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Sequences and Summation

A **sequence** IS A LIST of numbers in a specific order. For example, the positive integers $1, 2, 3, \cdots$ is a sequence, as is the list 4, 3, 3, 5, 4, 4, 3, 5, 5, 4 of the number of letters in the English words of the ten digits in order *zero*, *one*, $\cdots$, *nine*. Actually, the first is an example of an infinite sequence, the second is a finite sequence. The first sequence goes on forever; there is no last number. The second sequence eventually comes to a stop. In fact the second sequence has only ten items. A **term** of a sequence is one of the numbers that appears in the sequence. The first term is the first number in the list, the second term is the second number in the list, and so on.

13.1 Specifying sequences

A more general way to think of a sequence is as a function from some subset of $\mathbb{Z}$ having a least member (in most cases either $\{0, 1, 2, \cdots\}$ or $\{1, 2, \cdots\}$) with codomain some *arbitrary* set. In most mathematics courses the codomain will be a set of numbers, but that isn't necessary. For example, consider the finite sequence of initial letters of the words in the previous paragraph: $a, s, i, a, l, o, n, \cdots, a, s, o$. If the letter L is used to denote the function that forms this sequence, then $L(1) = a$, $L(2) = s$, and so on.

Computer science texts use the former and elementary math application texts use the later. Mathematicians use any such well-ordered domain set.

13.1.1 Defining a Sequence With a Formula

The examples of sequences given so far were described in words, but there are other ways to tell what objects appear in the sequence. One way is with a formula. For example, let $s(n) = n^2$, for $n = 1, 2, 3, \cdots$. As the values $1, 2, 3$ and so on are plugged into $s(n)$ in succession, the infinite sequence $1, 4, 9, 16, 25, 36, \cdots$ is built up. It is traditional to write s_n (or t_n, etc) instead of $s(n)$ when describing the terms of a sequence, so the formula above would usually be seen as $s_n = n^2$. Read that as *s sub n equals* n^2. When written this way, the n in the s_n is called a *subscript* or *index*. The subscript of s_{173} is 173.

Example 13.1. *What is the* 50^{th} *term of the sequence defined by the formula* $s_j = \frac{j+1}{j+2}$, *where* $j = 1, 2, 3, \ldots$? *We see that*

$$s_{50} = \frac{51}{52}.$$

Example 13.2. *What is the* 50^{th} *term of the sequence defined by the formula* $t_k = \frac{k+1}{k+2}$, *where* $k = 0, 1, 2, 3, \ldots$? *Since the indicies start at* 0, *the* $50th$ *term will be* t_{49}:

$$t_{49} = \frac{50}{51}.$$

13.1.2 Defining a Sequence by Suggestion

A sequence can also be specified by listing an initial portion of the sequence, and trust the reader to successfully perform the mind reading trick of guessing how the sequence is to continue based on the pattern suggested by those initial terms. For example, consider the sequence $7, 10, 13, 16, 19, 22, \cdots$. The symbol $\cdots$ means *and so on*. In other words, you *should* be able to figure out the way the sequence will continue. This method of specifying a sequence is dangerous of course. For instance, the number of terms sufficient for one person to spot the pattern might not be enough for another person. Also, maybe there are several different *obvious* ways to continue the pattern

Example 13.3. *What is the next term in the sequence* $1, 3, 5, 7 \cdots$? *One possible answer is* 9, *since it looks like we are listing the positive odd integers in increasing order. But another possible answer is* 8: *maybe we are* *listing each positive integer with an **e** in its name. You can probably think of other ways to continue the sequence.*

In fact, for any finite list of initial terms, there are always infinitely many more or less natural ways to continue the sequence. A reason can always be provided for absolutely any number to be the next in the sequence. However, there will typically be only one or two *obvious* simple choices for continuing a sequence after five or six terms.

