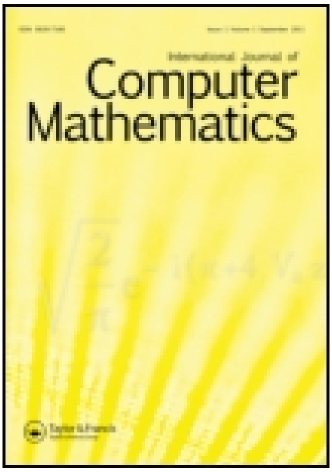

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Bézier curves and surfaces with shape parameters

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Bézier curves with n shape parameters and triangular Bézier surfaces with $3n(n+1)/2$ shape parameters are presented in this paper. The geometric significance of the shape parameters and the geometric properties of these curves and surfaces are discussed. The shapes of the curves and the surfaces can be modified intuitively, foreseeably and precisely by changing the values of the shape parameters.

Keywords: Bézier curves; triangular Bézier surfaces; shape parameters; basis functions; geometric significance

2000 AMS Subject Classification: 65D17; 65D18; 65U05

1. Introduction

Bézier curves and surfaces are widely used in CAGD, but once control points are given, the Bézier curve or surface is formed uniquely by the Bernstein basis functions and there are no means to adjust it anymore. The knot vector of B-spline curves and surfaces partially remedy this defect. As the well-known rational Bézier and B-spline curves and surfaces, the weights can be used to modify their shapes. The change of knots and weights is well-studied and the geometric effects are described in detail (Juhasz [7,8], Piegl [9]), but the possibilities of shape control by weights and especially by knots are highly restricted.

Several new approaches are given for the designing of curves. Chen and Wang [1,12] gave one kind of Bézier-like curve with one shape parameter, Han [4,5] presented the trigonometric polynomial curve with one shape parameter. The C-curves introduced by Zhang [15] also have one shape parameter and its properties were further discussed in [6]. The effect of the unique shape parameter is to push curves towards (or pulling away from) the control points.

Schmitt and Du [11] researched the bicubic rectangular Bézier patches with shape parameters. In [13], Wu and Xia extended Bézier curves and surfaces with multiple shape parameters to degree n. The curve basis functions are expressed as two categories via n, which is odd either or even. The surface basis functions are expressed as six categories. Due to the complicated expressions of basis functions, the number of shape parameters is hard to be counted, and the geometric significance of shape parameters cannot be clearly researched too.

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This paper simplifies and advances the work of Wu and Xia [13]. First, a Bézier curve with n shape parameters is given by one united expression, which is different from the two categories in [13], making the geometric significance of these shape parameters more clear. Second, using a method different to that of Wu and Xia, triangular Bézier surface with $3n(n+1)/2$ shape parameters is also constructed by one united expression. The number of shape parameters is different than those in [13] and the geometric effect of the shape parameters is better. With these simple expressions, in the both cases of curve and surface the geometric significance of the shape parameters is clearly discussed. Based on this result, the shape of the curve or surface can be modified intuitively, predictably and precisely by using one or several shape parameters.

2. Bézier curves with *n* shape parameters

2.1 *The construction of basis functions*

The original Bézier curves with control points P_i are defined by

$$P(t) = \sum_{i=0}^{n} B_{i,n}(t) P_i,$$

where

$$B_{i,n}(t) = \binom{n}{i} (1-t)^{n-i} t^i, \quad t \in [0,1]$$

is the ith Bernstein basis function of degree n.

