Math 208: Discrete Mathematics
Lesson 5: Lecture Video Notes

Topics

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Readings: Chapters 8-10

§8. Relations

8a. Relations

Two-place predicates such as

$$G(x, y) : x \text{ is greater than } y$$

or

$$B(x, y) : x \text{ is a brother of } y$$

or

$$F(x, y) : x\text{'s favorite book is } y$$

play a prominent role in mathematics. In fact, two-place predicates give rise to the notion of relations, a key mathematical concept.

Defn. Let $P(x, y)$ be a two-place predicate with the set A being the domain of discourse for x and the set B being the domain of discourse for y. We say P is a *relation from A to B*. The set A is referred to as the *domain* and the set B is called to *codomain*.

Remark. The order of variables is important here. We consider x to be the first variable and y to be the second variable.

Preview. This terminology is likely familiar from studying functions. In fact, we will see that functions are a special type of relation.

Motivation. A relation from A to B gives a way of capturing the "relationships" between elements of A and elements of B. An important special case is when $A = B$ so a relation captures the "relationships" amongst elements of A.

8b. Specifying a relation

There are several ways to specify a relation. We'll look at the following ways.

Ways to specify a relation.

- Give a verbal description
- As a set of ordered pairs (i.e. subset of $A \times B$)
- By a graph
- By a digraph (when $A = B$)
- By a $0 - 1$ matrix

Verbal description

Specify a relation with a two-place predicate.

Ex. Consider the two-place predicate

$$G(x, y) : x \text{ is greater than } y$$

where the domain and codomain are both real numbers. Then $(4, 2)$ is said to *satisfy the relation* G since $4 > 2$, however $(1, 2)$ does not satisfy the relation G since $1 \not> 2$.

As a set of ordered pairs

A relation can be specified by stating the domain A, codomain B, and listing the ordered pairs that satisfy the relation. Note that the set of ordered pairs satisfying the relation is a subset of $A \times B$.

Ex. Let $A = \{1, 2, 3\}$ and $B = \{w, x, y, z\}$. Define the relation R from A to B to be

$$R = \{(1, y), (1, z), (2, x), (3, z)\}.$$

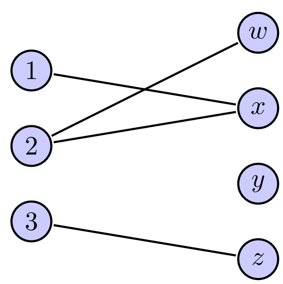
Observe the pair $(1, z)$ satisfies the relation R since $(1, z) \in R$, however $(2, w)$ does not since $(2, w) \notin R$.

Notation. The expression $(1, z) \in R$ is also commonly written as $1Rz$ or $R(1, z)$ is true. For abstract relations, the notation $1Rz$ is most common.

By a graph

Relations can be specified using a graph. Here a graph is a diagram consisting of vertices and edges.

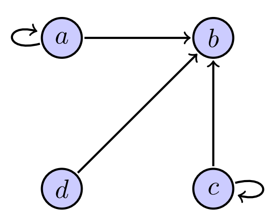
Ex. Identify the domain, codomain, and ordered pairs for the relation defined by the following (bipartite) graph.



By a digraph

When the domain and codomain are the same, i.e. $A = B$, it is no longer necessary to have separate vertices for each member of A and B. However, the edges must be directed to know which element is considered first and which is considered second. The tail of a directed edge indicates the first element; the head of the arrow indicates the second element in the ordered pair.

Ex. Identify the domain, codomain, and ordered pairs for the relation defined by the following digraph.



By a 0-1 matrix

A matrix can also be used to specify a relation R. The rows are indexed by the elements of the domain A (in some fixed order) and the columns are indexed by the elements of the codomain B (in some fixed order). Then a "1" is placed in the (a, b) position if $(a, b) \in R$ and "0" if $(a, b) \notin R$.

