ON THE STRUCTURE OF FINITE AND CONTINUOUS TRANSFORMATION GROUPS

ÉLIE CARTAN

Introduction

The notion of the *structure* of a finite and continuous transformation groups was presented by Lie from the beginning of the research with led the great Norwegian geometer to found his theory of groups, the theory which, by its fruitfulness, has renewed, so to speak, several branches of mathematical science. In his method of solving algebraic equations which admit a given group of substitutions, Galois reduces the problem to solving a series of auxiliary equations whose number and nature depend only on the *structure* of the group of substitutions considered. In the same way, Lie reduces the integration of a system of partial derivative equations which admits a *finite and continuous* group of transformations to the integration of a series of auxiliary systems, and the number of these systems, their nature, the way in which they relate to each other still depends only on the *structure* of the group considered. In another analogous manner, the theory of linear differential equations founded by Picard and Vessiot, and more generally that of differential equations which admit a *fundamental system* of integrals, highlight the importance of the *structure* of finite and continuous groups.

The proper problem of the structure of finite and continuous transformation groups could be stated as follows: *Find all the possible structures of groups with any number of parameters*; in this order of ideas, Lie determined, in 1874, all the structures of the groups with one, two, three and four parameters. But if we continue with that approach, we would only have an imperfect view of the question, and it is clear that the preceding work is only possible and useful if we know beforehand a certain number of general laws to guide the research. It is thus that Lie has long ago[1], by considerations drawn from his theory of integration of systems of partial differential equations, classified groups into *integrable and not integrable* and provided the means by which to recognize whether a given group is integrable or not. Among the non-integrable groups, there are some which play a promenant role, these are those which Lie has called *simple*; he has indeed shown that the systems of auxiliary differential equations to which the integration of a system of equations that admits a finite and continuous group is reduced are the equations of a *simple* group[2]; also he determined all the structures of simple groups of order r whose largest subgroups are of order $r-1$, $r-2$ or $r-3$,[3] and more recently, using a work

[1] *Comptes Rendus de l'Acad. des Sc.*, Christiania, 1874; *Archiv for Math. og Nat.*, 1878. t. 3. p.112-116.

[2] In particular *Math. Ann.*, t. 25, 1885, a paper by Lie, entitled: *Allgemeine Untersuchungen über Differentialgleichungen, die eine continuirliche, endliche Gruppe gestatten.*

[3] Lie, Acad. des Sc., Christiania, 1883; *Math. Ann.*, t. 25, loc. cit., p. 132-134.

of Page[4], that of simple groups whose largest subgroups are of order $r-4$. At the same time Lie identified four great classes of simple group structures, those of the general projective group in n variables, of the linear, complex projective group in $2n-1$ variables, and those of the projective group of a second degree surface in $2n$ and $2n-1$ variables[5].

Shortly after, Engel published two Notes relating to the structure of groups in the *Leipziger Berichte*, albeit they were in an entirely different vein. In the first of these notes (*Leipz. Ber.*, 1886, p. 83-94), he linked the problem of structure to the theory of a certain trilinear form; in the second (*Leipz. Ber.*, 1887, p. 89-99), he stated a very interesting theorem proposing the existence of a simple subgroup with three parameters within a group as a necessary and sufficient condition for the non-integrability of such group, and he recently gave a rigorous demonstration[6].

This was the state of the inquiry when Killing, who had already published two memoirs on the theory of groups[7], published in the *Mathematische Annalen*[8] a series of extensive works rich in new results. Killing uses there the notion of *characteristic equation* of a group, due also to Lie, and makes it the basis of his classification of groups according to what he calls their *rank*. The most important of his results are as follows: *Apart from the four great classes of groups found by Lie, there are only five possible structures of simple groups, which have respectively* **14, 52, 78, 133** *and* **248** *parameters.* Second, *any non-integrable group is either an integrable invariant subgroup and a simple subgroup or is made up of simple invariant subgroups.* The first result allows us to conceive the possibility of establishing all the systems of irreducible differential equations to which the integration of a system of partial differential equations admitting a finite and continuous group will be reduced; the second shows that, apart from the quadratures, we will only have to integrate irreducible systems *independent* of each other.

Unfortunately Killing's research lacks rigor, and in particular, with regard to groups which are not simple, he constantly makes use of a theorem which he does not prove in its generality; I point out in this work an example where this theorem is not verified, and, when the opportunity arises, a certain number of other errors of less importance. It was therefore to be hoped that Killing's research would be reproduced and his results rigorously demonstrated.

Thus, in a Thesis written in 1891 under the direction of Engel[9], Umlauf takes up the study of the characteristic equation, rigorously proves a certain number of

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[4]Leipziger Dissertation, *Amer. Journal*, t. 10 (1888).

[5]*Math. Ann.*, t. 25, p. 130 (1885); *Norw. Archiv*, t.10, p. 413 (1885); *Leipz. Ber.*, 1889. p. 325.

[6]*Leipz. Ber.*, 1893, p. 360-369.

[7]*Erweiterung des Raumbegriffes*, Braunsberg, 1884, and *Zur Theorie der Lie'schen Transformationsgruppen*, Braunsberg, 1886.

[8]*Die Zusammensetzung der stetigen endlichen Transformationsgruppen*; erster Theil, *Math. Ann.*, t. 31, p. 252-290 (1888); zweiter Theil, *Math. Ann.*, t. 33, p. 1-48 (1889); dritter Theil, *Math. Ann.*, t. 34, p. 57-122 (1889); vierter Theil, *Math. Ann.*, t. 36, p. 161-189 (1890).

[9]*Ueber die Zusammensetzung der endlichen continuierlichen Transformationsgruppen, insbesondere der Gruppen von Range Null*, von Karl Arthur Umlauf, Leipzig, 1891.

theorems stated by Killing, and specifically studies groups of rank zero; it shows that they belong to the great class of integrable groups and determines the structures of those of these groups which have less than seven parameters.

The aim of the present work is to review and supplement in certain points the research of Killing, by introducing appropriate rigor. I have already announced, in two Notes presented to the Academy of Sciences[10], that I had found the fundamental results stated by Killing, and, in a third Note published in the *Leipziger Berichte*[11], I indicated, in broad strokes, the portions of Killing's research relating to simple groups that we could make rigorous.

This work is divided into three parts. In the first, I recall the fundamental notions and the theorems necessary for the sequel, most of which have been known for a long time; then I expose the classification of groups, following Lie, into integrable and non-integrable, and I quickly recall the properties of integrable groups and of the rank zero groups which are closely related to them. The consideration of a certain quadratic form $\psi_2(e)$ leads me to a new criterion for the integrability of a group.

The second part is devoted to groups which admit no integrable invariant subgroup and which I call *semi-simple* using a Killing expression (*halbeinfach*). The consideration of the same quadratic form $\psi_2(e)$ immediately gives the fundamental property of these groups to decompose into simple invariant subgroups, which justifies their name. I then study the properties of the characteristic equation of such a group and show how Killing deduces all simple groups.

The third part is devoted to non-integrable groups. I start *a priori* from the existence of the largest integrable invariant subgroup, instead of obtaining it *a posteriori* and laboriously, like Killing. I demonstrate Engel's theorem of which I spoke earlier and state important and new results relating to the ranks of successive derived groups of a given group. The consideration of the quadratic form $\psi_2(e)$ provides me, without solving equations, algebraic or transcendent, the largest invariant integrable subgroup of a given group. A chapter is devoted to non-integrable groups of rank *one*. Finally, in the last chapter I study the groups which admit only an integrable invariant subgroup, groups whose theory is linked in a very close way to that of linear and semi-simple homogeneous groups; in particular I determine all the linear and homogeneous simple groups in n variables which are not isomorphic to any linear and homogeneous group with less than n variables, it is unnecessary to emphasize the significance of this determination relative to the reduction of systems of linear differential equations to systems of partial differential equations which admit a finite and continuous group. Finally, I say a word about a question no less important, which was the subject of a memoir by Killing[12], but whose development would have taken me to far afield, that of the determination of the largest subgroups of simple groups.

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[10] *Comptes Rendus*, t. 116, p. 784 sqq. (1893).

[11] *Leipz. Ber.*, 1893, p. 395-420.

[12] *Bestimmung der grössten Untergruppen von endlichen Transformationsgruppen*, von W. Killing in Braunsberg, *Math. Ann.*, t. 36, p. 239-254 (1890).

Allow me, in closing, to express to Sophus Lie all my gratitude for the interest he has kindly shown in my research. It is needless to say that I made, especially in the first part, the broadest borrowings from his excellent work on the theory of finite and continuous groups[13].

[13] *Theorie der Transformationsgruppen*, unter Mitwirkung von Prof. Dr. Friedrich Engel, bearbeitet von Sophus Lie, Leipzig. Erster Abschnitt, 1888; zweiter Abschnitt, 1890; dritter Abschnitt, 1893

First Part

Chapter I: Finite and Continuous Groups. Infinitesimal Transformations. Adjoint Group.

1. Given a set of transformations of n variables $x_1, x_2, \ldots, x_n$, depending on r arbitrary parameters $a_1, a_2, \ldots, a_r$,

$$x'_i = f_i(x_1, x_2, \ldots, x_n; a_1, a_2, \ldots, a_r), \quad (i = 1, 2, \ldots, n), \tag{1}$$

we say that these transformations form a group if, by first performing the transformation which corresponds to the parameters $a_1, a_2, \ldots, a_r$, then the transformation which corresponds to the parameters $b_1, b_2, \ldots, b_r$ we still obtain a transformation of the set (1), whatever the a and the b, in other words if we have relations of the form

$$\begin{aligned} f_i(f_1(x,a), \ldots, f_n(x,a); b_1, b_2, \ldots, b_r) \equiv f_i(x_1, \ldots, x_n; c_1, c_2, \ldots, c_r),& \\ (i = 1, 2, \ldots, n),& \end{aligned} \tag{2}$$

where the c are some functions of a and b:

$$c_k = \phi_k(a_1, a_2, \ldots, a_r; b_1, b_2, \ldots, b_r), \quad (k = 1, 2, \ldots, r). \tag{3}$$

The group defined by equation (1) is said to be *finite and continuous.*

2. If in equation (1) we can give the parameters values such that we have identically

$$f_i(x, a) \equiv x_i, \quad (i = 1, 2, \ldots, n),$$

we say that the group contains the identity transformation. Lie showed that, if the r parameters are *essential*, we can find in the group r infinitesimal transformations

$$x'_i = x_i + \xi_{ki}(x_1, \ldots, x_n)\delta t + \ldots, \quad (k = 1, 2, \ldots, r), \tag{4}$$

the unwritten terms being of the second order at least with respect to the infinitely small quantity δt. These r infinitesimal transformations are independent, that is to say that there is no relation with constant coefficients of the form

$$\sum_{k=1}^{r} c_k \xi_{ki}(x) \equiv 0, \quad (i = 1, 2, \ldots, n);$$

moreover the group contains all infinitesimal transformations of the form

$$x'_i = x_i + \sum_{k=1}^{r} e_k \xi_{ki}(x)\delta t + \ldots, \quad (i = 1, 2, \ldots, n),$$

where the e_k are arbitrary constants.

If we define the infinitesimal transformation (4) by the limit of the ratio $\frac{\delta f}{\delta t}$, where f denotes an arbitrary function of x, we have

$$\lim \frac{\delta f}{\delta t} = X_k f = \sum_{i=1}^{n} \xi_{ki}(x_1, x_2, \ldots, x_n) \frac{\partial f}{\partial x_i}$$

Each of the infinitesimal transformations $\sum e_k X_k f$ generates a group with one parameter; if we give e all possible values, we thus obtain a group of transformations with r *essential* parameters, *contained in the group* (1).

The r infinitesimal transformations $X_1 f, \ldots, X_r f$ satisfy the relations

$$[X_i, X_k] = X_i(X_k f) - X_k(X_i f) = \sum_{s=1}^{r} c_{iks} X_s f, \tag{5}$$

where c_{iks} denotes constants.

Conversely, given r independent infinitesimal transformations $X_1 f, \ldots, X_r f$, satisfying relations of the form (5), the one-parameter groups generated by the infinitesimal transformations $\sum e_k X_k f$, where the e take all possible values, form a group with r essential parameters.

From now on we will only deal with groups with r parameters generated by r independent infinitesimal transformations.

3. The constants c_{iks} introduced in equation (5) define what is called the *structure* of the group generated by the infinitesimal transformations $X_1 f, \ldots, X_r f$, or more concisely of the group $X_1 f, \ldots, X_r f$. This structure plays a fundamental role in a large number of questions, particularly in the theory of the integration of differential equations which admit a given finite and continuous group.

The theory of *structure* is dominated by the following theorem, which is due to Lie together with those above:

The necessary and sufficient condition for there to exist r independent infinitesimal transformations $X_1 f, \ldots, X_r f$ satisfying the relations (5):

$$[X_i, X_k] = \sum_{s=1}^{r} c_{iks} X_s f, \quad (i, k = 1, 2, \ldots, r), \tag{5}$$

is that the constants c_{iks} satisfy the relations

$$\begin{aligned} c_{iks} + c_{kis} &= 0, \\ \sum_{\rho=1}^{r} (c_{ik\rho} c_{h\rho s} + c_{kh\rho} c_{i\rho s} + c_{hi\rho} c_{k\rho s}) &= 0, \\ (i, k, h, s = 1, 2, \ldots, r)&, \end{aligned} \tag{6}$$

obtained from the Jacobi identity

$$[[X_i, X_k], X_h] + [[X_k, X_h], X_i] + [[X_h, X_i], X_k] = 0.$$

We will therefore have all possible structures of groups with r parameters, or, more concisely, *all structures of order r*, by solving equation (6) in the most general manner.

