2.11 hyperbolic set: $\{x \in R^2_+ \mid x_1 x \geq 1\}$

$X = (x_1, x_2)$ $Y = (y_1, y_2)$

set C is convex if any 2 points $x, y \in C$

any $\theta \in [0,1]$

$$z = (\theta x_1 + (1-\theta)y_1, \theta x_2 + (1-\theta)y_2)$$

given: $a^\theta b^{1-\theta} \leq \theta a + (1-\theta)b \rightarrow$ $\theta x_1 + (1-\theta)y_1 \geq x_1^\theta y_1^{1-\theta}$

and

$\theta x_2 + (1-\theta)y_2 \geq x_2^\theta y_2^{1-\theta}$

combined to be: $(\theta x_1 + (1-\theta)y_1)(x_2 + (1-\theta)y_2) \geq (x_1^\theta y_1^{1-\theta})(x_2^\theta y_2^{1-\theta})$

simplify : $(\theta x_1 x_2 + \theta x_1(1-\theta)y_2 + (1-\theta)y_1 \theta x_2 - (1-\theta)y_1(1-\theta)y_2 \geq (x_1^\theta x_2^\theta y_1^{1-\theta} y_2^{1-\theta})$

" : $(x_1^\theta x_2^\theta y_1^{1-\theta} y_2^{1-\theta}) \leq \theta_{x_1 x_2} + (1-\theta)_{y_1 y_2}$

$$1 \leq (x_1^\theta x_2^\theta y_1^{1-\theta} y_2^{1-\theta}) \leq \theta_{x_1 x_2} + (1-\theta)_{y_1 y_2}$$

thus $z \in S$ is $\boxed{\text{Convex}}$

Generalize

given $x = (x_1, \dots, x_n) \in S$ $y = (y_1, \dots, y_n) \in S$

$\prod_{i=1}^n x_i \geq 1$ and $\prod_{i=1}^n y_i \geq 1$

$z = \theta x + (1-\theta)y$, $\theta \in [0,1]$

$$\prod_{i=1}^n a^\theta b^{1-\theta} \leq \prod_{i=1}^n (\theta x_i + (1-\theta)y_i)$$

$$\prod_{i=1}^n a^\theta b^{1-\theta} \geq 1$$

thus

$$1 \leq \prod_{i=1}^n a^\theta b^{1-\theta} \leq \prod_{i=1}^n (\theta x_i + (1-\theta)y_i)$$

2.12 c, e, f, g

C A wedge is the intersection of 2 convex half spaces so it is convex, it could be a cone as well

E No, not normally.

$dist(x, S)$ and $dist(x, T)$ are convex if S and T are

the set: $x \mid dist(x, S) \leq dist(x, T)$

: $x \mid dist(x, S) - dist(x, T) \leq 0$

x not necessarily convex

ex: for $S = \{-1, 1\}$, $T = \{0\}$ not convex

F Convex. $x + S_2 \subseteq S_1$, $x + y \in S_1$ for all $y \in S_2$

$= \bigcap_{y \in S_2} \{x \mid x + y \in S_1\} = \bigcap_{y \in S_2} (S_1 - y)$ thus convex

G Convex : the set $\{x \mid \|x-a\|_2 \leq \theta \|x-b\|_2\}$

$\|x-a\|_2^2 = (x-a)^T(x-a) = x^Tx - 2a^Tx + a^Ta$

" for b

$$x^Tx - 2a^Tx + a^Ta \leq \theta^2(x^Tx - 2b^Tx + b^Tb)$$

rearrange and simplify: $x^Tx - \theta^2 x^Tx - 2a^Tx + 2\theta^2 b^Tx + a^Ta - \theta^2 b^Tb \leq 0$

$$(1-\theta^2)x^Tx-2(a-\theta^2b)^Tx+(a^Ta-\theta^2b^Tb)\leq 0$$

If $\theta=1$ $\quad 0\leq 0$

thus results in half space:

half space: $(b-a)^Tx\leq\frac{1}{2}(b^Tb-a^Ta) \Rightarrow -2(a-b)^Tx+(a^Ta-b^Tb)\leq 0$