Ex. Let $A = \{1, 2, 3\}$ and $B = \{w, x, y, z\}$. Give the 0-1 matrix for the relation

$$R = \{(1, z), (2, x), (2, y), (3, z)\}.$$

8c. Set operations with relations

Let R_1 and R_2 be two relations from A to B. Viewing these relations as subsets of $A \times B$, we can use set operations to produce another relation from A to B.

Ex. Let $A = \{1, 2, 3\}$ and $B = \{w, x, y, z\}$. Consider

$$R_1 = \{(1, z), (2, x), (2, y), (3, z)\}$$

and

$$R_2 = \{(1, w), (1, z), (2, y), (3, x)\}.$$

Compute:

(i) $R_1 \cup R_2$

(ii) $R_1 \cap R_2$

Observation. The set operations used in the last example have anologous bit operations on the corresponding 0-1 matrices.

We have

$$M_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad M_2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Compute:

(i) $M_1 \vee M_2$

(i) $M_1 \wedge M_2$

8d. Special relation operations

We present two special relation operations: inverses and composition.

Inverse of a relation

Defn. The inverse of a relation R from A to B is the relation R^{-1} from B to A where $(b, a) \in R^{-1}$ iff $(a, b) \in R$.

Ex. Let $A = \{1, 2, 3\}$ and $B = \{x, y, z\}$. If $R = \{(1, y), (2, x), (2, z), (3, x)\}$, compute R^{-1}.

Q. Is there any relationship between the 0-1 matrices associated to R and R^{-1}?

Composition of relations

Defn. Let S to a relation from A to B and R be a relation from B to C. The composition of S by R is the relation

$$R \circ S = \{(a, c) : a \in A, c \in C \text{ and } \exists b \in B \text{ such that } (a, b) \in S \text{ and } (b, c) \in R\}.$$

Ex. Let $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2, b_3, b_4\}$, and $C = \{c_1, c_2\}$. Consider the relations

$$S = \{(a_1, b_1), (a_2, b_3), (a_3, b_3), (a_2, b_4)\} \qquad \text{and} \qquad R = \{(b_1, c_2), (b_2, c_1), (b_4, c_1)\}.$$

Compute $R \circ S$. Hint: graphs are helpful.

Q. Is there any relationship between the 0-1 matrices associated to R, S and $R \circ S$?

Remark. $M_{R \circ S} = M_S \odot M_R$ where $\odot$ is the Boolean product. The Boolean product of two matrices is the oridnary product of matrices with entries reduced modulo 2.

§9. Properties of relations

Recall, one way to specify a relation R from A to B is as a set of ordered pairs (i.e. subset of $A \times B$). In this chapter, we are interested in relations with $A = B$. That is. we consider relations R from A to A (i.e. relations on A) and study several useful properties that such relations may or may not satisfy.

Ex. Let $A = \{1, 2, 3, 4\}$. Define the relation R on A to be

$$R = \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\} \subseteq A \times A$$

Q. In fact, R is a familiar relation. What is it (in words)?

Observation. Many familiar relations satisfy certain properties. We'll study five in this chapter:

- reflexive
- irreflexive
- symmetric
- antisymmetric
- transitive

9a. Reflexive

Defn. A relation R on A is reflexive iff for every $a \in A$, then $(a, a) \in R$.

Note. Sometimes, the reflexive property is expressed as: $(\forall a \in A)\,[aRa]$.

Ex. Determine whether or not each of the following relation is reflexive.

(a) Let $A = \mathbb{Z}$ and $R(x, y) : x + y$ is even.

(b) Let A be the set of all people and $B(x, y) : x$ is a brother of y.

(c) Let $A = \mathbb{Z}$ and $M(x, y) : x \leq y$.

(d) Let $A = \mathbb{Z}$ and $R(x, y) : x + y$ is a multiple of 4.

The reflexive property can be easily checked from the 0-1 matrix of a relation.

