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The Method of Characteristic Roots

There is no method that will solve all recurrence relations. However, for one particular type, there is a standard technique. The type is called a **linear recurrence relation with constant coefficients**. In such a recurrence relation, the recurrence formula has the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k} + f(n)$$

where $c_1, \cdots, c_k$ are constants with $c_k \neq 0$, and $f(n)$ is any function of n.

The **degree** of the recurrence is k, the number of terms we need to go back in the sequence to compute each new term. If $f(n) = 0$, then the recurrence relation is called homogeneous. Otherwise it is called **nonhomogeneous**.

In chapter 35, we noted that some simple non-homogeneous linear recurrence relations with constant coefficients can be solved by unfolding. This method is not powerful enough for more general problems. In this chapter we introduce a basic method that, in principle at least, can be used to solve any homogeneous linear recurrence relation with constant coefficients.

36.1 Homogeneous, constant coefficient recursions

We begin by considering the degree 2 case. That is, we have a recurrence relation of the form $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, for $n \geq 2$, where c_1 and c_2 are real constants. We must also have two initial conditions a_0 and a_1. That is, we are given a_0 and a_1 and the formula $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, for $n \geq 2$. Notice that $c_2 \neq 0$ or else we have a linear recurrence relation with constant coefficients and degree 1. What we seek is a **closed form formula** for a_n, which is a function of n alone, and which is therefore independent of the previous terms of the sequence.

36.1.1 *Basic example of the method*

Here's the technique in a specific example:

The problem we will solve is to find a formula for the terms of the sequence

$$a_0 = 4 \text{ and } a_1 = 8, \text{ with}$$
$$a_n = 4a_{n-1} + 12a_{n-2}, \text{for } n \geq 2.$$

The first thing to do is to ignore the initial conditions, and concentrate on the recurrence relation. And the way to solve the recurrence relation is to guess the solution. Well, actually, it is to guess the **form** of the solution[1]. For such a recurrence you should guess that the solution looks like $a_n = r^n$, for some constant r. In other words, guess the solution is simply the powers of some fixed number. The good news is that this guess will always be correct! You will always find some solutions of this form. When this guess is plugged into the recurrence relation and the equation is simplified, the result is an equation that can be solved for r. That equation is called the **characteristic equation** for the recurrence. In our example, when $a_n = r^n$ for each n, the result is $r^n = 4r^{n-1} + 12r^{n-2}$, and canceling r^{n-2} from each term, and rearranging the equation, we get $r^2 - 4r - 12 = 0$. That's the characteristic equation. The left side can be factored, and the equation then looks like $(r-6)(r+2) = 0$, and we see the solutions for r are $r = 6$ and $r = -2$. And, sure enough, if you check it out, you will see that $a_n = 6^n$ and $a_n = (-2)^n$ both satisfy the given

[1] An *educated* guess!

recurrence relation. In other words, we find that

$$6^n = 4 \cdot 6^{n-1} + 12 \cdot 6^{n-2}, \text{ for all } n \geq 2, \text{ and}$$
$$(-2)^n = 4 \cdot (-2)^{n-1} + 12 \cdot (-2)^{n-2}, \text{ for all } n \geq 2.$$

Using the characteristic equation, we have a method of finding some solutions to a recurrence relation. This method will not find all possible solutions however. *BUT...* if we find **all** the solutions to the characteristic equation, then they can be combined in a certain way to produce all possible solutions to the recurrence relation. The fact to remember is that if $r = a, b$ are the two solutions to the characteristic equation (for a recurrence of order two), then every possible solution to the linear homogenous recurrence relation must look like [2]

$$\alpha a^n + \beta b^n,$$

for some constants α, β. In the example we have been working on, every possible solution looks like[3]

$$a_n = \alpha(6)^n + \beta(-2)^n.$$

Once we have figured out the general solution to the recurrence relation, it is time to think about the initial conditions. In our case, the initial conditions are $a_0 = 4$ and $a_1 = 8$. The idea is to select the constants α and β of the general solution $a_n = \alpha 6^n + \beta(-2)^n$ so it will produce the correct two initial values. For $n = 0$ we see we need $4 = a_0 = \alpha 6^0 + \beta(-2)^0 = \alpha + \beta$, and for $n = 1$, we need $8 = a_1 = \alpha 6^1 + \beta(-2)^1 = 6\alpha - 2\beta$. Now, we solve the following pair of equations for α and β:

$$\alpha + \beta = 4,$$
$$6\alpha - 2\beta = 8.$$

Performing a bit of algebra, we learn that $\alpha = 2$ and $\beta = 2$. Thus the solution to the recurrence is

$$a_n = 2 \cdot 6^n + 2 \cdot (-2)^n.$$

[2] Actually, that is not quite true. There is a slight catch to be mentioned later (see section 36.2).

