OxWaSP: Probability & Approximation

Lecture 2: Perfect Simulation and Queues

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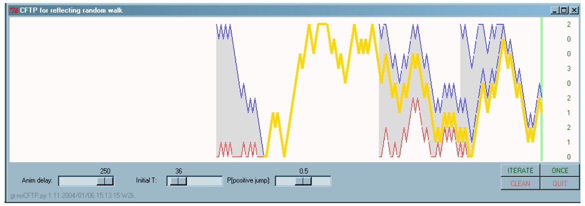
Perfect simulation: the basic idea.

Propp and Wilson (1996) Coupling from the Past (*CFTP*).
Very basic case: reflected simple random walk on state-space $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.



Ingredients:

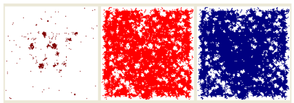
- Past → Present, not Present → Future;
- Monotonicity;
- Capture all starts using upper and lower processes;
- Re-use randomness.

Other examples of perfect simulation

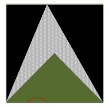
“Backwards from the past” constructions to analyse very general queues were introduced by Loynes 1962.

1. Dead Leaves example (WSK and Thönnes 1999):

2. Simple image analysis (Propp and Wilson 1996):



3. Use of small sets (Murdoch and Green 1998):



WSK (2015): “Introduction to *CFTP* using *R*”.

CFTP for "bounded" queues

1. Classic *CFTP* requires "bounded" state-space.
2. So focus initially on queues with finite capacity ("turn customers away when waiting room is full"). Seek to find exact draws from equilibrium.
3. Murdoch and Takahara (2006) study *CFTP* for: $M/M/1/c$, $M/M/c/c$, $GI/M/c/c$, $GI/G/c/c$. Here c means, if arrivals find c people already in the system then they go away never to return.
4. Integrated Masters Dissertation, Liu (2015): $M/D/1/c$ based on residual total workload. Problem: Need to alter upper process very carefully to allow for failure of monotonicity!

The trouble with M/D/1/c

Monotonicity can fail!

- Suppose Q_1 is at capacity, with current customer almost having completed service;
- and suppose Q_2 has one spare place, so Q_2 is "smaller" than Q_1.
- Someone arrives! There is room in Q_2 but not in Q_1.
- So Q_2 can jump above Q_1. This can break monotonicity.
- Fix:
  use $M/D/1/c+1$ for upper process,
  use $M/D/1/c-1$ for lower process.

  Failure of boundedness would be much trickier.

Classic CFTP and unbounded state-spaces

What to do if the state-space is "unbounded"? Example: the Lindley recursion for $GI/G/1$ provides a representation that *nearly* delivers *CFTP*, but unboundedness means we can't determine when to stop (see Foss and Tweedie 1998).

- "Unbounded" state-spaces can sometimes be small! (Key: is the chain uniformly ergodic? If so then classic *CFTP* is possible *in principle*; Foss and Tweedie 1998. Practical example from point process theory: Häggström, van Lieshout, and Møller 1999)
- One could truncate and hope. (Somewhat misses point of perfect simulation.)
- Murdoch (2000): induce uniform ergodicity by occasional proposals using independence sampler.
- Dominated *CFTP* (*domCFTP*): next section.

Basic idea for dominated *CFTP*

- Classic *CFTP* fails when we can't construct both upper and lower processes.
- So try to replace these by random processes in statistical stationarity ("dominating processes").
- If we can guarantee a coupling which maintains "sandwiching" for the resulting envelope processes, and if they coalesce to run as the target process, then we can rescue the *CFTP* algorithm.
- We'll need to be able to do the following:
  - draw from joint stationary distribution of the dominating processes;
  - simulate them backwards in time;
  - use their paths to simulate envelope processes forwards in time to coalesce as target processes;
  - check for coalescence, and ensure this happens on a reasonable time-scale.

