Mathematics for Machine Learning - Solutions
Chapter 2 - Linear Algebra

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2.1.
a) We first attempt to show the group properties:

1. Closure

   For the set $\mathbb{R} \setminus \{-1\}$ to be closed under $\star$, the latter should take two of its elements and yield a number different from -1.

   Let us assume that we can get -1 from applying the operator to some $a, b \in \mathbb{R} \setminus \{-1\}$. Then,

   $$ab + a + b = -1 \iff ab + a + b + 1 = 0 \iff (a+1)(b+1) = 0, \quad (1)$$

   so either $a = -1$ or $b = -1$. We have reached a contradiction; recall that we introduced the assumption that $a \neq -1$ and $b \neq -1$. Therefore, $\mathbb{R} \setminus \{-1\}$ is closed under $\star$.

2. Associativity

   $$\begin{aligned}(a \star b) \star c &= (ab + a + b) \star c \\ &= (ab + a + b)c + (ab + a + b) + c \\ &= a + ab + abc + ac + b + bc + c \\ &= (abc + ab + ac) + (bc + b + c) + a \\ &= a(b \star c) + a + (b \star c) \\ &= a \star (b \star c)\end{aligned} \quad (2)$$

3. Neutral element

   $\forall a \in \mathbb{R} \setminus \{-1\} : a \star 0 = a \cdot 0 + a + 0 = a$, and $0 \star a = 0 \cdot a + 0 + a = a$, so 0 is the neutral element of $(\mathbb{R} \setminus \{-1\}, \star)$.

4. Inverse element We try to find some x for which $\forall a \in \mathbb{R} \setminus \{-1\} : a \star x = x \star a = 0$. We have

   $$\begin{aligned}&a \star x = 0 \\ &\iff ax + a + x = 0 \\ &\iff x = -\frac{a}{a+1}.\end{aligned} \quad (3)$$

Since $a \neq -1$ necessarily, the solution for equation 3 always exists. Thus, x is the inverse element of $(\mathbb{R} \setminus \{-1\}, \star)$. *This concludes proof that* $(\mathbb{R} \setminus \{-1\}, \star)$ *is a group.*

5. Commutativity

   $\forall a, b \in \mathbb{R} \setminus \{-1\} : a \star b = ab + a + b = ba + b + a = b \star a.$

   Alternatively, we may observe that $a \star b = b \star a$, by the commutativity of addition and multiplication of real numbers.

We conclude that $(\mathbb{R} \setminus \{-1\}, \star)$ is an Abelian group.
b)

$$\begin{aligned} 3 \star x \star x = 15 &\iff (3x + 3 + x) \star x = 15 \\ &\iff 3x^2 + 3x + x^2 + 3x + 3 + x + x = 15 \\ &\iff 4x^2 + 8x - 12 = 0 \\ &\iff x = 1 \vee x = -3 \end{aligned} \tag{4}$$

Therefore, $x \in \{-3, 1\}$.
2.2. HELP.
2.3. Let $\boldsymbol{A}, \boldsymbol{B}$ be in $\mathcal{G}$ such that

$$\boldsymbol{A} = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}, \ \boldsymbol{B} = \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}, \text{ with } x, y, z, a, b, c \in \mathbb{R}$$

Then,

$$\boldsymbol{A} \cdot \boldsymbol{B} = \begin{bmatrix} 1 & a + x & c + bx + z \\ 0 & 1 & b + y \\ 0 & 0 & 1 \end{bmatrix}$$

1. Closure

   Since $a + x, b + y, c + bx + z \in \mathbb{R}$, then $\boldsymbol{A} \cdot \boldsymbol{B} \in \mathcal{G}$. Therefore, $\mathcal{G}$ is closed under $\cdot$ (matrix multiplication).

