The Geometry of Simple Groups

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Introduction

In a recent paper[1] developing and supplementing an earlier article published in collaboration with J.A. Shouten[2], I studied the spaces with affine connection, without curvature or torsion-free, representing continuous transformation groups. This study applied to the most general groups, and it was local. In the case of simple groups, the representative torsion-free spaces are Riemannian, either complex or real (with ds definite or indefinite). Complex spaces represent simple groups with complex parameters; real spaces represent simple groups with real parameters, unitary (if ds is definite), or non-unitary (if ds is indefinite). The last two cases are distinguished from each other by the property that the space is either closed or open.

Riemann spaces representative of simple real unitary groups fall into a more general and very important category of Riemann spaces, characterized by the property that their Riemannian curvature is preserved by parallel transport; their study is reduced to the study of those which I have called *irreducible* and which all relate to simple groups[3]. In each of these irreducible spaces, the Riemannian curvature has the same sign everywhere; in the same class there are both positive curvature spaces and negative curvature spaces. Given a simple structure, the representative spaces of the corresponding unitary real groups have positive curvature; negative curvature spaces of the same class are not representative of any group, but their group of displacements is isomorphic to the group of the given structure with complex parameters. If this is the simple structure with three parameters, the two spaces are the three-dimensional spaces with constant curvature, positive or negative; the first represents the group of rotations of ordinary space; the group of displacements of the second is isomorphic to the *complex* homographic group of one variable.

[1] É. Cartan, *La Géométrie des groupes de transformations.* (J. Math. pures et appl., t. 6, 1927, pp.1-119).

[2] É. Cartan and J.A. Schouten, On the Geometry of the group-manifold of simple and semi-simple groups. (Proc. Akad. Amsterdam, t. 29, 1926, pp. 803-815).

[3] The determination of all spaces is made in a paper, the first part of which has just appeared. (Bull. Soc. Math. de France, t. 54, 1924, pp. 214-264). See also É. Cartan, *Sur les espaces de Riemann dans lesquels le transport par parallélisme conserve la courbure.* (Rend. Acc. Lincei, (6), t. 3^1, 1926, pp. 544-547).

The detailed study of the most general irreducible Riemann spaces is of great interest, both from the point of view of group theory and from the geometric point of view. It will be the subject of a later brief. In the present memoir I am only concerned with the two particular classes mentioned above (spaces representative of simple real unitary groups and their correspondents with negative curvature). The study made here is no longer local; it relates rather to the properties of space coming under the *Analysis Situs*, to the distribution of geodesics, to the complete determination of their mixed groups of isometry, of their different Klein forms, etc. The questions which arise are, moreover, of a very different nature depending on whether the space has positive curvature or negative curvature.

The first introductory chapter is devoted to the topology of simple real unitary groups; it has its point of departure in the research of Weyl relating to the theory of semi-simple groups[4]; the question is taken up in full; Weyl's results are complete and the questions which arise are resolved to the end using one of my recent papers[5]

Part II is devoted to the spaces of real unitary groups; the distribution of geodesics is studied quite completely for the simply connected forms of these spaces; it reveals, associated with each point in space, the existence of a certain number of *antipodal varieties* (which can be reduced to points) which are in a way *necked varieties* for the closed geodesics resulting from the given point. There are as many as units as the rank of the group.

Part III is devoted to spaces with negative curvature whose group of displacements has a simple complex structure. They are all just related. We can thanks to them (and this is what will happen in all the other irreducible spaces with negative curvature) to solve important problems relating to their group of displacements. I will only point out the following result: simple complex groups have, from the point of view of Analysis Situs, the same properties as the corresponding unitary real groups, and they always admit a simply connected linear representative. This theorem itself results from a remarkable mode of generation of finite transformations of the complex gorup: to take just one example, each complex rotation of ordinary space is decomposable in one way and only one into one. real and a rotation of a purely imaginary angle around a real axis. Finally, the spaces in question also have their importance from a purely geometric point of view, but this importance will be manifested above all for the more general irreducible spaces with negative curvature.

I will assume that the fundamental principles of the theory of simple groups

[4]H. Weyl, *Theorie der Darstellung kontinuierlicher halb-einfacher Gruppen durch lineare Transformationen.* (Math. Zeitschr., t. 23, 1925, pp. 271-309; t. 24, 1925, pp. 328-395).

[5]É. Cartan, *Les tenseurs irréductibles et les groupes linéaires simples et semi-simples.* (Bull. Sc. Math., 2$^{\text{nd}}$ series, t. 49, 1925, pp. 130-152).

are known[6].