Let

$$x_{i,1} = \frac{\lambda_i}{n-i+1}, \quad x_{i,2} = -\frac{\lambda_{i+1}}{i+1}, \quad \lambda_0 = \lambda_{n+1} = 0, \quad -(n-i+1) < \lambda_i < i, \\ i = 0, 1, \ldots, n. \tag{1}$$

DEFINITION 1 *The Bernstein basis functions of degree n with n shape parameters are defined by*

$$\hat{B}_{i,n}(t) = B_{i,n}(t)(1 + x_{i,1}(1-t) + x_{i,2}t), \quad t \in [0,1], \ \ i = 0, 1, \ldots, n. \tag{2}$$

$\lambda_i (i = 1, \ldots, n)$ *are called shape parameters.*

PROPOSITION 1 *The new basis functions $\hat{B}_{i,n}(t)$ hold the following properties*

(1) $\hat{B}_{i,n}(t) \geq 0, t \in [0,1]$;
(2) $\sum_{i=0}^{n} \hat{B}_{i,n}(t) \equiv 1$;
(3) $\hat{B}_{i,n}(t) = \hat{B}_{n-i,n}(1-t)$, *when* $\lambda_i = -\lambda_{n-i+1}$;
(4) $\hat{B}_{i,n}(t) = B_{i,n}(t)$, *when* $\lambda_i = 0$.

Proof Since $B_{i,n}(t) \geq 0$ and $B_{i,n}(t) = B_{n-i,n}(1-t)$, it is easy to prove (1.1) , (1.3) and (1.4) from Equations (1) and (2). And,

$$\sum_{i=0}^{n} \hat{B}_{i,n}(t) = \sum_{i=0}^{n} B_{i,n}(t) + \sum_{i=0}^{n} (x_{i,1}(1-t) + x_{i,2}t) B_{i,n}(t) \\ = 1 + \sum_{i=0}^{n} \frac{\lambda_i}{n-i+1}(1-t) B_{i,n}(t) - \sum_{i=0}^{n} \frac{\lambda_{i+1}}{i+1} t B_{i,n}(t)$$

$$= 1 + \sum_{i=1}^{n} \frac{\lambda_i}{n-i+1}(1-t)B_{i,n}(t) - \sum_{i=0}^{n-1} \frac{\lambda_{i+1}}{i+1} t B_{i,n}(t)$$

$$= 1 + \sum_{i=1}^{n} \left(\frac{\lambda_i}{n-i+1}(1-t)B_{i,n}(t) - \frac{\lambda_i}{i} t B_{i-1,n}(t) \right)$$

$$= 1 + \sum_{i=1}^{n} \lambda_i \left(\frac{1}{n-i+1}(1-t)B_{i,n}(t) - \frac{1}{i} t B_{i-1,n}(t) \right)$$

$$= 1.$$

So (1.2) is proved. ■

2.2 *Properties of curves*

DEFINITION 2 *Bézier curves of degree n with n shape parameters are defined by*

$$\hat{P}(t) = \sum_{i=0}^{n} \hat{B}_{i,n}(t) P_i, \quad t \in [0, 1],$$

where P_i is a collection of control points in $\mathbb{R}^m$.

PROPOSITION 2 *The Bézier curves with n shape parameters hold the following properties*

(1) *interpolation at the endpoint and tangent at the end edge;*
(2) *convex hull property;*
(3) *geometric invariability and affine invariability;*
(4) *symmetry,* $\tilde{P}(1-t) = \sum_{i=0}^{n} \hat{B}_{i,n}(1-t) P_{n-i} = \hat{P}(t)$ *when* $\lambda_i = -\lambda_{n-i+1}$.
(5) $\hat{P}(t) = P(t)$ *when all* $\lambda_i = 0$.

With the properties of the basis functions and simple computations, Proposition 2 can be proved easily. All these properties of Bézier curves with n shape parameters are analogous to the original Bézier curves. Here,we focus on the effect of the shape parameters.

