Lecture 25 (Graphs 4)

Minimum Spanning Trees

CS61B, Spring 2025 @ UC Berkeley

Slides credit: Josh Hug

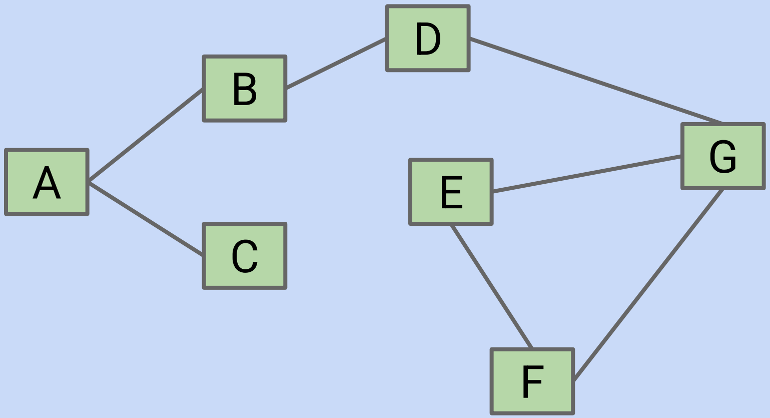
Graph Problem Warmup

Lecture 25, CS61B, Spring 2025

Warm-up Problem

Given a undirected graph, determine if it contains any cycles.

- May use any data structure or algorithm from the course so far.



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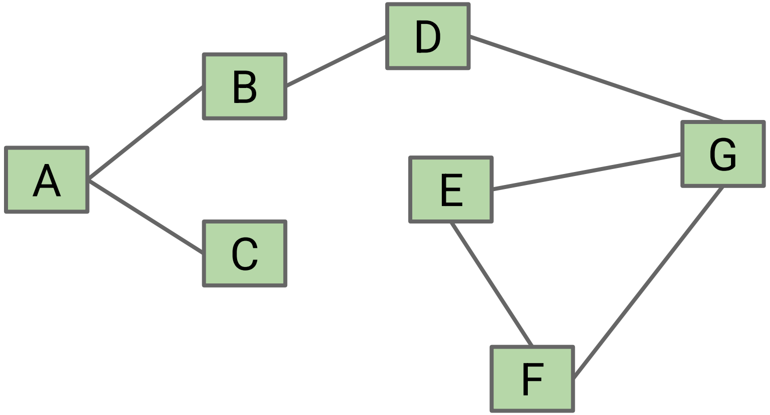

Approach 1: Do DFS from A (arbitrary vertex).

- Keep going until you see a marked vertex.
- Potential danger:
  - B looks back at A and sees marked.
  - Solution: Just don't count the node you came from.

Worst case runtime: O(V + E).

- With some cleverness, can give a tighter bound of O(V) (the number of edges we check is at most V, so O(V+E) = O(V))

Warm-up Problem

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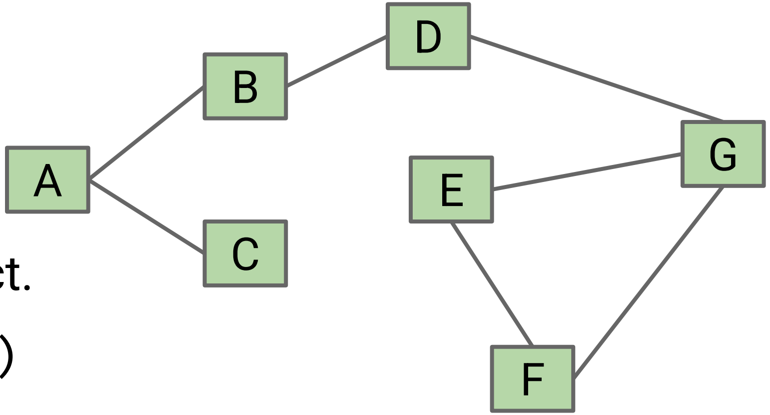

Approach 2: Use a WeightedQuickUnionUF object.

- For each edge `v-w`, check `connected(v, w)`
  - If not, `union(v, w)`.
  - If so, return true, there is a cycle

Worst case runtime: O(V + E α(V)) if we have path compression.

- Here α(V) is the [inverse Ackermann function] from Disjoint Sets.
- With similar reasoning from before, we can simplify to O(V α(V))

Minimum Spanning Trees Intro

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Spanning Trees

Given an **undirected** graph, a ***spanning tree*** T is a subgraph of G, where T:

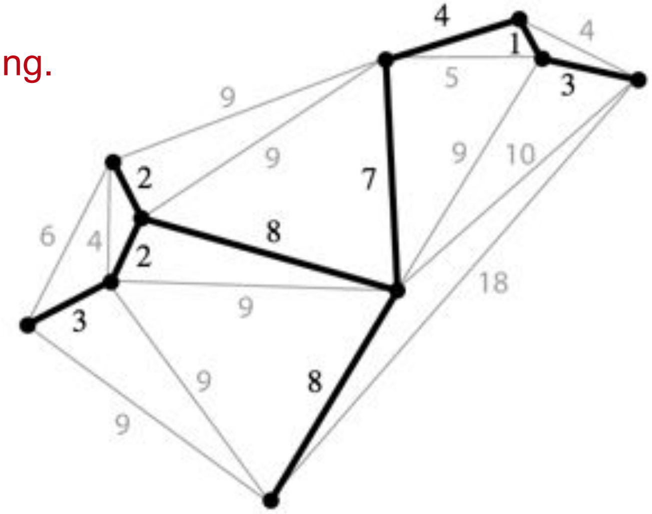
- Is connected.
- Is acyclic.
- Includes all of the vertices.

These two properties make it a tree.

This makes it spanning.

Example:

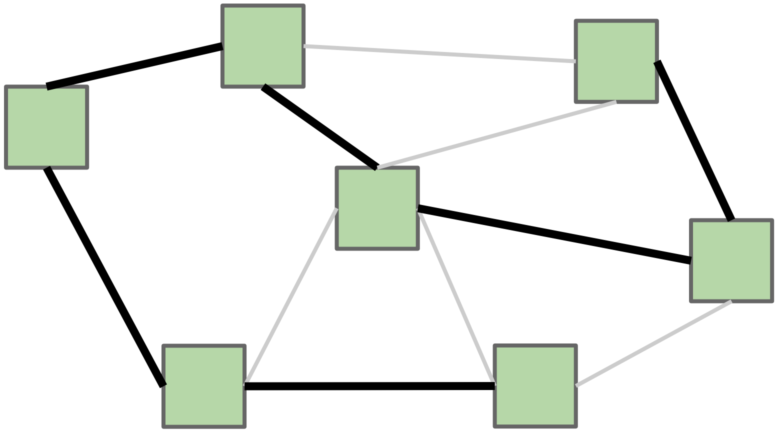
- Spanning tree is the black edges and vertices.