Fact 1. A relation R is reflexive iff the entries of the main diagonal of the matrix M_R consists of all 1s. (We assume the ordering of the elements of A for the rows and columns are the same here.)

Fact 2. A relation R is reflexive iff each vertex of the digraph of R has a loop (i.e. edge from a vertex v to itself).

Ex. Consider the relation $M(x, y) : x \leq y$ on $A = \{1, 2, 3, 4\}$.

9b. Irreflexive

Defn. A relation R on A is irreflexive iff for every $a \in A$, then $(a, a) \notin R$.

Note. Sometimes, the irreflexive property is expressed as: $\forall a \in A\,(a \not R a) \equiv \neg(\exists a \in A)\,[aRa]$.

Ex. Determine whether or not each of the following relation is irreflexive.

(a) Let $A = \mathbb{Z}$ and $R(x, y) : x + y$ is even.

(b) Let A be the set of all people and $B(x, y) : x$ is a brother of y.

(c) Let $A = \mathbb{Z}$ and $M(x, y) : x \leq y$.

(d) Let $A = \mathbb{Z}$ and $R(x, y) : x + y$ is a multiple of 4.

(e) Let $A = \mathbb{Z}$ and $S(x, y) : x^2 \leq y$.

Similar to the reflexive property, the irreflexive property can be easily checked from the 0-1 matrix of a relation.

Fact 1. A relation R is irreflexive iff the entries of the main diagonal of the matrix M_R consists of all 0s. (We assume the ordering of the elements of A for the rows and columns are the same here.)

Fact 2. A relation R is irreflexive iff the digraph of R has no loops (i.e. an edge from a vertex v to itself).

Ex. Consider the relation $S(x, y) : x < y$ on $A = \{1, 2, 3, 4\}$.

9c. Symmetric

Defn. A relation R on A is symmetric iff for all $a, b \in A$, $(a, b) \in R$ implies $(b, a) \in R$.

Note. Sometimes, the symmetric property is expressed as: $(\forall a, b \in A)\,[aRb \to bRa]$.

Fact 1. A relation R is symmetric iff each pair of entries (a, b) and (b, a) on the off diagonal of the 0-1 matrix M_R have the same value (0 or 1). That is, $M_R = M_R^{\mathrm{T}}$. Again, we assume the ordering of the elements of A for the rows and columns are the same here.

Fact 2. A relation R is symmetric iff in the digraph of R, if the directed edge from a to b appears, then so does the directed edge from b to a.

Ex. Determine whether or not each of the following relation is symmetric.

(a) Let $A = \mathbb{Z}$ and $R(x, y) : x + y$ is even.

(b) Let A be the set of all people and $B(x, y) : x$ is a brother of y.

(c) Let $A = \mathbb{Z}$ and $M(x, y) : x \leq y$.

Ex. Consider $A = \{1, 2, 3, 4, 5\}$ and $R(x, y) : x + y$ is a multiple of 3.

9d. Antisymmetric

Caution: The antisymmetric property is not the logical negation of the symmetric property.

Defn. A relation R on A is antisymmetric iff for all $a, b \in A$, $(a, b) \in R$ and $(b, a) \in R$ imply $a = b$. That is, the only objects that are each related to the other are objects that are the same.

Fact 1. A relation R is antisymmetric iff each pair of entries (a, b) and (b, a) on the off diagonal of the 0-1 matrix M_R has at most one entry with value 1. Again, we assume the ordering of the elements of A for the rows and columns are the same here.

Fact 2. A relation R is antisymmetric iff in the digraph of R, if the directed edge from a to b appears and the directed edge from b to a appears, then the directed edge must be a loop from a to a. Intuitively, there are no "two-way streets" between vertices in the digraph of R.

Ex. Determine whether or not each of the following relation is antisymmetric.

(a) Let $A = \mathbb{Z}$ and $R(x, y) : x + y$ is even.

(b) Let A be the set of all people and $B(x, y) : x$ is a brother of y.