13.2 Arithmetic sequences

The simple pattern suggested by the initial terms $7, 10, 13, 16, 19, 22, \cdots$ is that the sequence begins with a 7, and each term is produced by adding 3 to the previous term. This is an important type of sequence. The general form is $s_1 = a$ (a is just some specific number), and, from the second term on, each new term is produced by adding d to the previous term (where d is some fixed number). In the last example, $a = 7$ and $d = 3$. A sequence of this form is called an **arithmetic sequence**. The number d is called the **common difference**, which makes sense since d is the difference of any two consecutive terms of the sequence. It is possible to write down a formula for s_n in this case. After all, to compute s_n we start with the number a, and begin adding d's to it. Adding one d gives $s_2 = a + d$, adding two d's gives $s_3 = a + 2d$, and so on. For s_n we will add $n - 1$ d's to the a, and so we see $s_n = a + (n-1)d$. In the numerical example above, the 5^{th} term of the sequence ought to be $s_5 = 7 + 4 \cdot 3 = 19$, and sure enough it is. The 407^{th} term of the sequence is $s_{407} = 7 + 406 \cdot 3 = 1225$.

Example 13.4. *The 1^{st} term of an arithmetic sequence is 11 and the the 8^{th} term is 81. What is a formula for the n^{th} term?*

We know $a_1 = 11$ and $a_8 = 81$. Since $a_8 = a_1 + 7d$, where d is the common difference, we get the equation $81 = 11 + 7d$. So $d = 10$. We can now write down a formula for the terms of this sequence: $a_n = 11 + (n-1)10 = 1 + 10n$. Checking, we see this formula does give the required values for a_1 and a_8.

13.3 Geometric sequences

For an arithmetic sequence we added the same quantity to get from one term of the sequence to the next. If instead of adding we multiply each term by the same thing to produce the next term the result is called a **geometric sequence**.

Example 13.5. *Let $s_1 = 2$, and suppose we multiply by 3 to get from one term to the next. The sequence we build now looks like $2, 6, 18, 54, 162, \cdots$, each term being 3 times as large as the previous term.*

In general, if $s_1 = a$, and, for $n \geq 1$, each new term is r times the preceeding term, then the formula for the n^{th} term of the sequence is $s_n = ar^{n-1}$, which is reasoned out just as for the formula for the arithmetic sequence above. The quantity r in the geometric sequence is called the **common ratio** since it is the ratio of any term in the sequence to its predecessor (assuming $r \neq 0$ at any rate).

13.4 Summation notation

A sequence of numbers is an ordered list of numbers. A **summation** (or just **sum**) is a sequence of numbers added up. A sum with n terms (that is, with n numbers added up) will be denoted by S_n typically. Thus if we were dealing with sequence $1, 3, 5, 7, \cdots, 2n - 1, \cdots$, then $S_3 = 1 + 3 + 5$, and $S_n = 1 + 3 + 5 + \cdots + (2n - 1)$. For the arithmetic sequence $a, a + d, a + 2d, a + 3d, \cdots$, we see $S_n = a + (a + d) + (a + 2d) + \cdots + (a + (n - 1)d)$.

It gets a little awkward writing out such extended sums and so a compact way to indicate a sum, called **summation notation**, is introduced. For the sum of the first 3 odd positive integers above we would write $\sum_{j=1}^{3}(2j - 1)$. The Greek letter sigma (Σ) is supposed to be reminiscent of the word summation. The j is called the **index of summation** and the number on the bottom of the Σ specifies the starting value of j while the number above the Σ gives the ending value of j. The idea is that we replace j in turn by 1, 2 and 3, in each case computing the value of the expression following the Σ, and then add up the terms produced. In this example, when $j = 1$, $2j - 1 = 1$, when $j = 2$, $2j - 1 = 3$ and finally, when $j = 3$, $2j - 1 = 5$. We've reached the stopping value, so we have $\sum_{j=1}^{3}(2j-1) = 1 + 3 + 5 = 9$.

Notice that the index of summation takes only integer values. If it starts at 6, then next it is replaced by 7, and so on. If it starts at -11, then next it is replaced by -10, and then by -9, and so on.

The symbol used for the index of summation does not have to be j. Other traditional choices for the index of summation are i, k, m and n. So for example,

$$\sum_{j=0}^{4}(j^2+2) = 2 + 3 + 6 + 11 + 18,$$

and

$$\sum_{i=0}^{4}(i^2+2) = 2 + 3 + 6 + 11 + 18,$$

and

$$\sum_{m=0}^{4}(m^2+2) = 2 + 3 + 6 + 11 + 18,$$

and so on. Even though a different index letter is used, the formulas produce the same sequence of numbers to be added up in each case, so the sums are the same.