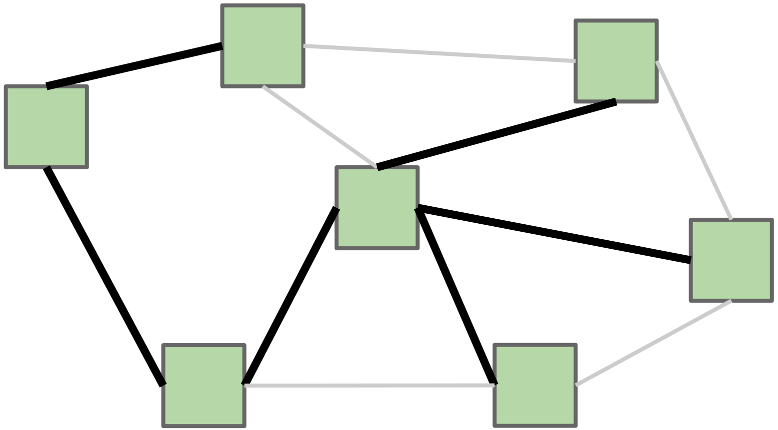


A ***minimum spanning tree*** is a spanning tree of minimum total weight.

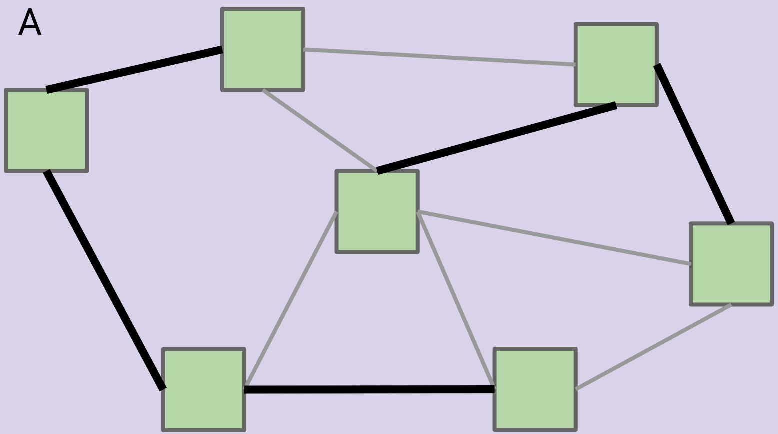
- Example: Network of power lines that connect a bunch of buildings.

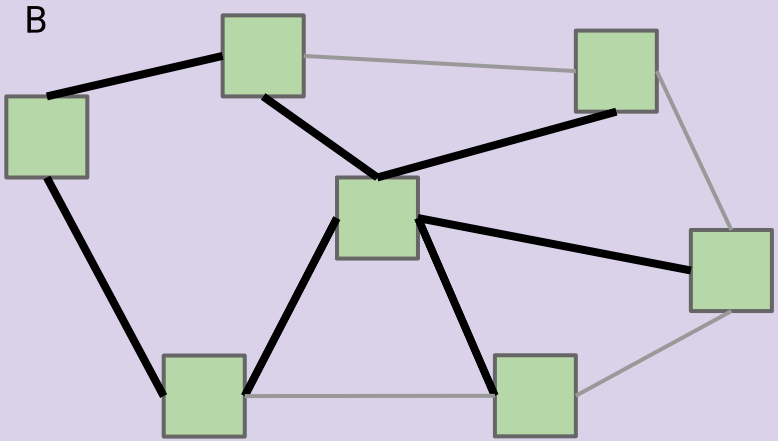
Spanning Trees

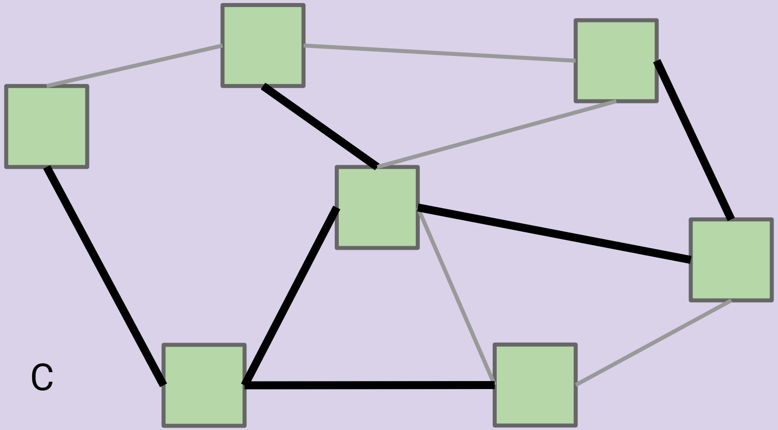




Which are Spanning Trees? http://yellkey.com/wife







MST Applications

Left: Old school handwriting recognition (link)

Right: Medical imaging (e.g. arrangement of nuclei in cancer cells)

For more, see: http://www.ics.uci.edu/~eppstein/gina/mst.html

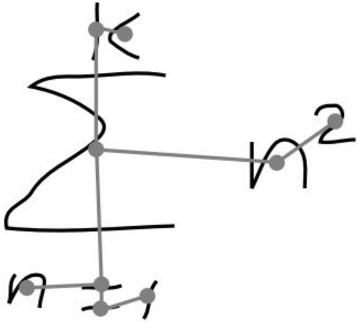
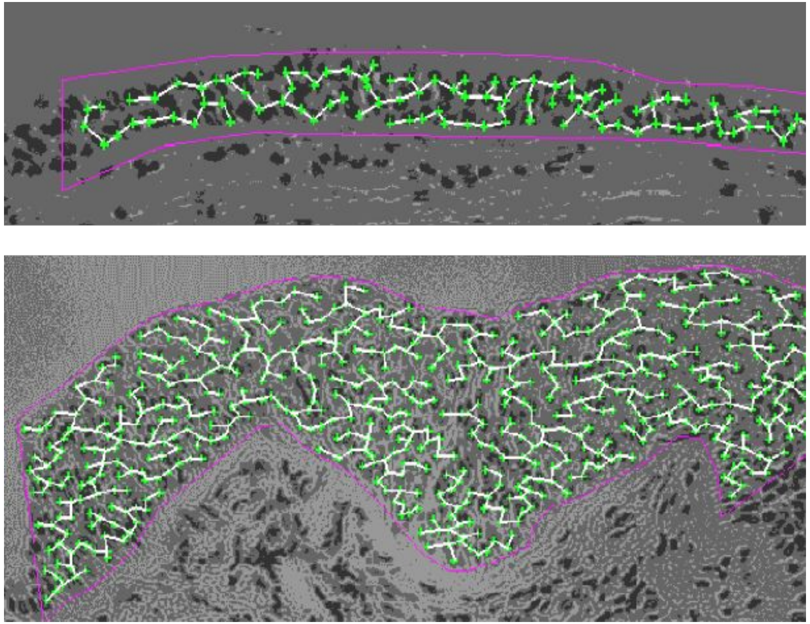


Figure 4-3: A typical minimum spanning tree



These slides are covered in the web videos, but we won't cover them live.