Given the group $X_1 f, \ldots, X_r f$, we can just as easily define it by r independent infinitesimal transformations $Y_1 f, \ldots, Y_r f$:

$$Y_i f = \sum_{\rho=1}^{r} h_{i\rho} X_\rho f, \quad (i = 1, 2, \ldots, r), \tag{7}$$

with the h being constants.

We then have relations analogous to (5):

$$[Y_i, Y_k] = \sum_{s=1}^{r} g_{iks} Y_s f, \quad (i, k = 1, 2, \ldots, r), \tag{5'}$$

where the g_{iks} are determined by the relations

$$\sum_{\omega=1}^{r} h_{\omega s} g_{ik\omega} = \sum_{\rho,\sigma=1}^{r} h_{i\rho} h_{k\sigma} c_{\rho\sigma s}, \tag{8}$$

and necessarily satisfy equation (6). It is clear that these new constants g_{iks} define a structure which is not to be considered as distinct from the c_{iks} structure. We can then take advantage of the indeterminancy of the r^2 constants h to simplify the solution of equation (6).

4. Two groups of order r which have the same structure are said to be *isomorphic*, whether the number of their variables is the same or not. In a more general manner, given a group of order r : $X_1 f, \ldots, X_r f$, with structure c_{iks}, and r infinitesimal transformations, *independent or not*, $Y_1 f, \ldots, Y_r f$, satisfying the relations

$$[Y_i, Y_k] = \sum_{s=1}^{r} c_{iks} Y_s f, \quad (i, k = 1, 2, \ldots, r), \tag{5}$$

these r transformations define a group which is said to be isomorphic to the first. The isomorphism is *holoedric* if the second group is of order r, *meriedric* if it is of order lower than r.

5. Given the most general infinitesimal transformation $\sum e_k X_k f$ of a group of order r, if we carry out on the primitive variables and on the transformed variables the same group transformation, for example the infinitesimal transformation $X_i f$, we obtain another infinitesimal group transformation, namely

$$\sum e_k X_k f + \left[\sum e_k X_k, X_i\right] \delta t + \ldots$$

or $\sum e'_k X_k f$, by taking

$$e'_k = e_k + \sum_{s=1}^{r} e_s c_{sik} \delta t + \ldots, \quad (k = 1, 2, \ldots, r). \tag{9}$$

If, in (9), we give all the possible values to the index i, we obtain r infinitesimal transformations of the variables $e_1, e_2, \ldots, e_r$, namely

$$E_i f = \sum_{s,k=1}^{r} e_s c_{sik} \frac{\partial f}{\partial e_k}, \quad (i = 1, 2, \ldots, r). \tag{10}$$

These r infinitesimal transformations, which are not necessarily independent, generate a group isomorphic to the primitive group, as we can verify by considering (6). This group is called the *adjoint group* of the primitive group. It depends only on the structure of the latter and indicates how the parameters $e_1, e_2, .., e_r$ of the infinitesimal transformation $\sum e_k X_k f$ are transformed when one carries out a group transformation on this infinitesimal transformation.

If we have a linear and homogeneous relation with constant coefficients among the Ef, then the relationship

$$\sum_{i=1}^{r} \lambda_i E_i f = \sum_{i,s,k=1}^{r} \lambda_i e_s c_{sik} \frac{\partial f}{\partial k} \equiv 0,$$

implies either

$$\sum_{i=1}^{r} \lambda_i c_{sik} = 0, \quad (s, k = 1, 2, \ldots, r),$$

or else

$$\left(\sum \lambda_i X_i, X_s\right) = 0, \quad (s = 1, 2, \ldots, r).$$

The infinitesimal transformation $\sum \lambda_i X_i f$ *commutes* with all the infinitesimal transformations of the group, or else is *distinct.* The converse is also true. *The adjoint group is therefore of order $r - \rho$, if the group gives admits ρ distinct independent infinitesimal transformations.*

6. The adjoint group is *intransitive*, which is to say that there exists a relation of the form

$$\sum_{k=1}^{r} \chi_k(e_1, e_2, \ldots, e_r) E_k f = 0;$$

in fact, we can immediately verify that

$$\sum_{k=1}^{r} e_k E_k f = 0. \tag{11}$$

We can still say that the complete system

$$E_i f = 0, \quad (i = 1, 2, \ldots, r)$$

admits at least one solution. If $F(e_1, e_2, \ldots, e_r)$ is such a solution, the function F does not change when we effect a transformation of the adjoint group on $e_1, e_2, \ldots, e_r$; thus it is an *invariant* of the adjoint group.

We can therefore state the following theorem:
The adjoint group of every group admits at least one invariant.

7. The adjoint group (10) $E_1 f, \ldots, E_r f$ was obtained by starting from the infinitesimal transformations $X_1 f, \ldots, X_r f$. If we had started from the transformations $Y f$ defined by equation (7), any infinitesimal transformation would have been defined by the parameters $\eta_1, \eta_2, \ldots, \eta_r$ of the expression $\sum \eta_k Y_k f$, and we would have obtained in these new variables a second adjoint group:

$$H_i f = \sum_{s,k=1}^{r} \eta_s g_{sik} \frac{\partial f}{\partial \eta_k}, \quad (i = 1, 2, \ldots, r), \tag{12}$$

where the g_{sik} are defined by $(5')$. We have between the two systems of parameters e and η of the same infinitesimal transformation the relation

$$\sum_{k=1}^{r} e_k X_k f = \sum_{k=1}^{r} \eta_k Y_k f,$$

which decomposes into

$$e_k = \sum_{\rho=1}^{r} h_{\rho k} \eta_\rho, \quad (k = 1, 2, \ldots, r). \tag{13}$$

It is obvious that one passes from the group Ef to the group Hf by carrying out the substitution (13) on e. It is easy to verify this directly, because we have

$$H_i f = \sum_{k=1}^{r} H_i e_k \cdot \frac{\partial f}{\partial e_k} = \sum_{s,\rho,k=1}^{r} \eta_s g_{si\rho} h_{\rho k} \frac{\partial f}{\partial e_k}$$

hence, considering (8) and then (13),

$$H_i f = \sum_{s,\mu,\nu,k=1}^{r} \eta_s h_{s\mu} h_{i\nu} c_{\mu\nu k} \frac{\partial f}{\partial e_k} = \sum_{\mu,\nu,k=1}^{r} e_\mu h_{i\nu} c_{\mu\nu k} \frac{\partial f}{\partial e_k} = \sum_{\nu=1}^{r} h_{i\nu} E_\nu f,$$

which is by virtue of equation (13). The relations thus obtained:

$$H_i f = \sum_{\nu=1}^{r} h_{i\nu} E_\nu f, \quad (i = 1, 2, \ldots, r) \tag{14}$$

are the same as those which define Yf as a function of Xf.

8. Given a group G of order r generated by the r independent infinitesimal transformations $X_1 f, \ldots, X_r f$, a *subgroup* of G is a group deduced from G by restricting the parameters of G by certain relations. Such a subgroup g of order m is generated by m independent infinitesimal transformations of the form

$$\mathscr{X}_i f = \alpha_{i1} X_1 f + \alpha_{i2} X_2 f + \ldots + \alpha_{ir} X_r f, \quad (i = 1, 2, \ldots, m), \tag{15}$$

the α_{ik} being mr suitably chosen constants. We see then that the search for subgroups of order m of G amounts to determining the most general constants α_{ik} in such a way that the brackets $[\mathscr{X}_i, \mathscr{X}_k]$ are deduced linearly from $\mathscr{X}_1 f, \ldots, \mathscr{X}_m f$:

$$[\mathscr{X}_i, \mathscr{X}_k] = \sum_{s=1}^{m} \gamma_{iks} \mathscr{X}_s f, \quad (i, k = 1, 2, \ldots, m).$$

We see that the solution to the problem depends only on the *structure* of the group.

We can also define the group g by the relations which satisfy the parameters $e_1, e_2, \ldots, e_r$ of its most general infinitesimal transformation. These relations are linear and homogeneous and are obtained, for example, by eliminating $\lambda_1, \lambda_2, \dots, \lambda_m$ between the r equations

$$e_k = \sum_{i=1}^{m} \lambda_i \chi_{ik}, \quad (k = 1, 2, \dots, r). \tag{16}$$

If we regard, with Lie, $e_1, e_2, \dots, e_r$ as the homogeneous coordinates of a point in a space with $r-1$ dimensions, the subgroup g of order m is represented in this space by a plane multiplicity with $m-1$ dimensions , M, which is called the *image* of the subgroup, any point of this multiplicity being given by equation (16). Imagine that we carry out on this multiplicity a transformation of the adjoint group. We obtain a new plane multiplicity M'. It is obviously the image of a subgroup g', obtained by carrying out on the x and the x' in the equations

$$x_i' = \phi_i(x_1, x_2, \dots, x_n; \alpha_1, \alpha_2, \dots, \alpha_m), \quad (i = 1, 2, \dots, n)$$

of the subgroup g the corresponding transformation of G. The two subgroups g and g' are said to be *homologous* in the total group G. We also say that they belong to the same *type* of subgroups.

If, in particular, whatever the transformation of the adjoint group that is carried out on M, M' coincides with M, the subgroup g is said to be *an invariant subgroup.* From §4, we see that if $\mathscr{X} f$ denotes any infinitesimal transformation of an invariant subgroup g and $X_i f$ any transformation of G, $[\mathscr{X}, X_i]$ is part of g, which is to say

$$[\mathscr{X}_i, X_k] = \sum_{s=1}^{m} b_{iks} \mathscr{X}_s f, \quad (i = 1, 2, \dots, m; k = 1, 2, \dots, r). \tag{17}$$

The converse is obvious.

9. An important invariant subgroup is formed by the infinitesimal transformations $[X_i, X_k]$. We have in fact, by virtue of the Jacobi identity,

$$[X_i, [X_\mu, X_\nu]] = \sum_{\rho=1}^{r} c_{i\mu\rho}[X_\rho, X_\nu] - \sum_{\rho=1}^{r} c_{i\nu\rho}[X_\rho, X_\mu], \quad (i, \mu, \nu = 1, 2, \dots, r).$$

So if the $[X_i, X_k]$ form less than r independent infinitesimal transformations, they define an invariant subgroup G_1 of G. This is called the *derived group* of G.[14] If the $[X_i, X_k]$ form r independent infinitesimal transformations, G *is its own derived group.*

The group G_2 derived from G_1 is the second derived group of G, and so on. We also say that G_1 is the first derived group of G.

[14] Lie, *Leipzinger Berichte*, 1888, p. 19. M. Killing (*Math. Ann.*, t. 31, p. 253-254), uses the expression "Hauptuntergruppe," or fundamental subgroup.

If g is an invariant subgroup of G, the derived group g_1 of g is also an invariant subgroup of G,[15] because of

$$[X_i, \mathscr{X}_k] = \sum_{s=1}^{m} b_{iks} \mathscr{X}_s f, \quad (i = 1, 2, \ldots, r; k = 1, 2, \ldots, m),$$

we deduce

$$[X_i[\mathscr{X}_\mu, \mathscr{X}_\nu]] = \sum_{\rho=1}^{m} b_{i\mu\rho}[\mathscr{X}_\rho, \mathscr{X}_\nu] - \sum_{\rho=1}^{m} b_{i\nu\rho}[\mathscr{X}_\rho, \mathscr{X}_\mu].$$

The difference $r - r_1$ of the orders of group G and of its derived group G_1 is equal to the number of independent linear invariants of the adjoint group of G;[16] because if e_1 is an invariant of the adjoint group, we have

$$E_\mu e_1 = \sum e_s c_{s\mu 1} = 0,$$

from which

$$c_{s\mu 1} = 0, \quad (s, \mu = 1, 2, \ldots, r);$$

and reciprocally, if $X_{m+1} f, \ldots, X_r f$ define G_1, we have

$$c_{s\mu i} = 0, \quad (s, \mu = 1, 2, \ldots, r; i = 1, 2, \ldots, m)$$

and $e_1, e_2, \ldots, e_m$ are invariants of the adjoint group.

10. Given a group G of order r defined by the r independent infinitesimal transformations $X_1 f, X_2 f, \ldots, X_r f$, we call, as earlier, $E_i f$ the infinitesimal transformations of the adjoint group:

$$E_i f = \sum_{s,k=1}^{r} e_s c_{sik} \frac{\partial f}{\partial e_k}, \quad (i = 1, 2, \ldots, r),$$

and set

$$G_i f = \frac{\partial f}{\partial e_i}, \quad (i = 1, 2, \ldots, r). \tag{18}$$

We then have

$$[E_i, G_k] = \sum_{s=1}^{r} c_{iks} G_s f. \tag{19}$$

That being the case, consider the family of infinitesimal transformations $\varepsilon_1 \mathscr{X}_1 f + \varepsilon_2 \mathscr{X}_2 f + \ldots + \varepsilon_m \mathscr{X}_m f$, where

$$\mathscr{X}_i f = \alpha_{i1} X_1 f + \ldots + \alpha_{ir} X_r f, \quad (i = 1, 2, \ldots, m).$$

Suppose that the infinitesimal transformation $\lambda_1 X_1 f + \ldots + \lambda_r X_r f$ leaves this family invariant, that is to say we have

$$\left[\sum \lambda_i X_i, \mathscr{X}_k\right] = \sum_{s=1}^{m} a_{ks} \mathscr{X}_s f, \quad (k = 1, 2, \ldots, m); \tag{20}$$

we then have, on setting

$$\mathscr{G}_i f = \alpha_{i1} G_i f + \ldots + \alpha_{ir} G_r f, \quad (i = 1, 2, \ldots, m),$$

[15] Cf. Killing, *Zusammensetsung von Gruppen*, *Math. Ann.*, t. 36, p. 167.

