Correlated Pseudo-Marginal Method

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Background

- What is the Correlated Pseudo Marginal (CPM) Method?
  - Metropolis Hastings (MH) $\rightarrow$ Pseudo-Marginal (PM) $\rightarrow$ CPM
  - Deligiannidis et al. (2015)
- Why do this?
  - In a Bayesian context, where MH target distribution can not be evaluated; using an unbiased estimator facilitates MH
  - Correlation improves efficiency of scheme
- Our `R` implementation: `https://github.com/JTT94/cpmmc`
  - Empirical Results / Does it work?

The Correlated Pseudo-Marginal Algorithm

For observation $y_{1:T}$ with likelihood function $p(y_{1:T}|\theta)$, let θ be the parameter of interest with prior density $p(\theta)$.

$$\pi(\theta) \propto p(y_{1:T}|\theta)p(\theta)$$

Let $U \sim m$ be the auxiliary random variable used to obtain a non-negative, unbiased estimator $\hat{p}(y_{1:T}|\theta, U)$ for the likelihood $p(y_{1:T}|\theta)$.

$$\bar{\pi}(\theta, u) = \pi(\theta)m(u)\frac{\hat{p}(y_{1:T}|\theta, u)}{p(y_{1:T}|\theta)}$$

The Correlated Pseudo-Marginal Algorithm

Algorithm 1 Correlated Pseudo-Marginal Algorithm

1: **Initialise** θ_0 and U_0
2: **for** Iteration $\ell \in 1, \ldots, L$ **do**
3: Sample $\theta' \sim q(\theta_{\ell-1}, \cdot)$
4: Sample $\epsilon \sim \mathcal{N}(0_M, I_M)$, set $U' = \rho U + \sqrt{1-\rho^2}\epsilon$
5: Compute the estimator $\hat{p}(y_{1:T}|\theta', U')$ of $p(y_{1:T}|\theta, U)$
6: With probability

$$\alpha_E\{(\theta_{\ell-1}, U_{\ell-1}), (\theta', U')\} = min\left\{1, \frac{\hat{p}(y_{1:T}|\theta', U')q(\theta', \theta_{\ell-1})p(\theta')}{\hat{p}(y_{1:T}|\theta_{\ell-1}, U_{\ell-1})q(\theta_{\ell-1}, \theta')p(\theta_{\ell-1})}\right\}$$

set $(\theta_\ell, U_\ell) \leftarrow (\theta', U')$, else $(\theta_\ell, U_\ell) \leftarrow (\theta_{\ell-1}, U_{\ell-1})$

Random Effects Model

$$X_t \overset{\text{i.i.d.}}{\sim} f_\theta(\cdot)$$
$$Y_t | X_t \sim g_\theta(\cdot | X_t)$$

For a realisation $Y_{1:T} = y_{1:T}$, the likelihood is

$$p(y_{1:T}) = \prod_{t=1}^{T} p(y_t|\theta)$$
$$p(y_t|\theta) = \int g_\theta(y_t|x_t, \theta) f_\theta(x_t) dx_t$$

Random Effects Model

Importance sampling estimator can be used if the integral is intractable.

$$
\begin{aligned}
\hat{p}(y_{1:T}|\theta, U) &= \prod_{t=1}^{T} \hat{p}(y_t|\theta, U_t) \\
\hat{p}(y_t|\theta, U_t) &= \frac{1}{N} \sum_{i=i}^{N} w(y_t, U_{t,i}; \theta)
\end{aligned}
$$

Assuming that there exists a deterministic map $\Xi : \mathbb{R}^p \times \Theta \to \mathbb{R}^k$ such that $X_{t,i} = \Xi(U_{t,i}, \theta) \sim q_\theta(\cdot|y_t)$ and $U_{t,i} \sim \mathcal{N}(0_p, I_p)$.

$$
w(y_t, U_{t,i}|\theta) = \frac{g_\theta(y_t|X_{t,i}) f_\theta(X_{t,i})}{q_\theta(X_{t,i}|y_t)}
$$

Implementation

19 lines (12 sloc) | 381 Bytes   Raw Blame History

Correlated Pseudo Marginal Monte Carlo

cpmmc

An R package to perform the Correlated Pseudo Marginal Method, see paper detailing methodology here

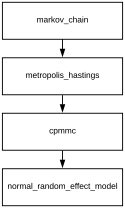
This package can be installed as follows:

```
devtools::install_github("JTT94/cpmmc")
```

Vignette help can be found as follows:

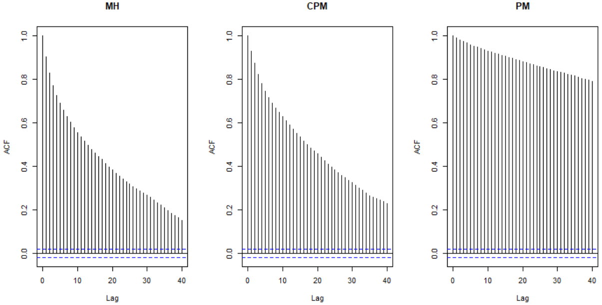
```
library(cpmmc)
vignette('cpmmc', package='cpmmc')
```

Class Diagram:




```
# Random Effect CPM (specific case inherited from cpmmc class)
rem_cpm <- normal_random_effect_model(data,
                                      theta_0 = theta_0,
                                      u_0 = u_0,
                                      rho = rho
                                      )
# Run models for nsim iterations
rem_cpm <- run_chain(rem_cpm, chain_length = nsim)
```

Simulation Study



Random effect model experiment with $\theta = 0.5$, $T = 8192$ and tuned ρ and N_T.
$f_\theta = \mathcal{N}(\theta, 1)$ and $g_\theta = \mathcal{N}(X, 1)$, where $X \sim f_\theta$.
Proposal: $\theta'|\theta \sim \mathcal{N}(\theta, 0.02^2)$, and prior $\theta \sim \mathcal{N}(0, 1)$

T	N_T	ρ	Average Acceptance ϱ_{MH}	ϱ_{CPM}	ϱ_{PM}	Inefficiency Score IF_{MH}	IF_{CPM}	IF_{PM}
1024	19	0.9894	0.85	0.45	0.0052	18.78	27.04	78.95
2048	28	0.9925	0.78	0.47	0.0044	11.96	21.65	76.56
4096	39	0.9947	0.72	0.44	0.0040	11.91	20.54	53.66

- $IF = 1 + 2\sum_{n=1}^{40} \phi_n(\theta)$, for chain θ
- ϕ_n is auto-correlation estimator with lag n
- ϱ is acceptance rate

References

George Deligiannidis, Arnaud Doucet, and Michael K Pitt. The correlated pseudo-marginal method. *arXiv preprint arXiv:1511.04992*, 2015.