Extra: Minimum Spanning Trees vs. Shortest Paths Trees

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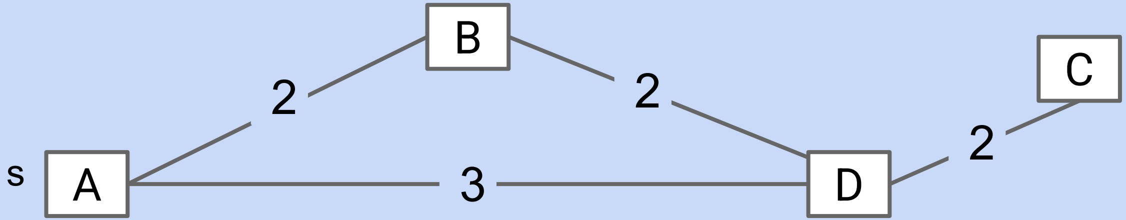
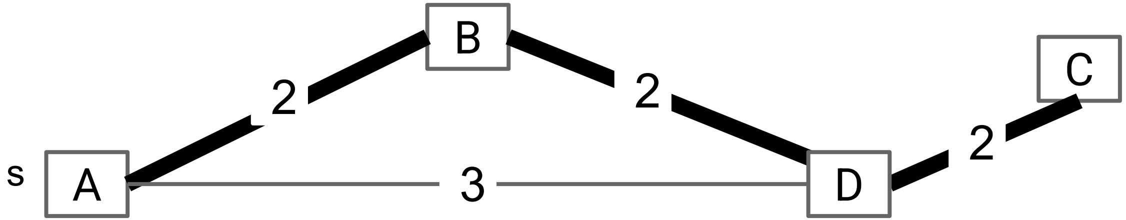
MST

Find the MST for the graph.



MST

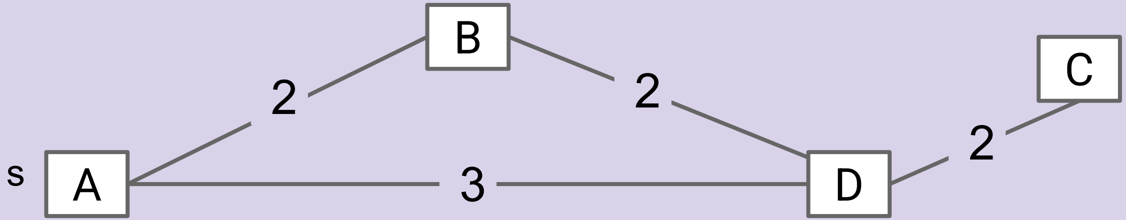
Find the MST for the graph.



MST vs. SPT

Is the MST for this graph also a shortest paths tree? If so, using which node as the starting node for this SPT?

A. A
B. B
C. C
D. D
E. No SPT is an MST.



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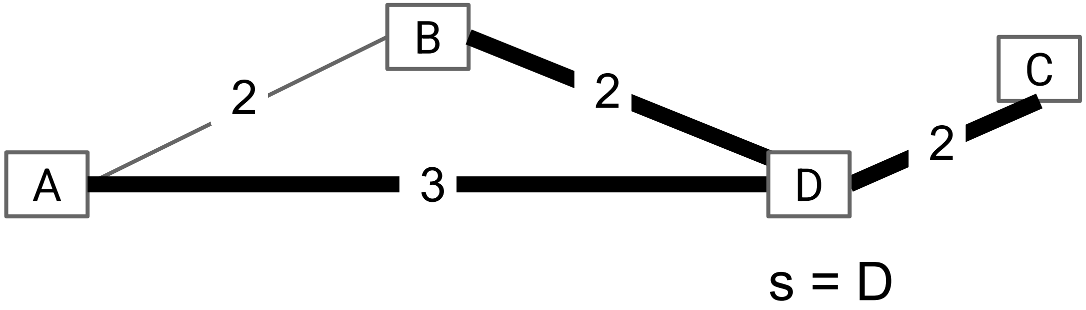

Doesn't work for s = D!

MST vs. SPT, http://yellkey.com/approach

Is the MST for this graph also a shortest paths tree? If so, using which node as the starting node for this SPT?

A. A
B. B
C. C
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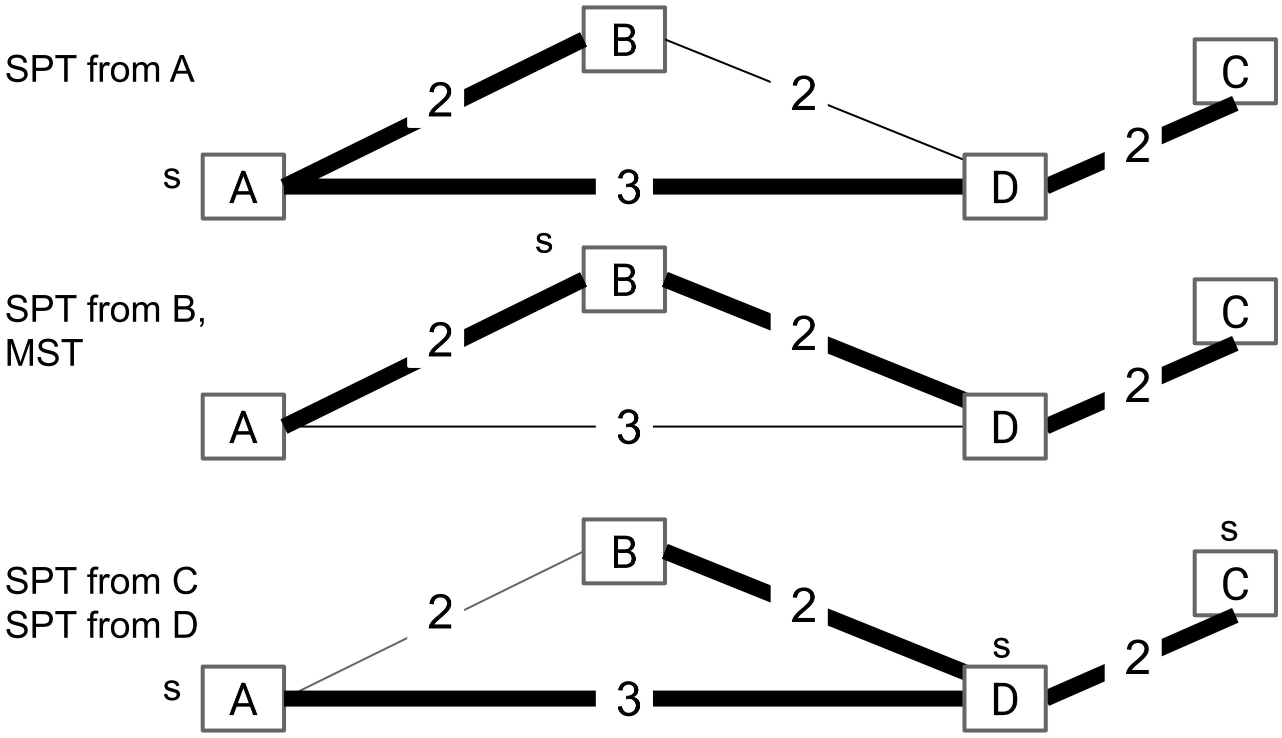


MST vs. SPT, http://yellkey.com/approach

A shortest paths tree depends on the start vertex:

- Because it tells you how to get from a source to EVERYTHING.