[16] Cf. Killing, *Math. Ann.*, t. 31, p. 268.

the relations

$$[\lambda_i E_i, \mathscr{G}_k] = \sum_{s=1}^{m} a_{ks} \mathscr{G}_s f, \quad (k = 1, 2, \ldots, m). \tag{20'}$$

On the other hand, let $F(e_1, e_2, \ldots, e_r)$ be an invariant of the adjoint group and set

$$\mathcal{E} f = \sum_{i=1}^{r} \lambda_i E_i f;$$

we then have

$$\begin{aligned} \mathcal{E}(\mathscr{G}_k F) &= \sum_{s=1}^{m} a_{ks} \mathscr{G}_s F, \\ \mathcal{E}(\mathscr{G}_h \mathscr{G}_k F) &= \sum_{s=1}^{r} a_{ks} \mathscr{G}_h \mathscr{G}_s F - \sum_{s=1}^{r} a_{hs} \mathscr{G}_k \mathscr{G}_s F, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &(h, k = 1, 2, \ldots, m). \end{aligned} \tag{21}$$

It follows that the infinitesimal transformation $\mathscr{X} f = \sum \lambda_i X_i f$ leaves invariant each of the families of infinitesimal transformations defined by one of the following systems of equations in $e_1, e_2, \ldots, e_r$:

$$\mathscr{G}_i F = 0, \quad (i = 1, 2, \ldots, m), \tag{22}$$

$$\mathscr{G}_i \mathscr{G}_j F = 0, \quad (i, j = 1, 2, \ldots, m), \tag{23}$$

$$\mathscr{G}_i \mathscr{G}_j \mathscr{G}_k F = 0, \quad (i, j, k = 1, 2, \ldots, m). \tag{24}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

It is the same for each infinitesimal transformation that leaves the family $\varepsilon_1 \mathscr{X}_1 f + \ldots + \varepsilon_m \mathscr{X}_m f$ invariant.

In particular, if $\mathscr{X}_1 f, \ldots, \mathscr{X}_m f$ form a subgroup of G, the same will be true for each of the transformations of this subgroup.

If $\mathscr{X}_1 f, \ldots, \mathscr{X}_m f$ form an invariant subgroup of G, the group G will leave invariant each of the families of transformations defined by the systems (22), (23), (24),.... If then $F(e_1, e_2, ..., e_r)$ is a homogeneous polynomial of degree m, the $(m-1)^{\text{th}}$ of these systems will define a new invariant subgroup of G.

Theorem. – *If the m independent infinitesimal transformations $\sum_{k=1}^{r} \alpha_{ik} X_k f$, $(i = 1, 2, \ldots, m)$ form an invariant subgroup of the group G of order $r : X_1 f, ..., X_r f$; and if F denotes an invariant of the adjoint group of G which is a homogeneous polynomial and of degree m in $e_1, e_2, \ldots, e_r$, the system of linear equations*

$$\mathscr{G}_{i_1} \mathscr{G}_{i_2} \ldots \mathscr{G}_{i_{m-1}} F = 0, \quad (i_1, \ldots, i_{m-1} = 1, 2, \ldots, m),$$

where

$$\mathscr{G}_i f = \sum_{k=1}^{r} \alpha_{ik} \frac{\partial f}{\partial e_k},$$

defines an invariant subgroup of G.

If the first invariant subgroup reduces to the group itself, the system of equations considered is obtained by setting to 0 all partial derivatives of F of order $m-1$ with respect to $e_1, e_2, \ldots, e_r$.

Chapter II: Characteristic Determinant. Characteristic Equation. Rank.

1. The characteristic equation of a group was introduced by Lie. Killing took it as the basis for his research on group structure and showed its great importance[17].

Given a group G generated by the r infinitesimal tranformations $X_1f, X_2f, \dots, X_rf$, we propose to find all *two*-parameter subgroups to which an arbitrary infinitesimal transformation of the group belongs. To accomplish this we will have to find the r constants $\lambda_1, \lambda_2, \dots, \lambda_r$ such that

$$\left[\sum e_i X_i,\ \sum \lambda_k X_k\right] = \rho \sum e_i X_i f + \omega \sum \lambda_k X_k f, \tag{25}$$

where ρ and ω are two new constants; naturally the λ cannot be proportional to the e. Lie has shown that this is always possible[18]. We can always assume that the infinitesimal transformation under consideration is X_1f and that $\lambda_1 = 0$. This leads to

$$\left[X_1, \sum_{k=2}^{r} \lambda_k X_k\right] = \sum_{k,s} \lambda_k c_{1ks} X_s f = \rho X_1 f + \omega \sum_{k=2}^{r} \lambda_k X_k f;$$

where ω is determined by the equation

$$\begin{vmatrix} c_{122} - \omega & c_{132} & \dots & c_{1r2} \\ c_{123} & c_{133} - \omega & \dots & c_{1r3} \\ \dots & \dots & \dots & \dots \\ c_{12r} & c_{13r} & \dots & c_{irr} - \omega \end{vmatrix} = 0,$$

and $\lambda_2, \lambda_3, \dots, \lambda_r$ are obtained by the equations

$$\sum_{k=2}^{r} c_{1ks} \lambda_k = \omega \lambda_s, \quad (s = 2, 3, \dots, r),$$

and finally ρ by the equation

$$\rho = \sum_{k=2}^{r} \lambda_k c_{1k1}.$$

In the specific case where we want ρ in equation (25) to be zero, i.e. we want an infinitesimal transformation $\sum \lambda_k X_k f$ that is invariant by a given transformation $\sum e_i X_i f$, ω is determined by the equation

$$\begin{vmatrix} \Sigma e_i c_{i11} - \omega & \Sigma e_i c_{i21} & \dots & \Sigma e_i c_{ir1} \\ \Sigma e_i c_{i12} & \Sigma e_i c_{i22} - \omega & \dots & \Sigma e_i c_{ir2} \\ \dots & \dots & \dots & \dots \\ \Sigma e_i c_{i1r} & \Sigma e_i c_{i2r} & \dots & \Sigma e_i c_{irr} - \omega \end{vmatrix} = \Delta(\omega) = 0, \tag{26}$$

The determinant $\Delta(\omega)$ is called the *characteristic determinant* of the group, and the equation

$$\Delta(\omega) = 0$$

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[17] *Die Zusammensetzung der stetigen, endlichen Transformationsgruppen*, von W. Killing. 1st part, *Math Ann.*, t. 31, p. 252 sqq.; 2nd part, t. 33, p.1 sqq.; 3rd part, t. 34, p. 57 sqq.; 4th part, t. 36, p. 161 sqq. I will designate in the following these different papers by the notation: Killing, Z. v. G., I (or II, or III, or IV).

[18] Lie, *Transformationsgruppen*, I, p. 590 and also *Archiv for Math. og Nat.* t. 1, 1876.

is the *characteristic equation* of the group.

In the preceeding expressions we assume that $e_1, e_2, \ldots, e_r$ are arbitrary parameters. If we assign to $e_1, e_2, \ldots, e_r$ the particular values $e_1^0, e_2^0, \ldots, e_r^0$, the determinant becomes the *characteristic determinant with respect to the transformation* $\sum e_i^0 X_i f$; in the same manner, if we express the parameters of the infinitesimal transformations of a certain subgroup g of order m, as a function of arbitrary $\lambda_1, \lambda_2, \ldots, \lambda_m$, we obtain the *characteristic determinant* (or the *characteristic equation*) *with respect to the subgroup* g, which should not be confused with the characteristic determinant of subgroup g, which is of the same order.

Note that for $\omega = 0$, the elements of the μ^{th} column of the characteristic determinant are the coefficients of $\frac{\partial f}{\partial e_1}, \ldots, \frac{\partial f}{\partial e_r}$ in the infinitesimal transformation $E_\mu f$ of the adjoint group:

$$E_\mu f = \sum_{s,i=1}^{r} e_s c_{s\mu i} \frac{\partial f}{\partial e_i}.$$

It follows, since

$$\sum_{\mu=1}^{r} e_\mu E_\mu f = 0,$$

that the characteristic determinant is zero when $\omega = 0$.

We will now consider

$$(-1)^r \Delta(\omega) = \omega^r - \psi_1(e)\omega^{r-1} + \psi_2(e)\omega^{r-2} + \ldots + (-1)^{r-1}\psi_{r-1}(e)\omega. \tag{27}$$

The coefficient $\psi_i(e)$ is a homogeneous polynomial in $e_1, e_2, \ldots, e_r$ of degree i. We also have

$$\begin{aligned} \psi_1(e) &= \sum_{i,k=1}^{r} e_i c_{ikk}, \\ \psi_1^2(e) - 2\psi_2(e) &= \sum_{i,j,k,h=1}^{r} e_i e_j c_{ikh} c_{jhk}, \\ &\ldots \end{aligned} \tag{28}$$

2. Suppose that we define the group by the r independent infinitesimal transformations $X_1' f, \ldots, X_r' f$:

$$X_i' f = \sum_{s=1}^{r} h_{is} X_s f, \quad (i = 1, 2, \ldots, r), \tag{29}$$

which amounts to performing on e the substitution

$$e_k = \sum_{i=1}^{r} h_{ik} e_i', \quad (k = 1, 2, \ldots, r). \tag{30}$$

So if we consider

$$[X_i', X_k'] = \sum_{s=1}^{r} c_{iks}' X_s' f,$$

we have, with reference to equation 1.8,

$$\sum_{\omega=1}^{r} h_{\omega s} c_{ik\omega} = \sum_{\rho,\sigma=1}^{r} h_{i\rho} h_{k\sigma} c_{\rho\sigma s}, \quad (i, k, s = 1, 2, \ldots, r). \tag{31}$$

Let $\Delta'(\omega)$ be the new determinant. If γ_{ij}' designates the element of Δ' at the i^{th} row and the j^{th} column, we have

$$\gamma_{ij}' = \sum_{s=1}^{r} e_s' c_{sij}' - \epsilon_{ij}\omega, \quad \epsilon_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{8'}$$

and in the same way for Δ:

$$\gamma_{ij} = \sum_{s=1}^{r} e_s c_{sij} - \epsilon_{ij}\omega. \tag{32}$$

I claim that we have, by virtue of the substitution (30),

$$\sum_{\rho=1}^{r} h_{i\rho}\gamma_{\rho j} = \sum_{\rho=1}^{r} h_{\rho j}\gamma_{i\rho}', \quad (i, j = 1, 2, \ldots, r). \tag{33}$$

Indeed, by virtue of (32) and (8′), this amounts to showing that

$$\sum_{\rho,s} h_{i\rho}(e_s c_{s\rho j} - \epsilon_{\rho j}\omega) = \sum_{\rho,s} h_{\rho j}(e_s' c_{si\rho}' - \epsilon_{i\rho}\omega).$$

The two terms in ω disappear immediately and, by using (30), the formula to be established becomes

$$\sum_{\rho,\omega,s} h_{i\rho} h_{s\omega} e_s' c_{\omega\rho j} = \sum_{\rho,s} h_{\rho j} e_s' c_{si\rho}',$$

which immediately comes back to equation (31).

Consider now the determinant D with general element

$$\Gamma_{ij} = \sum_{\rho=1}^{r} h_{i\rho}\gamma_{\rho j} = \sum_{\rho=1}^{r} h_{\rho j}\gamma_{i\rho}'.$$

We immediately see that D is equal to both the product of Δ and the determinant H of h_{ik} , and the product $\Delta' H$. As H is not equal to zero, it follows that

$$\Delta(\omega) = \Delta'(\omega)$$

Moreover, the degree of the principal minor of D is equal to both that of Δ and that of Δ'. We thus arrive at the following theorem, demonstrated by Umlauf[19]:

[19]See Umlauf, *Ueber die Zusammensetzung der endlichen continuierlichen Transformationsgruppen, insbesondere der Gruppen vom Range Null.* Leipzig, 1891, p. 15 sqq.

Theorem I. - *Given a group of order r defined simultaneously by the r independent infinitesimal transformations $X_1f, \dots, X_rf$, and the r transformations*

$$X_i'f = \sum h_{is} X_s f,$$

we pass from the first characteristic determinant of the group to the second, effectively by performing the substitution (30) on the e_i, and moreover the degree of the principal minor of the characteristic determinant does not change. In particular we have

$$\psi_i'(e_1', e_2', \dots, e_r') = \psi_i \left(\sum h_{k1} e_k', \dots, \sum h_{kr} e_k' \right).$$

From this theorem we can always, for example, assume that $\psi_1(e)$ reduces to e_1 or to 0.

3. The very origin of the characteristic determinant shows that the roots of the characteristic equation are invariants of the adjoint group. This is certain at least when all the roots are simple. This can be directly proven[20]. We outline the proof due to Engel, which is much simpler than Killing's, which proceeds step by step for each coefficient $\psi_1(e), \psi_2(e), \dots$.