(c) Let $A = \mathbb{Z}$ and $M(x, y) : x \leq y$.

Ex. Consider the relation $S(x, y) : x \leq y$ on $A = \{1, 2, 3, 4\}$.

9e. Transitive

Defn. A relation R on A is transitive iff for every $a, b, c \in A$, $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$. Equivalently, $R \circ R \subseteq R$ where $\circ$ denotes the composition of relations from Chapter 8.

Note. Sometimes, the transitive property is expressed as: $(\forall a, b, c \in A)\, [((aRb) \wedge (bRc)) \rightarrow (aRc)]$.

Fact 1. A relation R is transitive iff the 0-1 matrix of R, namely M_R, satisfies $M_R \odot M_R \leq M_R$ where the relation $\leq$ applies element-wise on M_R. As usual, we assume the ordering of the elements of A for the rows and columns are the same here.

Fact 2. A relation R is transitive iff in the digraph of R, if there a directed edge from a to b and another directed edge from b to c, then digraph contains the directed edge from a to c.

Ex. Determine whether or not each of the following relation is transitive.

(a) Let $A = \mathbb{Z}$ and $R(x, y) : x + y$ is even.

(b) Let A be the set of all people and $B(x, y) : x$ is a brother of y.

(c) Let $A = \mathbb{Z}$ and $M(x, y) : x \leq y$.

Ex. Consider the relation $R(x, y) : x < y$ on $A = \{-1, 2, 3, 4\}$.

9f. Examples

For each of the following examples, determine whether or not the relation is: reflexive, irreflexive, symmetric, antisymmetric, and/or transitive.

Ex. Define the relation N on the set of all living people by xNy iff x lives within one mile of y.

Ex. Let $A = \{1,2,3,4,5,6\}$. Define the "divides" relation R on A by

$$R = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,2),(2,4),(2,6),(3,3),(3,6),(4,4),(5,5),(6,6)\}.$$

Ex. Consider the relation $R(x,y) : x \leq y$ on $A = \{1,2,3,4\}$.

Remark. Motivated by the last example, any relation on a set which is reflexive, antisymmetric, and transitive is called an *ordering* relation.

§10. Equivalence relations

Equivalence relations are a certain type of relation which captures the essence of objects "being the same". Here's a motivating example.

Q. How many different kinds of cards are there in a 52 card deck?

There are many possible reasonable responses such as:

- 2 kinds (red or black)
- 4 kinds ($\heartsuit, \diamondsuit, \clubsuit, \spadesuit$)
- 13 kinds (A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K)
- 52 kinds (each card is different)
- 1 kind (cards are just cards after all)
- 10 kinds (A, 2, 3, 4, 5, 6, 7, 8, 9, and 10/J/Q/K) E.g. blackjack

In each of these possible responses, the individual cards are different but we may only be interested in a certain feature of the card such as suit. The other properties of individual cards such as rank may not be important. So, we may consider $2\heartsuit$ and $5\heartsuit$ to be the same (both hearts).

10a. Equivalence relation

Defn. A relation R on a set A is an *equivalence relation* if R is reflexive, symmetric, and transitive. Two objects which are related (i.e. the same) in an equivalence relation are said to be *equivalent*.

Memory Aid: Equivalence relation satisfies the RST properties.

Review.

- Reflexive: every object is equivalent to itself
- Symmetric: If the object x is equivalent to the object y, then y is equivalent to x.
- Transitive: If x is equivalent to y and y is equivalent to z, then x is equivalent to z.

Notation. E is often used to denote an equivalence relation.

Ex. The six reasonable responses for "same kind" of cards each defines an equivalence relation on the set of all 52 cards from a standard deck.

10b. Equivalence class of a relation

When studying an equivalence relation E, it is often of interest to know what is the "same" under E as a given object x. That is, what objects are equivalent to x under E?

Defn. The *equivalence class of* x is the set of all objects $y \in A$ that are equivalent to x and written as $[x] = \{y \in A : yEx\}$.

