Peer Effects and the Reflection Problem

Econometric Methods for Social Spillovers and Networks

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Peer Effects

- Social psychologists and sociologists have long studied the role peers play in shaping individual beliefs and behaviors.
- Economists have been especially interested in the role of peers and neighborhoods in the acquisition of human capital and in amplifying inequality (e.g., Loury, 2002; Ioannides and Loury, 2004)
- Manski's (1993) "reflection problem" attempted to codify a 1980s empirical social science literature on peer and neighborhood effects.
- While the paper is elegant and clear, it has nevertheless been misunderstood by much of the discipline.
  - Caution is warranted when reading the empirical literature, where many "identification" arguments are problematic and occasionally even incorrect.
  - The quality and credibility of empirical peer effects research remains uneven. Extant evidence on such effects is debated (e.g., Angrist, 2014).

Manski (1993, Review of Economic Studies)

- Each unit in the population of interest is characterized by the quadruple:

$$(Y, X, Z, U) \in \mathbb{R}^1 \times \mathbb{R}^J \times \mathbb{R}^K \times \mathbb{R}^1$$

with

  - Y the action of interest,
  - X attributes defining a unit's reference group (e.g., X might be vector of school/neighborhood indicator variables).
  - Z a vector of individual attributes
  - U individual-level heterogeneity (unobserved)

- Manski (1993) posits that Y varies according to the following "linear-in-means" model:

$$Y = \alpha + \beta \mathbb{E}[Y|X] + \mathbb{E}[Z|X]'\gamma + Z'\eta + U, \;\; \mathbb{E}[U|X, Z] = X'\delta. \quad (1)$$

Some comments of terminology

- Manski (1993) does not give an explicit game-theoretic formulation for his model.
- Nevertheless, from his introduction and citations it is clear that he views (1) as a reaction function (or best-reply function) *not* a regression function.
- Confusion about the implications of this fundamental distinction persist to this day (cf., Angrist, 2014).

Some comments of terminology (continued)

- A couple of suggestions before we delve into details:
  - refer to Y as an action or behavior not an outcome
    - we posit that agents' behaviors are influenced by the behaviors of their peers
    - observation: a "test score" is not a behavior, but amount of time spent studying is
  - keep in mind what you may already understand about the distinction between say, the regression function for quantity give price and a demand schedule.
  - using the word "action" instead of "outcome" helps to clarify thinking as well as suggests limitations to the scope of application.

<u>Micro-foundations</u>

- See references in introduction of Manski (1993) or Jackson and Zenou (2015).
- The utility agent i receives from from choosing action $Y_i = y_i$ is give by

$$u_i\left(y_i, \mathbb{E}\left[\left.Y_j\right| X_i\right]; Z_i, \mathbb{E}\left[\left.Z_j\right| X_i\right]\right) = v_i\left(Z_i, \mathbb{E}\left[\left.Z_j\right| X_i\right]\right) y_i - \frac{1}{2}y_i^2 + \beta\mathbb{E}\left[\left.Y_j\right| X_i\right] y_i$$

- Assume that $|\beta| < 1$ and define $v_i\left(Z_i, \mathbb{E}\left[\left.Z_j\right| X_i\right]\right)$ as

$$v_i\left(Z_i, \mathbb{E}\left[\left.Z_j\right| X_i\right]\right) = \alpha + Z_i'\eta + \mathbb{E}\left[\left.Z_j\right| X_i\right]'\gamma + U_i$$

- <u>Comment:</u> alternative is provided by quadratic "conformist" preferences (e.g., Akerlof, 1997); useful for thinking about stigma, conformity and " peer pressure".

Endogenous effects

- **endogenous**: the marginal utility associated with an increase in y_i is increasing in the average action of one's peers, $\mathbb{E}\left[\left.Y_j\right| X_i\right]$:

$$\frac{\partial^2 u_i}{\partial y_i \partial \mathbb{E}\left[\left.Y_j\right| X_i\right]} = \beta > 0.$$

- The returns to effort may be increasing in the average effort of one's peers (or teammates).
- If everyone in class is talking, the benefits of remaining quiet and trying to listen to the instructor are low. You might as well talk too!
- Many economists became interested in endogenous effects after reading Manski (1993).
- Unfortunately this interest was accompanied by a tremendous amount of confusion.