[3] This expression is called the **general solution** of the recurrence relation.

36.1.2 Initial steps: the characteristic equation and its roots

The steps in solving a recurrence problem are:

(1) Determine the characteristic equation.

(2) Find the solutions to the characteristic equation.

(3) Write down the general solution to the recurrence relation.

(4) Select the constants in the general solution to produce the correct initial conditions.

36.2 Repeated characteristic roots.

And now, about the little lie mentioned above: One catch with the method of characteristic equation occurs when the equation has repeated roots. Suppose, for example, that when the characteristic equation is factored the result is $(r-2)(r-2)(r-3)(r+5) = 0$. The characteristic roots are $2, 2, 3$ and -5. Here 2 is a repeated root. If we follow the instructions given above, then the general solution we would write down is

$$a_n = \alpha 2^n + \beta 2^n + \gamma 3^n + \delta(-5)^n. \tag{36.1}$$

However, this expression will **not** include all possible solutions to the recurrence relation. Happily, the problem is not too hard to repair: each time a root of the characteristic equation is repeated, multiply it by an additional factor of n in the general solution, and then proceed with step 4 as described earlier.

For our example, we modify one of the 2^n terms in equation 36.1. The correct general solution looks like

Notice the extra factor of n in the second term.

$$a_n = \alpha 2^n + \beta n \cdot 2^n + \gamma 3^n + \delta(-5)^n.$$

If $(r-2)$ had been a four fold factor of the characteristic equation[4], then the part of the general solution involving the 2's would look like

[4] in other words, if 2 had been a characteristic root four times

Each new occurrence of a 2 is multiplied by one more factor of n.

$$\alpha 2^n + \beta n \cdot 2^n + \gamma n^2 \cdot 2^n + \delta n^3 \cdot 2^n.$$

36.3 The method of characteristic roots more formally

Let's describe the method of **characteristic equation** a little more formally. First, the characteristic equation is denoted by $\chi(x) = 0$ [5]. Notice that the degree of $\chi(x)$ coincides with the degree of the recurrence relation. Notice also that the non-leading coefficients of $\chi(x)$ are simply the negatives of the coefficients of the recurrence relation. In general, the characteristic equation of $a_n = c_1 a_{n-1} + ... + c_k a_{n-k}$ is

$$\chi(x) = x^k - c_1 x^{k-1} - ... - c_{k-1}x - c_k = 0.$$

[5] For the general degree 2 case above we have $\chi(x) = x^2 - c_1 x - c_2$.

A number r (possibly complex) is a **characteristic root** if $\chi(r) = 0$. From basic algebra we know that r is a root of a polynomial if and only if $(x - r)$ is a factor of the polynomial. When $\chi(x)$ is a degree 2 polynomial, by the quadratic formula, either $\chi(x) = (x - r_1)(x - r_2)$, where $r_1 \neq r_2$, or $\chi(x) = (x - r)^2$, for some r.

Theorem 36.1. *Let c_1 and c_2 be real numbers. Suppose that the polynomial $\chi(x) = x^2 - c_1 x - c_2$ has two distinct roots r_1 and r_2. Then a sequence $a : \mathbb{N} \to \mathbb{R}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, for $n \geq 2$ if and only if $a_m = \alpha r_1^m + \beta r_2^m$, for all $m \in \mathbb{N}$, and for some constants α and β.*

The constants are determined by the initial conditions (see equation 36.2).

