# Mathematics for Machine Learning - Solutions

## Chapter 2 - Linear Algebra

João Tomás Caldeira

**2.1.**

**a)** We first attempt to show the group properties:

1. Closure

   For the set $\mathbb{R} \setminus \{-1\}$ to be closed under $\star$, the latter should take two of its elements and yield a number different from -1.

   Let us assume that we can get -1 from applying the operator to some $a, b \in \mathbb{R} \setminus \{-1\}$. Then,

   $$ab + a + b = -1 \iff ab + a + b + 1 = 0 \iff (a+1)(b+1) = 0, \quad (1)$$

   so either $a = -1$ or $b = -1$. We have reached a contradiction; recall that we introduced the assumption that $a \neq -1$ and $b \neq -1$. Therefore, $\mathbb{R} \setminus \{-1\}$ is closed under $\star$.

2. Associativity

   $$\begin{aligned}(a \star b) \star c &= (ab + a + b) \star c \\ &= (ab + a + b)c + (ab + a + b) + c \\ &= a + ab + abc + ac + b + bc + c \\ &= (abc + ab + ac) + (bc + b + c) + a \\ &= a(b \star c) + a + (b \star c) \\ &= a \star (b \star c)\end{aligned} \quad (2)$$

3. Neutral element

   $\forall a \in \mathbb{R} \setminus \{-1\} : a \star 0 = a \cdot 0 + a + 0 = a$, and $0 \star a = 0 \cdot a + 0 + a = a$, so 0 is the neutral element of $(\mathbb{R} \setminus \{-1\}, \star)$.

4. Inverse element We try to find some $x$ for which $\forall a \in \mathbb{R} \setminus \{-1\} : a \star x = x \star a = 0$. We have

   $$\begin{aligned}a \star x = 0& \\ &\iff ax + a + x = 0 \\ &\iff x = -\frac{a}{a+1}.\end{aligned} \quad (3)$$

Since $a \neq -1$ necessarily, the solution for equation 3 always exists. Thus, $x$ is the inverse element of $(\mathbb{R} \setminus \{-1\}, \star)$. *This concludes proof that* $(\mathbb{R} \setminus \{-1\}, \star)$ *is a group.*

5. Commutativity

   $\forall a, b \in \mathbb{R} \setminus \{-1\} : a \star b = ab + a + b = ba + b + a = b \star a.$

   Alternatively, we may observe that $a \star b = b \star a$, by the commutativity of addition and multiplication of real numbers.

We conclude that $(\mathbb{R} \setminus \{-1\}, \star)$ is an Abelian group.
**b)**

$$\begin{aligned} 3 \star x \star x = 15 &\iff (3x + 3 + x) \star x = 15 \\ &\iff 3x^2 + 3x + x^2 + 3x + 3 + x + x = 15 \\ &\iff 4x^2 + 8x - 12 = 0 \\ &\iff x = 1 \vee x = -3 \end{aligned} \tag{4}$$

Therefore, $x \in \{-3, 1\}$.
**2.2.** HELP.
**2.3.** Let $\boldsymbol{A}, \boldsymbol{B}$ be in $\mathcal{G}$ such that

$$\boldsymbol{A} = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}, \ \boldsymbol{B} = \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}, \text{ with } x, y, z, a, b, c \in \mathbb{R}$$

Then,

$$\boldsymbol{A} \cdot \boldsymbol{B} = \begin{bmatrix} 1 & a + x & c + bx + z \\ 0 & 1 & b + y \\ 0 & 0 & 1 \end{bmatrix}$$

1. Closure

   Since $a + x, b + y, c + bx + z \in \mathbb{R}$, then $\boldsymbol{A} \cdot \boldsymbol{B} \in \mathcal{G}$. Therefore, $\mathcal{G}$ is closed under $\cdot$ (matrix multiplication).