Exogenous or contextual Effects

- **exogenous:** the marginal utility associated with an increase in y_i varies with peer attributes $\mathbb{E}[Z_j | X_i]$:

$$\frac{\partial^2 u_i}{\partial y_i \partial \mathbb{E}[Z_j | X_i]} = \gamma.$$

- Highly educated parents' ($Z_i \uparrow$) may encourage greater effort at school, but the education level of peers' parents, $\mathbb{E}[Z_j | X_i]$, may be helpful too.

- We can think of $\mathbb{E}[Z_j | X_i]$ as a measure of communal social capital or "collective efficacy". See Coleman (1988) and Loury (2002).

- Exogenous/contextual effects have been widely studied in sociology (cf., Sharkey and Faber, 2014).

Correlated effects

- **correlated:** the marginal utility associated with an increase in y_i varies with (latent) variables which vary at the group level.
- Manski's (1993) notation is more general than the typical use case, but I will stick with using his notation for now (at the risk of some confusion).
- Write, using (1) above,

$$U_i = X_i'\delta + V_i, \ \mathbb{E}\left[V_i | X_i, Z_i\right] = 0.$$

- If X_i is a vector of group indicators then δ is a vector of group fixed-effects. These effects could capture differences in, say, school quality which have nothing to do with peer composition (e.g., the principal and teachers might be very good at some schools).
- We have

$$\frac{\partial^2 u_i}{\partial y_i \partial \left(X_i'\delta\right)} = 1 > 0.$$

- Students might exert more effort because their teachers inspire them to do so.

Reaction function

- Manski's (1993) has in mind a setting where peer groups are large (neighborhoods/schools).
- We'll look at the implications of smaller peer group settings in a moment.
- The first order conditional for utility maximization is

$$\alpha + Z_i'\eta + \mathbb{E}\left[Z_j|\, X_i\right]'\gamma + X_i'\delta + V_i - Y_i + \beta\mathbb{E}\left[Y_j|\, X_i\right] = 0. \qquad (2)$$

- This condition holds for every agent $i \in \mathbb{N}$ in the (large) peer group.
- Solving for Y_i yields the linear-in-means model

$$Y_i = \alpha + \beta\mathbb{E}\left[Y_j|\, X_i\right] + \mathbb{E}\left[Z_j|\, X_i\right]'\gamma + Z_i'\eta + X_i'\delta + V_i.$$

- The linear-in-means model is not a "regression function", it is a best-reply function! Internalizing this observation will save you a lot of confusion.

Equilibrium

- If everyone is "best responding", then – in equilibrium – the FOC (2) simultaneous holds for every agent $i \in \mathbb{N}$.
- If we average these first order conditions across all members of group $X = x$ we get

$$\mathbb{E}\left[\alpha + Z_i'\eta + \mathbb{E}\left[Z_j|\, X_i\right]'\gamma + X_i'\delta + V_i - Y_i + \beta\mathbb{E}\left[Y_j|\, X_i\right]\Big|\, X_i = x\right] = 0$$
$$\alpha + \mathbb{E}\left[Z_j|\, X_i = x\right]'(\eta + \gamma) + x'\delta - (1-\beta)\,\mathbb{E}\left[Y_j|\, X_i = x\right] = 0$$

- Solving for the average action in the group yields

$$\mathbb{E}\left[Y_j|\, X_i = x\right] = \frac{\alpha}{1-\beta} + \mathbb{E}\left[Z_j|\, X_i = x\right]'\frac{\eta+\gamma}{1-\beta} + x'\frac{\delta}{1-\beta}$$

- Plugging this back into $i's$ FOC and solving for Y_i gives an equilibrium action of

$$Y_i = \frac{\alpha}{1-\beta} + \mathbb{E}\left[Z_j|\, X_i\right]'\left(\frac{\gamma+\beta\eta}{1-\beta}\right) + Z_i'\eta + X_i'\left(\frac{\delta}{1-\beta}\right) + V_i$$

for every agent $i \in \mathbb{N}$.

(Non-)Identification

- The main result is Proposition 1 of Manski (1993) and (especially) the Corollary.
- In the leading use case X_i is a vector of group membership indicator variables. Groups are assumed mutually exclusive such that each agent belongs to one, and only one, reference group.
- In this case we can write

$$\mathbb{E}\left[\left. Z_j \right| X_i\right] = X_i' \pi$$

with π corresponding to the vector of group means of Z_i across all the groups.

