ICSolar Model

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1 Steady Model

Consider the model of air and water interaction consisting of an initial inlet region (denoted by 0) and a pair of regions, an open region with pipe followed by a module, denoted by (1,2) satisfying

$$\begin{aligned}
W_1 : &\quad \dot{m}_w C_{p,w}(T_{w,1} - T_{w,0}) - h_{wa}(T_{a,1} - T_{w,1}) = 0 & (1)\\
A_1 : &\quad \dot{m}_a C_{p,a}(T_{a,1} - T_{a,0}) - h_{wa}(T_{w,1} - T_{a,1}) - h_e(T_e - T_{a,1}) - h_i(T_i - T_{a,1}) = 0 & (2)\\
W_2 : &\quad \dot{m}_w C_{p,w}(T_{w,2} - T_{w,1}) - Q_w = 0 & (3)\\
A_2 : &\quad \dot{m}_a C_{p,a}(T_{a,2} - T_{a,1}) - Q_a = 0 & (4)
\end{aligned}$$

Where i and e are interior and exterior contributions. Each pair of these forms a 'module'. In this work, we use

$$\begin{aligned}
C_{p,w} &= 4.218 kJ/(kgK) & (5)\\
\dot{m}_w &= 0.0008483 kg/s & (6)\\
C_{p,a} &= 1.005 kJ/(kgK) & (7)\\
\dot{m}_a &= 0.384 kg/s & (8)\\
h_{wa} &= 4.823 \times 10^{-5} kW/(Km) & (9)\\
h_i &= 1.572 \times 10^{-4} kW/(Km) & (10)\\
h_e &= 4.837 \times 10^{-4} kW/(Km) & (11)\\
& & (12)
\end{aligned}$$

With Initial and Boundary Conditions of $T_{a,0} = 20C, T_i = 25.0C, T_e = 22.5C$. At this point, we set $Q_a = 0$ as the surrounding air acts like a reservoir and its effect is currently minimal. Our inputs are $T_{w,0}$ and $Q_{w,i}$ from experimental data. We also occasionally have access to $T_{a,0}$, the ambient air temperature.

2 Unsteady Model

Consider the steady model in 4 and introduce the time derivative, $mC_p\frac{\partial T}{\partial t}$ and rearrange to get

$$W_1: \quad m_{w,1}C_{p,w}\frac{\partial T}{\partial t} + \dot{m}_w C_{p,w}(T_{w,1} - T_{w,0}) - h_{wa}(T_{a,1} - T_{w,1}) = 0 \tag{13}$$

$$W_2: \quad m_{w,2}C_{p,w}\frac{\partial T}{\partial t} + \dot{m}_w C_{p,w}(T_{w,2} - T_{w,1}) - Q_w(t) = 0 \tag{14}$$

$$A_1: \quad m_{a,1}C_{p,a}\frac{\partial T}{\partial t} + \dot{m}_a C_{p,a}(T_{a,1} - T_{a,0}) - h_{wa}(T_{w,1} - T_{a,1}) - h_e(T_e - T_{a,1}) - h_i(T_i - T_{a,1}) = 0 \tag{15}$$

$$A_2: \quad m_{a,2}C_{p,a}\frac{\partial T}{\partial t} + \dot{m}_a C_{p,a}(T_{a,2} - T_{a,1}) - Q_a = 0 \tag{16}$$

To handle the mass term, we need the volume. We have a length of the first tube as $L_1 = 0.15m$ and $L_{3,5,...} = 0.3m$. The cross sectional area of the tube is based on inner diameter, $d = 0.003m$ and outer diameter of $d = 0.0142m$. The volume of the surrounding air we are interested in has a cross section of $0.4m \times 0.4m$. Using the density and the specific heat, we get that $m_a = 0.0576kg$, $m_w = 2.12 \times 10^{-3}kg$.

For regions with modules, we can create a small volume, $m_{a,2,4,6,...} \approx Cm_a$ and $m_{w,2,4,6,...}Cm_w$.

