Computational Physics II 5640, Spring 2017

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1 Finite-difference method for advection equation.

We implement the Lax finite difference scheme to solve the 1-D advection equation:

$$\frac{\partial u}{\partial t} = -v\frac{\partial u}{\partial x}. \tag{1}$$

We insist on solutions of the form:

$$u_j^{n+1} = \frac{1}{2}\left(u_{j+1}^n + u_{j-1}^n\right) - \frac{v\Delta t}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right), \tag{2}$$

where n and j are the time and space indices respectively. With this we can start from any initial condition, u_j^0, and work forward to get the entire solution. Here we start with $u_j^0 = exp(-x^2/25)$, with three different Δt's, where we can see the stability of the energy conservation is highly dependent on Δt.

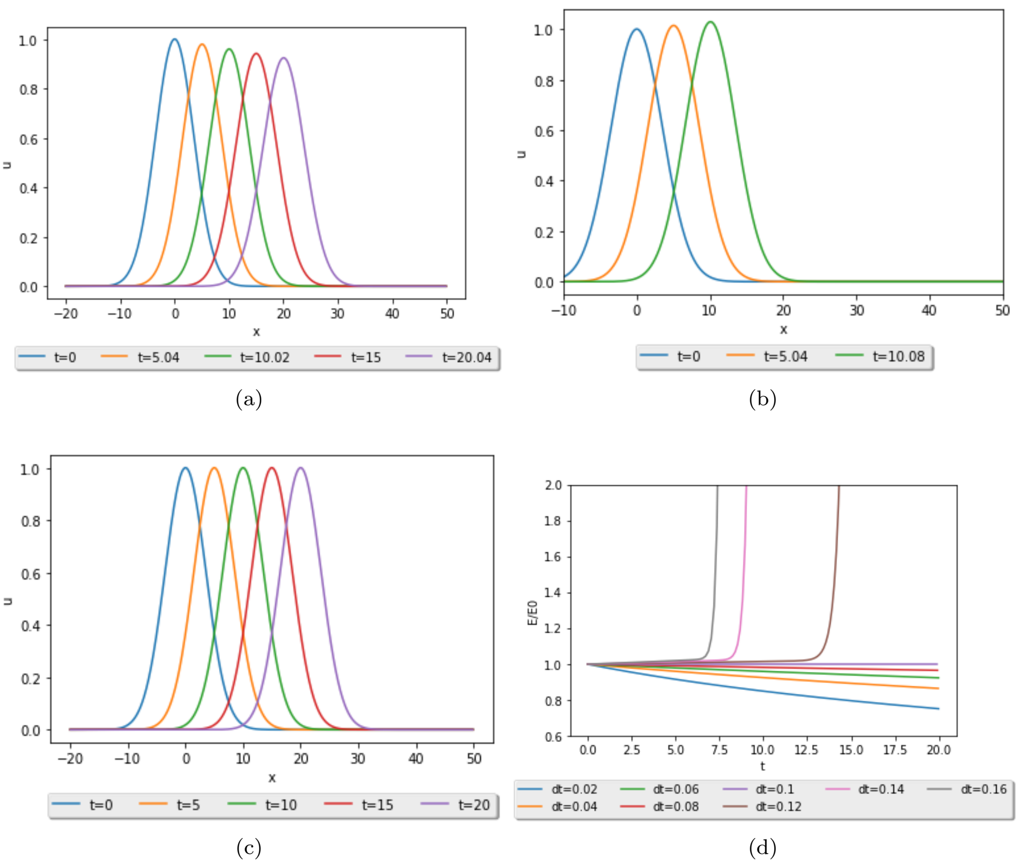


Figure 1: Energy stability in finite difference schemes. (a) $\Delta t = 0.06$, (b) $\Delta t = 0.14$, and (c) $\Delta t = 0.1$. $N = 100$, $v = 1$, and $\Delta x = 0.1$.

In figure 1 we get clear evidence that the energy stability of this method is precariously perched on $\Delta t = \Delta x/v$. In figure 1(d) we can see many simulations where this result is confirmed.

2 Crank-Nicholson method for diffusion equation.

Next we consider the diffusion equation:

$$\dot{u} = Du''. \tag{3}$$

we will use the implicit Crank-Nicholson method, where we insist on solutions of the form:

$$u_j^{n+1} = u_j^n + \frac{D\Delta t}{2\Delta x^2}\left[\left(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}\right) + \left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)\right], \tag{4}$$

which we can rearrange into the form:

$$\begin{bmatrix} (1+2r) & -r & & & 0 \\ & \ddots & \ddots & \ddots & \\ & -r & (1+2r) & -r & \\ & \ddots & \ddots & \ddots & \\ 0 & & & -r & (1+2r) \end{bmatrix} \times \begin{bmatrix} u_1^{n+1} \\ \vdots \\ \vdots \\ \vdots \\ x_J^{n+1} \end{bmatrix} = \begin{bmatrix} (1-2r) & r & & & 0 \\ & \ddots & \ddots & \ddots & \\ & r & (1-2r) & r & \\ & \ddots & \ddots & \ddots & \\ 0 & & & r & (1-2r) \end{bmatrix} \times \begin{bmatrix} u_1^{n} \\ \vdots \\ \vdots \\ \vdots \\ x_J^{n} \end{bmatrix}, \tag{5}$$