- Since $\mathbb{E}\left[\left. Z_j \right| X_i\right]$ is a linear function of X_i, there is no hope of distinguishing between peer effects ($\frac{\gamma+\beta\eta}{1-\beta} \neq 0$) on the one hand, and correlated effects or group-level heterogeneity on the other ($\delta \neq 0$).
- One can read Angrist (2014) as heuristic reprise of the Corollary to Proposition 1 in Manski (1993).

Finite-sized peer groups

- In the classic peer effects empirical paper agents belong to one of $c = 1, \ldots, N$ mutual exclusive reference groups (e.g., a classroom, school or neighborhood). In each group c there are T_c individuals (group sizes may vary).
- Algebra harder, but more applicable and easier to develop some key ideas/intuitions.
- Define the following block diagonal **adjacency matrix**

$$\mathbf{D} = \begin{pmatrix} \iota_{T_1}\iota_{T_1}' - I_{T_1} & 0 & \cdots & 0 \\ 0 & \iota_{T_2}\iota_{T_2}' - I_{T_2} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \iota_{T_N}\iota_{T_N}' - I_{T_N} \end{pmatrix}$$

- Here ι_T is a $T \times 1$ vector of ones and I_T the $T \times T$ identity matrix.
- We say i is connected to j if $D_{ij} = 1$ and that they are not connected otherwise.
- In this set-up all agents in the same group are "connected". Agents are not connected to themselves (no so called self-loops).

Finite-sized peer groups (continued)

- The row-normalized adjacency matrix is

$$\mathbf{G} = \begin{pmatrix} \frac{1}{T_1-1}\left(\iota_{T_1}\iota_{T_1}' - I_{T_1}\right) & 0 & \cdots & 0 \\ 0 & \frac{1}{T_2-1}\left(\iota_{T_2}\iota_{T_2}' - I_{T_2}\right) & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \frac{1}{T_N-1}\left(\iota_{T_N}\iota_{T_N}' - I_{T_N}\right) \end{pmatrix}$$

- We divide each element by its row sum. Note that $\mathbf{G}$ is row-stochastic (that observation will be helpful later).
- Let $n = \sum_{c=1}^{N} T_c$ be the total number of agents.
- Let $\mathbf{Y} = (Y_1, \ldots, Y_n)'$ be the $n \times 1$ vector of (equilibrium) agent actions.
  - We assume agents are ordered according to their peer groups in the same way as G is structured.
- Let $\mathbf{X}$ be a corresponding $n \times K$ matrix of agent-specific regressors.

Leave-own-out means

- Let $n(i) = \{j : D_{ij} = 1\}$ be the set of individuals in i's reference group (other than himself).
- Let $g(i) = c$ if individual i is in group c, etc.
- Let $\mathbf{G}_i$ be the i^{th} row of $\mathbf{G}$; note that

$$\mathbf{G_i Y} = \frac{1}{T_{g(i)} - 1} \sum_{j \neq i} D_{ij} Y_j = \bar{Y}_{n(i)}$$

  or the average action of agent i's peers.
- This is the so-called "leave-own-out" mean.

Utility

- Assume that the utility agent i receives from action profile $\mathbf{y}$, given peer group structure $(\mathbf{D})$ and agent attributes $(\mathbf{X})$, is

$$\begin{aligned} u_i\left(\mathbf{y};\mathbf{D},\mathbf{X}\right) &= v_i\left(\mathbf{D},\mathbf{X}\right)y_i - \frac{1}{2}y_i^2 + \beta\bar{y}_{n(i)}y_i \\ &= v_i\left(\mathbf{D},\mathbf{X}\right)y_i - \frac{1}{2}y_i^2 + \beta\mathbf{G}_i\mathbf{y}y_i. \end{aligned} \tag{3}$$

- Assume that $|\beta| < 1$ and define $v_i\left(\mathbf{D},\mathbf{X}\right)$ as

$$\begin{aligned} v_i\left(\mathbf{D},\mathbf{X}\right) &= X_i'\gamma + \bar{X}_{n(i)}'\delta + A_{g(i)} + U_i \\ &= X_i'\gamma + \left(\mathbf{G}_i\mathbf{X}\right)'\delta + A_{g(i)} + U_i. \end{aligned}$$