Also, the starting and ending points can for the index can be changed without changing the value of the sum provided care is taken to change the formula appropriately. Notice that

$$\sum_{k=1}^{3}(3k-1) = \sum_{k=0}^{2}(3k+2)$$

In fact, if the terms are written out, we see

$$\sum_{k=1}^{3}(3k-1) = 2 + 5 + 8$$

and

$$\sum_{k=0}^{2}(3k+2) = 2 + 5 + 8$$

Example 13.6. *We see that*

$$\sum_{m=-1}^{5} 2^m = 2^{-1} + 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 = \frac{127}{2}.$$

Example 13.7. *We find that*

$$\sum_{n=3}^{6} 2 = 2+2+2+2 = 8.$$

13.5 Formulas for arithmetic and geometric summations

There are two important formulas for finding sums that are worth remembering. The first is the sum of the first n terms of an arithmetic sequence.

$$S_n = a + (a+d) + (a+2d) + \cdots + (a+(n-1)d).$$

Here is a clever trick that can be used to find a simple formula for the quantity S_n: the list of numbers is added up twice, once from left to right, the second time from right to left. When the terms are paired up, it is clear the sum is $2S_n = n[a + (a+(n-1)d)]$. A diagram will make the idea clearer:

$$\begin{array}{lllll} a & +(a+d) & +(a+2d) & +\cdots+ & (a+(n-1)d) \\ +(a+(n-1)d) & +(a+(n-2)d) & +(a+(n-3)d) & +\cdots+ & a \\ \hline (2a+(n-1)d) & +(2a+(n-1)d) & +(2a+(n-1)d) & +\cdots+ & (2a+(n-1)d) \end{array}$$

The bottom row contains n identical terms, each equal to $2a + (n-1)d$, and so $2S_n = n\,[2a+(n-1)d)]$. Dividing by 2 gives the important formula, for $n = 1, 2, 3, \ldots$,

$$S_n = n\left(\frac{2a+(n-1)d}{2}\right) = n\left(\frac{a+(a+(n-1)d)}{2}\right). \tag{13.1}$$

An easy way to remember the formula is to think of the quantity in the parentheses as the average of the first and last terms to be added, and the coefficient, n, as the number of terms to be added.

Example 13.8. *The first* 20 *terms of the arithmetic sequence* $5, 9, 13, \cdots$ *is found to be*

$$S_{20} = 20\left(\frac{5+81}{2}\right) = 860.$$

For a geometric sequence, a little algebra produces a formula for the sum of the first n terms of the sequence. The resulting formula for $S_n = a + ar + ar^2 + \cdots + ar^{n-1}$, is

$$S_n = \frac{a - ar^n}{1-r} = a\left(\frac{1-r^n}{1-r}\right), \text{ if } r \neq 1.$$

Example 13.9. *The sum of the first ten terms of the geometric sequence $2, \frac{2}{3}, \frac{2}{9}, \cdots$ would be*

$$S_{10} = \frac{2 - 2\left(\frac{1}{3}\right)^{10}}{1 - \left(\frac{1}{3}\right)}$$

Notice that the numerator in this case is the difference of the first term we have to add in and the term *immediately following* the last term we have to add in.

The expression for S_{10} can be simplified as

$$S_{10} = \frac{2 - 2\left(\frac{1}{3}\right)^{10}}{1 - \left(\frac{1}{3}\right)} = 2\left(\frac{1 - \left(\frac{1}{3}\right)^{10}}{1 - \left(\frac{1}{3}\right)}\right) = 2\left(\frac{1 - \left(\frac{1}{3}\right)^{10}}{\frac{2}{3}}\right) = 3\left(1 - \frac{1}{3^{10}}\right) = 3 - \frac{1}{3^9}$$

Here is the algebra that shows the geometric sum formula is correct.