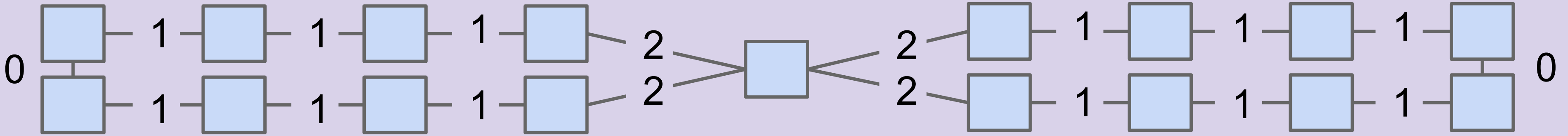
There is no source for a MST.

Nonetheless, the MST sometimes happens to be an SPT for a specific vertex.

Spanning Tree

Give a valid MST for the graph below.

- Hard B level question: Is there a node whose SPT is also the MST?

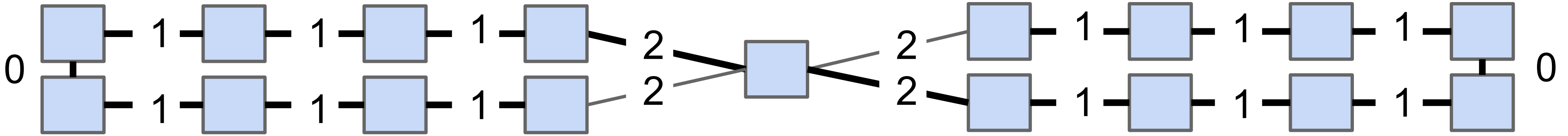


A. Yes
B. No

Spanning Tree

Give a valid MST for the graph below.

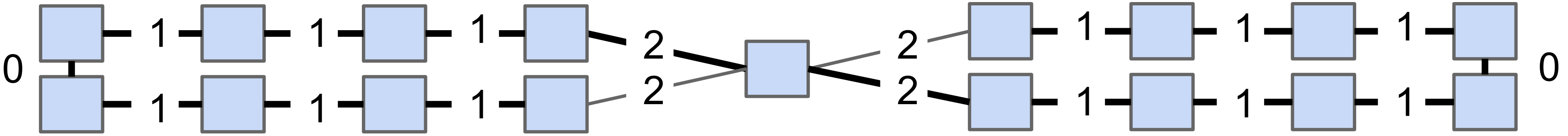
- Is there a node whose SPT is also the MST? [see next slide]



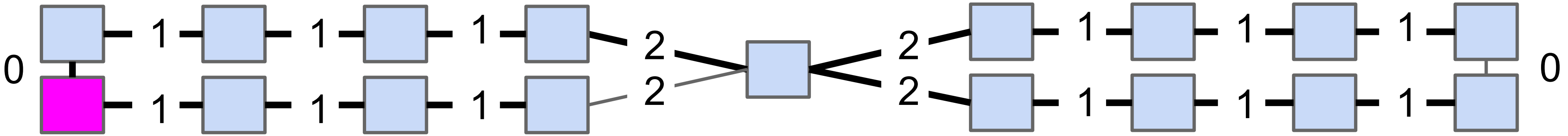
Spanning Tree

Give a valid MST for the graph below.

- Is there a node whose SPT is also the MST?
- **No!** Minimum spanning tree must include only 2 of the 2 weight edges, but the SPT always includes at least 3 of the 2 weight edges.



Example SPT from bottom left vertex:



The Cut Property

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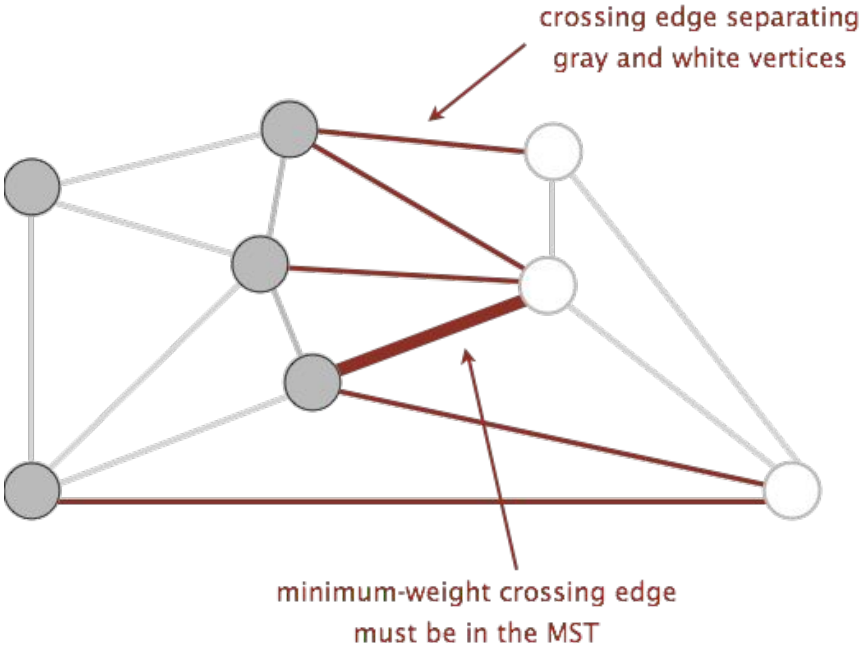
A Useful Tool for Finding the MST: Cut Property

- A ***cut*** is an assignment of a graph's nodes to two non-empty sets.
- A ***crossing edge*** is an edge which connects a node from one set to a node from the other set.

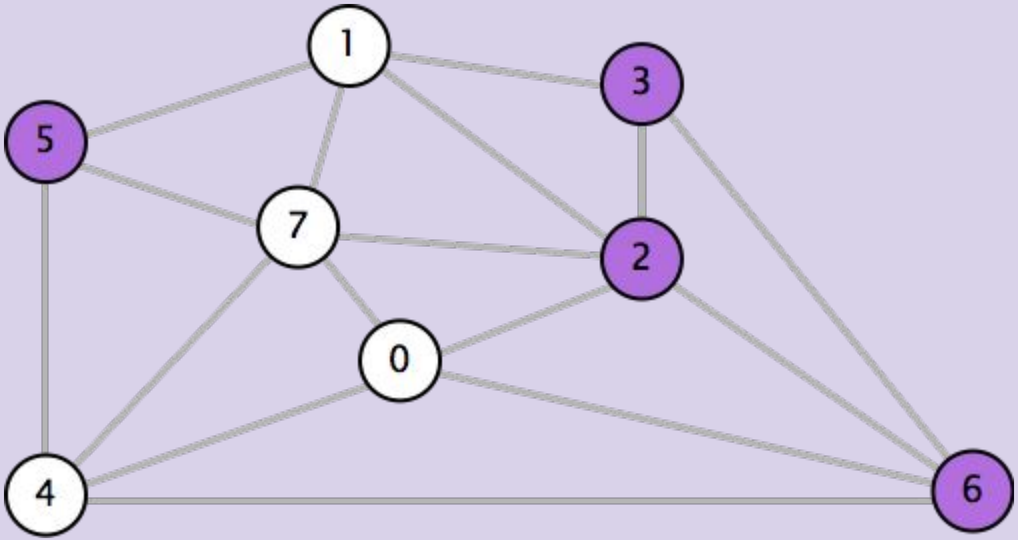


Cut property: Given any cut, minimum weight crossing edge is in the MST.