Equilibrium

- The observed action $\mathbf{Y}$ corresponds to a (complete information) Nash equilibrium (NE).
  - No agent can increase her utility by changing her action given the actions of all other agents in the network.
  - The econometrician observes the triple $(\mathbf{Y}, \mathbf{X}, \mathbf{D})$.
    - she does not observe $\mathbf{A} = (A_1, \ldots, A_N)'$, nor does she observe $\mathbf{U}$, the $n \times 1$ vector of individual-level heterogeneity terms.
    - agents do observe $(\mathbf{A}, \mathbf{U})$.
  - We assume that $\mathbb{E}\left[\mathbf{U} \middle| \mathbf{X}, \mathbf{A}\right] = 0$ (cf., strict exogeneity as in Chamberlain (1984))
    - However X_i and $A_{g(i)}$ may co-vary (i.e., there could be sorting into groups).

Endogenous and exogenous social effects

- **endogenous**: the marginal utility associated with an increase in y_i is increasing in the average action of one's peers, $\bar{y}_{n(i)}$:

$$\frac{\partial^2 u_i(\mathbf{y}, \mathbf{D}, \mathbf{X})}{\partial y_i \partial \bar{y}_{n(i)}} = \beta.$$

- **exogenous or contextual:** the marginal utility associated with an increase in y_i varies with peer attributes:

$$\frac{\partial^2 u_i(\mathbf{y}, \mathbf{D}, \mathbf{X})}{\partial y_i \partial \bar{X}'_{n(i)}} = \delta.$$

- We'll return to the policy implications of endogenous vs. contextual effects later in the course.
  - It turns out these differences are not so interesting in the special case we are studying now.

Correlated effects

- **correlated effects**: agents located in references with high values of $A_{g(i)}$ will choose higher actions.

$$\frac{\partial^2 u_i \left(\mathbf{y}, \mathbf{D}, \mathbf{X}\right)}{\partial y_i \partial A_{g(i)}} = 1.$$

- Endogenous, contextual and correlated effects all cause outcomes across members of a common network to covary.
- Attributing this covariance to true spillovers, whether endogenous or contextual, versus group-level heterogeneity is difficult.

Linear best replies

- F.O.C for optimal behavior generates best response functions of the form

$$Y_i = A_{g(i)} + \beta \bar{Y}_{n(i)} + X_i' \gamma + \bar{X}_{n(i)}' \delta + U_i$$

  for $i = 1, \ldots, N$.

- Called the **linear-in-means** model of social interactions Brock and Durlauf (2001)

- Basis of most empirical work on peer effects (e.g., Bertrand et al., 2000).

- The *reaction function* is

$$Y_i \left(\bar{y}_{n(i)} \right) = A_{g(i)} + \beta \bar{y}_{n(i)} + X_i' \gamma + \bar{X}_{n(i)}' \delta + U_i$$

  with holds *for all* $\bar{y}_{n(i)} \in \mathbb{Y}$.

Linear best replies (continued)

- An agent's best reply varies with

  1. the average action of those to whom she is directly connected $\bar{y}_{n(i)}$,
  2. her own observed attributes X_i,
  3. the average attributes of her direct peers $\bar{X}_{n(i)}$,
  4. the unobserved group effect, $A_{g(i)}$, and
  5. unobserved own attributes, U_i.

A system of simultaneous equations

- The n best reply functions define an $n \times 1$ system of (linear) simultaneous equations.
- A least squares fit of Y_i onto a constant, $\bar{Y}_{n(i)}$, X and $\bar{X}_{n(i)}$ will not provide consistent estimates of $\theta_0 = (A_0, \beta_0, \gamma_0', \delta_0')'$.
- Manski (1993) calls this feature of the linear-in-means model the **reflection problem**.
- This is true even if $\mathbb{E}\left[A_c \mid \mathbf{X}_c\right] = 0$.

Anatomy of the reflection problem

- Recall the index set of agent i's peers

$$n(i) = \{j \,:\, D_{ij} = 1\}$$

  with cardinality $T_{g(i)}$.