Let $S_n = a + ar + ar^2 + \cdots + ar^{n-1}$. Multiply both sides of that equation by r to get

$$rS_n = r(a + ar + ar^2 + \cdots + ar^{n-1}) = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n$$

Now subtract, and observe that most terms will cancel:

$$\begin{aligned} S_n - rS_n &= (a + ar + ar^2 + \cdots + ar^{n-1}) - (ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n) \\ &= a + (ar + ar^2 + \cdots + ar^{n-1}) - (ar + ar^2 + ar^3 + \cdots + ar^{n-1}) - ar^n \\ &= a - ar^n \end{aligned}$$

So $S_n(1 - r) = a - ar^n$. Assuming $r \neq 1$, we can divide both sides of that equation by $1 - r$, producing the promised formula[1]:

$$S_n = \frac{a - ar^n}{1 - r} = a\left(\frac{1 - r^n}{1 - r}\right), \text{ if } r \neq 1. \tag{13.2}$$

[1] Find a formula for S_n when $r = 1$.

13.6 Exercises

Exercise 13.1. *Guess the next term in the sequence* $1, 2, 4, 5, 7, 8, \cdots$. *What's another possible answer?*

Exercise 13.2. *What is the* 100^{th} *term of the arithmetic sequence with initial term* 2 *and common difference* 6*?*

Exercise 13.3. *The* 10^{th} *term of an arithmetic sequence is* -4 *and the* 16^{th} *term is* 47*. What is the* 11^{th} *term?*

Exercise 13.4. *What is the* 5^{th} *term of the geometric sequence with initial term* 6 *and common ratio* 2*?*

Exercise 13.5. *The first two terms of a geometric sequence are* $g_1 = 5$ *and* $g_2 = -11$*. What is the* g_5*?*

Exercise 13.6. *Which sequences are both a geometric sequence also an arithmetic sequence?*

Exercise 13.7. *Evaluate* $\sum_{j=1}^{4}(j^2 + 1)$.

Exercise 13.8. *Evaluate* $\sum_{k=-2}^{4}(2k - 3)$.

Exercise 13.9. *What is the sum of the first* 100 *terms of the arithmetic sequence with initial term* 2 *and common difference* 6*?*

Exercise 13.10. *What is the sum of the first five terms of the geometric sequence with initial term* 6 *and common ratio* 2*?*

Exercise 13.11. *Evaluate* $\sum_{i=0}^{4}\left(-\frac{3}{2}\right)^i$.

Exercise 13.12. *Express in summation notation:* $\dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{6} + \cdots + \dfrac{1}{2n}$, *the sum of the reciprocals of the first n even positive integers.*

13.7 Problems

Problem 13.1. *Guess the next term in the sequence* $1, 3, 5, 7, 8, 9 \cdots$. *What's another possible answer?*

Problem 13.2. *Guess the next term in the sequence* $1, 2, 2, 3, 2, 4, 2, 4, 3 \cdots$.

Problem 13.3. *A sequence begins* $1, 3, 9, 15$. *Could it be an arithmetic sequence? Could it be a geometric sequence?*

Problem 13.4. *What is the* 20^{th} *term of the arithmetic sequence with initial term* 4 *and common difference* 5*?*

Problem 13.5. *The* 8^{th} *term of an arithmetic sequence is* 20 *and the* 12^{th} *term is* 40. *What is the* 25^{th} *term?*

Problem 13.6. *What is the* 7^{th} *term of the geometric sequence with initial term* 3 *and common ratio* 4*?*

Problem 13.7. *Two terms of a geometric sequence are* $g_3 = 2$ *and* $g_5 = 72$. *There two possible values for* g_4. *What are those two values?*

Problem 13.8. *A geometric sequence has initial term* 3*, and common ration* 7. *Determine the smallest value of* n *so that the* n^{th} *term of the sequence is more that one million.*

Problem 13.9. *Evaluate* $\sum_{j=1}^{4}(j+1)^2$.

Problem 13.10. *Evaluate* $\sum_{k=-2}^{4}(2k+3)$.

Problem 13.11. *What is the sum of the first* 100 *terms of the arithmetic sequence with initial term* 2 *and common difference* 6*?*

Problem 13.12. *What is the sum of the first four terms of the geometric sequence with initial term* 3 *and common ratio* -2*?*

Problem 13.13. *What is the sum of the first four thousand terms of the geometric sequence with initial term* 3 *and common ratio* -1*?*

Problem 13.14. *You have two parents, and four grandparents, and eight great grandparents, for a total fourteen ancestors three generations back. How many ancestors do you have* 50 *generations back? (A generation is generally taken to be about* 30 *years, so* 50 *generations is about* 1500 *years.*

That would take us back to about the time the decimal system was invented in India. How can you explain the obviously impossible answer to this problem?)