Proof. *If $a_m = \alpha r_1^m + \beta r_2^m$ for all $m \in \mathbb{N}$, where α and β are some constants, then since $r_i^2 - c_1 r_i - c_2 = 0$, we have $r_i^2 = c_1 r_i + c_2$, for $i = 1$ and $n = 2$. Hence, for $n \geq 2$, we have*

$$\begin{aligned} c_1 a_{n-1} + c_2 a_{n-2} &= c_1(\alpha r_1^{n-1} + \beta r_2^{n-1}) + c_2(\alpha r_1^{n-2} + \beta r_2^{n-2}) \\ &= \alpha r_1^{n-2}(c_1 r_1 + c_2) + \beta r_2^{n-2}(c_1 r_2 + c_2) \text{ distributing and combining,} \\ &= \alpha r_1^{n-2} \cdot r_1^2 + \beta r_2^{n-2} \cdot r_2^2 \text{ by the remark above,} \\ &= \alpha r_1^n + \beta r_2^n = a_n. \end{aligned}$$

Conversely, if a is a solution of the recurrence relation and has initial terms a_0 and a_1, then one checks that the sequence $a_m = \alpha r_1^m + \beta r_2^m$ with

$$\alpha = \frac{a_1 - a_0 \cdot r_2}{r_1 - r_2}, \text{ and } \beta = \frac{a_0 r_1 - a_1}{r_1 - r_2} \tag{36.2}$$

also satisfies the relation and has the same initial conditions. The equations *for α and β come from solving the system of linear equations*

$$a_0 = \alpha(r_1)^0 + \beta(r_2)^0 = \alpha + \beta$$
$$a_1 = \alpha(r_1)^1 + \beta(r_2)^1 = \alpha r_1 + \beta r_2.$$

This system is solved using techniques from a prerequisite course. ♣

Example 36.2. *Solve the recurrence relation $a_0 = 2, a_1 = 3$ and $a_n = a_{n-2}$, for $n \geq 2$.*

Solution. *The recurrence relation is a linear homogeneous recurrence relation of degree 2 with constant coefficients $c_1 = 0$ and $c_2 = 1$. The characteristic polynomial is*

$$\chi(x) = x^2 - 0 \cdot x - 1 = x^2 - 1.$$

The characteristic polynomial has two distinct roots since

$$x^2 - 1 = (x-1)(x+1).$$

Let's say $r_1 = 1$ and $r_2 = -1$. Then, we find the system of equations:

$$2 = a_0 = \alpha 1^0 + \beta(-1)^0 = \alpha + \beta$$
$$3 = a_1 = \alpha 1^1 + \beta(-1)^1 = \alpha + \beta(-1) = \alpha - \beta.$$

Adding the two equations eliminates β and gives $5 = 2\alpha$, so $\alpha = 5/2$. Substituting this into the first equation, $2 = 5/2 + \beta$, we see that $\beta = -1/2$. Thus, our solution is

$$a_n = \frac{5}{2} \cdot 1^n + \frac{-1}{2}(-1)^n = \frac{5}{2} - \frac{1}{2} \cdot (-1)^n.$$

Example 36.3. *Solve the recurrence relation $a_1 = 3$, $a_2 = 5$, and,*

$$a_n = 5a_{n-1} - 6a_{n-2}, \text{ for } n \geq 3.$$

Solution. *Here the characteristic polynomial is*

$$\chi(x) = x^2 - 5x + 6 = (x-2)(x-3),$$

with roots $r_1 = 2$ and $r_2 = 3$. Now, we suppose that

$$a_m = \alpha 2^m + \beta 3^m, \text{ for all } m \geq 1.$$

The initial conditions give rise to the system of equations

$$\begin{aligned} 3 = a_1 &= \alpha 2^1 + \beta 3^1 = 2\alpha + 3\beta \\ 5 = a_2 &= \alpha 2^2 + \beta 3^2 = 4\alpha + 9\beta. \end{aligned}$$

If we multiply the top equation through by 2, we obtain

$$\begin{aligned} 6 &= 4\alpha + 6\beta \\ 5 &= 4\alpha + 9\beta. \end{aligned}$$

Subtracting the second equation from the first eliminates α and yields $1 = -3\beta$. So, we have found that $\beta = -1/3$. Substitution into the first equation yields $3 = 2\alpha + 3 \cdot (-1/3)$, so $\alpha = 2$. Thus

$$a_m = 2 \cdot 2^m - \frac{1}{3} \cdot 3^m = 2^{m+1} - 3^{m-1}, \text{ for all } m \geq 1.$$

36.4 The method for repeated roots

The other case we mentioned had a characteristic polynomial of degree two with one repeated root. Since the proof is similar we simply state the theorem.