2. Associativity

   Let us ake $\boldsymbol{A}, \boldsymbol{B}$ as defined above, and let $\boldsymbol{C}$ be in $\mathcal{G}$ such that

   $$\boldsymbol{C} = \begin{bmatrix} 1 & p & r \\ 0 & 1 & q \\ 0 & 0 & 1 \end{bmatrix}, \text{ with } p, q, r \in \mathbb{R}.$$

   Then,

$$
\begin{aligned}
(\boldsymbol{A}\cdot\boldsymbol{B})\cdot\boldsymbol{C} &= \begin{bmatrix} 1 & a+x & c+bx+z \\ 0 & 1 & b+y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & p & r \\ 0 & 1 & q \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & p+a+x & r+qa+qx+c+bx+z \\ 0 & 1 & q+b+y \\ 0 & 0 & 1 \end{bmatrix}.
\end{aligned} \tag{5}
$$

Similarly, $\boldsymbol{A}\cdot(\boldsymbol{B}\cdot\boldsymbol{C})$ yields the same result. Therefore, $\cdot$ is associative.

3. Neutral element

   $\mathbb{I}\cdot\boldsymbol{A} = \boldsymbol{A} = \boldsymbol{A}\cdot\mathbb{I}, \forall \boldsymbol{A} \in \mathcal{G}$.

   Therefore, $\mathbb{I}$ is the neutral element.

4. Inverse element

   We need to find the inverse such that

$$
\boldsymbol{A}\cdot\boldsymbol{A}^{-1} = \mathbb{I} \text{ (neutral element), and } \boldsymbol{A}^{-1}\cdot\boldsymbol{A} = \mathbb{I}.
$$

To find the *right* inverse $\boldsymbol{A}_r^{-1}$ of $\boldsymbol{A}$, which needs to satisfy $\boldsymbol{A}\cdot\boldsymbol{A}_r^{-1} = \mathbb{I}$, we solve the linear system $\boldsymbol{A}\cdot\boldsymbol{X} = \mathbb{I}$. We write the system in augmented notation $[\boldsymbol{A}|\mathbb{I}]$ and solve using Gaussian Elimination (G.E.), which yields

$$
[\boldsymbol{A}|\mathbb{I}] \overset{G.E.}{=} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -x & xy-z \\ 0 & 1 & 0 & 0 & 1 & -y \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right], \text{ so } \boldsymbol{A}_r^{-1} = \begin{bmatrix} 1 & -x & xy-z \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{bmatrix} \in \mathcal{G}.
$$

Since the inverse element is unique, if there is a left inverse $\boldsymbol{A}_l^{-1}$ (i.e., such that $\boldsymbol{A}_l^{-1}\boldsymbol{A} = \mathbb{I}$), then it is equal to the right inverse $\boldsymbol{A}_r^{-1}$. However, since matrix multiplication is not commutative, we need to check that $\boldsymbol{A}_r^{-1}\boldsymbol{A} = \mathbb{I}$ indeed:

$$
\begin{aligned}
\boldsymbol{A}_r^{-1}\boldsymbol{A} &= \begin{bmatrix} 1 & -x & xy-z \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & x-x & z-xy+xy-z \\ 0 & 1 & y-y \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbb{I}.
\end{aligned} \tag{6}
$$

Thus, every element of $\mathcal{G}$ has an inverse. *This concludes proof that $(\mathcal{G}, \cdot)$ is a group.*

5. Commutativity

We need only find some $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{G}$ for which $\boldsymbol{A} \cdot \boldsymbol{B} \neq \boldsymbol{B} \cdot \boldsymbol{A}$ to show that $\cdot$ is not commutative. Using $\boldsymbol{A}, \boldsymbol{B}$ as defined above, we have

$$\boldsymbol{B} \cdot \boldsymbol{A} = \begin{bmatrix} 1 & x+a & z+ax+c \\ 0 & 1 & y+b \\ 0 & 0 & 1 \end{bmatrix},$$

which differs from $\boldsymbol{A} \cdot \boldsymbol{B}$ in the element in the position $1,3$. In particular, e.g., for $A = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, we have

$$\boldsymbol{A} \cdot \boldsymbol{B} = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \neq \boldsymbol{B} \cdot \boldsymbol{A} = \begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore, $\cdot$ is not commutative, and thus $(\mathcal{G}, \cdot)$ is a non-Abelian group.