- For rest of today, we'll assume edge weights are unique.

Cut Property in Action: http://yellkey.com/maybe

Which edge is the minimum weight edge crossing the cut {2, 3, 5, 6}?

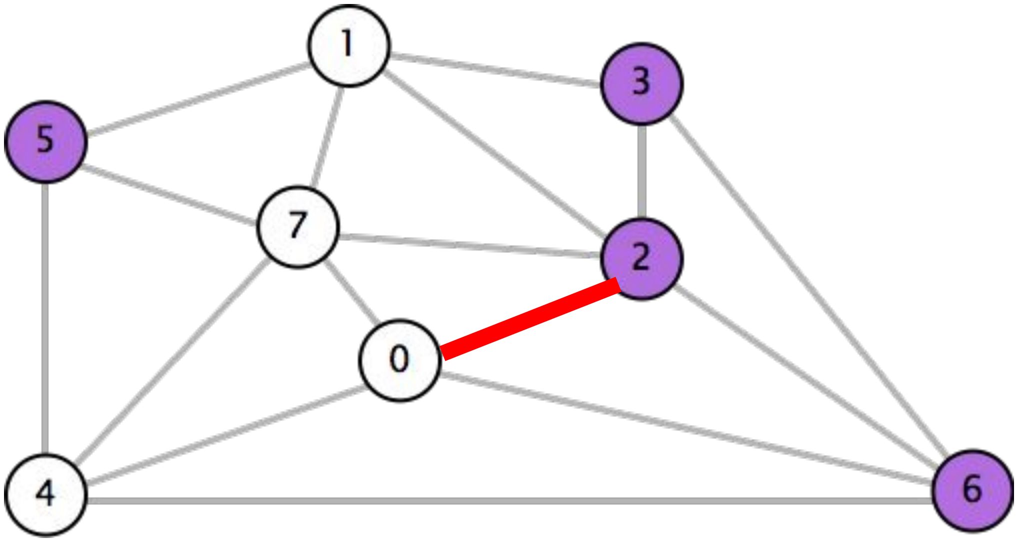


0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93

Cut Property in Action

Which edge is the minimum weight edge crossing the cut {2, 3, 5, 6}?

- 0-2. Must be part of the MST!



0-7	0.16
2-3	0.17
1-7	0.19
0-2	**0.26**
5-7	0.28
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4-7	0.37
0-4	0.38
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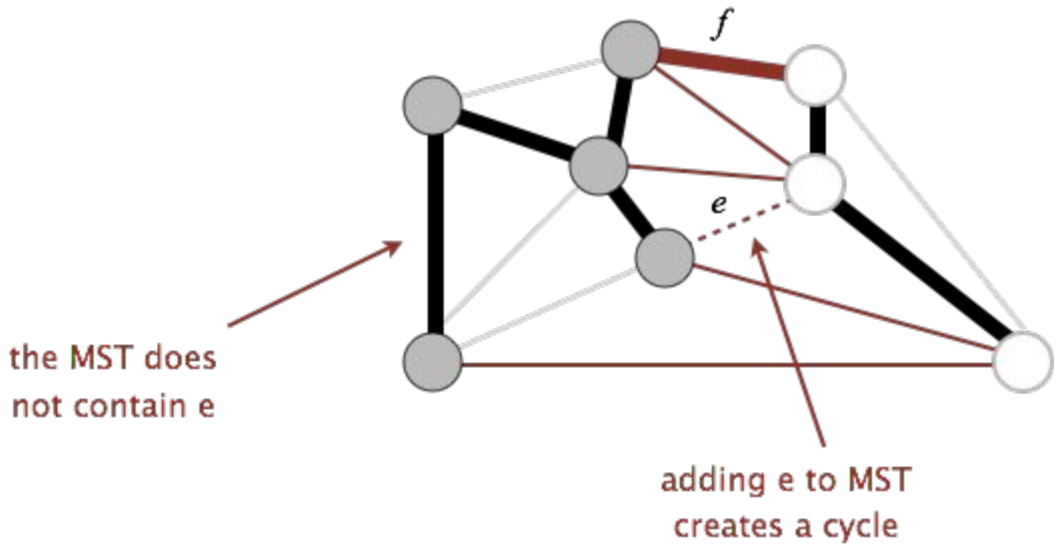
6-4	0.93

Cut Property Proof

Let X be the MST. Suppose that the minimum crossing edge *e* is not in X.

- Adding *e* to X creates a cycle.
- Some other edge *f* in X must also be a crossing edge and be part of the cycle.
- Removing *f* and adding *e* is a lower weight spanning tree.
- Contradiction!



Generic MST Finding Algorithm

Start with no edges in the MST.

- Find a cut that has no crossing edges in the MST.
- Add smallest crossing edge to the MST.
- Repeat until V-1 edges.

This should work, but we need some way of finding a cut with no crossing edges!

- Random isn't a very good idea.

Basic Prim's (Demo)

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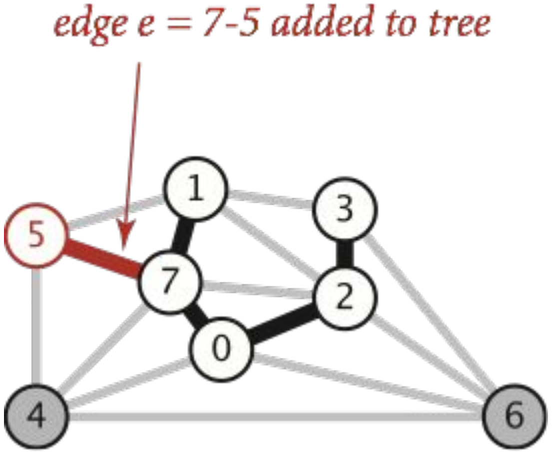
Prim's Algorithm

Start from some arbitrary start node.

- Repeatedly add shortest edge (mark black) that has one node inside the MST under construction.
- Repeat until V-1 edges.

Conceptual Prim's Algorithm Demo ([Link]())



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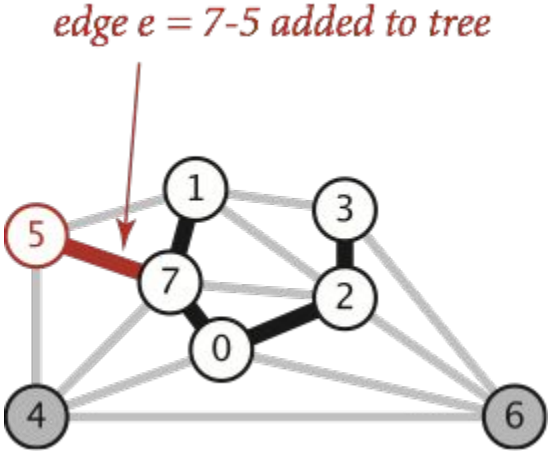

Conceptual Prim's Algorithm Demo ([Link]())

Why does Prim's work? Special case of generic algorithm.