Theorem 36.4. *Let c_1 and c_2 be real numbers with $c_2 \neq 0$ and suppose that the polynomial $x^2 - c_1 x - c_2$ has a root r with multiplicity 2, so that $x^2 - c_1 x - c_2 = (x - r)^2$. Then, a sequence $a : \mathbb{N} \to \mathbb{R}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, for $n \geq 2$ if and only if*

$$a_m = (\alpha + \beta m) r^m,$$

for all $m \in \mathbb{N}$, and for some constants α and β.

Example 36.5. *Solve the recurrence relation $a_0 = -1, a_1 = 4$ and $a_n = 4a_{n-1} - 4a_{n-2}$, for $n \geq 2$.*

Solution. *In this case we have $\chi(x) = x^2 - 4x + 4 = (x - 2)^2$. So, we* *may suppose that*

$$a_m = (\alpha + \beta m)2^m, \text{ for all } m \in \mathbb{N}.$$

The initial conditions give rise to the system of equations

$$\begin{aligned} -1 = a_0 &= (\alpha + \beta \cdot 0)2^0 = (\alpha) \cdot 1 = \alpha \\ 4 = a_1 &= (\alpha + \beta \cdot 1)2^1 = 2(\alpha + \beta) \cdot 2. \end{aligned}$$

Substituting $\alpha = -1$ *into the second equation gives* $4 = 2(\beta - 1)$, *so* $2 = \beta - 1$ *and* $\beta = 3$. *Therefore* $a_m = (3m - 1)2^m$, *for all* $m \in \mathbb{N}$.

36.5 The general case

Finally, we state[6] the general method of characteristic roots.

[6] without proof

Theorem 36.6. *Let* $c_1, c_2, ..., c_k \in \mathbb{R}$ *with* $c_k \neq 0$. *Suppose that the characteristic polynomial factors as*

$$\begin{aligned} \chi(x) &= x^k - c_1 x^{k-1} - c_2 x^{k-2} - ... - c_{k-1} x - c_k \\ &= (x - r_1)^{j_1} (x - r_2)^{j_2} \cdots (x - r_s)^{j_s} \end{aligned}$$

where $r_1, r_2, \ldots, r_s$ *are distinct roots of* $\chi(x)$, *and* $j_1, j_2, \ldots, j_s$ *are positive integers such that*

$$j_1 + j_2 + j_3 + ... + j_s = k.$$

Then a sequence $a : \mathbb{N} \to \mathbb{R}$ *is a solution of the recurrence relation*

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k} \quad \text{for } n \geq k$$

if and only if

$$a_m = p_1(m) r_1^m + p_2(m) r_2^m + ... + p_s(m) r_s^m \quad \text{for all } m \in \mathbb{N},$$

where

$$p_i(m) = \alpha_{0,i} + \alpha_{1,i} m + \alpha_{2,i} m^2 + ... + \alpha_{j_i - 1,i} m^{j_i - 1} \quad 1 \leq i \leq s$$

and the $\alpha_{l,i}$*'s are constants.*

There is a problem with the general case. It is true that given the recurrence relation we can simply write down the characteristic polynomial. However it can be quite a challenge to factor it as required by the theorem. Even if we succeed in factoring it we are faced with the tedious task of setting up and solving a system of k linear equations in k unknowns (the $\alpha_{l,i}$'s). While in theory such a system can be solved using the methods of elimination or substitution covered in a college algebra course, in practice, the amount of labor involved can become overwhelming. For this reason, computer algebra systems are often used in practice to help solve systems of equations, or even the original recurrence relation.

36.6 Exercises

Exercise 36.1. *For each of the following sequences find a recurrence relation satisfied by the sequence. Include a sufficient number of initial conditions to completely specify the sequence.*

(a) $a_n = 2n + 3, n \geq 0$

(b) $a_n = 3 \cdot 2^n, n \geq 1$

(c) $a_n = n^2, n \geq 1$

(d) $a_n = n + (-1)^n, n \geq 0$

Solve each of the following recurrence relations:

Exercise 36.2. $a_0 = 3, a_1 = 6$, *and* $a_n = a_{n-1} + 6a_{n-2}$, *for* $n \geq 2$.