where $r = D\Delta t/2\Delta x^2$. From here we can take the matrix product of the right side which gives us:

$$\begin{bmatrix} (1+2r) & -r & & & 0 \\ & \ddots & \ddots & \ddots & \\ & -r & (1+2r) & -r & \\ & \ddots & \ddots & \ddots & \\ 0 & & & -r & (1+2r) \end{bmatrix} \times \begin{bmatrix} u_1^{n+1} \\ \vdots \\ \vdots \\ \vdots \\ x_J^{n+1} \end{bmatrix} = \begin{bmatrix} d_1 \\ \vdots \\ \vdots \\ \vdots \\ d_J \end{bmatrix}, \tag{6}$$

which is a tridiagonal matrix equation, that can be solved using the method from reference [1].

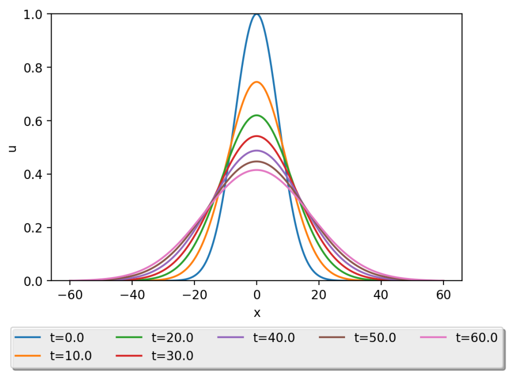
With initial state $u(x, 0) = exp(-x^2/w^2)$ We find an analytical solution:

$$u(x,t) = \frac{1}{\sqrt{1 + 4Dt/w^2}} exp\left(-\frac{x^2}{w^2 + 4Dt}\right), \tag{7}$$

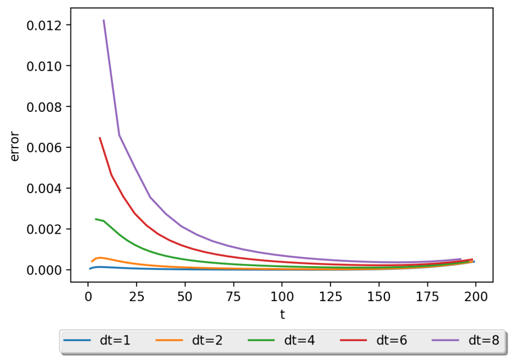
which we can show is a solution as:

$$\frac{\partial}{\partial t}u(x,t) = D\frac{\partial^2}{\partial t^2}u(x,t) = \left[\frac{1}{\sqrt{1+4Dt/w^2}} \times \frac{x^2 4D}{(w^2+4Dt)^2} + \frac{2D}{(w^2+4Dt)^{3/2}}\right] exp\left(-\frac{x^2}{w^2+4Dt}\right). \tag{8}$$

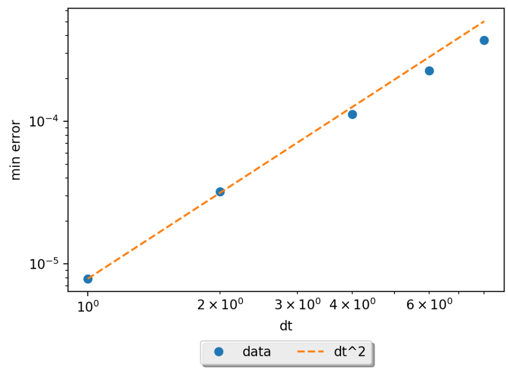
We can use this as a check of our numerical solution.



(a)



(b)



(c)

Figure 2: (b) error is defined as $|u(0,t) - u_0^n|/u(0,t)$. $D = 2$, $w = 10$, $J = 1199$,and $\Delta x = 0.1$

Here in figure 2(a) we see the expected behavior of diffusion over time. In figures 2(b) and (c) we see how the error goes as $\approx \Delta t^2$.