PROPOSITION 3 *Keeping $\lambda_i (i \neq j)$ constant, changing $\lambda_j (j = 1, \dots, n)$ from value λ_j^1 to value λ_j^2 in its domain, changing $\hat{P}(t)$ from $\hat{P}^1(t)$ to $\hat{P}^2(t)$ correspondingly, let $\Delta\lambda_j = \lambda_j^1 - \lambda_j^2$, $\Delta\hat{P}(t) = \hat{P}^1(t) - \hat{P}^2(t)$; then*

$$\Delta\hat{P}(t) = \Delta\lambda_j c_j (P_j - P_{j-1}),$$

where

$$c_j = \frac{n!}{j!(n-j+1)!}(1-t)^{n-j+1} t^j.$$

Proof Expanding $\hat{P}(t)$, the coefficient of λ_j is

$$\frac{n!}{j!(n-j+1)!}(1-t)^{n-j+1} t^j (P_j - P_{j-1}),$$

hence

$$\Delta\hat{P}(t) = \hat{P}^1(t) - \hat{P}^2(t) = (\lambda_j^1 - \lambda_j^2) \frac{n!}{j!(n-j+1)!}(1-t)^{n-j+1} t^j (P_j - P_{j-1}).$$

The proposition is proved. ■

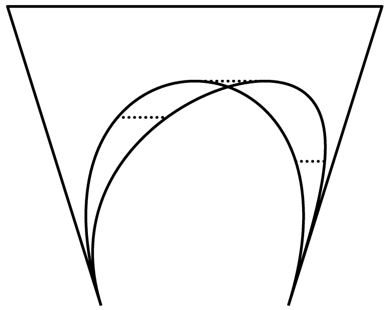

Figure 1. The intuitive effect of a shape parameter.

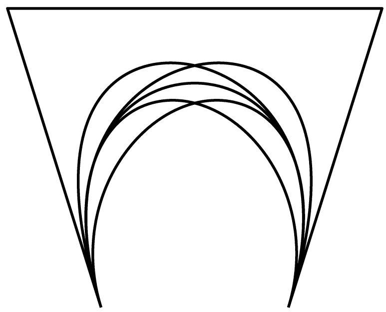

Figure 2. Five curves with different shape parameters of the same control points.

Proposition 3 shows us the intuitive effect of the shape parameters. As shape parameter λ_j changes from λ_j^1 to λ_j^2, the point $\hat{P}(t)$ on the curve moves along the direction of $\overrightarrow{P_{j-1}P_j}$(or $\overrightarrow{P_jP_{j-1}}$), and with the distance $\|\Delta\lambda_j c_j \overrightarrow{P_{j-1}P_j}\|$ (Figure 1).

Earlier methods affect the shape similarly to the rational curves, that is the curve is pushed towards (or pulled away from) the control points, while the shape parameter is changed. During that procedure, the manner in which each point on the curve moves is not specific. However, in our new method, the curve moves along the direction of the control legs, which is somehow a new dimension of shape control. With Proposition 3, the manner in which each point on the curve moves is intuitively foreseeable and can be described precisely (See the dotted line segment in Figure 1).

Especially, when all $\lambda_i = 0$, then $\hat{B}_{i,n}(t) = B_{i,n}(t)$, this curve is just the original Bézier curve (The middle curve in Figure 2). Moreover, with different shape parameters, the curve shape can be modified variously (Figure 2). This characteristic makes such curves more convenient for interactive modeling.

3. Bézier surfaces with shape parameters

3.1 *Rectangular surfaces*

The rectangular surfaces of tensor product with control points $P_{i,j}$ are defined by

$$\hat{S}(\mu, \nu) = \sum_{i=0}^{n} \sum_{j=0}^{m} \hat{B}_{i,n}(\mu) \hat{B}_{j,m}(\nu) P_{i,j},$$

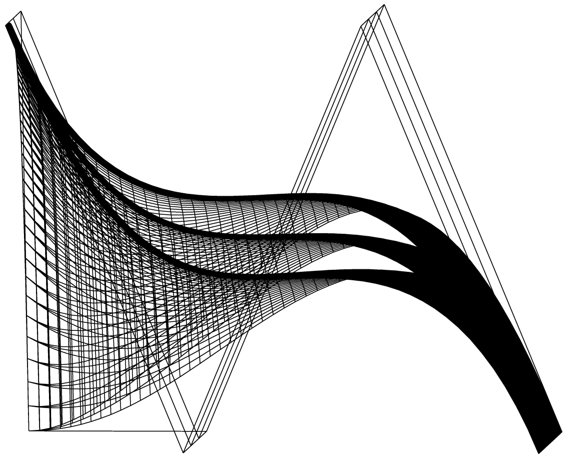

Figure 3. Three rectangular surfaces with different shape parameters of a single control-points-net.