If $\theta<1$ it is a ball: $(1-\theta^2)x^Tx-2(a-\theta^2b)^Tx+(a^Ta-\theta^2b^Tb)\leq 0$

$$(1-\theta^2)\left(x-\frac{a-\theta^2b}{1-\theta^2}\right)^T\left(x-\frac{a-\theta^2b}{1-\theta^2}\right)\leq\frac{(a-\theta^2b)^T(a-\theta^2b)}{(1-\theta^2)^2}-\frac{a^Ta-\theta^2b^Ta}{1-\theta^2}$$

thus: when $0\leq\theta<1$ a ball centered at $x_0\ \frac{a-\theta^2b}{1-\theta^2}$

2.15 $\quad a, b, f, g$

$$P=\{p\mid 1^Tp=1, p\geq 0\}, p\in\mathbb{R}^n$$

a $\quad\alpha\leq E\,f(x)\leq\beta$

$E\,f(x)\qquad E\,f(x)=\sum_{i=1}^{n}p_i f(a_i)$

$$\underbrace{\alpha\leq\sum_{i=1}^{n}p_i f(a_i)}\ \underbrace{\sum_{i=1}^{n}p_i f(a_i)\leq\beta}$$

each half space, thus intersection also convex

$\boxed{\text{convex}}$

B $\quad prob(x>a)\leq\beta$ linear inequality

$$\sum_{i:a_i\geq\alpha}p_i\leq\beta$$

$\boxed{\text{convex}}$

(f) $var(x) \leq \alpha$ where $var(x) = E[(x - E[x])]^2$ is the var of x

$$var(x) = E[x^2] - (E[x])^2$$

$$E[x] = \sum_{i=1}^{n} p_i a_i$$

$$E[x^2] = \sum_{i=1}^{n} p_i a_i^2$$

$$f(p) = \underbrace{\sum_{i=1}^{n} p_i a_i^2}_{\text{linear}} - \underbrace{\left(\sum_{i=1}^{n} p_i a_i^2\right)^2}_{\text{Quadratic} \rightarrow \text{convex}}$$

not necessarily convex

$\boxed{\text{not - convex}}$

(g) $var(x) \geq \alpha$ as in the previous question:

$$\sum_{i=1}^{n} p_i a_i^2 - \left(\sum_{i=1}^{n} p_i a_i^2\right)^2 \geq \alpha$$

$$b^T p \quad , \quad (a^T p)^2 = p^T (a a^T) p$$

$$\Downarrow$$

$$b^T p - p^T (a a^T) p \geq \alpha$$

$$b^T p - p^T A p \geq \alpha$$

$\boxed{\text{Convex}}$ $A = aa^T =$ positive semidefinite

2.19 a, b

$$f(x) = (Ax+b)/(c^T x+d) \qquad \text{dom } f = \{x \mid c^T x + d > 0\}$$

$$f^{-1}(C) = \{x \in \text{dom } f \mid f(x) \in C\}$$

$C \subseteq R$ give description of $f^{-1}(C)$

A half space $C = \{y \mid g^T y \leq h\}$ $g \neq 0$

$f(x) \in C$

$$f^{-1}(C) = \{x \mid g^T(Ax+b)/(c^T x+d) \leq h, \; c^T x + d > 0\}$$

$\Downarrow$

$g^T A x - h c^T x \leq hd - g^T b$

$\Downarrow$

$(g^T A - h c^T) x \leq hd - g^T b$

$$f^{-1}(C) = (A^T g - hc)^T x \leq hd - g^T b, \; c^T x + d > 0\}$$

B polyhedron $C = \{y \mid Gy \leq h\}$ same pretty much as above

$$C = \{y \mid G(Ax+b)/(c^T x+d) \leq h \quad c^T x + d > 0\}$$

$\Downarrow$

using same structure as above

$\Downarrow$

$$C = \{y \mid (GA - hc^T) x \leq hd - Gb, \; c^T x + d > 0\}$$

2.33 monotone negative cone

$K_{m+} = \{x \in R^n \mid x_1 \geq \dots \geq x_n \geq 0\}$

A

Proper cone, $\boxed{\text{closed}}$ cone

$\boxed{\text{not empty}}$

$\boxed{\text{pointed}}$ if $k \cap (-k) = 0$

: $(x_1 \geq \dots \geq x_n \geq 0)$

: $(-x_1 \geq \dots \geq -x_n \geq 0)$

thus, $x = 0$