3 Steady Model - Matrix Form

Lets write the equations in the form $\mathbf{Ax} = \mathbf{b}$

$$
\begin{aligned}
W_1: \quad & (\dot{m}_w C_{p,w} + h_{wa})T_{w,1} - h_{wa}T_{a,1} = && \dot{m}_w C_{p,w} T_{w,0} && (17)\\
A_1: \quad & (\dot{m}_a C_{p,a} + h_{wa} + h_e + h_i)T_{a,1} - h_{wa}T_{w,1} && = \dot{m}_a C_{p,a} T_{a,0} + h_e T_e + h_i T_i && (18)\\
W_2: \quad & \dot{m}_w C_{p,w} T_{w,2} - \dot{m}_w C_{p,w} T_{w,1} = && Q_w && (19)\\
A_2: \quad & \dot{m}_a C_{p,a} T_{a,2} - \dot{m}_a C_{p,a} T_{a,1} = && Q_a && (20)
\end{aligned}
$$

$$
\begin{aligned}
C_w &= \dot{m}_w C_{p,w} && (21)\\
C_a &= \dot{m}_a C_{p,a} && (22)\\
h_{win} &= h_e + h_i && (23)\\
Q_{win} &= h_e T_e + h_i T_i && (24)
\end{aligned}
$$

We can write this by defining 2×2 and 4×4 matrices along with a state vector

$$\mathbf{x}_i = [T_{w,i}, T_{a,i}, T_{w,i+1}, T_{a,i+1}]^T \tag{25}$$

for $i = 1, \ldots n - 1$, which corresponds to a tube/module pair, as

$$\mathbf{A}_0 = \begin{bmatrix} C_w & 0 \\ 0 & C_a \end{bmatrix} \tag{26}$$

$$\mathbf{A}_1 = \begin{bmatrix} h_{wa} & -h_{wa} \\ -h_{wa} & h_{wa} + h_{win} \end{bmatrix} \tag{27}$$

$$\mathbf{B}_0 = \begin{bmatrix} \mathbf{A}_0 + \mathbf{A}_1 & \mathbf{0} \\ -\mathbf{A}_0 & \mathbf{A}_0 \end{bmatrix} \tag{28}$$

$$\mathbf{B}_{-1} = \begin{bmatrix} \mathbf{0} & -\mathbf{A_0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{29}$$

and writing the boundary condition as $\mathbf{b}_0 = [\mathbf{x}_0^T \mathbf{A}_0, 0, 0]^T$, $\mathbf{x}_0 = [T_{w,0}, T_{a,0}]^T$ and $\mathbf{b}_i = [0, Q_{win}, Q_{w,i+1}, Q_{a,i+1}]^T$ to get $\mathbf{A}_n \mathbf{x} = \mathbf{b}$

$$\begin{bmatrix} \mathbf{B}_0 & & & \\ \mathbf{B}_{-1} & \mathbf{B}_0 & & \\ & \mathbf{B}_{-1} & \mathbf{B}_0 & \\ & & \ddots & \ddots \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_3 \\ \mathbf{x}_5 \\ \vdots \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_3 \\ \mathbf{b}_5 \\ \vdots \end{bmatrix} + \begin{bmatrix} \mathbf{b}_0 \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \end{bmatrix} \tag{30}$$

The eigenvalues of the system can be obtained from $\mathbf{B}_0$, itself, with algebraic multiplicity of n for $\mathbf{A}_n$ as

$$\lambda = C_w, C_a, \frac{1}{2}(C_a + C_w + h_{win} + 2h_{wa}) \pm \frac{1}{2}\sqrt{(C_a - C_w)^2 + 2h_{win}(C_a - C_w + h_{win}) + 4h_{win}} \tag{31}$$

and an obvious defectiveness of the eigenvalues leading to 4 linearly independent eigenvectors. To calculate the inverse of the system, introduce the matrix product

$$\mathbf{C} = \mathbf{B}_{-1}\mathbf{B}_0^{-1} \tag{32}$$

We have that

$$\begin{aligned}
\mathbf{B}_0^{-1} &= \begin{bmatrix} (\mathbf{A}_0 + \mathbf{A}_1)^{-1} & \mathbf{0} \\ (\mathbf{A}_0 + \mathbf{A}_1)^{-1} & \mathbf{A}_0^{-1} \end{bmatrix} && (33) \\
\mathbf{C} &= \begin{bmatrix} -\mathbf{A}_0(\mathbf{A}_0 + \mathbf{A}_1)^{-1} & -\mathbf{I} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} && (34) \\
\mathbf{B}_0^{-1}\mathbf{C} &= \begin{bmatrix} -(\mathbf{A}_0 + \mathbf{A}_1)^{-1}\mathbf{A}_0(\mathbf{A}_0 + \mathbf{A}_1)^{-1} & -(\mathbf{A}_0 + \mathbf{A}_1)^{-1} \\ -(\mathbf{A}_0 + \mathbf{A}_1)^{-1}\mathbf{A}_0(\mathbf{A}_0 + \mathbf{A}_1)^{-1} & -(\mathbf{A}_0 + \mathbf{A}_1)^{-1} \end{bmatrix} && (35) \\
\mathbf{B}_0^{-1}\mathbf{C}^n &= \begin{bmatrix} -(\mathbf{A}_0 + \mathbf{A}_1)^{-1}[\mathbf{A}_0(\mathbf{A}_0 + \mathbf{A}_1)^{-1}]^n & -(\mathbf{A}_0 + \mathbf{A}_1)^{-1}[\mathbf{A}_0(\mathbf{A}_0 + \mathbf{A}_1)^{-1}]^{n-1} \\ -(\mathbf{A}_0 + \mathbf{A}_1)^{-1}[\mathbf{A}_0(\mathbf{A}_0 + \mathbf{A}_1)^{-1}]^n & -(\mathbf{A}_0 + \mathbf{A}_1)^{-1}[\mathbf{A}_0(\mathbf{A}_0 + \mathbf{A}_1)^{-1}]^{n-1} \end{bmatrix} && (36)
\end{aligned}$$