We have

$$E_\mu \Delta = \sum_{i,j} E_\mu \gamma_{ij} \cdot \frac{\partial \Delta}{\partial \gamma_{ij}} = \sum_{\omega,s,i,j} e_\omega c_{\omega\mu s} c_{sij} \frac{\partial \Delta}{\partial \gamma_{ij}},$$

or, taking into account equation (6) (chap. I),

$$\begin{aligned} E_\mu \Delta &= \sum_{\omega,s,i,j} e_\omega (c_{\mu is} c_{\omega sj} + c_{\omega is} c_{s\mu j}) \frac{\partial \Delta}{\partial \gamma_{ij}} \\ &= \sum_{s,i,j} c_{\mu is} (\gamma_{sj} + \epsilon_{sj}\omega) \frac{\partial \Delta}{\partial \gamma_{ij}} - \sum_{s,i,j} c_{\mu sj} (\gamma_{is} + \epsilon_{is}\omega) \frac{\partial \Delta}{\partial \gamma_{ij}} \\ &= \left(\sum_i c_{\mu ii} - \sum_j c_{\mu jj} \right) \Delta = 0. \end{aligned}$$

We therefore have the following very important theorem:

Theorem II. - *The coefficients $\psi_1(e), \dots, \psi_{r-1}(e)$ of the characteristic equation are invariants of the adjoint group.*[21]

Since, according to this theorem, the roots of the characteristic equation are invariants of the adjoint group, if $\Delta(\omega)$ contains a polynomial factor

$$\delta(\omega) = \omega^m - \chi_1(e)\omega^{m-1} + \dots \pm \chi_m(e),$$

[20] Cf. Killing, Z. v. G., I, p. 259 sqq.

[21] If we refer to the previous chapter (§9), we see that the equation

$$\psi_1(e) = \sum_{i,k} e_i c_{ikk} = 0$$

defines an invariant subgroup of order $r-1$ of the group. This theorem was demonstrated by Lie (*Archiv for Math. og Nat.*, X, p. 88), and then by Engel (*Leipz. Ber.*, 1886, p. 89) and Killing (Z. v. G., I, p. 263).

the χ being polynomials in $e_1, e_2, \ldots, e_r$, it is clear that these polynomials χ will also be invariants of the adjoint group.

Corollary. - *If the characteristic determinant contains a homogeneous, integer polynomial of degree m in $\omega, e_1, \ldots, e_r$, with the coefficient of ω^m equal to unity, the coefficients of the other powers of ω are invariants of the adjoint group.*

4. Assume that we know an invariant subgroup g of a given group. We can always assume, from §2, that this subgroup is generated by

$$X_{m+1}f, \ldots, X_r f.$$

Thus we have

$$c_{i,m+j,k} = 0, \quad \begin{pmatrix} i = 1, 2, \ldots, r \\ j = 1, 2, \ldots, r-m \\ k = 1, 2, \ldots, m \end{pmatrix}.$$

It follows that in the characteristic determinant we have

$$\gamma_{m+i,j} = 0, \ \ (i = 1, 2, \ldots, r-m; j = 1, 2, \ldots, m).$$

Therefore the characteristic determinant decomposes into two parts:

$$\Delta = |\gamma_{ij}| \cdot |\gamma_{m+h,m+k}|, \ \ (i, j = 1, 2, \ldots, m; \ h, k = 1, 2, \ldots, r-m). \tag{34}$$

We see that the first factor depends only on $e_1, e_2, \ldots, e_m$ and not on $e_{m+1}, \ldots, e_r$. Regarding the second factor, if we make $e_1 = e_2 = \ldots = e_m = 0$, it is simply the characteristic determinant of the invariant subgroup. It follows from this that the characteristic determinant relating to the invariant subgroup is, in a neighborhood of ω^m, the characteristic determinant of this invariant subgroup.

On the other hand, recall the theorem of groups, due to Lie[22]:

If a group G admits an invariant subgroup g, there exists a group G' meriedric isomorphic to G, with identity transformation corresponding to the invariant subgroup g in G, and reciprocally, if G' is meriedric isomorphic to G, an invariant subgroup of g corresponds to its identity transformation.

The correspondence referred to in this statement is the one which gives the constants c_{iks} the same values for both groups.

Here, in particular, there exist r infinitesimal transformations $Y_1 f, \ldots, Y_m f, Y_{m+1} f, \ldots, Y_r f$, the last $r-m$ being identically zero, and satisfying the relations

$$[Y_i, Y_k] = \sum_{s=1}^{r} c_{iks} Y_s f.$$

These r transformations determine a group G' of order m: $Y_1 f, \ldots, Y_m f$, which we say *associates* with the invariant subgroup g. We see that the determinant $|\gamma_{ij}|$ is simply the characteristic determinant of the group G'.

[22]Lie, *Transformationsgruppen*, I, ch. 17.

Theorem III[23]**. -** *Let g be an invariant subgroup of order $r - m$ of a group G of order r, let G' be the isomorphic group of order m associated with g, the characteristic determinant of G contains as a factor the characteristic determinant of G'; moreover, the characteristic determinant of g is, in a neighborhood of ω^m, the determinant* relative *to this invariant subgroup.*

We have an analogous theorem for any subgroup.

Theorem IV. - *The characteristic determinant relative to any subgroup g contains as a factor the characteristic determinant of this subgroup*[24].

This follows because if $X_1f, \ldots, X_mf$ are the infinitesimal transformations of this subgroup, which we can always assume to be the case, we have, for all transformations $e_1X_1f + \ldots + e_mX_mf$,

$$\gamma_{i,m+j} = 0, \quad (i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, r - m).$$

5. Killing founded a classification of finite and continuous groups on the properties of the coefficients $\psi_i(e)$[25].

Definition. - *The number of coefficients of the characteristic equation that are independent is called the* rank *of a group.*

This definition is legitimate, by virtue of Theorem I, because it does not depend on the independent infinitesimal transformations chosen to define the group.

As the polynomials $\psi(e)$ are invariants of the adjoint group (th.II), the rank of the group is at most equal to the number of distinct solutions of the complete system

$$E_\mu f = \sum_{i=1}^{r} \gamma^0_{\mu i} \frac{\partial f}{\partial e_i}, \quad (\mu = 1, 2, \ldots, r),$$

where $\gamma^0_{\mu i}$ is the element $\gamma_{\mu i}$ of the characteristic determinant after setting $\omega = 0$; after which the rank is at most equal to the order of the principal minor of $\Delta(0)$. If m is this order, it is quite clear that we have

$$\psi_{r-1}(e) = \psi_{r-2}(e) = \ldots = \psi_{r-m+1}(e) = 0,$$

because $\psi_i(e)$ is a sum of minors with i rows and i columns of $\Delta(0)$. We can therefore state the following theorem:

Theorem V[26]**.-** *The rank of a group is at most equal to the number of identically zero roots of its characteristic equation.*

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[23] See Umlauf, loc. cit., p. 34.
[24] Cf. Umlauf, loc. cit. p. 32.
[25] Killing, Z. v. G., I, p. 266 sqq.
[26] Cf. Killing, loc. cit., p. 267

In particular if $\psi_{r-1}(e)$ is not identically zero, the rank is equal to one. The converse is not true. We will see later than in the case $\psi_{r-1}(e) \neq 0$, we can always reduce this polynomial to one of the forms e_1^{r-1} or $(e_1^2 - e_2 e_3)^{\frac{r-1}{2}}$.

If all the polynomials ψ are null, the group is of rank zero.

With reference to Theorem III, we see that the rank of a group G' meriedric isomorphic to group G is at most equal to the rank of group G; because essentially rank is nothing more than the number of independent roots of the characteristic equation.

6. Killing studied in particular groups for which $\psi_{r-1}(e)$ is not identically zero[27]. I will immediately address the general case in which the characteristic equation admits a multiple zero root.

In general, if the characteristic equation relative to a transformation $X_1 f$ admits a root ω_0 different from zero, there always exists a transformation $X_2 f$ distinct from the first and such that

$$[X_1, X_2] = \omega_0 X_2 f.$$

If $\omega_0 = 0$, is this always the case?

First, if the group is rank zero, $X_1 f$ is part of at least one subgroup with two parameters $X_1 f, X_2 f$, and we have

$$[X_1, X_2] = a X_1 f + b X_2 f.$$

If one of the two numbers a and b were not zero, b for example, we would have

$$\left[X_1, X_2 + \frac{a}{b} X_1\right] = b \left[X_2 f + \frac{a}{b} X_1 f\right],$$

so that the charateristic equation relative to $X_1 f$ would admit the root b, which cannot be. Therefore $a = b = 0$ and

$$[X_1, X_2] = 0.$$

It is different when the group is rank $l > 0$. Let F be an invariant of the adjoint group, homogeneous and of degree m in $e_1, e_2, \dots, e_r$. Assume that the principal minor of $\Delta(0)$ is first order for some infinitesimal transformation $e_1^0 X_1 f + \dots + e_r^0 X_r f$. Set

$$E_\mu f = \sum_{i=1}^{r} \alpha_{\mu i} \frac{\partial f}{\partial e_i}, \quad (\mu = 1, 2, \dots, r).$$

Calling A_{ij} the minor of $\Delta(0)$ relative to α_{ij}, we have

$$\sum_{i=1}^{r} \alpha_{\rho i} A_{\sigma i} = 0, \quad (\rho, \sigma = 1, 2, \dots, r), \tag{35}$$

$$\sum_{i=1}^{r} \alpha_{i\rho} A_{i\sigma} = 0, \quad (\rho, \sigma = 1, 2, \dots, r). \tag{36}$$

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[27] Killing, Z. v. G., I, p. 276-285. An error slipped into this first part, which is rectified in the third part, p. 80 sqq.

On the other hand

$$\sum_{i=1}^{r} \alpha_{\rho i} \frac{\partial F}{\partial e_i} = 0, \quad (\rho, \sigma = 1, 2, \ldots, r), \tag{37}$$

$$\sum_{i=1}^{r} \alpha_{i\rho} e_i = 0, \quad (\rho, \sigma = 1, 2, \ldots, r), \tag{38}$$

As one of the minors, $A_{\alpha\beta}$ for example, is different from zero, we can pose, by comparing (36) and (38),

$$A_{i\sigma} = e_i \lambda_\sigma, \quad (i, \sigma = 1, 2, \ldots, r), \tag{39}$$

from which, using (35),

$$e_\sigma \sum_{i=1}^{r} \lambda_i \alpha_{\rho i} = 0, \quad (\rho, \sigma = 1, 2, \ldots, r).$$

This results in

$$\sum_{i=1}^{r} \alpha_{\rho i} \lambda_i = 0, \quad (\rho = 1, 2, \ldots, r). \tag{40}$$

The λ are not all zero because $\lambda_\beta \neq 0$, the comparison of (36) and (40) allows us to state

$$\frac{\partial F}{\partial e_i} = \rho \lambda_i, \quad (i = 1, 2, \ldots, r), \tag{41}$$

which leads to[28]

$$mF = \sum_{i=1}^{r} e_i \frac{\partial F}{\partial e_i} = \rho \sum_{i=1}^{r} e_i \lambda_i = \rho \sum_{i=1}^{r} A_{ii} = \rho \psi_{r-1}(e). \tag{42}$$

Therefore if an infinitesimal transformation cancels $\psi_{r-1}(e)$ without canceling all the minors of the first order of $\Delta(0)$, it cancels all the homogeneous invariants with non-zero degree of the adjoint group.

If the characteristic equation relative to a certain infinitesimal transformation $X_1 f$ admits k zero roots $(1 < k < r)$, this transformation certainly does not cancel the invariant $\psi_{r-k}(e)$, and consequently it cancels all the minors of the first order of $\Delta(0)$; i.e., there is a commuting transformation $X_2 f$ (such that $[X_1, X_2] = 0$).

But there is more; equation (42) is valid regardless of $e_1, e_2, \ldots, e_r$. Therefore if the group is rank $l > 0$ or $\psi_{r-1}(e) \neq 0$, then $l = 1$ or all the first order minors of $\Delta(0)$ are identically zero, and in that case every infinitesimal transformation commutes with at least one other.

Recalling the earlier statements for the case $l = 0$, we can state the following theorem:

Theorem VI. - *If the polynomial $\psi_{r-1}(e)$ of the characteristic equation of a group is identically zero, every infinitesimal transformation of the group commutes with*

[28] Cf. Lie, *Transformationsgruppen*, III, p. 676 sqq., where similar conditions are applied, and where there is a polynomial $g(e_1, e_2, \ldots, e_r)$ of degree $r - 1$, which is simply $\psi_{r-1}(e)$

at least one other infinitesimal transformation.

If $\psi_{r-1}(e) \neq 0$ there is nothing more we can say. For example, in the group with three parameters $X_1 f, X_2 f, X_3 f$:

$$[X_1, X_2] = X_1 f, \ \ [X_1, X_3] = 2X_2 f, \ \ [X_2, X_3] = X_3 f,$$

we have

$$\psi_{r-1}(e) = \psi_2(e) = 4e_1 e_3 - e_2^2;$$

$X_1 f$ cancels $\psi_2(e)$, but does not commute with any other infinitesimal transformation of the group.

Killing proved the previous theorem by noting that if $[X_1, X_2] = X_1 f$, the characteristic equation relative to $X_1 f$ has only zero roots, because

$$c_{122} = c_{123} = \ldots = c_{12r} = 0, \quad c_{121} = 1,$$

and therefore

$$[E_2, \psi_i]_{\substack{e_1=1 \\ \cdots \\ e_r=0}} = \left(\frac{\partial \psi_i}{\partial e_1}\right)_{\substack{e_1=1 \\ \cdots \\ e_r=0}} = \frac{1}{i}(\psi_i)_{\substack{e_1=1 \\ e_2=0 \\ \cdots \\ e_r=0}} = 0 :$$

$X_1 f$ cancels all the polynomials ψ.
I do not insist on the rest of the proof and content myself with reference to Killing's work[29].