where $\hat{B}_{i,n}(\mu) and \hat{B}_{j,m}(\nu)$ are defined as in Equation (2). The surface of degree $n \times m$ contains $n + m$ shape parameters. Due to the style of tensor product, their properties can be researched similarly to the curves; here we only give one example (Figure 3). Let us now pay more attention to the triangular surfaces.

3.2 *Triangular surfaces*

3.2.1 *Construction of basis functions*

The original triangular surfaces with control points $P_{i,j,k}$ are defined by

$$T(\mu, \nu) = \sum_{i+j+k=n} B_{i,j,k}(\mu, \nu) P_{i,j,k},$$

where

$$B_{i,j,k}(\mu, \nu) = \frac{n!}{i!j!k!} \mu^i \nu^j \omega^k,$$

$$\mu, \nu \geq 0, \quad \mu + \nu + \omega = 1, \quad i, j, k \in \mathbb{N}, \quad i + j + k = n$$

are the triangular Bernstein basis functions of degree n. Their positions are displayed as Figure 4 ($n = 3$) and Figure 5.

Figure 5 is the map of basis functions (Figure 4 is the case of $n = 3$); the position of each basis function is identified by its subscript $\{i, j, k\}$. (The line segment fixed by two adjoining basis is called a leg; for example, $\overline{B_{i,j,k} B_{i-1,j,k+1}}$ etc.) It is therefore easy to identify the number of legs: *e.g.* there are $3n(n + 1)/2$ legs in Figure 5 (18 legs in Figure 4).

Though the construction of new basis functions of triangular surfaces is a little more complicated than that of curves, as long as it is contrasted with the map of basis functions (Figure 5), the entire sequence of steps would be clear and intuitive.

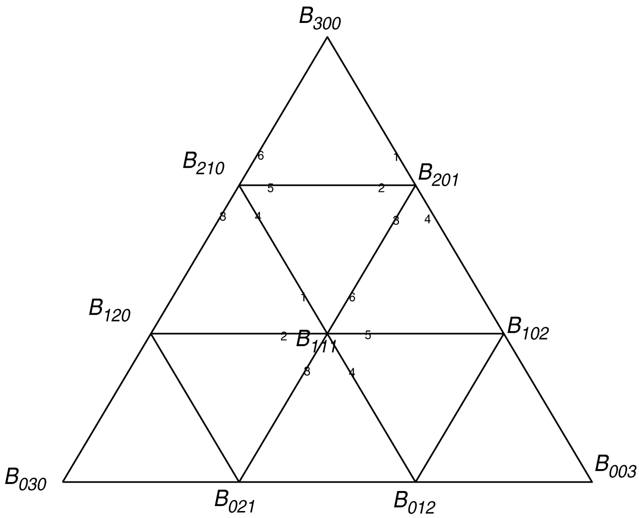

Figure 4. The map of basis functions ($n = 3$).

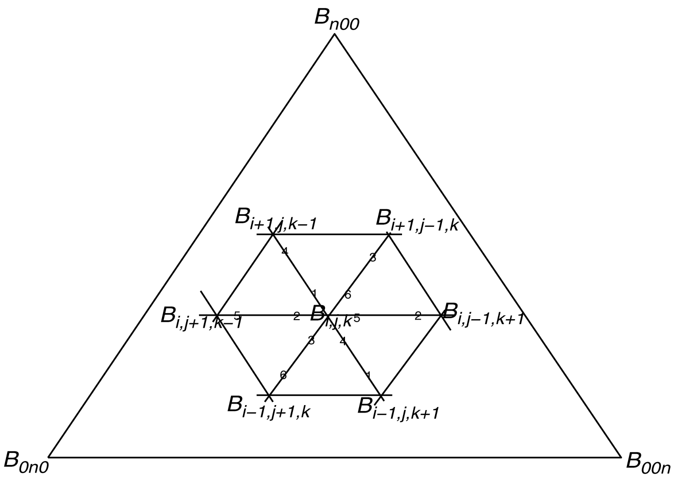

Figure 5. The map of basis functions.

