Active polar tissue: ext. force balance and weak formulation

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1 Strong (PDE) formulation

The steady-state boundary value problem for the polarity field is given by:

$$L_c^2 \nabla^2 \mathbf{p} = \mathbf{p} \qquad \text{in } \Omega(t), \tag{1}$$

$$\mathbf{p} = \mathbf{n} \qquad \text{on } \Gamma(t). \tag{2}$$

With $\mathbf{p}$ resolved, the reduced momentum balance gives the following boundary value problem for the flow $\mathbf{v}$:

$$\nabla \cdot \sigma^{\mathrm{s}} + \mathbf{f} = 0 \qquad \text{in } \Omega(t), \tag{3}$$

$$\sigma^{\mathrm{s}} \cdot \mathbf{n} = 0 \qquad \text{on } \Gamma(t), \tag{4}$$

where

$$\sigma^{\mathrm{s}} = \eta \left(\nabla \mathbf{v} + \nabla \mathbf{v}^{\top} \right) - \zeta \mathbf{p}\mathbf{p}, \qquad f = -\xi \mathbf{v} + \zeta_i \mathbf{p}, \tag{5}$$

that is:

$$\sigma^{\mathrm{s}}_{\alpha\beta} = \eta \left(\partial_\alpha v_\beta + \partial_\beta v_\alpha \right) - \zeta p_\alpha p_\beta, \qquad f_\alpha = -\xi v_\alpha + \zeta_i p_\alpha.$$

The coupled system above is evolved in time via the kinematic condition, stating that the velocity of the sharp interface V_{n} is determined by the fluid flow:

$$V_{\mathrm{n}} = \mathbf{v} \cdot \mathbf{n} \qquad \text{on } \Gamma(t). \tag{6}$$

2 External force balance and cm velocity

Let us integrate eq. (3) over the domain:

$$0 = \int_\Omega \nabla \cdot \sigma^{\mathrm{s}} + \int_\Omega \mathbf{f} = \int_\Gamma \sigma^{\mathrm{s}} \cdot \mathbf{n} + \int_\Omega \mathbf{f} = \int_\Omega \mathbf{f}, \tag{7}$$

where we used the divergence theorem and the stress free condition, eq. (4). Indeed, this simple exercise amounts to the net balance of external forces (friction and active traction) that act on the tissue:

$$\int_\Omega \mathbf{f} = -\xi \int_\Omega \mathbf{v} + \zeta_i \int_\Omega \mathbf{p} = 0.$$

To calculate the cm velocity of the projected 2D domain $\Omega(t)$ (ignoring the third dimension), we first develop the derivative of $A\,\mathbf{x}_{\text{cm}}$ (with A the domain area):

$$\begin{aligned}\frac{d}{dt}(A\,\mathbf{x}_{\text{cm}}) =& \frac{d}{dt}\int_{\Omega(t)} \mathbf{x} = \int_{\Gamma(t)} \mathbf{x} V_{\text{n}} = \int_{\Gamma(t)} \mathbf{x}(\mathbf{v}\cdot\mathbf{n}) = \int_{\Gamma(t)} \{x\mathbf{v}\cdot\mathbf{n},\, y\mathbf{v}\cdot\mathbf{n}\} \\ =& \int_{\Omega(t)} \{\nabla\cdot(x\mathbf{v}),\, \nabla\cdot(y\mathbf{v})\} = \int_{\Omega(t)} \{v_x + x\nabla\cdot\mathbf{v},\, v_y + y\nabla\cdot\mathbf{v}\} \\ =& \int_{\Omega(t)} \mathbf{v} + \int_{\Omega(t)} \mathbf{x}(\nabla\cdot\mathbf{v}) = \frac{\zeta_i}{\xi}\int_{\Omega(t)} \mathbf{p} + \int_{\Omega(t)} \mathbf{x}(\nabla\cdot\mathbf{v}).\end{aligned} \tag{8}$$