3 Korteweg – de Vries equation.

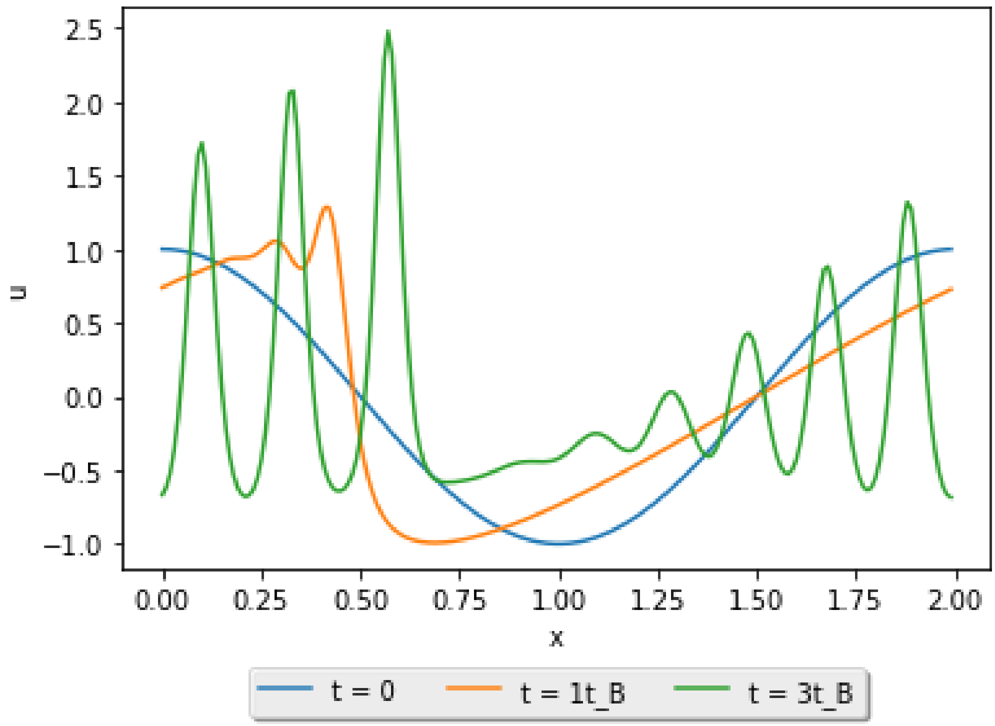
Here we show the explicit leapfrog solution of the so-called KdV equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \delta^2 \frac{\partial^3 u}{\partial x^3}. \tag{9}$$

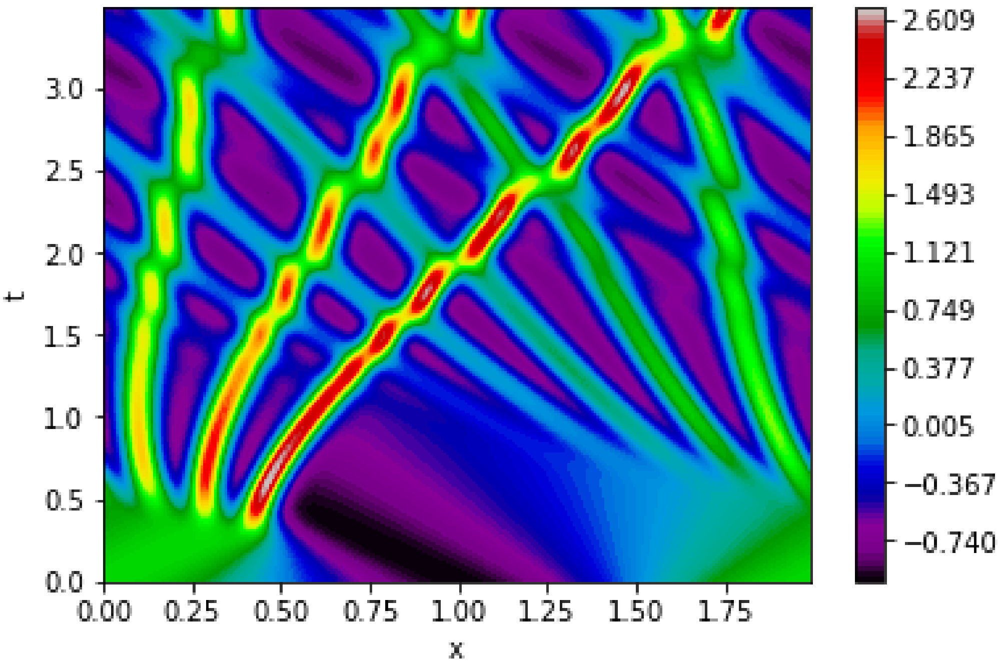
We insist on solutions of the form:

$$u_j^{n+1} = u_j^{n-1} - \frac{\Delta t}{3\Delta x}\left(u_{j+1}^n + u_j^n + u_{j-1}^n\right)\left(u_{j+1}^n - u_{j-1}^n\right) + \frac{\delta^2 \Delta t}{\Delta x^3}\left(u_{j+2}^n - 2u_{j+1}^n + 2u_{j-1}^n - u_{j-2}^n\right), \tag{10}$$

where δ is the amplitude of dispersion. In figure 3 we present the time evolution of our equation with initial state $u(x, 0) = \cos(\pi x)$, with periodic boundary conditions. We can see in figure 3(a) that there is a special time $t_B = 1/\pi$, or the breakdown time, where the singularities start to form in x, and in (b) we see how those eight singularities form and their particle-like trajectories.



(a)



(b)

Figure 3: $\Delta t = 2e - 7$, $\Delta x = 0.1$, and $\delta = 0.022$

References

[1] *https://en.wikipedia.org/wiki/Tridiagonal_matrix_algorithm*