Chapter I.

The Topology of Simple Unitary Groups

I. The Fundamental Polyhedron of the Adjoint Group

1. We know[7] that to each type of simple group of order r belongs a real unitary form, with r real parameters, characterized by the property that the sum of the squares of the characteristic roots of an arbitrary infinitesimal transformation is a negative definite quadratic form $-\phi(e)$. The unitary groups are, for the four major classes of simple groups, respectively isomorphic:

A) to the unimodular linear group of a positive definite Hermitian form

$$x_1\bar{x}_1 + x_2\bar{x}_2 + \ldots + x_{l+1}\bar{x}_{l+1};$$

B) D) to the linear group of a positive definite quadratic form in $n = 2l + 1$ (type B) or $n = 2l$ (type D) variables;

C) to the linear group leaving invariant a definite positive Hermitian form;

$$x_1\bar{x}_1 + x_2\bar{x}_2 + \ldots + x_{2l}\bar{x}_{2l};$$

and an outer quadratic form

$$[x_1x_2] + [x_3x_4] + \ldots + [x_{2l-1}x_{2l}].$$

In the preceding notation, l denotes the *rank* of the group, which will be discussed below.

2. Let G be a simple unitary group, Γ its adjoint group. Γ is a linear group with r real variables

$$e_1, e_2, \ldots, e_r,$$

which leaves the positive definite quadratic form $\phi(e)$ invariant. Each transformation can be represented by a matrix T of order r, determinant equal to 1.

Any matrix T can, in an infinite number of ways, be generated by an infinitesimal transformation Y of Γ. Among the characteristic roots of Y, l are zero, the other $r - l$ are two-by-two equal and opposite; they are linear combinations

[6] In this regard, the reader may refer to my Thesis (Paris, Nony, 1894) or to the previously cited paper of Weyl; reading the paper cited in footnote ([1]), will not be useless either.

[7] É. Cartan, *Les tenseurs irréductibles et les groupes linéaires simples et semi-simples.* (Bull. Sc. Math., 2[nd] series, t. 49, 1925, pp. 135).

with integer coefficients determined by l of them (called *fundamental*). All these roots are purely imaginary: we will designate them, with Weyl, by the notation

$$2\pi i\phi_\alpha;$$

the quantities ϕ_α are the *angular parameters* of the transformation Y.

The matrix T generated by Y admits l characteristic roots equal to 1, the others are the quantities $e^{2\pi i\phi_\alpha}$.

3. If an infinitesimal transformation Y is general, that is to say does not admit more than l characteristic zero roots, there exist $l-1$ other infinitesimal transformations independent of each other and independent of Y, which enjoy the property of commuting among themselves and commuting with Y. We thus obtain an Abelian subgroup γ of order l. If

$$e_1Y_1 + e_2Y_2 + \ldots + e_lY_l$$

is the most general infinitesimal transformation of γ, the angular parameters ϕ_α are linear forms in $e_1, e_2, \ldots, e_l$ of which l are linearly independent.

If we now start from a singular infinitesimal transformation Y, that is to say admitting more than l characteristic zero roots, there exists more than l independent transformations exchangeable with Y. We can ask ourselves if, among these transformations, there exists at least one which is not singular. This is what we will demonstrate.

Let Y_1 be a particular transformation commuting with Y; let Y_2 be a particular transformations commuting with Y and Y_1 (and linearly independent of Y and Y_1), and so on. Suppose that we can thus find λ independent transformations

$$Y, Y_1, \ldots, Y_{\lambda-1},$$

commuting among themselves and such that no other transformation of the group commutes at the same time with each of them. The characteristic ω_α roots of the transformation

$$eY + e_1Y_1 + \ldots + e_{\lambda-1}Y_{\lambda-1}$$

are *linear*[8] forms in $e, e_1, \ldots, e_{\lambda-1}$. Among these linear forms, there are at least λ independent ones; otherwise there would exist a non-zero transformation $\sum e_iY_i$ having all its characteristic roots zero; this is impossible, since the sum of the squares of the characteristic roots of an arbitrary transformation of the group is a *definite* form. Of the characteristic roots of an arbitrary transformation a minimum of λ are independant, this proves, by the very definition of the rank, that λ is at most equal to l. But on the other hand λ cannot be less than l,

[8]This is because the transformations commute with each other.

since l of the ω_α roots are zero and to each linear form ω_α correspond one or more transformations X such that we have

$$\left[\sum e_i Y_i, X\right] = \omega_\alpha X,$$

whatever the coefficients $e, e_1, \ldots, e_{\lambda-1}$. There would therefore exist $l-\lambda$ transformations independent of Y and commuting with Y, which is contrary to the hypothesis.