The first term on the RHS can be developed further:

$$\frac{\zeta_i}{\xi}\int_{\Omega(t)} \mathbf{p} = \frac{\zeta_i}{\xi} L_c^2 \int_{\Omega(t)} \{\nabla\cdot\nabla p_x,\, \nabla\cdot\nabla p_y\} = \frac{\zeta_i}{\xi} L_c^2 \int_{\Gamma(t)} \{\nabla p_x\cdot\mathbf{n},\, \nabla p_y\cdot\mathbf{n}\} = \frac{\zeta_i}{\xi} L_c^2 \int_{\Gamma(t)} \partial_{\text{n}}\mathbf{p}.$$

Not sure that this is useful since we don't have $\partial_{\text{n}}\mathbf{p}$ explicitly.

On the other hand, one has

$$\frac{d}{dt}(A\,\mathbf{x}_{\text{cm}}) = \dot{A}\mathbf{x}_{\text{cm}} + A\mathbf{v}_{\text{cm}}, \tag{9}$$

where

$$\dot{A} = \frac{d}{dt}\int_{\Omega(t)} 1. = \int_{\Gamma(t)} V_{\text{n}} = \int_{\Gamma(t)} \mathbf{v}\cdot\mathbf{n} = \int_{\Omega(t)} \nabla\cdot\mathbf{v}, \qquad \mathbf{x}_{\text{cm}} = A^{-1}\int_{\Omega(t)} \mathbf{x}.$$

Ultimately, we obtain:

$$\mathbf{v}_{\text{cm}} = A^{-1}\left(\frac{\zeta_i}{\xi}\int_{\Omega(t)} \mathbf{p} + \int_{\Omega(t)} \mathbf{x}(\nabla\cdot\mathbf{v}) - A^{-1}\left(\int_{\Omega(t)} \nabla\cdot\mathbf{v}\right)\int_{\Omega(t)} \mathbf{x}\right). \tag{10}$$

Can it be proven that the last two terms under parenthesis balance out? This would be trivial for a 2D incompressible problem. If not, is it possible to get $\nabla\cdot\mathbf{v}$ in terms of $\mathbf{p}$?

3 Weak (variational) formulation

The variational formulation of the inhomogeneous Dirichlet problem, eqs. (1)–(2), is best derived for a modified homogeneous problem for $\mathbf{p}_H = \mathbf{p} - \mathbf{p}_D$:

$$\mathbf{p}_H - L_c^2\nabla^2\mathbf{p}_H = -\mathbf{p}_D + L_c^2\nabla^2\mathbf{p}_D \qquad \text{in } \Omega(t), \tag{11}$$

$$\mathbf{p}_H = 0 \qquad \text{on } \Gamma(t). \tag{12}$$

where $p_{D\alpha} \in H^1(\Omega)$ such that $\gamma_0(p_{D\alpha}) = n_\alpha$ (meaning $\mathbf{p}_D = \mathbf{n}$ on Γ). Clearly, one has $p_{H\alpha} \in H_0^1(\Omega)$ (i.e., $\gamma_0(p_{H\alpha}) = 0$).

Let us multiply (scalar product) eq.(11) by the test functions $\mathbf{q}$, such that $q_\alpha \in H_0^1(\Omega)$, and integrate over the domain:

$$\int_\Omega p_{H\alpha}q_\alpha - L_c^2\int_\Omega (\nabla\cdot\nabla p_{H\alpha})q_\alpha = -\int_\Omega p_{D\alpha}q_\alpha + L_c^2\int_\Omega (\nabla\cdot\nabla p_{D\alpha})q_\alpha. \tag{13}$$