- White vertices: Everything connected to start.
- Green vertices: Everything not connected to start.
- Always added smallest weight edge connecting white to green vertices.

Optimized Prim's (Demo)

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Prim's Algorithm Implementation

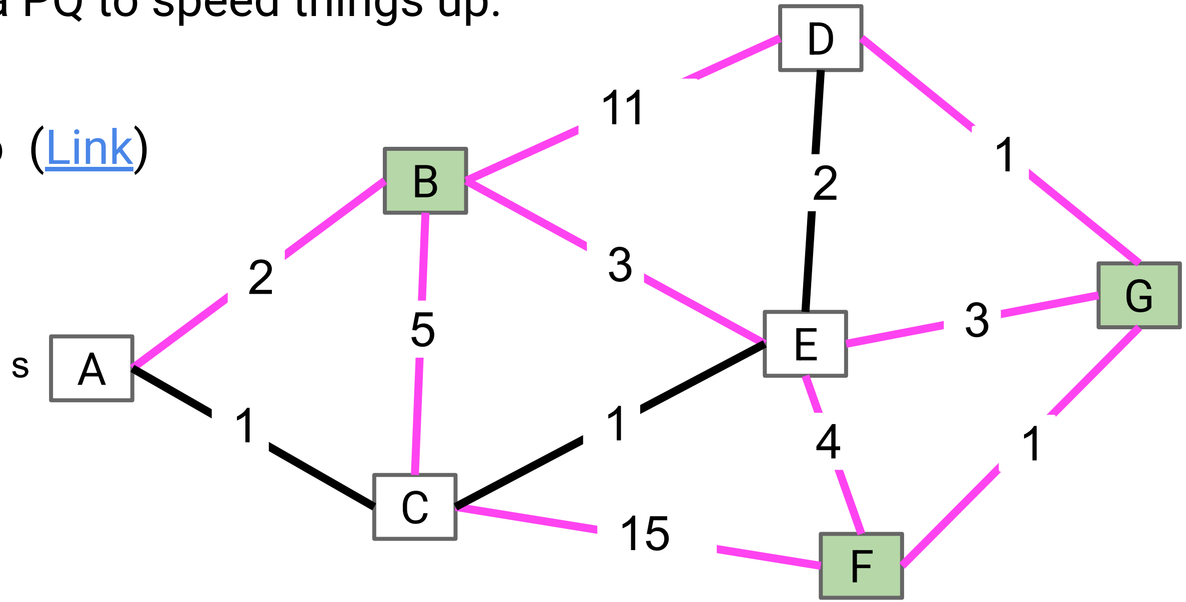
The natural implementation of the conceptual version of Prim's algorithm is highly inefficient.

- Example: Iterating over all magenta edges shown is unnecessary and slow.

Can use some cleverness and a PQ to speed things up.

Realistic Implementation Demo (Link)

- Very similar to Dijkstra's!



Prim's Algorithm Code and Runtime

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Prim's vs. Dijkstra's

Prim's and Dijkstra's algorithms are exactly the same, except Dijkstra's considers "distance from the source", and Prim's considers "distance from the tree."

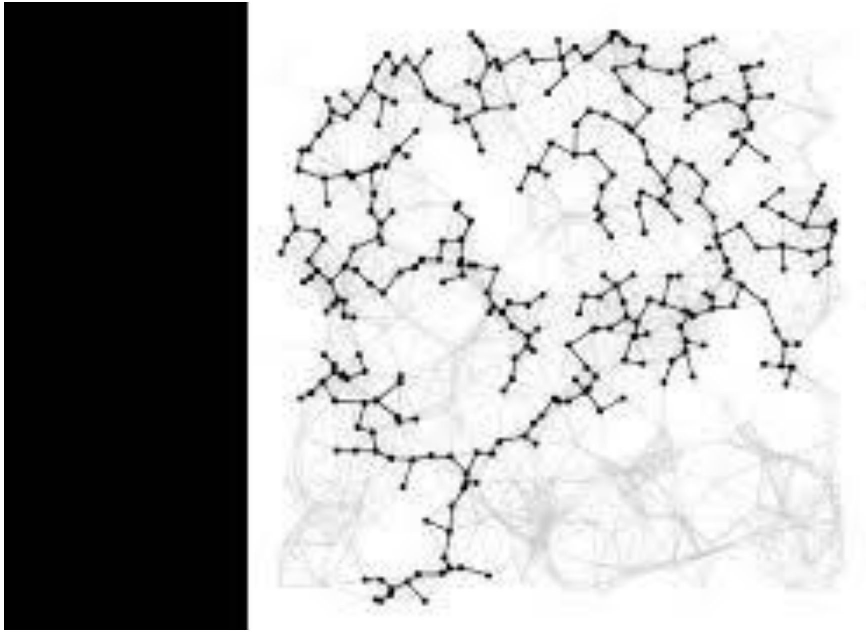
Visit order:

- Dijkstra's algorithm visits vertices in order of distance from the source.
- Prim's algorithm visits vertices in order of distance from the MST under construction.

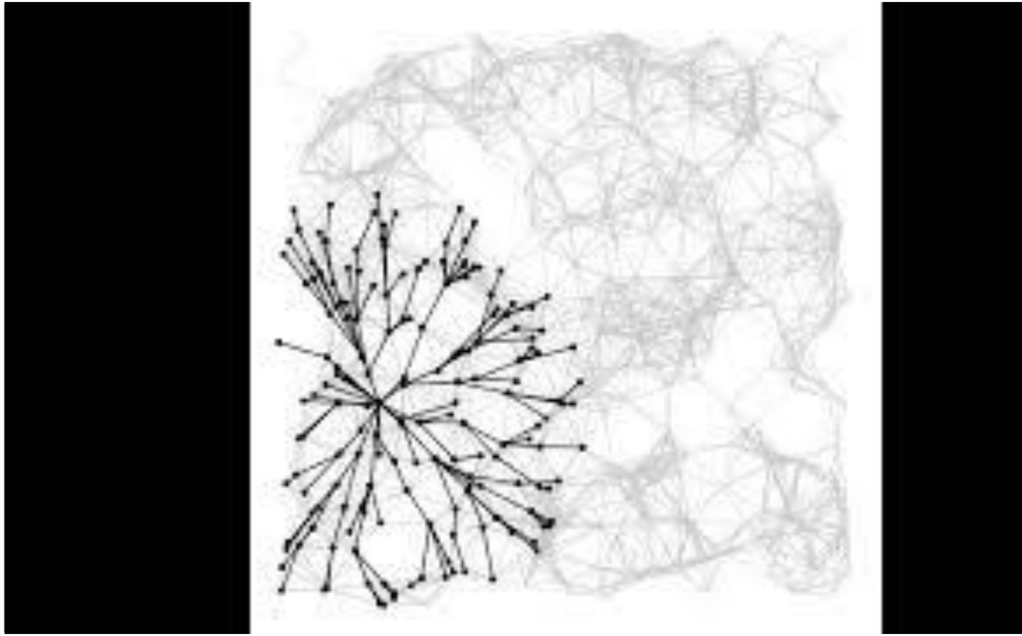
Relaxation:

- Relaxation in Dijkstra's considers an edge better based on distance to source.
- Relaxation in Prim's considers an edge better based on distance to tree.