Choosing row (or column) $i = 1, 2, \ldots, n$, the general formula for $\mathbf{A}_n^{-1}$ is

$$\mathbf{A}_n^{-1} = \begin{bmatrix} \mathbf{B}_0^{-1} & & & & & \\ -\mathbf{B}_0^{-1}\mathbf{C} & \mathbf{B}_0^{-1} & & & & \\ \mathbf{B}_0^{-1}\mathbf{C}^2 & -\mathbf{B}_0^{-1}\mathbf{C} & \mathbf{B}_0^{-1} & & & \\ \vdots & \ddots & \ddots & & & \\ (-1)^{i-1}\mathbf{B}_0^{-1}\mathbf{C}^{i-1} & \ldots & -\mathbf{B}_0^{-1}\mathbf{C} & \mathbf{B}_0^{-1} & & \\ \vdots & & & \ddots & \ddots & \\ (-1)^{n-1}\mathbf{B}_0^{-1}\mathbf{C}^{n-1} & & \ldots & & -\mathbf{B}_0^{-1}\mathbf{C} & \mathbf{B}_0^{-1} \end{bmatrix} \tag{37}$$

The solution to the system is then

$$\mathbf{x} = \mathbf{A}_n^{-1}\mathbf{b} \tag{38}$$

4 Uncertainty Quantification

To do basic uncertainty quantification, llets define the heat generated as a uncorrelated gaussian random variable

$$Q_{w,i+1} = \bar{Q}_{w,i+1} + \tilde{Q}_{w,i+1} \tag{39}$$

where $\tilde{Q}_{w,i+1}$ is a gaussian PDF with mean 0 and variance σ_{i+1}^2 as

$$\tilde{Q}_{w,i+1} = \frac{1}{\sigma_{i+1}\sqrt{2\pi}} \exp\left[\frac{x^2}{2\sigma_{i+1}^2}\right] \tag{40}$$

The covariance matrix of $\mathbf{b}$ is block diagonal, with each block containing

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_i^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{41}$$

The system can be written as

$$\mathbf{A}_n \mathbf{x} = \mathbf{b} + \tilde{\mathbf{b}} \tag{42}$$

and the solution can be written as a gaussian PDF with mean $\mathbf{A}_n^{-1}\bar{\mathbf{x}}$ and variance $\Sigma_x = \mathbf{A}_n^{-1}\Sigma_b\mathbf{A}_n^{-T}$. Using our equation for $\mathbf{A}_n^{-1}$ and noting that this is a Cholesky decomposition, and maintaining block form,

$$\Sigma_x = \mathbf{A}_n^{-1}\Sigma_b\mathbf{A}_n^{-T} \tag{43}$$

$$\Sigma_x = \mathbf{A}_n^{-1}\Sigma_b^{\frac{1}{2}}(\mathbf{A}_n^{-1}\Sigma_b^{\frac{1}{2}})^T \tag{44}$$

$$\Sigma_{x,ii} = \Sigma_{b,i} + \sum_{j=1}^{i-1} \mathbf{A}_{n,ij}^{-1}\Sigma_{b,j}\mathbf{A}_{n,ij}^{-T} \tag{45}$$

$$\Sigma_{x,ij} = \mathbf{A}_{n,ij}^{-1}\Sigma_{b_j} + \sum_{k=1}^{j-1} \mathbf{A}_{n,ik}^{-1}\Sigma_{b,k}\mathbf{A}_{n,jk}^{-T} \tag{46}$$

The diagonals of this matrix are

$$\Sigma_{x,ii} = \Sigma_{b,i} + \sum_{j=1}^{i-1} \mathbf{A}_{n,ij}^{-1}\Sigma_{b,j}\mathbf{A}_{n,ij}^{-T} \tag{47}$$

$$\tag{48}$$