  - Y_i is a component of the best response functions of all $j \in \{j \,:\, j \in n(i)\}$.
  - therefore U_i will be correlated with all $Y_j \in \{Y_j \,:\, j \in n(i)\}$.
  - $\Rightarrow U_i$ will covary with $\bar{Y}_{n(i)}$!

- Think of a textbook linear supply and demand model.
- This is just a linear simultaneous equations system in a very soft disguise.

Reduced form

- Write the system of best replies for group c as:

$$\mathbf{Y}_c = A_c \iota_{T_c} + \mathbf{X}_c \gamma + \mathbf{G}_c \mathbf{X}_c \delta + \beta \mathbf{G}_c \mathbf{Y}_c + \mathbf{U}_c. \tag{4}$$

- If $|\beta| < 1$, then $I_{T_c} - \beta \mathbf{G}_c$ is strictly (row) diagonally dominant & hence non-singular.

- To see this re-write $I_{T_c} - \beta \mathbf{G}_c$ as

$$\begin{aligned} I_{T_c} - \beta \mathbf{G}_c &= I_{T_c} - \beta \left[\frac{1}{T_c - 1} \left(\iota_{T_c} \iota'_{T_c} - I_{T_c} \right) \right] \\ &= \frac{T_c - 1 + \beta}{T_c - 1} I_{T_c} - \frac{\beta}{T_c - 1} \iota_{T_c} \iota'_{T_c} \end{aligned}$$

- Then, using the Henderson and Searle (1981) expression,

$$(\mathbf{A} + b\mathbf{u}\mathbf{v})^{-1} = \mathbf{A}^{-1} - \frac{b}{1 + b\mathbf{v}'\mathbf{A}^{-1}\mathbf{u}} \mathbf{A}^{-1}\mathbf{u}\mathbf{v}'\mathbf{A}^{-1}$$

we can compute the inverse $(I_{T_c} - \beta \mathbf{G}_c)^{-1}$.

Reduced form (aka algebra done by many graduate students)

- This probably took me a month back in 2003....

$$
\begin{aligned}
(I_{T_c} - \beta \mathbf{G}_c)^{-1} =& \frac{T_c - 1}{T_c - 1 + \beta} I_{T_c} \\
& - \frac{\left(-\frac{\beta}{T_c - 1}\right)}{1 + \left(-\frac{\beta}{T_c-1}\right) \iota'_{T_c} \frac{T_c - 1}{T_c - 1 + \beta} \iota_{T_c}} \left[\frac{T_c - 1}{T_c - 1 + \beta} \iota_{T_c} \iota'_{T_c} \frac{T_c - 1}{T_c - 1 + \beta} \right] \\
=& \frac{T_c - 1}{T_c - 1 + \beta} I_{T_c} + \frac{\frac{\beta}{T_c - 1}}{1 - \frac{\beta T_c}{T_c - 1 + \beta}} \left(\frac{T_c - 1}{T_c - 1 + \beta} \right)^2 \iota_{T_c} \iota'_{T_c} \\
=& \frac{T_c - 1}{T_c - 1 + \beta} \left[I_{T_c} + \frac{1}{\frac{T_c - 1 - \beta(T_c - 1)}{T_c - 1 + \beta}} \left(\frac{\beta}{T_c - 1 + \beta} \right) \iota_{T_c} \iota'_{T_c} \right] \\
=& \frac{T_c - 1}{T_c - 1 + \beta} \left[I_{T_c} + \left(\frac{\beta}{1 - \beta} \right) \frac{1}{T_c - 1} \iota_{T_c} \iota'_{T_c} \right]
\end{aligned}
$$

Reduced form (aka algebra done by many graduate students)

- It turns out one can avoid all this algebra by exploiting a little Matrix Analysis (e.g., Horn and Johnson, 2013).
- We'll use some of these tricks when we look at more complex, networked, reference group structures.
- For now the algebra us a good exercise...(and is historically this is how these results were derived)

Reduced form (aka algebra done by many graduate students)

- Solving for the equilibrium action vector as a function of $\mathbf{D}_c$, $\mathbf{X}_c$, A_c and $\mathbf{U}_c$ alone yields

$$\begin{aligned}\mathbf{Y_c} \;=\; & A_c\left(I_{T_c}-\beta\mathbf{G}_c\right)^{-1}\iota_{T_c}+\left(I_{T_c}-\beta\mathbf{G}_c\right)^{-1}\left(\mathbf{X}\gamma+\mathbf{G}_c\mathbf{X}\delta\right)\\ & +\left(I_{T_c}-\beta\mathbf{G}_c\right)^{-1}\mathbf{U}_c.\end{aligned}$$

- We'll now try to exploit the special structure of out set-up to simplify this expression further.