7. That being the case, let us call with Killing k the number of identically zero roots of the characteristic equation, and assume $k > 1$. Take a *general* infinitesimal transformation, i.e., for which $\psi_{r-k}(e) \neq 0$. We can always assume that $X_1 f$ is such a transformation. Then, by Theorem VI, there exists at least one other commuting transformation $X_2 f$ such that:

$$[X_1, X_2] = 0.$$

The characteristic determinant relative to $X_1 f$ is therefore

$$\Delta(\omega, 1, 0, \ldots, 0) = \omega^2 |c_{1ij} - \epsilon_{ij}\omega|, \ \ (i, j = 3, 4, \ldots, r).$$

If $k > 2$, we can determine an infinitesimal transformation $X_3 f$ independent of $X_1 f$ and $X_2 f$, such that by combining it with $X_1 f$, we obtain a transformation $aX_1 f + bX_2 f$, such that:

$$[X_1, X_3] = c_{131} X_1 f + c_{132} X_2 f;$$

therefore

$$\Delta(\omega, 1, 0, \ldots, 0) = -\omega^3 |c_{1ij} - \epsilon_{ij}\omega|, \ \ (i, j = 4, 5, \ldots, r).$$

If $k > 3$, we can also find a transformation $X_1 f$ such that

$$[X_1, X_4] = c_{141} X_1 f + c_{142} X_2 f + c_{143} X_3 f,$$

and so on, until

$$[X_1, X_k] = c_{1k1} X_1 f + c_{1k2} X_2 f + \ldots + c_{1kk-1} X_{k-1} f.$$

The characteristic determinant then reduces to

$$\Delta(\omega, 1, 0, \ldots, 0) = (-1)^k \omega^k |c_{1ij} - \epsilon_{ij}\omega|, \ \ (i, j = k+1, \ldots, r).$$

[29]See Killing, Z. v. G., I, §5, p. 269 et II, §10, p. 4

As the characteristic equation relative to $X_1 f$ admits only k zero roots, we have

$$|c_{1ij}| \neq 0, \quad (i, j = k+1, \ldots, r),$$

and therefore it is impossible to find an infinitesimal transformation

$$e_{k+1} X_{k+1} f + \ldots + e_r X_r f,$$

such that

$$[X_1, e_{k+1} X_{k+1} + \ldots + e_r X_r] = \alpha_1 X_1 f + \alpha_2 X_2 f + \ldots + \alpha_k X_k f.$$

We have therefore made correspond to the general transformation $X_1 f$, $k-1$ other transformations, $X_2 f, X_3 f, \ldots, X_k f$ I say that the k transformations $X_1 f, \ldots, X_k f$ form a group. I suppose, in fact, that we have shown that all the brackets $[X_i, X_j]$, where $i = 1, 2, \ldots, h$; $j = 1, 2, \ldots, k$, $(h < k)$, depend linearly on $X_1 f, X_2 f, \ldots, X_k f$; I say that it is still so for $i = 1, 2, \ldots, h+1$. Indeed we have

$$[X_1, [X_{h+1}, X_{h+2}]] = \sum_{s=1}^{h} c_{1,h+1,s} [X_s, X_{h+2}] - \sum_{s=1}^{h} c_{1,h+2,s} [X_s, X_{h+1}],$$

hence it follows that $[X_{h+1}, X_{h+2}]$ combined with $X_1 f$, depends only on $X_1 f, \ldots, X_k f$. Therefore, according to a remark made above, $[X_{h+1}, X_{h+2}]$ also depends only on $X_1 f, \ldots, X_k f$.

We will have the same

$$[X_1, [X_{h+1}, X_{h+3}]] = \sum_{s=1}^{h} c_{1,h+1,s} [X_s, X_{h+3}] - \sum_{s=1}^{h+2} c_{1,h+3,s} [X_s, X_{h+1}],$$

which proves the proposition for $[X_{h+1}, X_{h+3}]$, and so on.

Since the theorem is true for $h = 1$, it follows that it is generally true, and hence the k *infinitesimal transformations* $X_1 f, X_2 f, \ldots, X_k f$ *form a group.*

I say this group is rank zero. Indeed, let us consider the characteristic equation relating to the subgroup $X_1 f, ..., X_k f$. It is written

$$(43) \quad \begin{vmatrix} \Sigma e_i c_{i11} - \omega & \ldots & \Sigma e_i c_{ik1} \\ \ldots & \ldots & \ldots \\ \Sigma e_i c_{i1k} & \ldots & \Sigma e_i c_{ikk} - \omega \end{vmatrix} \cdot \begin{vmatrix} \Sigma e_i c_{i,k+1,k+1} - \omega & \ldots & \Sigma e_i c_{i,r,k+1} \\ \ldots & \ldots & \ldots \\ \Sigma e_i c_{i,k+1,r} & \ldots & \Sigma e_i c_{irr} - \omega \end{vmatrix} = 0$$

the summations being extended from 1 to k. This equation admits k identically zero roots, because this is so, by hypothesis, with the general characteristic equation. However, the second determinant does not admit a zero root for

$$e_1 = 1, e_2 = 0, \ldots, e_k = 0;$$

therefore, it does not admit any more within a certain domain around this system of values. Consequently the first determinant is, inside this domain, divisible by ω^k, which is to say that we have identically

$$(44) \qquad \left| \sum_{s=1}^{k} e_s c_{sij} - \epsilon_{ij} \omega \right| = (-1)^k \omega^k, \quad (i, j = 1, 2, \ldots, k).$$

In other words, *the sub-group* $X_1 f, \ldots, X_k f$ *is of rank zero.*

Let us now take any general transformation belonging to this subgroup, that is to say $e_1X_1f + \ldots + e_kX_kf$. We can never have an equality of form

$$[e_1X_1 + \ldots + e_kX_k, \lambda_{k+1}X_{k+1} + \ldots + \lambda_rX_r] = \beta_1X_1f + \ldots + \beta_kX_kf$$

without all λ being zero. It follows that the subgroup of order k which corresponds to this infinitesimal transformation does not contain any transformation independent of $X_1f, \ldots, X_kf$, that is to say it coincides with the subgroup $X_1f, \ldots, X_kf$.

Theorem VII[30]**.** - *Given a group of order r whose characteristic equation admits k identically zero roots, and k only, any general transformation of this group is part of a subgroup of order k and rank zero. This subgroup is not contained in any larger subgroup of rank zero.*

8. Returning to the general transformation X_1f and to the subgroup of rank zero, $X_1f, \ldots, X_kf$, of which it is a part. The infinitesimal transformations $X_{k+1}f, \ldots, X_rf$ have until now been subject only to the single condition of being independent of $X_1f, \ldots, X_kf$. We can take advantage of this to cancel all the coefficients

$$c_{1,k+i,j} \quad (i = 1, 2, \ldots, r-k;\; j = 1, 2, \ldots, k).$$

In fact, we pose

$$X'_{k+i}f = X_{k+i}f + \sum_{s=1}^{k} a_{k+i,s}X_sf, \quad (i = 1, 2, \ldots, r-k).$$

Thus we have

$$[X_1, X'_{k+i}] = \sum_{\rho=1}^{r-k} c_{1,k+i,k+\rho}\left[X'_{k+\rho}f - \sum_{s=1}^{k} a_{k+\rho,s}, X_sf\right] + \sum_{s=1}^{k} c_{1,k+i,s}X_sf$$
$$+ \sum_{\rho=1}^{k} a_{k+i,\rho}\left(\sum_{s=1}^{\rho-1} c_{1\rho s}X_sf\right), \quad (i = 1, 2, \ldots, r-k).$$

We therefore need to solve the following systems of equations, obtained by setting the coefficients of $X_kf, X_{k-1}f, \ldots, X_1f$ to 0, in the second element:

$$\sum_{\rho=1}^{r-k} c_{1,k+i,k+\rho}a_{k+\rho,k} = c_{1,k+i,k}, \qquad (i = 1, 2, \ldots, r-k)$$

$$\sum_{\rho=1}^{r-k} c_{1,k+i,k+\rho}a_{k+\rho,k-1} = c_{1,k+i,k-1} - a_{k+i,k}c_{1,k,k-1}, \qquad (id.)$$

$$\sum_{\rho=1}^{r-k} c_{1,k+i,k+\rho}a_{k+\rho,k-2} = c_{1,k+i,k-2} - \sum_{s=k-1}^{k} a_{k+i,s}c_{1,s,k-2}, \qquad (id.)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\sum_{\rho=1}^{r-k} c_{1,k+i,k+\rho}a_{k+\rho,1} = c_{1,k+i,1} - \sum_{s=2}^{k} a_{k+i,s}c_{1,s,1}, \qquad (id.)$$

[30] See Killing, Z. v. G., II, p. 7 and 8, and also Umlauf, loc. cit., p. 24-32

We see that the first system completely determines the quantities $a_{k+i,k}$; this system being solved, the second determines the quantities $a_{k+i,k-1}$, and so on.

We can therefore assume that we have relations of the form

(45) $\quad [X_1, X_{k+i}] = c_{1,k+i,k+1}X_{k+1}f + \ldots + c_{1,k+i,r}X_rf, \quad (i = 1, 2, \ldots, r-k).$

I say further that if we have $r-k$ infinitesimal transformations of the group, $Y_{k+1}f, \ldots, Y_rf$, where

$$Y_{k+i}f = \sum_{s=1}^{r} \lambda_{k+i,s}X_sf, \quad (i = 1, 2, \ldots, r-k),$$

such that

(46) $\quad [X_1, Y_{k+i}] = \sum_{\rho=1}^{r-k} c_{1,k+i,k+\rho}Y_{k+\rho}f + \sum_{\rho=1}^{r-k} \alpha_{i,k+\rho}X_{k+\rho}f, \quad (i = 1, 2, \ldots, r-k),$

the constants $c_{1,k+i,k+\rho}$ being the same as in equation (45), the transformations $Y_{k+1}f, \ldots, Y_rf$ depend only on $X_{k+1}f, \ldots, X_rf$. Indeed, we equate, in the two sides of (46), the coefficients of $X_kf, X_{k-1}f, \ldots, X_1f$, assuming that the left side, $[X_1, Y_{k+i}]$, is replaced by the value

$$[X_1, Y_{k+i}] = \sum_{\rho,s=1}^{r} \lambda_{k+i,\rho}c_{1\rho s}X_sf.$$

We will first have

$$0 = \sum_{\rho=1}^{r-k} c_{1,k+i,k+\rho}\lambda_{k+\rho,k}, \quad (i = 1, 2, \ldots, r-k),$$

which gives $\lambda_{k+i,k} = 0$; then

$$0 = \sum_{\rho=1}^{r-k} c_{1,k+i,k+\rho}\lambda_{k+\rho,k-1}, \quad (i = 1, 2, \ldots, r-k),$$

which gives $\lambda_{k+i,k-1} = 0$, and so on, which is what we needed to prove.

That being the case, we will have

$$[X_1, [X_2, X_{k+i}]] = [X_2, [X_1, X_{k+i}]] = \sum_{s=1}^{r-k} c_{1,k+i,k+s}[X_2, X_{k+s}], \; (i = 1, 2, \ldots, r-k),$$

which shows that $[X_3, X_{k+i}]$ depends only on $X_{k+1}f, \ldots, X_rf$. We continue in this way step by step. We thus arrive at the following formula:

(47) $\quad [X_i, X_{k+j}] = \sum_{s=1}^{r-k} c_{i,k+j,k+s}X_{k+s}f, \quad (i = 1, 2, \ldots, k;\; j = 1, 2, \ldots, r-k).$

The subgroup $X_1f, \ldots, X_kf$ leaves the family of infinitesimal transformations $e_{k+1}X_{k+1}f + \ldots + e_rX_rf$ invariant.

Theorem VIII[31]. - *If $X_1f, \ldots, X_kf$ designates the largest subgroup of rank zero*

[31] Cf. Umlauf, loc. cit., p. 31.

containing the general transformation X_1f, there exists $r-k$ independent transformations of the first k: $X_{k+1}f, \dots, X_rf$, and such that the subgroup $X_1f, \dots, X_kf$ leaves the family of transformations $e_{k+1}X_{k+1}f + \dots + e_rX_rf$ invariant.

9. We will always assume that in the subgroup of rank zero $X_1f, \dots, X_kf$, the transformation X_1f is general. Then the characteristic equation relating to X_1f admits $r-k$ roots other than zero. Suppose that among these $r-k$ roots, there are p distinct ones, which we will denote by $a_1, a_2, \dots, a_p$; let $m_1, m_2, \dots, m_p$ be their degrees of multiplicity. So by proceeding in the same way as in §7, we will see that we can assume $X_{k+1}f, \dots, X_rf$ are chosen in the following way. We set

$$k + m_1 = k_1, \quad k_1 + m_2 = k_2, \quad \dots, \quad k_{p-1} + m_p = k_p = r.$$

We then have

$$\begin{aligned}
&[X_1, X_{k_{i-1}+1}] = a_iX_{k_{i-1}+1}f,\\
&[X_1, X_{k_{i-1}+2}] = a_iX_{k_{i-1}+2}f + c_{1,k_{i-1}+2,k_{i-1}+1}X_{k_{i-1}+1}f,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&[X_1, X_{k_i}] = a_iX_{k_i}f + c_{1,k_i,k_{i-1}}X_{k_{i-1}}f + \dots + c_{1,k_i,k_{i-1}+1}X_{k_{i-1}+1}f.\\
&\qquad (i = 1, 2, \dots, p), \ (k_0 = k).
\end{aligned} \tag{48}$$

Moreover, by reasoning already done in §8, we see that if an infinitesimal transformation Yf satisfies, for example, the relation

$$[X_1, Y] = a_1Yf + \alpha_1X_{k+1}f + \dots + \alpha_{k_1-k}X_{k_1}f.$$

Yf only depends on $X_{k+1}f, \dots, X_{k_1}f$. As a result, all brackets

$$[X_i, X_{k+1}], [X_i, X_{k+2}], \dots, [X_i, X_{k1}], \quad (i = 1, 2, \dots, k)$$

only depend on $X_{k+1}f, \dots, X_{k_1}f$. in other words the subgroup of rank zero leaves the family of transformations $e_{k+1}X_{k+1}f + ... + e_{k_1}X_{k_1}$ invariant.