Exercise 36.3. $a_0 = 4, a_1 = 7$, *and* $a_n = 5a_{n-1} - 6a_{n-2}$, *for* $n \geq 2$.

Exercise 36.4. $a_2 = 5, a_3 = 13$, *and* $a_n = 7a_{n-1} - 10a_{n-2}$, *for* $n \geq 4$.

Exercise 36.5. $a_1 = 3, a_2 = 5$, *and* $a_n = 4a_{n-1} - 4a_{n-2}$, *for* $n \geq 3$.

Exercise 36.6. $a_0 = 1, a_1 = 6$, *and* $a_n = 6a_{n-1} - 9a_{n-2}$, *for* $n \geq 2$.

Exercise 36.7. $a_1 = 2, a_2 = 8$, *and* $a_n = a_{n-2}$, *for* $n \geq 3$.

Exercise 36.8. $a_0 = 2, a_1 = 5, a_2 = 15$, *and* $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$, *for* $n \geq 3$.

Exercise 36.9. *Find a closed form formula for the terms of the Fibonacci sequence:* $f_0 = 0$, $f_1 = 1$, *and for* $n \geq 2$, $f_n = f_{n-1} + f_{n-2}$.

36.7 Problems

Problem 36.1. *Solve* $a_0 = 1$, *and* $a_n = 2a_{n-1}$, *for* $n \geq 1$ *using the characteristic equation method.*

Problem 36.2. *Solve* $a_0 = 2, a_1 = 5$ *and* $a_n = a_{n-1} + 6a_{n-2}$, *for* $n \geq 2$.

Problem 36.3. $a_0 = 3, a_1 = 7$, *and* $a_n = 6a_{n-1} - 5a_{n-2}$, *for* $n \geq 2$.

Problem 36.4. $a_2 = 5, a_3 = 13$, *and* $a_n = 3a_{n-1} + 10a_{n-2}$, *for* $n \geq 4$.

Problem 36.5. $a_1 = 3, a_2 = 5$, *and* $a_n = 8a_{n-1} - 16a_{n-2}$, *for* $n \geq 3$.

Problem 36.6. *$a_1 = 2, a_2 = 8$, and $a_n = 4a_{n-2}$, for $n \geq 3$.*

Problem 36.7. *$a_0 = 0, a_1 = 1, a_2 = 2$, and $a_n = -a_{n-1} + 4a_{n-2} + 4a_{n-3}$, for $n \geq 3$.*

Problem 36.8. *$a_0 = 0$, $a_1 = 1$, and for $n \geq 2$, $a_n = 2a_{n-1} + a_{n-2}$.*

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Solving Nonhomogeneous Recurrences

When a linear recurrence relation with constant coefficients for a sequence $\{s_n\}$ looks like

$$s_n = c_1 s_{n-1} + c_2 s_{n-2} + \cdots + c_k s_{n-k} + f(n),$$

where $f(n)$ is some (nonzero) function of n, then the recurrence relation is said to be **nonhomogeneous**. For example, $s_n = 2s_{n-1} + n^2 + 1$ is a nonhomogeneous recurrence. Here $f(n) = n^2 + 1$. The methods used in the last chapter are not adequate to deal with nonhomogeneous problems. But it wasn't all a waste since those methods do provide one step in the solution of nonhomogeneous problems.

37.1 Steps to solve nonhomogeneous recurrence relations

Step (1): Replace the $f(n)$ by 0 to create a homogeneous recurrence relation,

$$s_n = c_1 s_{n-1} + c_2 s_{n-2} + \cdots + c_k s_{n-k}.$$

Now solve this and write down the general solution[1]. For example, in the case of no repeated roots, the general solution will look something like:

$$s_n = a_1 r_1^n + a_2 r_2^n + \cdots + a_k r_k^n,$$

where the constants $a_1, a_2, \cdots, a_k$ are to be determined.

[1] We learned to do this in chapter 36.