2.4.

a) The matrices have dimensions (3×2) and (3×3), respectively. Therefore, the matrix product is not defined for these two matrices.

b)

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 & 5 \\ 10 & 9 & 11 \\ 16 & 15 & 17 \end{bmatrix}$$

c)

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 5 & 7 & 9 \\ 11 & 13 & 15 \\ 8 & 10 & 12 \end{bmatrix}$$

d)

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 4 & 1 & -1 & -4 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & -1 \\ 2 & 1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 14 & 6 \\ -21 & 2 \end{bmatrix}$$

e)

$$\begin{bmatrix} 0 & 3 \\ 1 & -1 \\ 2 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 4 & 1 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 12 & 3 & -3 & -12 \\ -3 & 1 & 2 & 6 \\ 6 & 5 & 1 & 0 \\ 13 & 12 & 3 & 2 \end{bmatrix}$$

2.5. For each of the systems of linear equations, we start by writing them in augmented matrix notation and perform Gaussian Elimination to obtain reduced row-echelon form.

a)

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 2 & 5 & -7 & -5 & -2 \\ 2 & -1 & 1 & 3 & 4 \\ 5 & 2 & -4 & 2 & 6 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 0 & 3 & -5 & -3 & -4 \\ 0 & 0 & -2 & 2 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

We can see from the last row that the equation system is inconsistent and thus has no solution. Therefore, $S = \emptyset$.

b)

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 3 \\ 1 & 1 & 0 & -3 & 0 & 6 \\ 2 & -1 & 0 & 1 & -1 & 5 \\ -1 & 2 & 0 & -2 & -1 & -1 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 0 \end{array}\right]$$

The third row is a tautology, and therefore redundant in the system. We remove it, rearrange the rows, and apply the "minus 1 trick":

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 0 & -2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{array}\right]$$

The right-hand side of the system is a particular solution, while the two columns with -1 on the diagonal show the directions of the solution space. The solution set is thus

$$S = \left\{ \boldsymbol{x} \in \mathbb{R}^5 : \boldsymbol{x} = \begin{bmatrix} 3 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} + \lambda_1 \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 3 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \lambda_1, \lambda_2 \in \mathbb{R} \right\}$$

2.6.

We start by writing the system of linear equations in augmented matrix form and apply Gaussian Elimination to obtain reduced row-echelon form:

$$\left[\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 \end{array}\right]$$

Applying the "minus 1 trick", we get

$$\left[\begin{array}{cccccc|c} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{array}\right]$$

The right-hand side of the system is a particular solution, while the three columns with -1 on the diagonal show the directions of - and thys span - the solution space. The solution set is then

$$S = \left\{ \boldsymbol{x} \in \mathbb{R}^6 : \boldsymbol{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + \lambda_1 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R} \right\}.$$

2.7. We have $\boldsymbol{A}\boldsymbol{x} = 12\boldsymbol{x} \iff \boldsymbol{A}\boldsymbol{x} - 12\boldsymbol{x} = \boldsymbol{0} \iff (\boldsymbol{A} - 12\mathbb{I})\boldsymbol{x} = \boldsymbol{0}$. Writing this system of linear equations in augmented form, we have

$$\left[\begin{array}{ccc|c} -6 & 4 & 3 & 0 \\ 6 & -12 & 9 & 0 \\ 0 & 8 & -12 & 0 \end{array}\right]$$

We can also write the constraint $\sum_{i=1}^{3} x_i = 1 \iff x_1 + x_2 + x_3 = 1$ as another row (equation) in the above matrix, which we also solve to reduced row-echelon form:

$$\left[\begin{array}{ccc|c} -6 & 4 & 3 & 0 \\ 6 & -12 & 9 & 0 \\ 0 & 8 & -12 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Therefore, we have a unique solution $\begin{bmatrix} 3 & 3 & 2 \end{bmatrix}^\top$.

2.8. IN PROGRESS...