First, let us construct a linear function $f_{i,j,k}$ attached to $B_{i,j,k}$. Let

$$f_{i,j,k}(\mu,\nu,\omega) = 1 + \lambda^1_{i,j,k}\frac{\mu}{i+1} + \lambda^2_{i,j,k}\frac{\nu}{j+1} + \lambda^3_{i,j,k}\frac{\nu}{j+1} + \lambda^4_{i,j,k}\frac{\omega}{k+1} + \lambda^5_{i,j,k}\frac{\omega}{k+1} + \lambda^6_{i,j,k}\frac{\mu}{i+1} \tag{3}$$

where the subscript $\{i, j, k\}$ still means its position, the superscript $l(l = 1, 2, 3, 4, 5, 6)$ can be considered as a different direction around $B_{i,j,k}$ (Figure 5); and the constants $\lambda^l_{i,j,k}$ are subjected to the following conditions:

(a) $\lambda^1_{i,j,k} = \lambda^2_{i,j,k} = 0$ when $k = 0$, $\lambda^3_{i,j,k} = \lambda^4_{i,j,k} = 0$ when $i = 0$, $\lambda^5_{i,j,k} = \lambda^6_{i,j,k} = 0$ when $j = 0$;
(b) $\lambda^1_{i,j,k}, \lambda^6_{i,j,k} > -(i+1)/2$, $\lambda^2_{i,j,k}, \lambda^3_{i,j,k} > -(j+1)/2$, $\lambda^4_{i,j,k}, \lambda^5_{i,j,k} > -(k+1)/2$; and
(c) $\lambda^1_{i,j,k} = -\lambda^4_{i+1,j,k-1}$, $\lambda^2_{i,j,k} = -\lambda^5_{i,j+1,k-1}$, $\lambda^3_{i,j,k} = -\lambda^6_{i-1,j+1,k}$.

What do these mean and why are they needed? We will explain it immediately after Definiton 3.

DEFINITION 3 *The triangular Bernstein basis functions of degree n with $3n(n+1)/2$ shape parameters are defined by*

$$\hat{B}_{i,j,k}(\mu,\nu) = B_{i,j,k}(\mu,\nu) f_{i,j,k}(\mu,\nu,\omega), \tag{4}$$

where

$$\mu, \nu \geq 0, \quad \mu+\nu+\omega = 1, \quad i,j,k \in \mathbb{N}, \quad i+j+k = n$$

and $\lambda^l_{i,j,k}$ are called shape parameters.

Now, let us explain the meaning of the conditions. At the same time, we will obtain the properties of the basis functions defined by Equation (4).

For (a), when $k=0$, for example, $B_{2,1,0}$ in Figure 4, $B_{i,j,k}$ is on the outside edge $\overline{B_{n,0,0}B_{0,n,0}}$, its directions 1 and 2 do not exist (Figures 4 and 5). Then let $\lambda^1_{i,j,k} = \lambda^2_{i,j,k} = 0$ when $k=0$ is reasonable, and so also for the other two.

For (b), because $f_{i,j,k}$ can be rewritten as

$$f_{i,j,k} = 1 + (\lambda^1_{i,j,k} + \lambda^6_{i,j,k})\frac{\mu}{i+1} + (\lambda^2_{i,j,k} + \lambda^3_{i,j,k})\frac{\nu}{j+1} + (\lambda^4_{i,j,k} + \lambda^5_{i,j,k})\frac{\omega}{k+1},$$

the condition (b) is adequate for $f_{i,j,k} > 1 + (-\mu) + (-\nu) + (-\omega) = 0$; then $\hat{B}_{i,j,k} \geq 0$ is also true.