It follows that the characteristic determinant relating to the subgroup of rank zero $X_1f, \dots, X_kf$ decomposes into a product of $p+1$ determinants:

$$\begin{vmatrix} \Sigma e_ic_{i11} - \omega & \dots & \Sigma e_ic_{ik1} \\ \dots & \dots & \dots \\ \Sigma e_ic_{i1k} & \dots & \Sigma e_ic_{ikk} - \omega \end{vmatrix} \cdot \begin{vmatrix} \Sigma e_ic_{i,k+1,k+1} - \omega & \dots & \Sigma e_ic_{i,k_1,k+1} \\ \dots & \dots & \dots \\ \Sigma e_ic_{i,k+1,k_1} & \dots & \Sigma e_ic_{i,k_1,k_1} - \omega \end{vmatrix} \dots,$$

the summations being extended from 1 to k. The first of these determinants reduces as we have already seen to $(-w)^k$; the second is reduced to

$$e_1 = 1, \ e_2 = \dots = e_k = 0, \text{ at } (a_1 - \omega)^{m_1}, \tag{49}$$

and so on. If one of these determinants, the second for example, were not, for any $e_1, e_2, \dots, e_k$, a perfect power with respect to ω, there would be at least one general transformation of the form $e_1X_1f + \dots + e_kX_kf$ for which the roots of the characteristic equation derived from this determinant would not all be equal. Then we could push the decomposition of the characteristic determinant further. We can assume that we have the following relation:

$$\begin{vmatrix} \Sigma e_ic_{i,k+1,k+1} - \omega & \dots & \Sigma e_ic_{i,k_1,k+1} \\ \dots & \dots & \dots \\ \Sigma e_ic_{i,k+1,k_1} & \dots & \Sigma e_ic_{i,k_1,k_1} - \omega \end{vmatrix} = (\omega_1 - \omega)^{m_1}, \tag{50}$$

ω_1 being a certain function of e_k, and the same for the other $p-1$ determinants. In this way the characteristic equation relating to the subgroup $X_1f, \dots, X_kf$ admits $p+1$ distinct roots, namely:

$$0, \omega_1, \omega_2, \dots, \omega_p.$$

We have, moreover, by virtue of (50),

$$\sum_{i=1}^{k} e_i \left(\sum_{\rho=1}^{m_1} c_{i,k+\rho,k+\rho} \right) = m_1 \omega_1, \tag{51}$$

which shows that these roots $\omega_1, \omega_2, \dots, \omega_p$ are homogeneous linear functions of $e_1, e_2, \dots, e_k$, that is to say

$$\omega_i = \omega_i^{(1)} e_1 + \omega_i^{(2)} e_2 + \dots + \omega_i^{(k)} e_k, \quad (i = 1, 2, \dots, p). \tag{52}$$

Theorem IX. - *If $e_1X_1f + \dots + e_kX_kf$ is the most general infinitesimal transformation of the largest subgroup of rank zero which contains a given general transformation, the characteristic determinant relative to this subgroup decomposes into a product of linear and homogeneous factors in $\omega, e_1, e_2, \dots, e_k$.*

This theorem is proved by Killing only under the assumption that the transforms $X_1f, \dots, X_kf$ commute with each other[32]. This fact arises, according to him, as soon as the given group does not contain a distinguished infinitesimal transformation[33]; however in his 3rd paper (*Math. Ann.*, t. 34, p. 66) he restricts the theorem to the case where the group is its own derived group and where there is no distinguished transformation. We will come back to this later and we will give an example where the theorem is not verified.

10. From now on we will designate the subgroup of order k and rank zero which contains a given general transformation as a *subgroup γ relative to this transformation.* The roots $0, \omega_1, \omega_2, \dots, \omega_p$ of the characteristic equation relative to this subgroup will be designated, as long as there is no fear of confusion, as *roots of the characteristic equation.* The infinitesimal transformations $X_{k+1}f, \dots, X_{k_1}f$ (see §9) will be said to b*elong to the root* ω_1 (with respect to the γ subgroup); by analogy the transformations $X_1f, \dots, X_kf$ of the γ subgroup will be said to *belong to the root* 0.

The choice of independent infinitesimal r transformations

$$X_1f, \dots, X_kf, X_{k+1}f, \dots, X_{k_1}f, \dots, X_rf,$$

made as set forth in the preceding paragraphs, will constitute what we will call *a reduced form of the group relating to the subgroup γ*: $X_1f, \dots, X_kf$.

From now on to simplify the notation, we will denote by $X_{01}f, \dots, X_{0k}f$ the infinitesimal transformations of the γ subgroup; by $X_{11}f, \dots, X_{1m_1}f$ those which *belong*

[32]Killing, Z. v. G., II, p. 9-12.

[33]Killing, Z. v. G., II, p. 8 and 9; the proof, which is based on the theory of groups of rank zero, is made only in the case where the roots of the characteristic equation are all simple.

to the root $\omega_1, \ldots$, by $X_{p1}f, \ldots, X_{pm_p}f$ those which *belong* to the root ω_p. We will agree by symmetry to set $\omega_0 = 0, m_0 = k$. Then

$$m_0 + m_1 + \ldots + m_p = r.$$

All that we have just said applies naturally to the particular cases of $k = r$ (groups of rank zero) and of $k = 1$; in the latter case, the γ subgroup contains only one infinitesimal transformation.

11. Given an arbitrary infinitesimal transformation belonging to the root ω_α, $\mathscr{X}_{\alpha 1}f$, consider a transformation of the γ subgroup, $X_{01}f$. If $[X_{01}, \mathscr{X}_{\alpha 1}]$ is not equal to $\omega_\alpha^{(1)}\mathscr{X}_{\alpha 1}f$, we will set

$$[X_{01}, \mathscr{X}_{\alpha 1}] = \omega_\alpha^{(1)}\mathscr{X}_{\alpha 1}f + \mathscr{X}_{\alpha 2}f$$

and similarly

$$[X_{01}, \mathscr{X}_{\alpha 2}] = \omega_\alpha^{(1)}\mathscr{X}_{\alpha 2}f + \mathscr{X}_{\alpha 3}f$$

and so on until

$$[X_{01}, \mathscr{X}_{\alpha h}] = \omega_\alpha^{(1)}\mathscr{X}_{\alpha h}f,$$

where the integer number h certainly cannot be larger than m_a, the degree of multiplicity of the root ω_a. However, if an invariant subgroup contains the $\mathscr{X}_{\alpha 1}f$ transformation, it will also contain $\mathscr{X}_{\alpha 2}f, \mathscr{X}_{\alpha 3}f, \ldots, \mathscr{X}_{\alpha h}f$.

Suppose then that an invariant subgroup contains an infinitesimal transformation of the form

$$\mathscr{X}f = \mathscr{X}_{\alpha 1}f + \mathscr{X}_{\beta 1}f + \mathscr{X}_{\delta 1}f + \ldots,$$

where $\mathscr{X}_{\alpha 1}f, \mathscr{X}_{\beta 1}f$ are infinitesimal transformations which belong respectively to the roots $\omega_\alpha, \omega_\beta, \omega_\delta, \ldots$. If, as we can always assume, the $p+1$ roots of the characteristic equation relative to $X_{\alpha 1}f$ are all distinct, the invariant subgroup will contain

$$\mathscr{X}'f = [X_{01}, \mathscr{X}] - \omega_\alpha^{(1)}\mathscr{X}f = \mathscr{X}_{\alpha 2}f + ((\omega_\beta^{(1)} - \omega_\alpha^{(1)}\mathscr{X}_{\beta 1}f + \mathscr{X}_{\beta 2}f) + \ldots.$$

If I prove that the invariant subgroup contains each of the transformations

$$\mathscr{X}_{\alpha 2}f,\ (\omega_\beta^{(1)} - \omega_\alpha^{(1)})\mathscr{X}_{\beta 1}f + \mathscr{X}_{\beta 2}f, \ldots,$$

then it will contain

$$(\omega_\beta^{(1)} - \omega_\alpha^{(1)})\mathscr{X}_{\beta 2}f + \mathscr{X}_{\beta 3}f, \ldots, (\omega_\beta^{(1)} - \omega_\alpha^{(1)})\mathscr{X}_{\beta h'}f$$

(h' denoting the number analogous to h), and therefore, with $\omega_\beta^{(1)} \neq \omega_\alpha^{(1)}$, it will also contain

$$\mathscr{X}_{\beta, h'-1}f, \ldots, \mathscr{X}_{\beta 1}f.$$

The invariant subgroup will contain $\mathscr{X}_{\beta 1}f$, $\mathscr{X}_{\delta 1}f, \ldots$ and consequently $\mathscr{X}_{\alpha 1}f$. But we can perform on $\mathscr{X}'f$ the same operation as on $\mathscr{X}f$ until there are no more transformations belonging to the root ω_α, then until there is no longer any transformation belonging to ω_β, etc. Finally, we will reduce the problem to the case where there is a single transformation belonging to a determined root; it then obviously belongs to the invariant subgroup.

Theorem X[34]**.** -*Given a group G of order r in a reduced form relative to a given γ subgroup, any invariant subgroup of G is completely determined by those of its infinitesimal transformations which belong to the different roots of the relative characteristic equation to the gamma subgroup, in the sense that all the other transformations of the invariant subgroup are linearly deduced from them.*

12. Given two transformations, one belonging to the root ω_a, the other to the root ω_b, what can be said of their bracket?

Suppose first that $\omega_a + \omega_b$ is not a root of the characteristic equation. So we can always assume that $\omega_a^{(1)} + \omega_b^{(1)}$ is none of the quantities $0, \omega_1^{(1)}, \omega_2^{(1)}, \ldots, \omega_p^{(1)}$. We can then consider the transformations $X_{\alpha 1}f, \ldots, X_{\alpha m_\alpha}f$ chosen in such a way that

$$[X_{01}, X_{\alpha i}] = \omega_\alpha^{(1)} X_{\alpha i}f + \lambda_{i1}X_{\alpha 1}f + \lambda_{i2}X_{\alpha 2}f + \ldots + \lambda_{i,i-1}X_{\alpha,i-1}f, \ (i = 1, 2, \ldots, m_\alpha),$$

and the same for $X_{\beta 1}f, \ldots, X_{\beta m_\beta}f$. As such, we will have

$$[X_{01}, [X_{\alpha 1}, X_{\beta 1}]] = (\omega_\alpha^{(1)} + \omega_\beta^{(1)})[X_{\alpha 1}, X_{\beta 1}],$$

and since the characteristic equation relating to $X_{01}f$ does not admit the root $\omega_\alpha^{(1)} + \omega_\beta^{(1)}$ we deduce $[X_{\alpha 1}, X_{\beta 1}] = 0$.

We then have, taking this result into account,

$$[X_{01}, [X_{\alpha 1}, X_{\beta 2}] = (\omega_\alpha^{(1)} + \omega_\beta^{(1)})[X_{\alpha 1}, X_{\beta 2}],$$

hence $[X_{\alpha 1}, X_{\beta 2}] = 0$, and so on. We see, step by step, that all the brackets $[X_{\alpha i}, X_{\beta j}]$ are zero.

Now suppose that $\omega_\alpha + \omega_\beta$ is a root, ω_δ for example. We can assume that all the roots $0, \omega_1^{(1)}, \ldots, \omega_p^{(1)}$ are distinct. So we will have

$$[X_{01}[X_{\alpha 1}, X_{\beta 1}]] = \omega_\delta^{(1)}[X_{\alpha 1}, X_{\beta 1}],$$

which shows that $[X_{\alpha 1}, X_{\beta 1}]$ belongs to the root ω_δ and is deduced linearly from $X_{\delta 1}f, X_{\delta 2}f, \ldots, X_{\delta m_\delta}f$. Moreover

$$[X_{01}, [X_{\alpha 2}, X_{\beta 1}]] = \omega_\delta^{(1)}[X_{\alpha 2}, X_{\beta 1}] + \lambda_{21}[X_{\alpha 1}, X_{\beta 1}],$$

hence, from a previous remark (§9), we see that $[X_{\alpha 2}, X_{\beta 1}]$ still belongs to the root ω_δ, and so on.

Theorem XI[35]**.** -*The bracket of two infinitesimal transformations belonging respectively to the roots ω_α and ω_β is zero if $\omega_\alpha + \omega_\beta$ is not a root, and belongs to the root ω_δ if $\omega_\alpha + \omega_\beta = \omega_\delta$.*

The indices α, β, δ are arbitrary in the sequence $0, 1, 2, \ldots, p$. If in particular $\omega_\beta = -\omega_\alpha$, the brackets $[X_{\alpha i}, X_{\beta j}]$ belong to the γ subgroup.

[34]Killing, Z. v. G., III, p. 72-73; the theorem is proved in the case where all the roots are simple.

[35]See Killing, Z. v. G., I, p. 280 sqq.; II, p. 14; III, p. 69.

