# OxWaSP: Exercises for Probability & Approximation Lecture 1

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Here follow (mostly) simple exercises aimed at testing comprehension.

## 1 Review of Methods of Convergence

### 1.1 Almost sure convergence

1. Write out an expression of the event of convergence $[X_n \to X]$ in terms of countable unions and complements of events of the form $[|X_n - X| > z]$.
2. Construct the definition of "almost sure Cauchy convergence", and confirm that almost sure Cauchy convergence implies almost sure convergence to a limiting random variable $X$.

### 1.2 Convergence in probability

1. Use the answer to Exercise 1.1(1) to show that a.s. convergence implies convergence in prob.
2. Construct an example in which almost sure convergence is not implied by convergence in probability. (HINT: use $X_n = \mathbb{I}[A_n]$, where $\mathbb{I}[A_n]$ is the indicator of an event $A_n$, and the $A_1, A_2, \ldots$ have probabilities tending to zero, but not tending to zero too fast. Then use the Borel-Cantelli lemmas.)
3. Explain "any sequence converging in probability fast enough will converge almost surely".

### 1.3 Convergence in $p$-mean

1. Prove that convergence in $p$-norm implies convergence in probability, even if $0 < p < 1$.
2. Produce a counterexample to the assertion that convergence in probability implies convergence in $p$-norm. (HINT: use $X_n = n\,\mathbb{I}[A_n]$ for suitable events $A_n$.)
3. For which $p_1$ and $p_2$ does convergence in $p_1$-norm imply convergence in $p_2$-norm, for two different positive $p_1$, $p_2$? Give counterexamples to show that you have identified the complete range of possibilities. (HINT: use $X_n = n^\alpha\,\mathbb{I}[A_n]$ for suitable events $A_n$, suitable $\alpha > 0$.)

### 1.4 Uniform integrability

1. Construct an example of $X_n \to 0$ in probability such that $\mathbb{E}[X_n] = 1 \not\to 0$.
2. Show $\mathbb{E}[|X|] < \infty$ is equivalent to $\lim_{K\to\infty} \mathbb{E}[|X|; |X| > K] = 0$.
(HINT: write $\mathbb{E}[|X|] = \mathbb{E}[|X|; |X| \leq K] + \mathbb{E}[|X|; |X| > K]$.)

### 1.5 Weak convergence

1. Construct random variables $X_n$ which are defined on the same probability space, converge weakly, but do not converge either almost surely, or in probability, or in $p$-norm for $p \geq 1$.
(HINT: consider independent identically distributed sequences.)
2. Suppose $X$ has a distribution function $F(x) = \mathbb{P}[X \leq x]$ which is continuous and is strictly increasing. Then its inverse function $F^{-1}$ is well-defined. Find a random variable $U$ such that $F^{-1}(U)$ has the same distribution as $X$.

3. Use the above exercise 1.5(2) to show how to construct an almost surely converging sequence $Y_1$, $Y_2, \ldots$, with the same distributions as a weakly converging sequence $X_1, X_2, \ldots$.

4. Suppose $X_n \to X$ in total variation. Show that it suffices to show

$$\sup_{A \subseteq S} (\mathbb{P}[X_n \in A] - \mathbb{P}[X \in A]) \quad \to \quad 0\,. \tag{1}$$

5. Suppose in addition that $X_n$, $X$ all take values in some finite range of points $x_1, x_2, \ldots, x_k$. Show

$$\sup_{A \subseteq S} (\mathbb{P}[X_n \in A] - \mathbb{P}[X \in A]) \quad = \quad \frac{1}{2} \sum_{i=1}^{k} \Big| \mathbb{P}[X_n = x_i] - \mathbb{P}[X = x_i] \Big|\,. \tag{2}$$

6. Use the corresponding metrics to show, convergence in total variation implies convergence in truncated Wasserstein distance (equivalent to weak convergence).

7. Exhibit an example of random variables converging weakly but not in total variation.
(HINT: computer approximations to random variables uniformly distributed over $[0, 1]$.)

# 2 Some Classical Convergence Theorems

## 2.1 Weak Law of Large Numbers

1. Prove the WLLN for the case when the identically distributed random variables need not be independent, but are uncorrelated, using Chebychev inequality and First Borel-Cantelli Lemma.

## 2.2 Strong Law of Large Numbers

1. Suppose $X_1, \ldots, X_{n-1}$ are independent $(\pm 1)$-valued random variables, equally likely to take values $\pm 1$. Set $X_n = X_1 \times \ldots \times X_{n-1}$. Show that any strict subsequence of the random variable sequence $X_1, \ldots, X_n$ is independent, but that the whole sequence $X_1, \ldots, X_n$ is not independent.

2. Consider independent and identically distributed Cauchy random variables $X_1, X_2, \ldots$(so probability density is $\frac{1}{\pi}\frac{1}{1+x^2}$). Either show mathematically that the SLLN does not hold, or produce a simulation which indicates the same.

## 2.3 Central Limit Theorem

1. Show that the Lyapunov condition implies the Lindeberg condition.

2. The vector-valued case of Lindeberg's CLT is a bit subtle. Lindeberg's condition might become

$$\sum_{i=1}^{n} \mathbb{E}\left[ \left\| \frac{X_i}{s_n} \right\|^2 \,;\, \left\| \frac{X_i}{s_n} \right\| \geq \varepsilon \right] \quad \to \quad 0\,, \tag{3}$$

where $\|X_i\|$ is the Euclidean norm of $X_i$, and now $s_n^2 = \mathbb{E}[\|X_i\|^2]$ (we assume $\mathbb{E}[X_i] = 0$ as before). Construct a sequence of independent two-dimensional Gaussian random variables which satisfy (3), but do *not* satisfy a CLT.
WSK and Le (2011, §4, "Euclidean Interlude") discuss "Central Approximation" in this case.

# References

WSK and H. Le (2011). Limit theorems for empirical Fréchet means of independent and non-identically distributed manifold-valued random variables. *Brazilian Journal of Probability and Statistics 25*(3), 323–352.