For (c), at first, let us consider

$$\sum_{i+j+k=n} \hat{B}_{i,j,k} = \sum_{i+j+k=n} B_{i,j,k} f_{i,j,k} = \sum_{i+j+k=n} B_{i,j,k} + \Lambda = 1 + \Lambda,$$

where

$$\Lambda = \sum_{i+j+k=n} B_{i,j,k}\left(\lambda^1_{i,j,k}\frac{\mu}{i+1} + \lambda^2_{i,j,k}\frac{\nu}{j+1} + \lambda^3_{i,j,k}\frac{\nu}{j+1} + \lambda^4_{i,j,k}\frac{\omega}{k+1}\right.$$
$$\left. + \lambda^5_{i,j,k}\frac{\omega}{k+1} + \lambda^6_{i,j,k}\frac{\mu}{i+1}\right).$$

Contrasting each term in Λ with the leg in Figure 5, and together with condition (a), after rearranging the terms in Λ, we have

$$\Lambda = \Lambda_1 + \Lambda_2 + \Lambda_3,$$

where

$$\Lambda_1 = \sum_{\substack{i+j+k=n \\ k\neq 0}} \left(B_{i,j,k}\lambda^1_{i,j,k}\frac{\mu}{i+1} + B_{i+1,j,k-1}\lambda^4_{i+1,j,k-1}\frac{\omega}{k}\right),$$

$$\Lambda_2 = \sum_{\substack{i+j+k=n \\ k\neq 0}} \left(B_{i,j,k}\lambda^2_{i,j,k}\frac{\nu}{j+1} + B_{i,j+1,k-1}\lambda^5_{i,j+1,k-1}\frac{\omega}{k}\right),$$

$$\Lambda_3 = \sum_{\substack{i+j+k=n \\ i\neq 0}} \left(B_{i,j,k}\lambda^3_{i,j,k}\frac{\nu}{j+1} + B_{i-1,j+1,k}\lambda^6_{i-1,j+1,k}\frac{\mu}{i}\right).$$

Furthermore

$$\Lambda_1 = \sum_{\substack{i+j+k=n\\k\neq 0}} \left(\lambda^1_{i,j,k}\frac{n!}{i!j!k!}\mu^i\nu^j\omega^k\frac{\mu}{i+1} + \lambda^4_{i+1,j,k-1}\frac{n!}{(i+1)!j!(k-1)!}\mu^{i+1}\nu^j\omega^{k-1}\frac{\omega}{k}\right)$$
$$= \sum_{\substack{i+j+k=n\\k\neq 0}} \frac{n!}{(i+1!)j!k!}\mu^{i+1}\nu^j\omega^k\left(\lambda^1_{i,j,k} + \lambda^4_{i+1,j,k-1}\right)$$

with the first equation in Condition (c); so

$$\Lambda_1 = 0.$$

And $\Lambda_2 = \Lambda_3 = 0$ can be proved in the same way. Then $\Lambda = 0$, and $\sum_{i+j+k=n} \hat{B}_{i,j,k} = 1$.

Hence, the following proposition is obtained.

Proposition 4 *The basis functions $\hat{B}_{i,j,k}(\mu,\nu)$ hold the following properties*

(1) $\hat{B}_{i,j,k}(\mu,\nu) \geq 0$;
(2) $\sum_{i+j+k=n} \hat{B}_{i,j,k}(\mu,\nu) \equiv 1$; *and*
(3) $\hat{B}_{i,j,k}(\mu,\nu) = B_{i,j,k}(\mu,\nu)$ *when all* $\lambda^l_{i,j,k} = 0$.

Two points are worth noticing. The first, though there are two shape parameters on each leg in Figure 5 and the number of legs is $3n(n+1)/2$, with the Condition (c), the shape parameter amount is $3n(n+1)/2$.