Prim's vs. Dijkstra's (visual)



Prim's Algorithm



Dijkstra's Algorithm

Demos courtesy of Kevin Wayne, Princeton University

Prim's Implementation (Pseudocode, 1/2)

```
public class PrimMST {
  public PrimMST(EdgeWeightedGraph G) {
    edgeTo = new Edge[G.V()];
    distTo = new double[G.V()];
    marked = new boolean[G.V()];
    fringe = new SpecialPQ<Double>(G.V());

    distTo[s] = 0.0;
    setDistancesToInfinityExceptS(s);
    insertAllVertices(fringe);

    /* Get vertices in order of distance from tree. */
    while (!fringe.isEmpty()) {
      int v = fringe.delMin();
      scan(G, v);
    }
  }
  ...
```

Fringe is ordered by distTo tree. Must be a specialPQ like Dijkstra's.

Get vertex closest to tree that is unvisited.

Scan means to consider all of a vertex's outgoing edges.

Prim's Implementation (Pseudocode, 2/2)

```
while (!fringe.isEmpty()) {
  int v = fringe.delMin();
  scan(G, v);
}
```

Important invariant, fringe must be ordered by current best known distance from tree.

```
private void scan(EdgeWeightedGraph G, int v) {
  marked[v] = true;
  for (Edge e : G.adj(v)) {
    int w = e.other(v);
    if (marked[w]) { continue; }
    if (e.weight() < distTo[w]) {
      distTo[w] = e.weight();
      edgeTo[w] = e;
      pq.decreasePriority(w, distTo[w]);
    }
  }
}
```

Vertex is closest, so add to MST.

Already in MST, so go to next edge.

Better path to a particular vertex found, so update current best known for that vertex.

Prim's Runtime

```
while (!fringe.isEmpty()) {
  int v = fringe.delMin();
  scan(G, v);
}
```

```
private void scan(EdgeWeightedGraph G, int v) {
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      distTo[w] = e.weight();
      edgeTo[w] = e;
      pq.decreasePriority(w, distTo[w]);
    }
  }
}
```

What is the runtime of Prim's algorithm?

- Assume all PQ operations take O(log(V)) time.
- Give your answer in Big O notation.

Prim's Algorithm Runtime

Priority Queue operation count, assuming binary heap based PQ:

- Insertion: V, each costing O(log V) time.
- Delete-min: V, each costing O(log V) time.
- Decrease priority: O(E), each costing O(log V) time.
    - Operation not discussed in lecture.

Overall runtime: O(V*log(V) + V*log(V) + E*log(V)).

- Assuming E > V, this is just O(E log V) (Same as Dijkstra's).

	# Operations	Cost per operation	Total cost
PQ add	V	O(log V)	O(V log V)
PQ delMin	V	O(log V)	O(V log V)
PQ decreasePriority	O(E)	O(log V)	O(E log V)

Basic Kruskal's (Demo)

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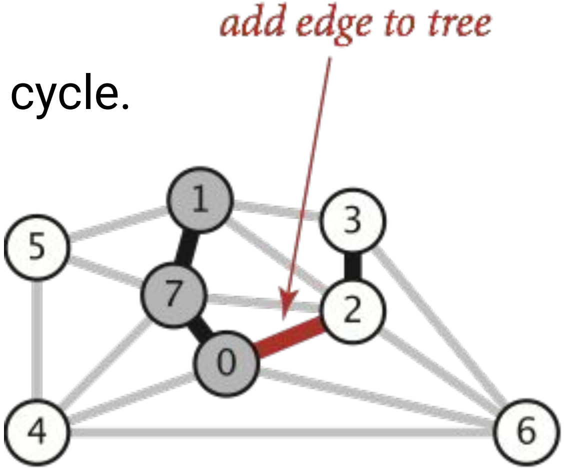
Kruskal's Algorithm

Initially mark all edges gray.

- Consider edges in increasing order of weight.
- Add edge to MST (mark black) unless doing so creates a cycle.
- Repeat until V-1 edges.



Basic Kruskal's Algorithm Demo (Link)

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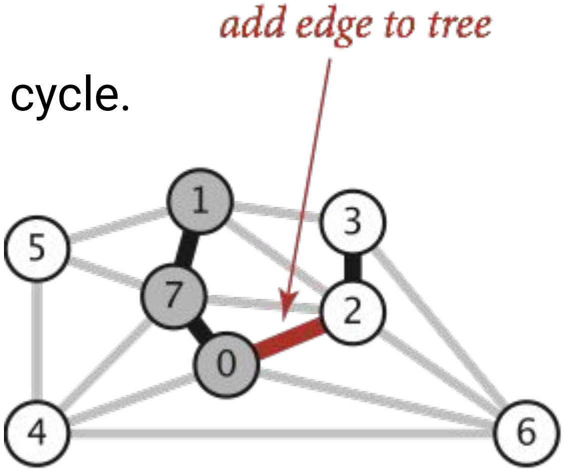

Basic Kruskal's Algorithm Demo (Link)

Why does Kruskal's work? Special case of generic MST algorithm.

- Suppose we add edge e = v->w.
- Side 1 of cut is all vertices connected to v, side 2 is everything else.
- No crossing edge is black (since we don't allow cycles).
- No crossing edge has lower weight (consider in increasing order).

Optimized Kruskal's (Demo)

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Kruskal's Algorithm

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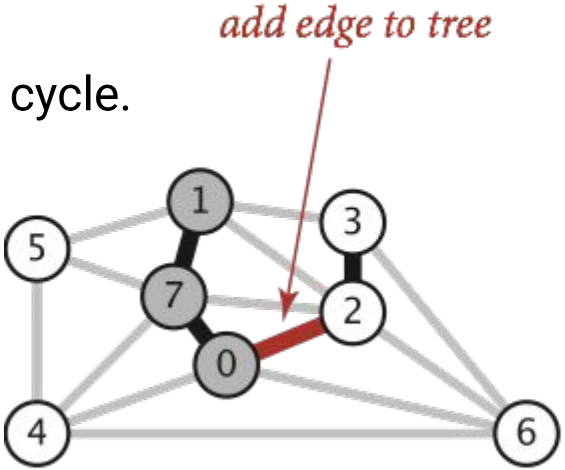

Basic Kruskal's Algorithm Demo ([Link]())

Optimized Kruskal's Algorithm Demo ([Link]())