Birth-death chain

Nonlinear birth and death processes: I

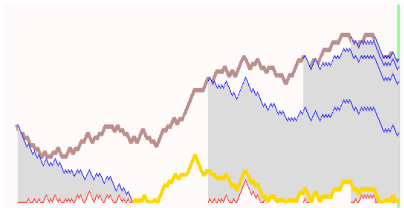
Adapted from WSK (1997).

- X is nonlinear immigration-death process:
  $X \to X - 1$ at rate μX;
  $X \to X + 1 \ldots \quad \alpha_X$ where $\alpha_X \leq \alpha_\infty < \infty$.
  No maximum (not uniformly ergodic), so no classic *CFTP*!
- *Bound* by *linear* immigration-death process Y:
  $Y \to Y - 1$ at rate μY;
  $Y \to Y + 1 \ldots \quad \alpha_\infty$.
- Produce X from Y by censoring births and deaths:
  if $Y \to Y - 1$ then $X \to X - 1$ with c.prob. X/Y;
  if $Y \to Y + 1 \ldots \quad X \to X + 1 \ldots \quad \alpha_X/\alpha_\infty$.

The domCFTP construction

Nonlinear birth and death processes: II

- Given trajectory of Y, build trajectories of X starting at every $0 \leq X_0 \leq Y_0$ and then staying below Y.
- Because Y is *reversible*, with *known equilibrium* (detailed balance!) we can simulate Y backwards, then run forwards with sandwiched X realizations.
- Identify "golden thread" of X started arbitrarily far back in time. (Animation describes analogous random walk.)



Dominated CFTP: a recipe

Basic ingredients:

- dominating process:
  - draw from equilibrium;
  - simulate backwards in time.
- sandwiching:
  $\text{Lower}_1 \precsim \text{Lower}_2 \precsim \ldots \precsim \text{Targets} \precsim \ldots \precsim \text{Upper}_2 \precsim \text{Upper}_1$
- coalescence:
  eventually a Lower and an Upper process must coalesce to produce a Target process.
- computability:
  is the price tag of Perfection too high?

Generalities

- Corcoran and Tweedie (2001) produce a general recipe for *domCFTP* for Harris chains. Roughly speaking:
  - Identify small set to which dominating process returns;
  - Check for regeneration at each visit;
  - Check domination maintained if no regeneration.
- WSK (2004) used this idea to show *domCFTP* is possible in principle for all geometrically ergodic chains:
  - Use Foster-Lyapunov criterion to build set-valued dominating process;
  - Model using a discrete-time $D/M/1$ queue!
  - Ensure ergodicity by sub-sampling;
  - Exploit regeneration (uses small-set theory).
- Connor and WSK (2007b, 2007a) extends this to some non-geometrically ergodic positive-recurrent chains.

Super-stable M/G/c and *domCFTP* (I)

Sigman (2011) pioneered *domCFTP* for multi-server queues.
Key step: find amenable dominating process.

- Restrict to super-stable case (queue remains stable if we remove all but one server).
- Workload of $M/G/1$ queue as stable dominating process.
- Same $M/G/1$ workload if PS not FCFS.
- But $M/G/1[PS]$ is dynamically reversible (so we can reverse time in equilibrium).
- Recover $M/G/1[FCFS]$ from workload $M/G/1[PS]$.
- $M/G/c$: *FCFS* workload smaller than $M/G/1[FCFS]$ (FCFS is optimally efficient: Wolff 1977, 1987).
- In particular, domination is sample-wise if each service time is assigned at initiation of service.
- Coalescence forced when $M/G/1[PS]$ empties. (Finite mean if finite second moment of service time.)

Super-stable M/G/c and *domCFTP* (II)

Natural objections to what has been achieved so far

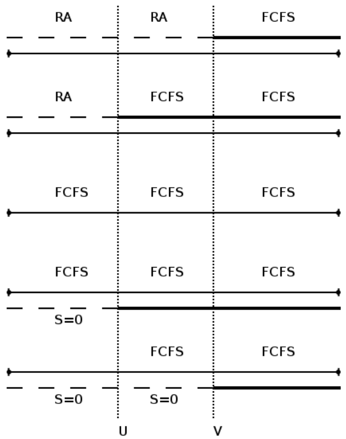
The Sigman (2011) result is pioneering, but only partial.