The second, together with (b) and (c), the domain of each shape parameter is

$$-\frac{i+1}{2} < \lambda^1_{i,j,k} < \frac{k}{2}, \quad -\frac{j+1}{2} < \lambda^2_{i,j,k} < \frac{k}{2}, \quad -\frac{j+1}{2} < \lambda^3_{i,j,k} < \frac{i}{2},$$
$$-\frac{k+1}{2} < \lambda^4_{i,j,k} < \frac{i}{2}, \quad -\frac{k+1}{2} < \lambda^5_{i,j,k} < \frac{j}{2}, \quad -\frac{i+1}{2} < \lambda^6_{i,j,k} < \frac{j}{2}.$$

3.2.2 *The properties of surfaces*

Definition 4 *Given control points $P_{i,j,k}$, the triangular Bézier surfaces with $3n(n+1)/2$ shape parameters are defined by*

$$\hat{T}(\mu,\nu) = \sum_{i+j+k=n} \hat{B}_{i,j,k}(\mu,\nu)P_{i,j,k}.$$

Proposition 5 *The triangular Bézier surfaces $\hat{T}(\mu,\nu)$ hold the following properties*

(1) *interpolation at the corner-point and tangent at the corner-plane (in fact, the three edges of the surface are the kind of curves in Definition 2)*;
(2) *convex hull property*;
(3) *geometric invariability and affine invariability*; *and*
(4) $\hat{T}(\mu,\nu) = T(\mu,\nu)$ *when all* $\lambda^l_{i,j,k} = 0$.

These properties of the surfaces can be easily obtained from the properties of the basis functions. The focus here remains on the function of shape parameters.

For convenience, let us investigate the geometric significance of shape parameter $\lambda^l_{i,j,k}$ when $l = 1$. In the other cases, the result is analogical.

PROPOSITION 6 *Let shape parameter $\lambda^1_{i,j,k}$ change from value λ_1 to λ_2 (simultaneously $\lambda^4_{i+1,j,k-1}$ changes; from value $-\lambda_1$ to $-\lambda_2$) keeping other shape parameters unchanging, the surface $\hat{T}(\mu, \nu)$ is moved from $\hat{T}_1(\mu, \nu)$ to $\hat{T}_2(\mu, \nu)$; simultaneously $\triangle\lambda = \lambda_1 - \lambda_2$, $\triangle\hat{T} = \hat{T}_1(\mu, \nu) - \hat{T}_2(\mu, \nu)$; then*

$$\triangle\hat{T} = \triangle\lambda c^*(P_{i,j,k} - P_{i+1,j,k-1}). \tag{5}$$

c^* is a value correlative with $i, j, k, \mu, \nu, \omega$.

Proof Straightforward computation of $\hat{T}_1(\mu, \nu) - \hat{T}_2(\mu, \nu)$ gives

$$\triangle\hat{T} = (\lambda_1 - \lambda_2)\frac{n!}{(i+1)!j!k!}\mu^{i+1}\nu^i\omega^k(P_{i,j,k} - P_{i+1,j,k-1}) = \triangle\lambda c^*(P_{i,j,k} - P_{i+1,j,k-1}). \qquad \blacksquare$$

Proposition 6 shows us the specific geometric significance of shape parameters. when $\triangle\lambda > 0$, the surface $\hat{T}(\mu, \nu)$ moves along the direction of $\overrightarrow{P_{i+1,j,k-1}P_{i,j,k}}$(if $\triangle\lambda < 0$, the direction is reverse), and with the distance $||\triangle\lambda c^*(P_{i+1,j,k-1} - P_{i,j,k})||$ (Figure 7).

And this proposition also reveals that making use of shape parameters, the surface shape can be modified along the direction of each leg on the control-points-net, and with explicit distance. By the earlier methods, the surface shape is changed by pushing (or pulling) along one uncertain direction; the specific manner in which each point moves on the surface is not clear. In this new method, because the direction of the leg on control-points-net is displayed distinctly, the manner in which each point moves on the surface is clear and intuitively foreseeable.