Step (2): Next, find one **particular solution** to the original nonhomogeneous recursion. In other words, one specific sequence that obeys the recursive formula (ignoring the initial conditions). A method for finding a particular solution that works in many cases is to guess! Actually, it is to make an educated guess. Reasonable guesses depend on the form of $f(n)$. There is an algorithm that will produce the correct guess, but it is so complicated it isn't worth learning for the few simple examples we will be doing. Instead, rely on the following guidelines to guess the form of a particular solution.

Roughly, the plan is the guess a particular solution that is the most general function of the same type as $f(n)$. Specifically, table 37.1 shows reasonable guesses.

$f(n)$	Particular Solution Guess
c (a constant)	A (constant)
n	$An + B$
n^2	$An^2 + Bn + C$
n^3	$An^3 + Bn^2 + Cn + D$
2^n	$A2^n$
r^n (r constant)	Ar^n

Table 37.1: Particular solution patterns

These guesses can be *mixed-and-matched*. For example, if

$$f(n) = 3n^2 + 5^n,$$

then a reasonable candidate particular solution would be

$$An^2 + Bn + C + D5^n.$$

Once a guess has been made for the form of a particular solution, that guess is plugged into the recurrence relation, and the coefficients $A, B, \cdots$ are determined. In this way a specific particular solution will be found.

It will sometimes happen that when the equations are set up to determine the coefficients of the particular solution, an inconsistent system will appear. In such a case, as with repeated characteristic roots, the trick is (more-or-less) to multiply the guess for the particular solution by n, and try again.

Step (3): Once a particular solution has been found, add the particular solution of step (2) to the general solution of the homogeneous recurrence found in step (1). If we denote a particular solution by $h(n)$, then the total general solution looks like

$$s_n = a_1 r_1^n + a_2 r_2^n + \cdots + a_k r_k^n + h(n).$$

Step (4): Invoke the initial conditions to determine the values of the coefficients $a_1, a_2, \cdots, a_k$ just as we did for the homogeneous problems in chapter 36.

The major oversight made solving a nonhomogeneous recurrence relation is trying to determine the coefficients $a_1, a_2, \cdots, a_k$ before the particular solution is added to the general solution. This mistake will usually lead to inconsistent information about the coefficients, and no solution to the recurrence will be found.

37.2 Examples

Example 37.1. *Let's solve the Tower of Hanoi recurrence using this method.*

The recurrence is $H_0 = 0$, and, for $n \geq 1$, $H_n = 2H_{n-1} + 1$. We know the closed form formula for H_n is $2^n - 1$ already, but let's work it out using the method outlined above.

Step (1): Find the general solution of related homogeneous recursion (indicated by the superscript (h)): $H_n^{(h)} = 2H_{n-1}^{(h)}$. That will be $H_n^{(h)} = A2^n$.

Step (2): Guess the particular solution (indicated by superscript (p)): $H_n^{(p)} = B$, a constant. Plugging that guess into the recurrence gives $B = 2B + 1$, and so we see $B = -1$.

Step (3): Hence, the general solution to the Tower of Hanoi recurrence is

$$H_n = H_n^{(h)} + H_n^{(p)} = A2^n - 1.$$

Step (4): Now, use the initial condition to determine A: When $n = 0$, we want $0 = A2^0 - 1$ which means $A = 1$. Thus, we find the expected result:

$$H_n = 2^n - 1, \text{ for } n \geq 0.$$

Example 37.2. *Here is a more complicated example worked out in detail to exhibit the method. Let's solve the recurrence*

$$\begin{aligned} s_1 &= 2, \quad s_2 = 5 \quad \textit{and,} \\ s_n &= s_{n-1} + 6s_{n-2} + 3n - 1, \quad \textit{for } n \geq 3. \end{aligned}$$

Step (1): *Find the general solution of $s_n = s_{n-1} + 6s_{n-2}$. After finding the characteristic equation, and the characteristic roots, the general solution turns out to be $s_n = a_1 3^n + a_2(-2)^n$.*