1. Coalescence is ensured by running backwards in time till dynamically-reversed $M/G/1[PS]$ empties.
   But this will be inefficient if the target $M/G/c$ workload is such that $M/G/1$ is nearly unstable.
2. Worse, the interesting case for $M/G/c$ is exactly when the $M/G/1$ is not stable.
   (When else would you pay for more than one server?!)
3. Sigman (2012) describes an importance-sampling approach when the corresponding $M/G/1$ is not stable.
   Unfortunately mean termination time is infinite.

Stable M/G/c and *domCFTP* (I)

Connor and Kendall (2015): dominate with $M/G/c[RA]$.
RA = "random assignment", so c copies of $M/G/1$, independent fractional arrival processes.

- Evidently stable if and only if $M/G/c$ is stable!
- Domination! (extend Asmussen 2003, Chapter XII)



Stable M/G/c and *domCFTP* (I)

Algorithm 1 (coalescence when $M/G/c[RA]$ empties):
NB: Processes run backwards are crowned with a tilde.

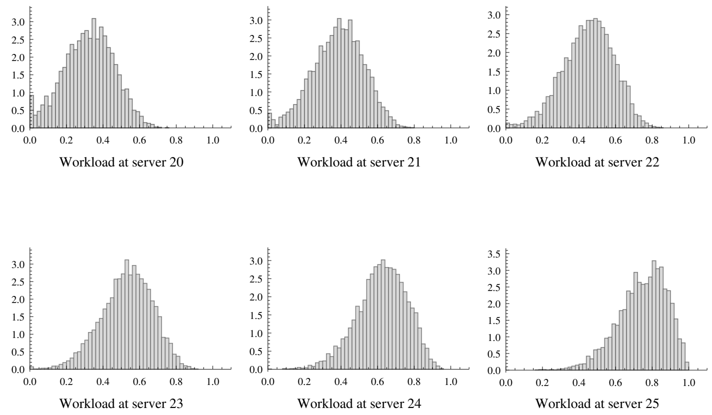
1. *Consider* $(M/G/1[PS])^c$ *process* $\tilde{Y}$, *run backwards in statistical equilibrium. Draw from* $\tilde{Y}(0)$.
2. *Simulate c components of reversed-time* $(\tilde{Y}(\tilde{t}) : \tilde{t} \geq 0)$ *over* $[0, \tilde{\tau}]$, *where* $\tilde{\tau}$ *is smallest reversed time such that all components are empty at* $\tilde{\tau}$.
3. *Use* $(\tilde{Y}(\tilde{t}) : \tilde{t} \in [0, \tilde{\tau}])$ *to construct (dynamic) time reversal, and so build* $(Y(t) : \tau \leq t \leq 0)$, *an* $M/G/c[RA]$ *process (set* $\tau = -\tilde{\tau}$*).*
4. *Use Y to evolve X, an* $M/G/c[FCFS]$ *process, over* $[\tau, 0] = [-\tilde{\tau}, 0]$, *started in the empty state.*
   Now return $X(0)$.