The integer λ being equal to l, and the number of identically zero ω_α linear forms not exceeding l, it suffices to give the coefficients e_i numerical values not canceling any of the non-identically zero ω_α forms to obtain a *general* transformation; the transformation given Y is thus part of the γ subgroup defined by this general transformation.

4. Any infinitesimal transformation Y is therefore part of at least one abelian subgroup γ, containing an infinite number of general transformations. All the γ subgroups are, moreover, homologous to each other in the continuous adjoint group Γ ([9]), so that any transformation of Γ is homologous to a transformation of a particular γ subgroup.

That said, let us look at the l fundamental angular parameters $\phi_1, \phi_2, \ldots, \phi_l$ of an arbitrary infinitesimal transformation as the Cartesian coordinates of a point in a space of l dimensions. We will choose the unit vectors of coordinates so that the positive definite quadratic form

$$\sum_\alpha \phi_\alpha^2$$

represents, to a nearly constant factor, the square, of the distance from a point to the origin.

Any point $M(\phi_i)$ represents an infinitesimal transformation of each γ subgroup, and consequently an infinite number of homologous infinitesimal transformations between them. If none of the parameters ϕ_α is zero, these transformations are general and consequently form a set ∞^{r-l}. If h of the parameters ϕ_α are zero (h even), each transformation represented by M is invariant by a subgroup with $l+h$ parameters and therefore admits ∞^{r-l-h} homologues.

Within a given γ subgroup, an infinitesimal transformation admits a certain number of homologues; their representative points are obtained by carrying out on the $r-l$ parameters ϕ_α (regarded as letters) a finite group $\mathcal{G}'$ of substitutions;

[9]This is because the *general* infinitesimal transformations Y, each of which defines a γ subgroup, form a *connected* set; indeed, as we will see in a moment, the singular transformations fill, in the domain of the group, one or more manifolds with 3 dimensions less than this domain.

these substitutions preserve the linear relations with integer coefficients which exist between the angular parameters[10]. Geometrically the group $\mathcal{G}'$, operating on representative points M, is a group of rotations and symmetries, generated by $\frac{r-l}{2}$ symmetries with respect to the hyperplanes $\phi_\alpha = 0$ ([11]).

If we consider the $\frac{r-l}{2}$ hyperplanes $\phi_\alpha = 0$ led by the origin, they divide the space into a number of undefined regions (polyhedral angles) convex. Each of them represents the *fundamental domain* (D) of the group $\mathcal{G}'$, and any infinitesimal transformation of γ is homologous to one transformation and only one inside this region. Any convex region, bounded by a certain number of $\phi_\alpha = 0$ hyperplanes, *and such that no other of these hyperplanes crosses it*, can be taken as a fundamental domain. We will verify later that all these regions admit exactly l hyperplane faces.

Any point inside the fundamental domain (D) represents ∞^{r-l} homologous infinitesimal transformations; any point located on one of its faces, or one of its edges, etc., represents at most ∞^{r-l-2} homologous transformations.

5. Moving on to finite transformations, or to the T matrices of the Γ group. Let Y be one of its infinitesimal generating transformations, belonging to a certain γ subgroup; we can represent T and Y by the same point M. Now inside the same γ subgroup, the matrix T can be generated by an infinite number of infinitesimal transformations different from Y; they are those obtained by adding to the fundamental angular parameters ϕ_i arbitrary integers. Let us then consider the lattice (R) of points with integer coordinates ϕ_i.

The same matrix T is represented by an infinite number of points, homologous among themselves with respect to the lattice (R).

Suppose that $l + 2k$ of the characteristic roots of T are equal to 1; we then prove that, T being invariant by a subgroup with $l + 2k$ parameters of Γ, there exist ∞^{r-l-2k} matrices homologous to T. The hypothesis made amounts to saying that, among the angular parameters ϕ_α of Y, $2k$ are integers. The matrix T can therefore be invariant by a group larger than its generating transformation Y.

6. The set of operations of group $\mathcal{G}'$ and of translations $\mathcal{T}$ which leave the lattice (R) invariant generates a group $\mathcal{G}_1$ of displacements and symmetries. Two points M homologous with respect to $\mathcal{G}_1$ represent matrices T homologous

[10]There can be substitutions enjoying this property without belonging to G'. See E. Cartan, *Le principe de dualité et la théorie des groupes simples et semi-simples.* (Bull. Sc. Math., 2nd series, t. 49, 1925, pp. 365-366).

