.037×10^{-6}	1.201×10^{-5}	6.889×10^{-6}	1.723×10^{-5}
8	3.314×10^{-9}	3.025×10^{-7}	4.796×10^{-9}	4.591×10^{-7}
10	1.973×10^{-12}	4.086×10^{-9}	2.755×10^{-12}	6.391×10^{-9}
12	1.110×10^{-15}	3.463×10^{-11}	2.104×10^{-15}	5.528×10^{-11}
14	4.441×10^{-16}	2.031×10^{-13}	7.014×10^{-16}	3.289×10^{-13}
16	4.441×10^{-16}	1.110×10^{-15}	1.693×10^{-15}	1.598×10^{-15}
18	4.441×10^{-16}	4.441×10^{-16}	1.563×10^{-15}	1.172×10^{-15}

Table 5: Comparison between the errors of different methods in Example 5.3.

Error	BGM	BPGM	Sinc-Galerkin [21]	Septic spline [23]	Decomposition [24]
E_p	4.4×10^{-16}	4.4×10^{-16}	9.2×10^{-6}	2.1×10^{-4}	1.3×10^{-4}
E_r	1.6×10^{-14}	1.2×10^{-16}	0.1×10^{-3}	1.8×10^{-3}	—

Example 5.3. Consider the sixth-order BVP (see, [21, 23, 24])

$$\begin{gathered} u^{(6)}(x) - u(x) = -6e^{x}, \quad x \in [0,1], \\ u(0) = 1, \qquad u'(0) = 0, \qquad u''(0) = -1, \\ u(1) = 0, \qquad u'(1) = -e, \qquad u''(1) = -2e, \end{gathered} \tag{5.3}$$

with the exact solution $u(x) = (1-x)e^{x}$.

Table 4 lists the maximum pointwise error and maximum absolute relative error of $u - u_N$ using BGM and BPG with various choices of N. Table 5 exhibits a comparison between the error obtained by using BGM, BPGM, and Sinc-Galerkin in [21], septic splines in [23] and modified decomposition in [24]. From this Table, one can check that our methods are more accurate.

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