Integrating the second term on each side by parts:

$$\int_\Omega p_{H\alpha}q_\alpha + L_c^2\int_\Omega \nabla p_{H\alpha}\cdot\nabla q_\alpha - L_c^2\int_\Gamma q_\alpha(\nabla p_{H\alpha}\cdot\mathbf{n}) = -\int_\Omega p_{D\alpha}q_\alpha - L_c^2\int_\Omega \nabla p_{D\alpha}\cdot\nabla q_\alpha + L_c^2\int_\Gamma q_\alpha(\nabla p_{D\alpha}\cdot\mathbf{n}), \tag{14}$$

and since $\gamma_0(q_\alpha) = 0$,

$$\int_\Omega p_{H\alpha}q_\alpha + L_c^2\int_\Omega \nabla p_{H\alpha}\cdot\nabla q_\alpha = -\int_\Omega p_{D\alpha}q_\alpha - L_c^2\int_\Omega \nabla p_{D\alpha}\cdot\nabla q_\alpha. \tag{15}$$

To conclude, the modified homogeneous weak problem consists in finding $p_{H\alpha} \in H_0^1(\Omega)$ such that eq. (15) holds for any test function $q_\alpha \in H_0^1(\Omega)$. The solution for the polarity field is then given by $\mathbf{p} = \mathbf{p}_H + \mathbf{p}_D$. Note for FEM implementation: the bilinear terms are organized on the LHS and the linear (explicit) terms are on the RHS.

Let us now derive the weak formulation of the momentum balance problem, eqs. (3)–(4). Note that, while this PDE problem for $\mathbf{u}$ appears more complicated than that for $\mathbf{p}$, the boundary condition in eq. (4) is actually far more convenient to substitute in the weak formulation ($\sigma^{\mathrm{s}}\cdot\mathbf{n}$ appears naturally after integrating by parts). Multiplying (scalar product) eq. (3) by the test functions $\mathbf{u}$ and integrating over the domain:

$$\int_\Omega (\nabla\cdot\sigma^{\mathrm{s}})\cdot\mathbf{u} + \int_\Omega \mathbf{f}\cdot\mathbf{u} = 0. \tag{16}$$

Integrating the first term by parts:

$$\int_\Omega (\nabla\cdot\sigma^{\mathrm{s}})\cdot\mathbf{u} = \int_\Omega (\nabla\cdot\sigma^{\mathrm{s}}_\alpha)u_\alpha = -\int_\Omega \sigma^{\mathrm{s}}_\alpha\cdot\nabla u_\alpha + \int_\Gamma (\sigma^{\mathrm{s}}_\alpha\cdot\mathbf{n})u_\alpha = -\int_\Omega \sigma^{\mathrm{s}} : \nabla\mathbf{u} + \int_\Gamma (\sigma^{\mathrm{s}}\cdot\mathbf{n})\cdot\mathbf{u}.$$

The last term on the RHS vanishes owing to eq. (4) and thus:

$$-\int_\Omega \sigma^{\mathrm{s}} : \nabla\mathbf{u} + \int_\Omega \mathbf{f}\cdot\mathbf{u} = 0, \tag{17}$$

where σ^{s} and $\mathbf{f}$ are defined in eq. (5) and $(\nabla\mathbf{u})_{\alpha\beta} = \partial_\beta u_\alpha$.

For a convenient FEM implementation, let us rearrange eq. (17) such that the bilinear terms (passive −viscosity and −friction) appear on the LHS while the explicit linear terms (active contractility and traction) appear on the RHS:

$$\eta\int_\Omega (\nabla\mathbf{v} + \nabla\mathbf{v}^\top) : \nabla\mathbf{u} + \xi\int_\Omega \mathbf{v}\cdot\mathbf{u} = \zeta\int_\Omega \mathbf{pp} : \nabla\mathbf{u} + \zeta_i\int_\Omega \mathbf{p}\cdot\mathbf{u}. \tag{18}$$

To conclude, the problem consists in finding $v_x,\, v_y \in H^1(\Omega)$ such that eq. (18) holds for any test functions $u_x,\, u_y \in H^1(\Omega)$.