13. In the latter case, we will find a remakable relation between the roots of the characteristic equation relating to such a transformation $[X_{\alpha i}, X_{\beta j}]$ of the subgroup γ ($\omega_\beta = -\omega_\alpha$).

To simplify the notation, let $\mathscr{X}_\alpha f$ and $\mathscr{X}_{\alpha'} f$ be two transformations belonging respectively to the roots ω_α and $\omega_{\alpha'} = -\omega_\alpha$. Let ω_β be any root of the characteristic equation relating to the subgroup γ; among the roots of the form $\omega_\beta + m\omega_\alpha$, m being a positive or negative integer, let $\omega_\beta + h\omega_\alpha$ be the one for which m is the largest and $\omega_\beta - h'\omega_\alpha$ the one for which m is the smallest. We set

$$\begin{array}{llll} \omega_\beta + \omega_\alpha = \omega_\gamma, & \omega_\gamma + \omega_\alpha = \omega_\delta, & \ldots, & \omega_\chi + \omega_\alpha = \omega_\lambda = \omega_\beta + h\omega_\alpha, \\ \omega_\beta - \omega_\alpha = \omega_\mu, & \omega_\mu + \omega_\alpha = \omega_\nu, & \ldots, & \omega_\rho - \omega_\alpha = \omega_\sigma = \omega_\beta - h'\omega_\alpha, \end{array}$$

Finally, set

$$\begin{aligned} [\mathscr{X}_\alpha, \mathscr{X}\alpha'] &= \sum_{i=1}^{k} e_{0i} X_{0i} f, \\ [\mathscr{X}_\alpha, X_{\beta i}] &= \sum_{j=1}^{m_\gamma} a_{(\beta i)(\gamma j)} X_{\gamma j} f, \quad (i = 1, 2, \ldots, m_\beta), \\ [\mathscr{X}_{\alpha'}, X_{\gamma i}] &= \sum_{j=1}^{m_\beta} b_{(\gamma i)(\beta j)} X_{\beta j} f, \quad (i = 1, 2, \ldots, m_\gamma), \end{aligned}$$

and analogous relationships. Then write the Jacobi identities[36]

$$\overline{\mathscr{X}_\alpha \mathscr{X}_{\alpha'} X_{\beta i}} = \overline{\mathscr{X}_\alpha \mathscr{X}_{\alpha'} X_{\gamma j}} = \ldots = 0.$$

We will then have, in particular

$$\sum_{\omega=1}^{k} e_{0\omega} c_{(0\omega)(\sigma i)(\sigma i)} = -\sum_{j=1}^{m_\rho} a_{(\sigma i)(\rho j)} b_{(\rho j)(\sigma i)}, \quad (i = 1, 2, \ldots, m_\sigma),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\sum_{\omega=1}^{k} e_{0\omega} c_{(0\omega)(\mu i)(\mu i)} = -\sum_{j=1}^{m_\beta} a_{(\mu i)(\beta j)} b_{(\beta j)(\mu i)} + \sum_{j=1}^{m_\nu} b_{(\mu i)(\nu j)} a_{(\nu j)(\mu i)}, \; (i = 1, 2, \ldots, m_\mu),$$

$$\sum_{\omega=1}^{k} e_{0\omega} c_{(0\omega)(\beta i)(\beta i)} = -\sum_{j=1}^{m_\gamma} a_{(\beta i)(\gamma j)} b_{(\gamma j)(\beta i)} + \sum_{j=1}^{m_\mu} b_{(\beta i)(\mu j)} a_{(\mu j)(\beta i)}, \; (i = 1, 2, \ldots, m_\beta),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\sum_{\omega=1}^{k} e_{0\omega} c_{(0\omega)(\lambda i)(\lambda i)} = +\sum_{j=1}^{m_\chi} b_{(\lambda i)(\chi j)} a_{(\chi j)(\lambda i)}, \quad (i = 1, 2, \ldots, m_\lambda),$$

Add element by element all the equalities we get in this manner. This gives

$$\sum_{\omega=1}^{k} e_{0\omega} (m_\sigma \omega_\sigma^{(\omega)} + \ldots + m_\mu \omega_\mu^{(\omega)} + m_\beta \omega_\beta^{(\omega)} + \ldots + m_\lambda \omega_\lambda^{(\omega)}) = 0,$$

[36] I define the abbreviation $\overline{X_i X_j X_k} = [[X_i, X_j], X_k] + [[X_j, X_k], X_i] + [[X_k, X_i], X_j]$.

and

$$(53) \quad (m_\sigma + m_\rho + \ldots + m_\beta + m_\gamma + \ldots + m_\lambda) \sum_{i=1}^{k} e_{0i}\omega_\beta^{(i)}$$
$$= (h'm_\sigma + \ldots + m_\mu - m_\gamma - \ldots - hm_\lambda) \sum_{i=1}^{k} e_{0i}\omega_\alpha^{(i)}.$$

As the coefficient $m_\sigma + m_\rho + \ldots$ is certain to be different from 0, we see that we can state the following theorem:

Theorem XII[37]. *If we consider the transformation of the subgroup γ obtained by combining two transformations belonging respectively to equal roots with opposite signs ω_α and $-\omega_\alpha$, the roots of the characteristic equations relating to this transformation are real and commensurable multiples of one of them, namely the root ω_α itself.*

If the root ω_α relating to this transformation is zero, all the others are zero[38]; this is certainly what occurs if $\alpha = 0$, that is to say if we combine two gamma transformations.

Corollary[39]. - *The characteristic equation relating to the group derived from the γ subgroup has all its roots equal to zero.*

14. Finally a formula analogous to formula (53) is obtained by considering the roots of the form $m\omega_\alpha$, m being a positive integer. If h is the largest possible value of the integer m, and setting

$$\omega_{\alpha_i} = \omega_\alpha, \quad \omega_{\alpha_2} = 2\omega_\alpha, \quad \ldots, \quad \omega_{\alpha_h} = h\omega_\alpha,$$

we have

$$(54) \quad (m_{\alpha_1} + m_{\alpha_2} + \ldots + m_{\alpha_h} \sum_{\rho=1}^{k} c_{(\alpha i)(\alpha' j)(0\rho)}\omega_\alpha^{(\rho)} = \sum_{\omega,\rho} c_{(\alpha\omega)(\alpha' f)(0\rho)} c_{(0\rho)(\alpha i)(\alpha\omega)},$$
$$(i = 1, 2, \ldots, m_\alpha;\ j = 1, 2, \ldots, m_{\alpha'})$$

We see in particular that if we have, for any ρ

$$[X_{0\rho}, X_{\alpha i}] = \omega_\alpha^{(\rho)} X_{\alpha i} f,$$

we obtain the result

$$(m_{\alpha_1} + m_{\alpha_2} + \ldots + m_{\alpha_h} - 1) \sum c_{(\alpha i)(\alpha' j)(0\rho)}\omega_\alpha^{(\rho)} = 0;$$

or if the root ω_α is simple, there is no other root of the form $m\omega_\alpha$ $(m > 1)$; or else the characteristic equation relative to $[X_{\alpha i}, X_{\alpha' j}]$ has, for any j, all its roots equal to zero[40].

15. The first eight paragraphs of this chapter are, apart from a few small changes

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[37]Cf. Killing, Z. v. G., II, p. 15 and 16.
[38]Cf. Killing, Z. v. G., III, p. 80, 99.
[39]Cf. Killing, Z. v. G., III, p. 59 sqq.
[40]Cf. Killing, Z. v. G., III, p. 89.

(§6), only a summary of the first part of the cited thesis by Umlauf; they contain the proofs of theorems due to Killing, but where precision and rigor were lacking. It is in large part to Engel that the proofs I have mentioned are due.

The theorems discussed in the last paragraphs also belong to Killing, although Theorem XII is nowhere stated in all of its generality. But Killing is limited in his papers to the case where all the non-zero roots of the characteristic equation are distinct; or at least when he examines the general case, he assumes that the transformations of the γ subgroup all commute with each other, which, according to him, occurs whenever the group is its own derived group or admits no distinguished transformation. However, this last assertion is incorrect, and, even if it were correct, Killing would not have proved in full generality Theorems IX, X, XI and XII, which are fundamental.

Chapter III: Classification of Groups Following Lie. Integrable Groups. Groups of Rank Zero.

1. Lie has long since[41] divided groups into two large classes, *integrable* groups and *non-integrable* groups.

We say that a group is *integrable* when the order of its successive derivative groups is constantly decreasing until one of them reduces to the identity transformation. A group is *non-integrable* when from a certain rank, all its derived groups are the identity, while being of an order greater than zero.

It follows immediately from this definition that *any subgroup of an integrable group is integrable, that if the group derives from a given group is integrable, the group itself is integrable*, that *the order of the derived group of a given integrable group is smaller than that of the group.*

If G is an integrable group of order r, G_1, its derived group of order $r_1 < r$, G admits at least one invariant subgroup of order $r-1$; it suffices to add to G_1 $r - r_1 - 1$ arbitrary infinitesimal transformations independent of each other. Such an invariant subgroup being itself integrable, admits an invariant subgroup with $r-2$ parameters, and so on.

Theorem I. - *If a group of order r is integrable, it admits an invariant subgroup of order $r-1$, having in turn an invariant subgroup of order $r-2$, and so on.*

This is also the first definition that Lie gave of integrable groups.

The converse of the theorem is obvious, because if a group of order r admits an invariant subgroup with $r-1$ parameters, its derived group is a subgroup of this invariant subgroup.

There is more. Lie, starting from the fact that any infinitesimale, linear and homogeneous transformation in $x_1, x_2, \ldots, x_n$ leaves invariant at least one point $x_1 : x_2 : \ldots : x_n$, and that any multiplicity invariant by an invariant subgroup g of a group G is transformed by G in a multiplicity invariante by g, proving that any integrable group, linear and homogenous, leaves invariant at least one point, one line passing through this point, one plane in two dimensions passing through this line, etc. Applying this to the adjoint group of an integrable group, we arrive at the next Theorem, due to Lie[42]

Theorem II. - *Any integrable group of order r contains an invariant subgroup of order 1, which is contained in an invariant subgroup of order 2, the latter in a*

[41] *Archiv for Math. og Nat.*, 1878, t. 3, p. 112-116. The notion of an integrable group was first introduced by Lie in 1874 in the Comptes Rendus de l'Acad. des Sc. de Christiania; the name itself of the integrable group is found for the first time in the *Leipziger Berichte*, 1889.

[42] V. Lie, *Transformationsgruppen, III*, p. 678 sqq.

sub-group of order 3, and so on; so that we can find r independent infinitesimal transformations $X_1f, X_2f, \ldots, X_rf$ such that

$$(55) \qquad [X_i, X_{i+k}] = \sum_{s=1}^{i} c_{i,i+k,s} X_s f, \quad (i = 1, 2, \ldots, r;\ k = 1, 2, \ldots, r - i).$$

In the context of theorem I, the sub-group of order $q(q = 1, 2, \ldots, r-1)$ is invariant only in the subgroup of order $q + 1$.

2. Equation (55) immediately gives us the form of the characteristic equation of an integrable group.

Indeed we have

$$(56) \qquad \Delta(\omega) = -\omega \prod_{i=1}^{r-1} \left(\sum_{\rho=i-1}^{r} e_\rho c_{\rho ii} - \omega \right).$$

The characteristic determinant of an integrable group decomposes into a product of factors linear in $\omega, e_1, e_2, \ldots, e_r$.

If we refer to the corollary of Theorem II of chap. II, we see that these linear forms $\Sigma e_\rho c_{\rho ii} (i = 1, 2, \ldots, r)$ are the invariants of the adjoint group. Therefore, they cancel each other out identically for the derived group (I, 9). So (II, 3, th.III):

Theorem III. - *A group derived from an integrable group is of zero rank.*

Killing (Z, v.G., I, p. 269) clearly states the property of the derived group to be of rank zero, when the coefficients of the characteristic equation of the group can be expressed as a function of l linear forms in $e_1, e_2, \ldots, e_r$, l being the rank of the group; but nowhere does he say that this fact is present for all integrable groups.

3. Engel is the first to prove the converse of Theorem III, namely that any group of rank zero is integrable[43]. Here is his proof, slightly modified.

Let us first notice that, according to Th. IV of Chap. II, *any subgroup of a group of rank zero is itself of rank zero.*

Moreover, let us recall the following theorem, proved by Lie[44], namely that *if a group of order r contains an integrable subgroup of order $q < r$, this subgroup is at least contained in a subgroup of order $q + 1$.*

That being the case, to prove that any group of rank zero is integrable, it suffices to show that any group of rank zero of order r contains at least one invariant subgroup of order $r - 1$. Now this is true for $r = 2$, because then we have $[X_1, X_2] = 0$. Suppose this is proved for $r = 1, 2, \ldots, s$. I claim that the theorem is true for $r = s + 1$. Indeed, let us consider in G_{s+1} of rank zero an arbitrary infinitesimal transformation. It is contained at least in a necessarily integrable two-parameter

[43]Umlauf, loc. cit., p. 35 sqq.

[44]Lie, *Transformationsgruppen*, I, p. 596 sqq., and III, p. 682.

subgroup, say g_2; g_2 is therefore contained in a subgroup g_3 of rank zero and therefore integrable if $3 \leq s$, g_3 in a subgroup g_4, and so on. Finally we arrive at a subgroup g_s of order $s = r - 1$. I claim that this subgroup of order $r - 1$ is invariant. For this it suffices to show that any one of its transformations combined with a certain transformation not being part of g_s gives rise to a transformation being part of g_s. Now let $X_1 f$ be any transformation of g_s. We can find (II, 7) $r - 1$ other transformations $X_2 f, \dots, X_{r-1} f$ such that we have

$$[X_1, X_i] = \sum_{j=1}^{i-1} c_{1ij} X_j f, \quad (i = 2, 3, \dots, r).$$

Let $X_q f$ be the first of these transformations which is not part of g_s. We then have

$$[X_1, X_q] = c_{1q1} X_1 f + c_{1q2} X_2 f + \cdots + c_{1,q,q-1} X_{q-1} f,$$

which is to say that $[X_1, X_q]$ is part of g_s. C.Q.F.D.