Ex. Let A be the set of US states. Define E such that two states are equivalent if they begin with the same letter. Compute [Arizona] and [Ohio].

Ex. Consider the card example. Compute $[5\heartsuit]$ under:

(a) E_1: cards with the same suit are equivalent

(b) E_2: cards with the same rank are equivalent

(c) E_3: cards with the same color are equivalent

Observation. An equivalence relation E on A groups the objects of A into piles, namely one pile for each kind. This process is more commonly known in mathematics as E *partitions* the set A into equivalence classes.

10c. Examples

Ex. Define R on $\mathbb{N} = \{0, 1, 2, \dots\}$ by aRb iff $a = b$.

Ex. Let A be all logical propositions and define E on A by pEq iff $p \equiv q$.

Ex. Let A be all people in the world and define aRb on A by aRb iff a and b have the same initial letter of their first name.

Ex. Define P on $\mathbb{Z}$ by xPy iff x and y are both even or both odd. If aPb, we say a and b have the same *parity*.

The next theorem captures an important property of the equivalence classes of an equivalence relation.

Theorem. Let E be an equivalence relation on a set A and let $a \in A$. Then there is exactly one equivalence class to which a belongs.

Proof. Let $a \in A$. Recall, the equivalence class of a is defined as

$$[a] = \{y \in A : yEa\}.$$

Since E is reflexive, aEa and so $a \in [a]$ by definition of the equivalence class of a. Thus, a belongs to at least one equivalence class.

To show a belongs to a unique equivalence class, suppose $a \in [b]$. We need to show the set equality, $[a] = [b]$. To do this, we show $[a] \subseteq [b]$ and $[b] \subseteq [a]$.

To see $[a] \subseteq [b]$, we need to show $c \in [a]$ implies $c \in [b]$. Assume $c \in [a]$. Then cEa. Since $a \in [b]$ by hypothesis, aEb. Now cEa and aEb, so cEb since E is transitive. Thus $c \in [b]$ by definition of equivalence class of b.

The details to show $[b] \subseteq [a]$ are given in the textbook in section 10.3. Be sure to read through the argument.

10d. Partitions

Defn. A *partition* of a set A is a collection of nonempty, pairwise disjoint subsets of A, so that A is the union of the subsets in the collection. The subsets forming a partition are called the *parts of the partition*.

Ex. Consider the set $A = \{1, 2, 3, 4, 5, 6\}$. Then $\{\{1, 6\}, \{2, 5\}, \{3, 4\}\}$ is a partition of A. Each part of the partition consists of opposite faces of a standard 6-sided die.

Theorem. (Restated) The equivalence classes of an equivalence relation on a set partition the set into nonempty disjoint pieces.

10e. Digraph of an equivalence relation

The digraphs of an equivalence relation E has a distinct structure. Using the RST properties, the digraph of E consists of connected components, one for each part of the partition. Each component contains all possible directed edges. There are no directed edges between objects from different equivalence classes.

Ex. Let $A = \{1, 2, 3, 4, 5\}$ and define $R(x, y) : x + y$ is even. Compute the digraph of R.

Ex. Let $A = \{\text{Alice, Adam, Bob, Brenda, Bart, Charlie}\}$ and define E by xEy if x and y have the same initial letter in their name. Compute the digraph of E.

10f. Matrix representation of an equivalence relation

The 0-1 matrix associated to an equivalence relation E has a distinct structure. If the elements of A are ordered based on their equivalence classes under E, then the 0-1 matrix has a *block decomposition* where each block consists of all 1's and there are 0's elsewhere.

Ex. Let $A = \{1, 2, 3, 4, 5\}$ and define $R(x, y) : x + y$ is even. Compute the 0-1 matrix of R.

Ex. Let $A = \{\text{Alice, Adam, Bob, Brenda, Bart, Charlie}\}$ and define E by xEy if x and y have the same initial letter in their name. Compute the 0-1 matrix of E.