Problem 13.15. *Evaluate* $\sum_{i=0}^{4} \left(\frac{3}{2}\right)^{i}$.

Problem 13.16. *Express in summation notation:* $\frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}$, *the sum of the reciprocals of the first* n *odd positive integers.*

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Recursively Defined Sequences

Besides specifying the terms of a sequence with a formula, such as $a_n = n^2$, an alternative is to give an initial term, usually something like b_1, (or the first few terms, b_1, b_2,$b_3 \ldots$) of a sequence, and then give a rule for building new terms from old ones. In this case, we say the sequence has been defined **recursively**.

Example 14.1. *For example, suppose $b_1 = 1$, and for $n > 1$, $b_n = 2b_{n-1}$. Then the 1^{st} term of the sequence will be $b_1 = 1$ of course. To determine b_2, we apply the rule $b_2 = 2b_{2-1} = 2b_1 = 2 \cdot 1 = 2$. Next, applying the rule again, $b_3 = 2b_{3-1} = 2b_2 = 2 \cdot 2 = 4$. Next $b_4 = 2b_3 = 8$. Continuing in this fashion, we can form as many terms of the sequence as we wish: $1, 2, 4, 8, 16, 32, \cdots$. In this case, it is easy to guess a formula for the terms of the sequence: $b_n = 2^{n-1}$.*

In general, to define a sequence recursively, (1) we first give one or more initial terms (this information is called the **initial condition(s)** for the sequence), and then (2) we give a rule for forming new terms from previous terms (this rule is called the **recursive formula**).

Example 14.2. *Consider the sequence defined recursively by $a_1 = 0$, and, for $n \geq 2$, $a_n = 2a_{n-1} + 1$. The five terms of this sequence are*

$$0, \quad 2 \cdot 0 + 1 = 1, \quad 2 \cdot 1 + 1 = 3, \quad 2 \cdot 3 + 1 = 7, \quad 2 \cdot 7 + 1 = 15 \quad \cdots$$

In words, we can describe this sequence by saying the initial term is 0 and each new term is one more than twice the previous term. Again, it is *easy to guess a formula that produces the terms of this sequence:* $a_n = 2^{n-1} - 1$.

Such a formula for the terms of a sequence is called a **closed form formula** to distinguish it from a recursive formula.

14.1 *Closed form formulas*

There is one big advantage to knowing a closed form formula for a sequence. In example 14.2 above, the closed form formula for the sequence tells us immediately that $a_{101} = 2^{100} - 1$, but using the recursive formula to calculate a_{101} means we have to calculate in turn $a_1, a_2, \cdots a_{100}$, making 100 computations. The closed form formula allows us to jump directly to the term we are interested in. The recursive formula forces us to compute 99 additional terms we don't care about in order to get to the one we want. With such a major drawback why even introduce recursively defined sequences at all? The answer is that there are many naturally occurring sequences that have simple recursive definitions but have no reasonable closed form formula, or even no closed form formula at all in terms of familiar operations. In such cases, a recursive definition is better than nothing.

14.1.1 *Pattern recognition*

There are methods for determining closed form formulas for some special types of recursively defined sequences. Such techniques are studied later in chapter 35. For now we are only interested in understanding recursive definitions, and determining some closed form formulas by the method of *pattern recognition* (aka *guessing*).

14.1.2 *The Fibonacci Sequence*

The most famous recursively defined sequence is due to Fibonacci. There are two initial conditions: $f_0 = 0$ and $f_1 = 1$. The recursive rule is, for $n \geq 2$, $f_n = f_{n-1} + f_{n-2}$. In words, each new term is the sum of the two terms that precede it. So, the **Fibonacci sequence**

The index starts at **zero**, by tradition.

begins

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ 233,\ \cdots$$

There is a closed form formula for the Fibonacci Sequence, but it is not at all easy to guess:

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

14.1.3 *The Sequence of Factorials*

For a positive integer n, the symbol $n!$ is read **n factorial** and it is defined to be the product of all the positive integers from 1 to n. In order to make many formulas work out nicely, the value of 0! is defined to be 1.