Especially, when all shape parameters equal zero, the surface is just the original Bézier surface (the blue one in Figure 7). With different shape parameters, the suface shape can be modified variously (Figure 7). This characteristic makes such surfaces more useful for interactive modeling.

4. Examples

Two concrete examples are given in this section. The first is about curves, and the other is about triangular surfaces.

Example 1 (Figure 6) By cubic Bézier curves with shape parameters of same control points, the letter "**s**" is designed in several styles. The common control points are {0, 3, 0}, {5, 3, 0}, {0, 0, 0}, {5, 0, 0}; the shape parameters' values are displayed in Table 1.

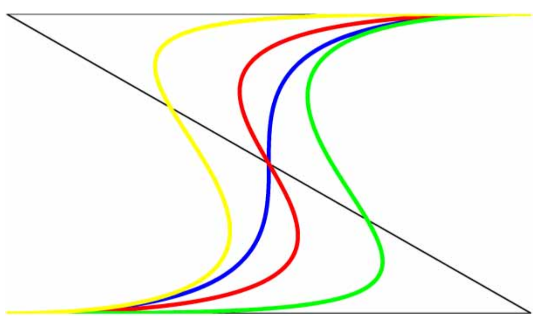

Figure 6. Four cubic Bézier curves with different shape parameters of the same control points.

Table 1. The values of the shape parameters.

	λ_1	λ_2	λ_3
Blue	0	0	0
Red	0.9	0	−0.9
Green	0.9	1.9	−0.9
Yellow	0.9	−1.9	−0.9

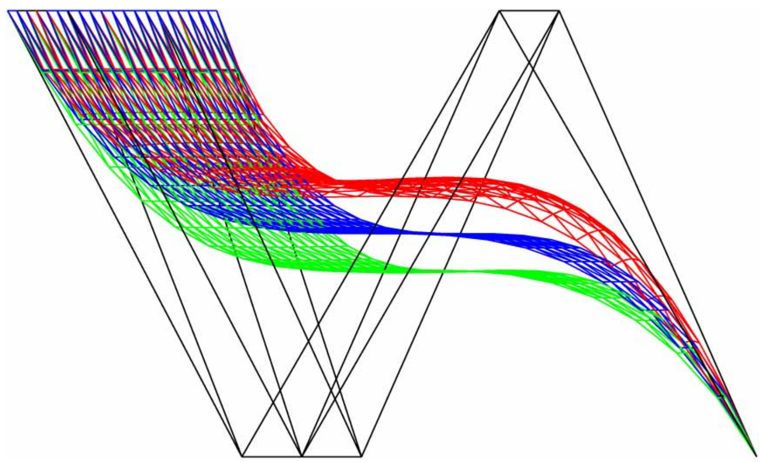

Figure 7. Three cubic triangular Bézier surfaces with different shape parameters of a single control-points-net.

Table 2. The values of the shape parameters.

	$\lambda^4_{210} = -\lambda^1_{111}$	$\lambda^3_{210} = -\lambda^6_{120}$	$\lambda^4_{201} = -\lambda^1_{102}$	$\lambda^3_{201} = -\lambda^6_{111}$	others
Red	0.9	0.9	0.9	0.9	0
Blue	0	0	0	0	0
Green	−0.49	−0.9	−0.9	−0.49	0

Example 2 (Figure 7) By cubic triangular Bézier surfaces with shape parameters of a single control-points-net, the triangular face of a chair is designed in three styles. The common control points are {0, 0, 0}, {2, 1, 1}, {4, 2, 0}, {6, 2.5, 1}, {1, 2, 1}, {3, 3, 0}, {5, 3.5, 1}, {2, 4, 0}, {3.5, 5, 1}, {2.5, 6, 1}; the shape parameters' values are displayed in Table 2.

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