[11]This interpretation of the group $\mathcal{G}'$ as a group of rotations and symmetries, as well as that of its generating operations, is due to Weyl, *Theorie der Dartellung kontinuierlicher halbeinfacher Gruppen durch lineare Transformationen.* (Math. Zeitschr., 24, 1925, pp. 367-371). The group $\mathcal{G}'$ is the group (S) of Weyl.

with respect to Γ.

We will now consider the set of hyperplanes (Π) obtained by equating one of the angular parameters ϕ_α to an arbitrary integer. All these hyperplanes share space in an infinity of convex polyhedra; let (P) be one of them, which we can suppose to be inside the undefined fundamental domain (D). We will show that any matrix T can be represented by at least one point of (P).

Let us start from the remark, due to Weyl[12], that the matrices represented by a given point of the same hyperplane (Π), admitting at least $l+2$ characteristic roots equal to 1, form in the space of the group at most an $r-l-2$ dimensional manifold; consequently the matrices represented by the different points of the hyperplanes (Π) form a finite number of manifolds with

$$(r-l-2)+(l-1)=r-3$$

dimensions. It is therefore possible to go from one point to another in the space of the group, i.e. to go continuously from any matrix T to any other matrix T', avoiding the singular matrices. Let M_0 be a particular point inside (P), let T_0 be one of the matrices represented by M_0, and let T be any matrix. By passing from T_0 to T in order to avoid singular matrices, the corresponding representative point, starting from M_0, will remain inside (P)[13], and consequently there exists indeed inside (P) a point M representative of T. In particular the unit matrix must be represented, this means that at least one of the vertices of (P) belongs to the lattice (R). We can therefore assume, by one of the translations $\mathcal{T}$, that the polyhedron (P) has one of its vertices at the origin O.

7. We are now in a position, taking Weyl's general point of view, to study the topology of the space of the adjoint group Γ, in particular to find out whether there exist in this space any firm contours not reducible to a point by continuous deformation.

TODO

31. With the isometry group of the simply connected space $\mathcal{E}$ being transitive, the study of geodesics from any point is reduced to that of geodesics from the point of origin O, which corresponds to the identical transformation of G. These geodesics correspond to the different one parameter subgroups of G; to look for a geodesic joining O to a given point A is to look for an infinitesimal transformation generating the finite transformation represented by A. It follows immediately that *by any two points of $\mathcal{E}$ there always passes a geodesic (and even an infinity if $l>1$)*. We will leave aside in what follows the case $l=1$, which corresponds to the three-dimensional spherical space (and to the

[12] Math. Zeitschr., t. 24, 1925, p. 379.

[13] We can rigorously show that when the matrix T varies in a continuous way without ever being singular, the representative point M can also be followed by continuity, without any ambiguity.

elliptical space).

Any direction resulting from O represents an infinitesimal transformation of G, which belongs at least to an abelian subgroup γ (§3). The gamma transformations provide in space $\mathcal{E}$ an l-dimensional manifold E_l passing through O. This manifold has zero Riemannian curvature, since the rotation associated with an elementary parallelogram whose sides represent the infinitesimal transformations U and V has the effect of giving, at the vector X, the geometric increase $[[U,V],X]$, and is everywhere zero if the bracket $[U,V]$ is zero: this is the case for any two transformations of γ.

The manifold E_l is therefore locally Euclidean; it is moreover totally geodesic, since it represents a subgroup of G. One can analytically define a point A of E_l by the l fundamental angular parameters $\phi_1, \ldots, \phi_l$ of the infinitesimal transformation Y of γ which generates the finite transformation represented by A; the distance OA, measured on the corresponding geodesic, is equal, to a nearly constant factor, to the square root of the sum $\sum \phi_\alpha^2$ extended to $r-l$ angular parameters. We thus see that *the representation used in Chapter I in a Euclidean space with l dimensions is nothing more than an application on this Euclidean space of the locally Euclidean manifold E_l.*

There is, however, an important difference. The variety E_l is not an undefined Euclidean space; if we develop it on the l-dimensional Euclidean space, it gives the lattice $(\bar{R})$ of parallelepipedes, *each of which fully represents E_l.*

We see from this that any point of E_l can be joined to O by an infinite number of geodesics, located entirely in E_l, and which have for images, in Euclidean space, the lines joining the origin O to the different homologous points of a point given with respect to the group $\bar{\mathcal{T}}$ of the translations of the network $(\bar{R})$. Those whose directional parameters are rational are closed. All the geodesics, located in E_l, joining O to A are moreover isolated; none cut itself.