Any group of rank zero is therefore integrable. As a result, we can give its structure the form (55), where now all the constants cjii are zero:

$$[X_i, X_{i+j}] = \sum_{s=1}^{i-1} c_{i,i+j,s} X_s f. \tag{57}$$

Theorem IV. - *The necessary and sufficient condition for a group to be integrable is that its derived group is of zero rank.*

We see from equation (57) that *any group of rank zero contains at least one distinguished infinitesimal transformation*[45], namely $X_1 f$, and that *its derived group is of order r-2 at most.*

4. We can give an extremely simple criterion to recognize if a group is integrable or not. *It suffices,* in fact, *that the group derived from a group G of order r identically cancels the coefficient $\psi_2(e)$ of the characteristic equation of G for G to be integrable.*

Because if G were not integrable, it would have an invariant subgroup, namely one of its successive derived groups, which would be its own derived group. Now for this invariant subgroup, $\psi_2(e)$ would be identically zero. Moreover, it would be the same for $\psi_1(e)$, since the group derived from G cancels any linear invariant of the adjoint group and in particular ψ_1. We would therefore have a certain group of order q for which the characteristic equation would be of the form

$$\omega^q - \psi_3(e)\omega^{q-3} + \psi_4(e)\omega^{q-4} \cdots \pm \psi_{q-1}(e)\omega = 0, \tag{58}$$

and which would be its own derived group. Now, let us put this group in the reduced form relating to any *general* transformation. By using the notations of Ch. II, §10 sqq., we see that all the transformations of the γ subgroup: $X_{01}f, X_{02}f, \dots, X_{0k}f$, are deduced linearly from the brackets $[X_{0i}, X_{0j}]$ and $[X_{\alpha i}, X_{\alpha' j}]$, or $\omega_{\alpha'} = -\omega_\alpha$, $(\alpha = 1, 2, \dots, p)$. Now, according to Theorem XII (II, 13), for each of these transformations $[X_{0i}, X_{0j}]$, $[X_{\alpha i}, X_{\alpha' j}]$, the roots of equation (58) are *real* multiples of

[45] Cf. Umlauf, loc. cit., p. 40, Satz 12.

one of them, which, according to the theory of equations, is only possible if *all these roots are null*. We can therefore assume that for k independent transformations of the γ subgroup, the characteristic equation has all null roots, and hence it is the same for the characteristic equation relating to the γ subgroup itself. This is only possible if $k = r$, that is to say if the group is of rank zero. But then it is integrable and cannot be its own derived group.

Theorem V. - *The necessary and sufficient condition for a group of order r to be integrable is that the transformations of its derived group identically cancel the coefficient $\psi_2(e)$ of ω^{r-2} in its characteristic equations.*

Note that if $X_{m+1}f, X_{m+2}f, \ldots, X_rf$ form the group derived from an integrable group G, $\psi_2(e)$ depends only on $e_1, e_2, \ldots, e_3$, and reciprocally, according to Theorem V, if $\psi_2(e)$ depends only on $e_1, e_2, \ldots, e_m$, the group is integrable. So the equations

$$\frac{\partial\psi_2}{\partial e_{m+1}} = \frac{\partial\psi_2}{\partial e_{m+2}} = \cdots = \frac{\partial\psi_2}{\partial e_r} = 0$$

can be reduced to identities. So in the general case the group will be integrable if the equations

$$\sum_{\rho=1}^{r} c_{ik\rho}\frac{\partial\psi_2}{\partial e_\rho} = 0, \quad (i, k = 1, 2, \ldots, r)$$

can be reduced to identities, which, taking into account the expression of $psi_2(e)$ and the relations between the c_{iks}, can be written

$$(59) \qquad \sum_{\lambda,\mu,\nu=1}^{r} c_{i\lambda\mu}c_{j\mu\nu}c_{k\nu\lambda} = \sum_{\lambda,\mu,\nu=1}^{r} c_{i\mu\lambda}c_{j\nu\mu}c_{k\lambda\nu}, \quad (i, j, k = 1, 2, \ldots, r).$$

Equations (59) *express the necessary and sufficient conditions for the group to be integrable.*

5. Let us refer to the theorem stated at the end of chapter 1 and apply it to the invariant $\psi_2(e)$ of the adjoint group. We see that we can state the following propositions, consequences of Theorem V:

The linear equations

$$\frac{\partial\psi_2}{\partial e_i} = 0, \quad (i = 1, 2, \ldots, r)$$

define an integrable invariant subgroup.

If $\alpha_{i1}X_1f + \alpha_{i2}X_2f + \cdots + \alpha_{ir}X_rf$, $(i = 1, 2, \ldots, m)$ are m independent infinitesimal transformations of an invariant subgroup g of order m of the group G: $X_1f, X_2f, \ldots, X_rf$, the equations

$$\alpha_{i1}\frac{\partial\psi_2}{\partial e_1} + \alpha_{i2}\frac{\partial\psi_2}{\partial e_2} + \ldots + \alpha_{ir}\frac{\partial\psi_2}{\partial e_r} = 0, \quad (i = 1, 2, \ldots, m)$$

define an invariant subgroup g' of G, and the transformations common to this invariant subgroup g' and g form an integrable invariant subgroup of G, because for any transformation common to g and g', ψ_2 vanishes.

In particular, *if g is the group derived from G, g' is an integrable invariant subgroup.* We will see later that this is the largest integrable invariant subgroup of G; it is defined by the equations

$$\sum_{\rho=1}^{r} c_{ik\rho}\frac{\partial\psi_2}{\partial e_\rho} = 0, \quad (i,k = 1,2,\ldots,r). \tag{60}$$

6. Returning to any group of order r, integrable or not, put in the reduced form relative to a given γ subgroup. Let $e_{\alpha 1}X_{\alpha 1}f + \ldots + e_{\alpha m_\alpha}X_{\alpha m_\alpha}f$ always be the most general infinitesimal transformation belonging to the ω_α root. Let $E_{01}f, E_{02}f, \ldots, E_{0k}f$ be the subgroup of the adjoint group that corresponds to the γ subgroup.

This is a linear and homogeneous group of zero rank, and therefore integrable, which exchanges the points of the space $e_{\alpha 1}, e_{\alpha 2}, \ldots, e_{\alpha m_\alpha}$ with each other. Therefore, according to a remark already made (§1), it leaves a point of this space invariant, a line passing through this point, etc. In other words we can always choose $X_{\alpha_1}f, \ldots, X_{\alpha m_\alpha}f$ in such a way that we have

$$[X_{0i}, X_{\alpha j}] = \sum_{\rho=1}^{j} c_{(0i)(\alpha j)(\alpha\rho)}X_{\alpha\rho}f, \quad (i = 1,2,\ldots,k;\ j = 1,2,\ldots,m_\alpha) \tag{61}$$

and moreover we have

$$c_{(0i)(\alpha j)(\alpha j)} = \omega_\alpha^{(i)}. \tag{62}$$

In particular we have, regardless of i,

$$[X_{0i}, X_{\alpha 1}] = \omega_\alpha^{(i)}X_{\alpha 1}f, \quad (i = 1,2,\ldots,k).$$

Killing makes regular use of equations (61), which he does not prove, except in the case where all the transformations of the subgroup γ commute between themselves. But even in this case the demonstration he gives is not rigorous[46]. He says in effect that if, for each transformation $\mathscr{X}_0 f$ of γ, there are for example at least three transformations $X_{\alpha 1}f, X_{\alpha 2}f, X_{\alpha 3}f$, such that

$$[\mathscr{X}_0, X_{\alpha 1}] = \omega_\alpha X_{\alpha 1}f, \qquad [\mathscr{X}_0, X_{\alpha 2}] = \omega_\alpha X_{\alpha 2}f, \qquad [\mathscr{X}_0, X_{\alpha 3}] = \omega_\alpha X_{\alpha 3}f,$$

and if there are only three in general, for example for $\mathscr{X}_0 f$, we will have, for any i,

$$[X_{0i}, X_{\alpha 1}] = \omega_\alpha^{(i)}X_{\alpha 1}f, \qquad [X_{0i}, X_{\alpha 2}] = \omega_\alpha^{(i)}X_{\alpha 2}f, \qquad [X_{0i}, X_{\alpha 3}] = \omega_\alpha^{(i)}X_{\alpha 3}f.$$

However, this is not correct: it suffices to take $k = 2$, and

$$\begin{aligned}
&[X_{01}, X_{\alpha 1}] = X_{\alpha 1}f, \quad [X_{01}, X_{\alpha 2}] = X_{\alpha 2}f, \quad &&[X_{01}, X_{\alpha 3}] = X_{\alpha 3}f + X_{\alpha 1}f,\\
&[X_{02}, X_{\alpha 1}] = X_{\alpha 1}f, \quad [X_{02}, X_{\alpha 2}] = X_{\alpha 2}f + X_{\alpha 1}f, \quad &&[X_{02}, X_{\alpha 3}] = X_{\alpha 3}f.
\end{aligned}$$

7. The results stated and demonstrated in §1, 2 and 3 are due in large part to Lie; nevertheless the proof of the property of zero-rank groups being integrable was given for the first time by Engel. As for §4 and 5, they relate to entirely new results.

[46] V. Killing, Z.v.G., II, p. 10, et III, p. 68.

Killing (Z.v.G., I, p. 285 sqq.) also made a special study of zero-rank groups; but his reasoning is lacking in rigor and he claims inaccurate theorems[47].

[47]Killing, Z.v.G., I, p. 288; regarding this see Umlauf, loc. cit., p. 45, note.

Second Part

Chapter IV: Semi-Simple Groups. Properties of the Characteristic Equations of a Simple Group.

1. Among the non-integrable groups, a very important class is formed by the groups of order $r > 1$ which do not admit an invariant integrable subgroup. We will call these *semi-simple* groups. Among semi-simple groups, those of order $r > 1$ which do not admit an invariant subgroup are said to be *simple*. It follows from this that a simple group is its own derivative group and therefore is not integrable.

Recall that, according to the previous chapter, for a group to be semi-simple, the equations $\frac{\partial \psi_2}{\partial e_i}$ must lead to $e_1 = e_2 = \ldots = e_r = 0$. *The coefficient $\psi_2(e)$ of the characteristic equation of a semi-simple group is therefore reducible to a sum of r independent squares.*

Conversely, if the discriminant of the quadratic form $\psi_2(e)$ is different from zero, the group G does not admit any integrable invariant subgroup; because if it were to admit one, g, by taking the successive derivative groups of g, we would end up finding one, g', invariant in G and forms either of a single infinitesimal transformation, or of infinitesimal transformation exchangeable between them. If $X_{m+1}f, X_{m+2}f, \ldots, X_r f$ are these transformations, we immediately verify that the characteristic determinant does not depend on $e_{m+1}, e_{m+2}, \ldots, e_r$, and therefore $\psi_2(e)$ is reducible to a sum of m independent squares at most.

TODO

Chapter V: Simple Integer Systems of Order l. Simple Groups of Rank l.

1. This chapter is devoted to the calculations I indicated in the previous chapter.

We first have to determine all the systems of inequivalent simple integers a_{ij} of order l. This determination will be simplified by the remark already made (IV, 9) that any simple system of order p contains at least one simple subsystem of order $p-1$; if therefore we have determined all the inequivalent simple systems of order $p-1$, we have, for each of them, only $2(p-1)$ integers to be determined so as to satisfy the conditions of Theorem X (IV, 8). We will then have, according to what was said in the previous chapter (IV, 10) to determine $2(p-1)$ new integers so that the total determinant is zero and to verify that one of the p new roots can obtained by means of the other $p-1$.

We start with the simplest values of p.

For $p=1$ there is only one system $a_{11}=-2$, which forms the two roots $\pm\omega_1$.

Let us start from this system and seek first to determine a_{12} and a_{21} in such a way that the determinant satisfies

$$a_{11}a_{22}-a_{12}a_{21}=4-a_{12}a_{21}=0;$$

requiring either $a_{12}=a_{21}=\pm 2$, or $a_{12}=4a_{21}=\pm 4$, or $a_{21}=a_{12}=\pm 4$. The last two cases give either $\omega_1=\mp 2\omega_2$, or $\omega_2=\mp 2\omega_1$, and both are excluded. The case $\omega_2=\pm\omega_1$ remains; so we cannot have any other root than $\pm\omega_1$.

We now determine a_{12} and a_{21} in such a way that

$$0<a_{11}a_{22}-a_{12}a_{21}\leq 4$$

and that the obtained system of order 2 is simple; thus it is necessary $a_{12}a_{21}=1,2$ or 3. Since we can always change w2 to -w2 if necessary, and also exchange w1 and w2, we see that there are three simple systems of order 2:

1°	$a_{11}=a_{22}=-2,$	$a_{12}=a_{21}=-1;$	
2°	$a_{11}=a_{22}=a_{12}=-2,$	$a_{21}=-1;$	
3°	$a_{11}=a_{22}=-2,$	$a_{12}=-3,$	$a_{21}=-1.$

TODO

Translated by JONATHAN TROUSDALE