Stable M/G/c and *domCFTP* (II)

Algorithm 2 (use upper and lower processes):

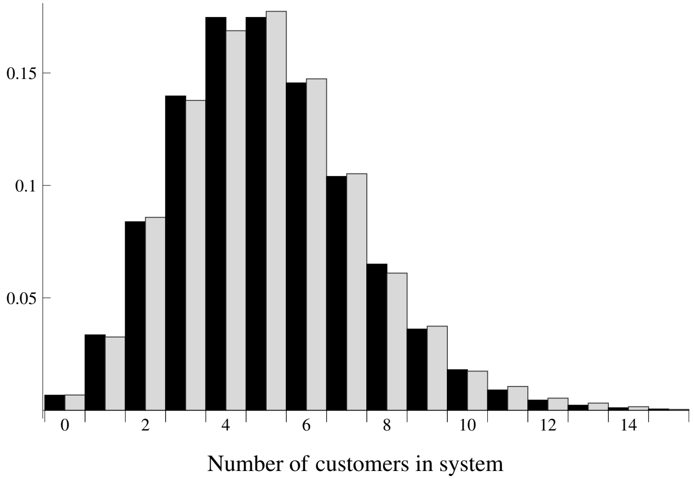
1. *Consider a $(M/G/1[PS])^c$ process $\widetilde{Y}$, run backwards in statistical equilibrium. Draw from $\widetilde{Y}(0)$.*
2. *Fix suitable $\widetilde{T} = -T$. Evolve queue for server j (independently of other servers) until first time $\widetilde{\tau}_j \geq \widetilde{T}$ that this server is empty, for $j = 1, \ldots, c$.*
3. *Construct $M/G/1[FCFS]$ Y_j over corresponding reversed time interval $[-\widetilde{\tau}_j, 0]$, for $j = 1, \ldots, c$.*
4. *Produce lists $\mathcal{L}^*_T$, $\mathcal{L}_T$ of service durations, arrival times.*
5. *Construct upper sandwiching process, $U_{[T,0]}$ over $[T, 0]$.*
6. *Construct lower sandwiching process, $L_{[T,0]}$ over $[T, 0]$.*
7. *Check for coalescence.* Otherwise, extend.

Results of *domCFTP* for $M/G/c$ (I)



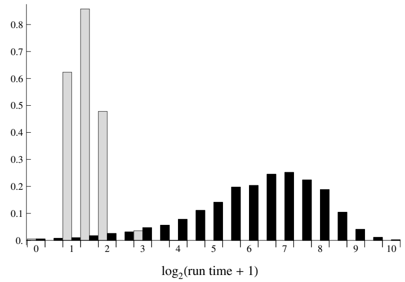
Equilibrium distribution of final 6 coordinates of Kiefer-Wolfowitz workload vector:
arrival rate $\lambda = c = 25$, service durations Uniform($[0, 1]$).
(5000 draws, Algorithm 2.)

Results of *domCFTP* for $M/G/c$ (II)



Number of customers for $M/M/c$ queue in equilibrium when $\lambda = 10$, $\mu = 2$ and $c = 10$. Black bars summarize theoretical number of customers in system; light grey bars summarize 5000 draws, Algorithm 2. χ^2-test: p-value 0.62.

Results of *domCFTP* for $M/G/c$ (III)



Distribution of time taken to detect coalescence under Algorithms 1, 2 applied to $M/M/c$ queue, 5000 runs, $\lambda = 10$, $\mu = 2$ $c = 10$.
Black bars show distribution of $\log_2(\widetilde{\tau} + 1)$ for Algorithm 1 ($\widetilde{\tau}$ is first time at which $\widetilde{Y}$ empties).
Light grey bars show distribution of $\log_2(\widetilde{T} + 1)$ for Algorithm 2, where $\widetilde{T}$ is smallest time needed to detect coalescence using binary back-off.
(Algorithm 2 speed-up also suggested by calculations.)

Conclusion

It is highly feasible to produce exact simulations of stable $M/G/c$ queues (if service distribution has finite second moment) using *domCFTP*.

Further possibilities:

- $M/G/c$ behaviour is not encoded by first two moments of service distribution (Gupta et al. 2009).
- General input processes: Blanchet and Wallwater (2014), Blanchet et al. (2015).
- Omnithermal *domCFTP* for range c (Connor 2016).
- Networks? Sigman (2013), Blanchet and Dong (2013).
- Compartment models? Connor and WSK are working on SIR epidemics.
- "Efficient simulation"?

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