For example, $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$.

A recursive formula can be given for $n!$. The initial term is $0! = 1$, and the recursive rule is, for $n \geq 1$, $n! = n[(n-1)!]$. Hence, the first few factorial values are:

$$\begin{aligned}
1! &= 1[0!] = 1 \cdot 1 = 1,\\
2! &= 2[1!] = 2 \cdot 1 = 2,\\
3! &= 3[2!] = 3 \cdot 2 = 6,\\
4! &= 4[3!] = 4 \cdot 6 = 24,\\
&\vdots
\end{aligned}$$

We sometimes write a *general* formula for the factorial as

Why is this **not** a closed form formula?

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots n, \text{ for } n > 0.$$

The sequence of factorial grows very quickly. Here are the first few terms:

$$1,\ 2,\ 6,\ 24,\ 120,\ 720,\ 5040,\ 40320,\ 362880,\ 3628800,\ 39916800,\ 479001600,\ 6227020800,\ \cdots$$

14.2 *Arithmetic sequences by recursion*

Consider the terms of an arithmetic sequence with initial term a and common difference d:

$$a,\ (a+d),\ (a+2d),\cdots,\ (a+(n-1)d),\cdots.$$

These terms may clearly be found by adding d to the current term to get the next. That is, the arithmetic sequence may be defined recursively as (1) $a_1 = a$, and (2) for $n \geq 2$, $a_n = a_{n-1} + d$.

14.3 Exercises

Exercise 14.1. *List the first five terms of the sequence defined recursively by* $a_1 = 3$*, and, for* $n \geq 2$*,* $a_n = a_{n-1}(2 + a_{n-1})$.

Exercise 14.2. *List the first seven terms of the sequence defined recursively by* $a_0 = 1$*,* $a_1 = 1$*, and, for* $n \geq 2$*,* $a_n = 1 + a_{n-1}a_{n-2}$.

Exercise 14.3. *List the first ten terms of the sequence defined recursively by* $a_0 = 1$*, and, for* $n \geq 1$*,* $a_n = 1 + a_{\lfloor \frac{n}{2} \rfloor}$.

Exercise 14.4. *List the first ten terms of the sequence defined recursively by* $a_0 = 1$*, and for* $n \geq 1$*,* $a_n = 2n - a_{n-1} - 1$*, and guess a closed form formula for* a_n.

Exercise 14.5. *The first few terms of a sequence are*

$$1, 11, 21, 1211, 111221, 312211, 13112221, 1113213211.$$

There is an easy recursive rule for building the terms of this sequence. Guess the next term.

Exercise 14.6. *Let* d *be a fixed real number. For a positive integer* n*, the symbol* nd *means the sum of* n d*'s. Give a recursive definition of* nd *analogous to the definition of* $n!$ *given in this chapter.*

14.4 Problems

Problem 14.1. *List the first five terms of the sequence defined recursively by* $a_1 = 2$*, and, for* $n \geq 2$*,* $a_n = a_{n-1}^2 - 1$.

Problem 14.2. *List the first five terms of the sequence defined recursively by* $a_1 = 2$*, and, for* $n \geq 2$*,* $a_n = 3a_{n-1} + 2$*. Guess a closed form formula for the sequence.*

Hint: This is a lot like example 14.2.

Problem 14.3. *List the first five terms of the sequence with initial terms* $u_0 = 2$ *and* $u_1 = 5$*, and, for* $n \geq 2$*,* $u_n = 5u_{n-1} - 6u_{n-2}$*. Guess a closed form formula for the sequence.*

Hint: The terms are simple combinations of powers of 2 and powers of 3.

Problem 14.4. *Let* r *be a fixed real number different from* 0*. For a positive integer* n*, the symbol* r^n *means the product of* n r*'s. For convenience,* r^0 *is defined to be* 1*. Give a recursive definition of* r^n *analogous to the definition of* $n!$ *given in this chapter.*

Problem 14.5. *Give a recursive definition of the geometric sequence with initial term* 3 *and common ratio* 2.

Problem 14.6. *Generalize problem 5: give a recursive definition of the geometric sequence with initial term* a *and common ratio* r.