2. Associativity

   Let us ake $\boldsymbol{A}, \boldsymbol{B}$ as defined above, and let $\boldsymbol{C}$ be in $\mathcal{G}$ such that

   $$\boldsymbol{C} = \begin{bmatrix} 1 & p & r \\ 0 & 1 & q \\ 0 & 0 & 1 \end{bmatrix}, \text{ with } p, q, r \in \mathbb{R}.$$

   Then,

$$
\begin{aligned}
(\boldsymbol{A}\cdot\boldsymbol{B})\cdot\boldsymbol{C} &= \begin{bmatrix} 1 & a+x & c+bx+z \\ 0 & 1 & b+y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & p & r \\ 0 & 1 & q \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & p+a+x & r+qa+qx+c+bx+z \\ 0 & 1 & q+b+y \\ 0 & 0 & 1 \end{bmatrix}.
\end{aligned} \tag{5}
$$

Similarly, $\boldsymbol{A}\cdot(\boldsymbol{B}\cdot\boldsymbol{C})$ yields the same result. Therefore, $\cdot$ is associative.

3. Neutral element

   $\mathbb{I}\cdot\boldsymbol{A} = \boldsymbol{A} = \boldsymbol{A}\cdot\mathbb{I}, \forall \boldsymbol{A}\in\mathcal{G}$.

   Therefore, $\mathbb{I}$ is the neutral element.

4. Inverse element

   We need to find the inverse such that

$$
\boldsymbol{A}\cdot\boldsymbol{A}^{-1} = \mathbb{I} \text{ (neutral element), and } \boldsymbol{A}^{-1}\cdot\boldsymbol{A} = \mathbb{I}.
$$

To find the *right* inverse $\boldsymbol{A}_r^{-1}$ of $\boldsymbol{A}$, which needs to satisfy $\boldsymbol{A}\cdot\boldsymbol{A}_r^{-1} = \mathbb{I}$, we solve the linear system $\boldsymbol{A}\cdot\boldsymbol{X} = \mathbb{I}$. We write the system in augmented notation $[\boldsymbol{A}|\mathbb{I}]$ and solve using Gaussian Elimination (G.E.), which yields

$$
[\boldsymbol{A}|\mathbb{I}] \overset{G.E.}{=} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -x & xy-z \\ 0 & 1 & 0 & 0 & 1 & -y \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right], \text{ so } \boldsymbol{A}_r^{-1} = \begin{bmatrix} 1 & -x & xy-z \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{bmatrix} \in \mathcal{G}.
$$

Since the inverse element is unique, if there is a left inverse $\boldsymbol{A}_l^{-1}$ (i.e., such that $\boldsymbol{A}_l^{-1}\boldsymbol{A} = \mathbb{I}$), then it is equal to the right inverse $\boldsymbol{A}_r^{-1}$. However, since matrix multiplication is not commutative, we need to check that $\boldsymbol{A}_r^{-1}\boldsymbol{A} = \mathbb{I}$ indeed:

$$
\begin{aligned}
\boldsymbol{A}_r^{-1}\boldsymbol{A} &= \begin{bmatrix} 1 & -x & xy-z \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & x-x & z-xy+xy-z \\ 0 & 1 & y-y \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbb{I}.
\end{aligned} \tag{6}
$$

Thus, every element of $\mathcal{G}$ has an inverse. *This concludes proof that $(\mathcal{G}, \cdot)$ is a group.*

5. Commutativity

   We need only find some $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{G}$ for which $\boldsymbol{A} \cdot \boldsymbol{B} \neq \boldsymbol{B} \cdot \boldsymbol{A}$ to show that $\cdot$ is not commutative. Using $\boldsymbol{A}, \boldsymbol{B}$ as defined above, we have

$$\boldsymbol{B} \cdot \boldsymbol{A} = \begin{bmatrix} 1 & x+a & z+ax+c \\ 0 & 1 & y+b \\ 0 & 0 & 1 \end{bmatrix},$$

   which differs from $\boldsymbol{A} \cdot \boldsymbol{B}$ in the element in the position $1, 3$. In particular, e.g., for $A = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, we have

$$\boldsymbol{A} \cdot \boldsymbol{B} = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \neq \boldsymbol{B} \cdot \boldsymbol{A} = \begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   Therefore, $\cdot$ is not commutative, and thus $(\mathcal{G}, \cdot)$ is a non-Abelian group.

**2.4.**
**a)**
IN PROGRESS...