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Kruskal's vs. Prim's

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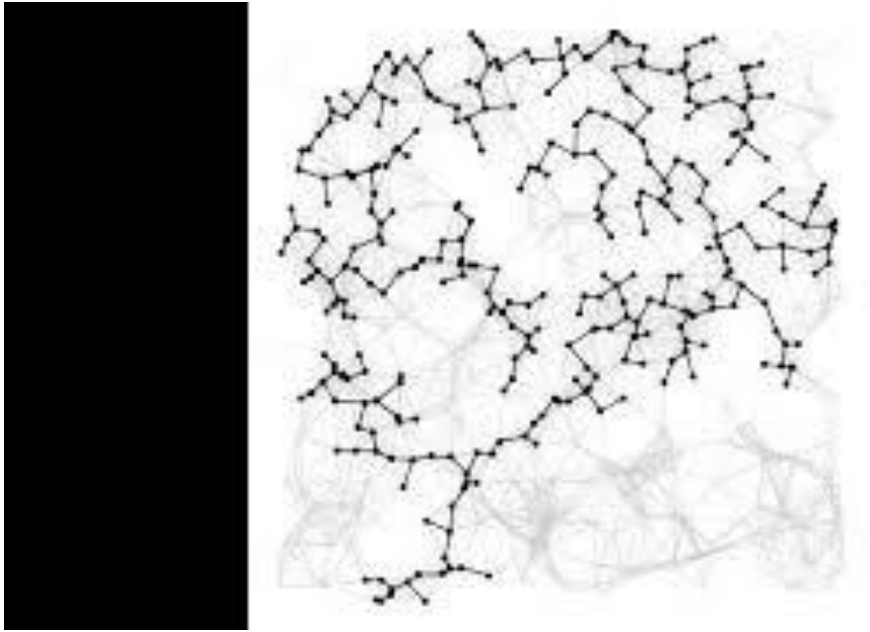
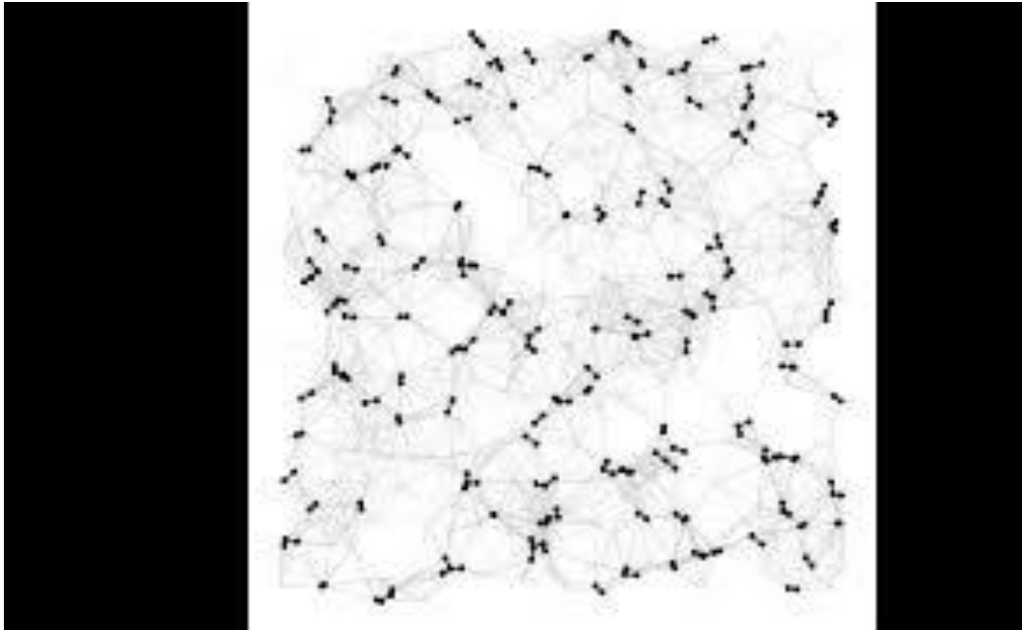
Prim’s vs. Kruskal’s (visual)



Prim’s Algorithm



Kruskal’s Algorithm

Demos courtesy of Kevin Wayne, Princeton University

Kruskal's Algorithm Code and Runtime

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Kruskal's Implementation (Pseudocode)

```
public class KruskalMST {
  private List<Edge> mst = new ArrayList<Edge>();

  public KruskalMST(EdgeWeightedGraph G) {
    MinPQ<Edge> pq = new MinPQ<Edge>();
    for (Edge e : G.edges()) {
      pq.insert(e);
    }
    WeightedQuickUnionPC uf =
             new WeightedQuickUnionPC(G.V());
    while (!pq.isEmpty() && mst.size() < G.V() - 1) {
      Edge e = pq.delMin();
      int v = e.from();
      int w = e.to();
      if (!uf.connected(v, w)) {
        uf.union(v, w);
        mst.add(e);
} } } }
```

What is the runtime of Kruskal's algorithm?

- Assume all PQ operations take O(log(V)) time.
- Assume all WQU operations take O(log* V) time.
- Give your answer in Big O notation.

Kruskal's Runtime

Kruskal's algorithm on previous slide is O(E log E).

Fun fact: In HeapSort lecture, we will discuss how do this step in O(E) time using "bottom-up heapification".

Operation	Number of Times	Time per Operation	Total Time
Insert	E	O(log E)	O(E log E)
Delete minimum	O(E)	O(log E)	O(E log E)
union	O(V)	O(log* V)	O(V log* V)
isConnected	O(E)	O(log* V)	O(E log* V)

Note 1: If we use a pre-sorted list of edges (instead of a PQ), then we can simply iterate through the list in O(E) time, so overall runtime is O(E + V log* V + E log* V) = O(E log* V).

Note 2: E < V^2, so log E < log V^2 = 2 log V, so O(E log E) = O(E log V). So while Kruskal's algorithm will be slower than Prim's algorithm for a worst-case unsorted set of edges, it won't be asymptotically slower.

Shortest Paths and MST Algorithms Summary

Problem	Algorithm	Runtime (if E > V)	Notes
Shortest Paths	Dijkstra's	O(E log V)	Fails for negative weight edges.
MST	Prim's	O(E log V)	Analogous to Dijkstra's.
MST	Kruskal's	O(E log E)	Uses WQUPC.
MST	Kruskal's with pre-sorted edges	O(E log* V)	Uses WQUPC.

Question: Can we do better than O(E log V)? See bonus slides.

These slides are covered in the web videos, but we won't cover them live.

Extra: MST Algorithm History

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170 Spoiler: State of the Art Compare-Based MST Algorithms

year	worst case	discovered by
1975	*E log log V*	Yao
1984	*E log* V*	Fredman-Tarjan
1986	*E log (log* V)*	Gabow-Galil-Spencer-Tarjan
1997	*E α(V) log α(V)*	Chazelle
2000	*E α(V)*	Chazelle
2002	*optimal (link)*	Pettie-Ramachandran (paper)
???	*E ???*	???

(Slide Courtesy of Kevin Wayne, Princeton University)