- Begin by noting that:

$$\begin{aligned}\left(I_{T_c}-\beta\mathbf{G}_c\right)^{-1}\iota_{T_c} &= \frac{T_c-1}{T_c-1+\beta}\left[\iota_{T_c}+\left(\frac{\beta}{1-\beta}\right)\frac{T_c}{T_c-1}\right]\iota_{T_c}\\ &= \frac{T_c-1}{T_c-1+\beta}\left[\frac{(1-\beta)\left(T_c-1\right)+\beta T_c}{(1-\beta)\left(T_c-1\right)}\right]\iota_{T_c}\\ &= \frac{1}{T_c-1+\beta}\left[\frac{\left(T_c-1\right)-\beta T_c+\beta+\beta T_c}{(1-\beta)}\right]\iota_{T_c}\\ &= \frac{1}{1-\beta}\iota_{T_c}\end{aligned}$$

Reduced form (aka algebra done by many graduate students)

- We can also evaluate:

$$\begin{aligned}
\left(I_{T_c} - \beta \mathbf{G}_c\right)^{-1} \mathbf{U}_c &= \frac{T_c - 1}{T_c - 1 + \beta} \left[I_{T_c} + \left(\frac{\beta}{1 - \beta} \right) \frac{1}{T_c - 1} \iota_{T_c} \iota_{T_c}' \right] \mathbf{U}_c \\
&= \frac{T_c - 1}{T_c - 1 + \beta} \left[\mathbf{U}_c + \left(\frac{\beta}{1 - \beta} \right) \frac{T_c}{T_c - 1} \bar{\mathbf{U}}_c \right] \\
&= \frac{T_c - 1}{T_c - 1 + \beta} \mathbf{U}_c + \frac{T_c}{T_c - 1 + \beta} \frac{\beta}{1 - \beta} \bar{\mathbf{U}}_c.
\end{aligned}$$

Reduced Form (aka algebra done by many graduate students)

- Next observe that:

$$
\begin{aligned}
\left(I_{T_c} - \beta \mathbf{G}_c\right)^{-1} \mathbf{G}_c =& \frac{T_c - 1}{T_c - 1 + \beta} \left[I_{T_c} + \left(\frac{\beta}{1 - \beta} \right) \frac{1}{T_c - 1} \iota_{T_c} \iota_{T_c}' \right] \\
& \times \frac{1}{T_c - 1} \left(\iota_{T_c} \iota_{T_c}' - I_{T_c} \right) \\
=& \frac{1}{T_c - 1 + \beta} \left[I_{T_c} + \left(\frac{\beta}{1 - \beta} \right) \frac{1}{T_c - 1} \iota_{T_c} \iota_{T_c}' \right] \left(\iota_{T_c} \iota_{T_c}' - I_{T_c} \right) \\
=& \frac{1}{T_c - 1 + \beta} \left[\left(1 + \frac{\beta T_c}{(1 - \beta)(T_c - 1)} \right) \iota_{T_c} \iota_{T_c}' \right] \\
& - \frac{1}{T_c - 1 + \beta} \left[I_{T_c} + \left(\frac{\beta}{1 - \beta} \right) \frac{1}{T_c - 1} \iota_{T_c} \iota_{T_c}' \right] \\
=& \frac{1}{(1 - \beta)(T_c - 1)} \iota_{T_c} \iota_{T_c}' \\
& - \frac{1}{T_c - 1} \left(I_{T_c} - \beta \mathbf{G}_c \right)^{-1}.
\end{aligned}
$$

Reduced form (aka algebra done by many graduate students)