32. Instead of representing E_l by the fundamental parallelepiped of the network $(\bar{R})$, it is better to represent it by the polyhedron $(\mathcal{D})$ introduced in §26. The operations of the group G' correspond to rotations of the continuous isotropy group of $\mathcal{E}$ which bring the variety E_l into coincidence with itself, (but without leaving all the points of E_l fixed); in each of these rotations, the different $(l+1)$-hedra (P) in which the polyhedron $(\mathcal{D})$ is decomposed are transformed into each other; they are moreover in the same number as the operations of G' and each of them is a fundamental domain for the discontinuous group G'.

Figure 5 thus represents one of the plane manifolds E_2 of the simply connected 8-dimensional space of the simple group of type A of rank 2. The opposite edge of the regular hexagon must be regarded as identical, their points corresponding to each other by the translation which brings these two edges into coincidence. The three translations corresponding to the three pairs of opposite edges generate the group $\bar{\mathcal{T}}$ of the translations of the lattice $(\bar{R})$ (it is basically the holonomy group of the Clifford plane E_2 with respect to the Euclidean plane). The three vertices O_1 constitute only one point of E_2; the same is true of the three O_2 vertices. The figure highlights three closed geodesics distended from O; each will pass successively through the points O_1 and O_2, which divide it into three equal parts. We have one of these geodesics starting for example from O in the horizontal direction of the figure to O_1; this geodesic continues with one of the horizontal dimensions O_1 O_2, then ends with the horizontal radius O_2 O which leads back to the starting point. These three geodesics, of the same length, intersect in O at angles of 120° (if we take them in the direction which leads first to point O_1). Of course, there is an infinite number of other closed geodesics in E_2, but they are longer.

Figure 6 shows the domain $(\mathcal{D})$ representative of a variety E_2 of the simply connected space with 10 dimensions of the simple unitary group of the type B of rank 2. The opposite sides of the square correspond by translation and must be considered as identical. The four vertices of the square represent the same point of E_2; as for the midpoints of the edges, they represent two distinct points A and A'. In the figure there are four closed geodesics coming from O; two of them, which intersect at O at a right angle, are divided in their middle by the point O_1; two others (whose length is the same as the preceding ones in the ratio $\frac{1}{\sqrt{2}}$) are divided in their middle, one by point A, the other by point A'. There are also in the figure two other closed geodesics passing through O_1 and shared in their middle, one by A, the other by A'.

FIG. 7 represents a variety E_2 of the space with 14 dimensions of the simple group of type G. The vertices of the regular hexagon represent two distinct points A and A' of E_2; the midpoints of the edges represent three other distinct points B, B', B''. The figure represents three closed geodesics originating from O and passing successively through points A and A' which divide them into three equal parts; they are on the other hand cut in their middle by one of the points B, B' or B''. The figure shows three other closed geodesics cut in their middle by one of these three points, but passing neither through A nor through A'. The compared lengths of these two types of geodesics are 3 and $\sqrt{3}$.

33. Let us return to the general study of geodesics. If a given point A other than O belongs to a single manifold E_l, all the geodesics joining O to A are in this manifold; they are isolated and there is an infinite number of them.

If the point A belongs to an infinity of manifolds E_l, the finite transformation of G represented by A is exchangeable with more than l independent infinitesimal transformations, that is to say $l + \lambda$. We will have the number λ by looking for how many, among the $r - l$ angular parameters ϕ_α of A, are integers. The point A is then invariant by a subgroup $g_{l+\lambda}$ of the continuous group of rotations around O. It therefore belongs to ∞^λ distinct E_l manifolds, and in each of them there exists a countably infinite number of geodesics joining O to A. Suppose that the direction in O of one of these geodesics is invariant by a subgroup $g_{l+\lambda}$ of the group of rotations around O; the corresponding geodesic will belong to ∞^{μ} distinct E_l varieties. If $\mu = \lambda$, this geodesic will be isolated in space $\mathcal{E}$ (among the set of geodesics joining O to A. If $\mu < \lambda$, this geodesic will belong to a continuous variety with $\lambda - \mu + 1$ dimensions of geodesics joining O to A, and this variety will be obtained by applying to the given geodesic all the rotations of the group $g_{l+\lambda}$ which leaves fixed the points O and A.

Finally, *certain geodesics joining O to A may not be isolated*; this case arises if the group of rotations which leave the point A invariant is greater than the group of rotations which leave the direction in O of the geodesic invariant; the dimension of the continuous manifold of which geodesics are a part is the difference increased by 1 between the orders of these two groups.