Step (2): *To find a particular solution let's guess that there is a solution $h(n)$ that looks like $h(n) = an + b$, where a and b are to be determined. To find values of a and b that work, we substitute this guess for a solution into the original recurrence relation. In this case, the result of plugging in the guess ($s_n = h(n) = an + b$) gives us:*

$$an + b = a(n-1) + b + 6(a(n-2) + b) + 3n - 1.$$

which can be rearranged to

$$(6a + 3)n + (-13a + 6b - 1) = 0.$$

If this equation is to be correct for all n, then, in particular, it must be correct when $n = 0$ and when $n = 1$, and that tells us that

$$\begin{aligned} -13a + 6b - 1 &= 0 \quad \textit{and,} \\ 6a + 3 - 13a + 6b - 1 &= 0. \end{aligned}$$

Solving this pair of equations we find $a = -\frac{1}{2}$ and $b = -\frac{11}{12}$. And, sure enough, if you plug this alleged solution into the original recurrence, you will see it checks.

Step (3): *Write down the general solution to the original nonhomogeneous problem by adding the particular solution of step (2) to the general solution from step (1) getting:*

$$s_n = a_1 3^n + a_2(-2)^n + \left(-\frac{1}{2}\right)n + \left(-\frac{11}{12}\right).$$

Step (4): *Now a_1, a_2 can be calculated: For $n = 1$, the first initial condition gives*

$$2 = a_1 3^1 + a_2(-2)^1 + \left(-\frac{1}{2}\right) 1 + \left(-\frac{11}{12}\right),$$

and for $n = 2$, we get

$$5 = a_1 3^2 + a_2(-2)^2 + \left(-\frac{1}{2}\right) 2 + \left(-\frac{11}{12}\right).$$

Solving these two equations for a_1 and a_2, we find that $a_1 = \frac{11}{12}$ and $a_2 = -\frac{1}{3}$.

So the solution to the recurrence is

$$s_n = \frac{11}{12} 3^n - \frac{1}{3}(-2)^n + \left(-\frac{1}{2}\right) n + \left(-\frac{11}{12}\right).$$

37.3 Exercises

Use the general solutions for the related homogeneous problems of chapter 36 to help solve the following nonhomogeneous recurrence relations with initial conditions.

Exercise 37.1. $a_0 = 3, a_1 = 6$ *and* $a_n = a_{n-1} + 6a_{n-2} + 1$, *for* $n \geq 2$.

Exercise 37.2. $a_2 = 5, a_3 = 13$ *and* $a_n = 7a_{n-1} - 10a_{n-2} + n$, *for* $n \geq 4$.

Exercise 37.3. $a_1 = 3, a_2 = 5$ *and* $a_n = 4a_{n-1} - 4a_{n-2} + 2^n$, *for* $n \geq 3$.

Exercise 37.4. $a_0 = 1, a_1 = 6$ *and* $a_n = 6a_{n-1} - 9a_{n-2} + n$, *for* $n \geq 2$.

Exercise 37.5. $a_0 = 2, a_1 = 5, a_2 = 15$, *and* $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3} + 2n + 1$, *for* $n \geq 3$.

37.4 Problems

Problem 37.1. *Solve* $a_0 = 1$, *and* $a_n = 2a_{n-1} + 1$, *for* $n \geq 1$.

Problem 37.2. *Solve* $a_0 = 2, a_1 = 5$ *and* $a_n = a_{n-1} + 6a_{n-2} + 2$, *for* $n \geq 2$.

Problem 37.3. $a_0 = 3, a_1 = 7$, *and* $a_n = 6a_{n-1} - 5a_{n-2} + n$, *for* $n \geq 2$.

Problem 37.4. $a_2 = 5, a_3 = 13$, *and* $a_n = 3a_{n-1} + 10a_{n-2} + n + 2$, *for* $n \geq 4$.

Problem 37.5. $a_1 = 3, a_2 = 5$, *and* $a_n = 8a_{n-1} - 16a_{n-2}n^2$, *for* $n \geq 3$.

Problem 37.6. $a_1 = 2, a_2 = 8$, *and* $a_n = 4a_{n-2} + 2^n$, *for* $n \geq 3$.

Problem 37.7. $a_0 = 0, a_1 = 1, a_2 = 2$, *and* $a_n = -a_{n-1} + 4a_{n-2} + 4a_{n-3} + 2n$, *for* $n \geq 3$.

Problem 37.8. $a_0 = 0$, $a_1 = 1$, *and for* $n \geq 2$, $a_n = 2a_{n-1} + a_{n-2} + 1$.