- The previous result allows us to write:

$$
\begin{aligned}
\left(I_{T_c} - \beta \mathbf{G}_c\right)^{-1} \mathbf{G}_c \mathbf{X}_c =& \frac{T_c}{(1-\beta)(T_c - 1)} \bar{\mathbf{X}}_c \\
& - \frac{1}{T_c - 1 + \beta} \mathbf{X}_c \\
& - \frac{1}{T_c - 1 + \beta} \frac{T_c}{T_c - 1} \frac{\beta}{1-\beta} \bar{\mathbf{X}}_c \\
=& \frac{T_c}{(1-\beta)(T_c - 1)} \left(1 - \frac{\beta}{T_c - 1 + \beta}\right) \bar{\mathbf{X}}_c \\
& - \frac{1}{T_c - 1 + \beta} \mathbf{X}_c \\
=& \frac{1}{1-\beta} \left(\frac{T_c}{T_c - 1 + \beta}\right) \bar{\mathbf{X}}_c \\
& - \frac{1}{T_c - 1 + \beta} \mathbf{X}_c
\end{aligned}
$$

Reduced form finale

- Putting the previous results together gives, after a little further manipulation,

$$
\begin{aligned}
\mathbf{Y_c} =& \frac{A_c}{1-\beta}\iota_{T_c} + \frac{T_c - 1}{T_c - 1 + \beta}\mathbf{X}_c\left(\gamma - \frac{\delta}{T_c - 1}\right) \\
&+ \left(\frac{T_c}{T_c - 1 + \beta}\right)\bar{\mathbf{X}}_c\left(\frac{\beta\gamma + \delta}{1-\beta}\right) \\
&+ \frac{T_c - 1}{T_c - 1 + \beta}\mathbf{U}_c + \frac{T_c}{T_c - 1 + \beta}\frac{\beta}{1-\beta}\bar{\mathbf{U}}_c
\end{aligned}
$$

- A few observations:
  1. For $T_c \to \infty$ we simplify to Manski (1993).
  2. An obvious takeaway of Manski (1993) is that the OLS fit of Y_i onto $\bar{Y}_{n(i)}$ is gibberish...
  3. ...if that was not obvious, then hopefully it is now
  4. cf., demand curve vs. regression of quantity onto price.

- How do we identify and estimate the parameters' of the agent's reaction function $\theta = (\beta, \gamma', \delta')'$.

Lee (2007)

- Apply the within-group transform to the reduced form.
- This yields

$$
\begin{aligned}
Y_i - \bar{Y}_{g(i)} =& \frac{T_c - 1}{T_c - 1 + \beta} \left(X_i - \bar{X}_{g(i)} \right) \left(\gamma - \frac{\delta}{T_c - 1} \right) + \frac{T_c - 1}{T_c - 1 + \beta} \left(U_i - \bar{U}_{g(i)} \right) \\
=& \left(X_i - \bar{X}_{g(i)} \right)' \left[\frac{\gamma - \frac{\delta}{T_c - 1}}{1 + \frac{\beta}{T_c - 1}} \right] + \frac{T_c - 1}{T_c - 1 + \beta} \left(U_i - \bar{U}_{g(i)} \right)
\end{aligned}
$$

- So taking the model at face value, a series of within-group regression, one for each group size, identifies θ as long as T_c varies enough.
- See Boucher et al. (2014) for an application with Canadian educational data.
- Note we only require strict exogeneity of X_i in this case.
- Concern is that this takes the model very "literally".
- Also requires that $T_{g(i)}$ and U_i vary independently.

Brock and Durlauf (2001)

- If $\delta_k = 0$ for some $k = 1 \ldots K$ with $K = \dim(X_i)$, then $\bar{X}_{k,n(i)}$ can serve as an instrument for $\bar{Y}_{k,n(i)}$ in an attempt to directly estimate the reaction function by instrumental variables.

- For this to "work" the relevant element of X_i needs to vary independently of $A_{g(i)}$; but if agents sort into groups it might be that $\mathbb{C}\left(X_i, A_{g(i)}\right) \neq 0$
  - for example, high levels of past achievement could be correlated with unobserved components of teacher or school quality.

- Same issues arises when considering identification of contextual effects.

Sacerdote (2001)

- Sorting into peer groups is probably the most difficult issue faced by empirical researchers studying peer effects.
- The reflection problem is, in many ways, an easier problem.
- Of course the combination of the two problems is a challenge.
- Random formation of peer groups, as in Sacerdote (2001) and others, ensures that $\mathbb{C}\left(X_i, A_{g(i)}\right) = 0$.
- We'll try to return to some of these issues later in the course.

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