CSCI-PHYS 3090 Spring 2023 **Name (Print):** Joel Carlson

Midterm 1 Examination, Feb 14, 2023

CSCI-PHYS 3090

Time allowed: 1.5 hours

Answer ALL questions

Total Points 100

Do not start the exam until told to do so

- This exam contains 5 questions, not all worth the same points.
- You must show working to receive the full points for each question.
- If you need to continue on extra sheets of paper, please label clearly where the work continues.

Question 1: Short Answer (20 points)

(a) (5 points) What does it mean for two quantum states $|x\rangle = \begin{bmatrix} x_0 \\ \vdots \\ x_n \end{bmatrix}$ and $|y\rangle = \begin{bmatrix} y_0 \\ \vdots \\ y_n \end{bmatrix}$ to be **orthogonal**? Give an example of two single qubit states that are orthogonal and are not computational basis states.

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

thad their result is the idenity matrix

(b) (5 points) Consider two quantum states $|\alpha\rangle$ and $|\beta\rangle$ and a linear unitary operator $\hat{U}$. Show that the action of $\hat{U}$ preserves the inner product, i.e., that $U|\alpha\rangle$ and $U|\beta\rangle$ has the same inner product as $|\alpha\rangle$ and $|\beta\rangle$. Then give an example of a 2×2 unitary matrix that is not the identity matrix.

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is a unitary matrix

$|\beta\rangle \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} c & a \\ d & b \end{bmatrix}$$

$$|\alpha\rangle = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}' \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$$

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \begin{bmatrix} 1a+3c, & 2a+4c \\ 1b+3d, & 2b+4d \end{bmatrix}$$

$$\begin{bmatrix} c & a \\ d & b \end{bmatrix}^T = \begin{bmatrix} c & d \\ a & b \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} a+3c & b+3d \\ 2a+4c_2 & 2b+4d \end{bmatrix}$$

It isn't quite right but the Unitary matrix done to both will be the same as multiply a normal math equation by 1. Such that the result comes out to be the same.

(c) (5 points) Given states

$$|x\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}, \quad \text{and} \quad |y\rangle = \frac{1}{\sqrt{3}}\begin{bmatrix}i\\-i\\-1\end{bmatrix},$$

find the **tensor product state** $|x\rangle \otimes |y\rangle$.

$|x\rangle \otimes |y\rangle$

$\begin{bmatrix}1\\-1\end{bmatrix} i = \begin{matrix}i\\-i\end{matrix}$

$\begin{bmatrix}1\\-1\end{bmatrix} -i = \begin{matrix}-i\\i\end{matrix}$

$\begin{bmatrix}1\\-1\end{bmatrix} -1 = \begin{matrix}-1\\1\end{matrix}$

$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{3}}\right)$

$$= \boxed{\begin{pmatrix} i\cdot\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{3}}\right) \\ -i\cdot\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{3}}\right) \\ -i\cdot\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{3}}\right) \\ i\cdot\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{3}}\right) \\ -1\cdot\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{3}}\right) \\ 1\cdot\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{3}}\right) \end{pmatrix}}$$

(d) (5 points) Given two Pauli operators

$$\hat{X} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \quad \text{and} \quad \hat{Z} = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix},$$

find the **tensor product operator** $\hat{X} \otimes \hat{Z}$

$0 \cdot \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} \quad 1 \cdot \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$

$1 \cdot \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} \quad 0 \cdot \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$

$$= \begin{bmatrix}0 & 0 & 1 & 0\\0 & 0 & 0 & -1\\1 & 0 & 0 & 0\\0 & -1 & 0 & 0\end{bmatrix}$$

Question 2: Quantum Norm and Measurements (10 points)

A two qubit system is given in terms of the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ as

$$|\phi\rangle = A\left(|00\rangle + 2i\,|01\rangle + |10\rangle - 3\,|11\rangle\right)$$

(a) (5 points) Determine the constant A such that $|\phi\rangle$ is a valid quantum state (i.e., normalized).

$i^2 = (\sqrt{-1})^2 = (-1)$

$$\sqrt{1^2 + (2i)^2 + 1^2 + (-3)^2} = 1 - 4 + 1 + 9 = -2 + 9 = \sqrt{7}$$

$A = 1/\sqrt{7}$

$|00\rangle + 2i|01\rangle + |10\rangle - 3|11\rangle$

(b) (5 points) The two possible results of a measurement of the *first* qubit (i.e., the left of the two qubits) of $|\phi\rangle$ in the computational basis are 0 and 1. Complete, in the table below, with what probabilities these occur, and what is the two-qubit state that results.

Measurement result	Probability to occur	State after measurement
$\frac{1}{1} + \frac{1}{(2i)}^2 =$ 0	$\frac{-4}{-4} + \frac{1}{-4} = \frac{-3}{-4} = 3/4$	$\frac{3\left(\|00\rangle + 2i\,\|01\rangle\right)}{4}$
1	$1/(1-3)^2$ $1/4$	$\left(\|10\rangle - 3\|11\rangle\right)/4$

I was un sure if it should be

$3/4\,(|0\rangle + 2i|1\rangle)$

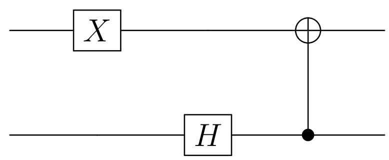
$1/4\,(|0\rangle - 3|1\rangle)$

of work

I decided it was wrong, but for proof

Question 3: Circuit Diagrams (35 points)

Consider the quantum circuit shown below:



(a) (5 points) Each of these three gate operations corresponds to an action (a matrix) that operates on the composite input state of the two qubits (a vector). What is the dimension of this vector?

$$\text{Cnot} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

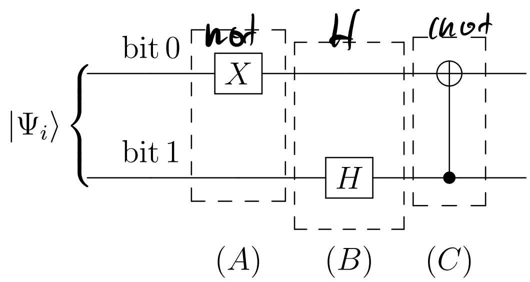
$X = 2\times2$ and $H = 2\times2$

$H \otimes$

$\text{Cnot} = 4\times4$

$H \otimes \text{cnot} = \boxed{8\times8}$

This circuit, $|\Psi_f\rangle = U_{\text{circuit}}\,|\Psi_i\rangle$, consists of three consecutive gate operations; (A) a NOT gate (Pauli-X) on bit 0, (B) a Hadamard $\hat{H}$ gate on bit 1, and (C) a controlled-NOT between bits 1 and 0.



$X\,|\Psi_i\rangle \otimes H|\Psi_i\rangle$

(b) (15 points) For *each* of the three gate operations (A), (B), and (C) shown in the dashed boxes above, write on the next page the expression that describes the gate. Your answer should contain a combination of symbols such as $\hat{H}$, $\hat{X}$, $\hat{X}_{ij}$, (where $i, j \in \{0, 1\}$), and $\mathbb{I}_2$ (the 2×2 identity operator), and the tensor product symbol $\otimes$. Do NOT write out the explicit forms for the matrices, we want the answer here in terms of the given elementary symbols as just described. Check that your gate operators all have the correct dimension corresponding to your answer given in part (a).

Gate A: $X|0\rangle =$

$|1\rangle =$

Gate B: $X|0\rangle =$

$H|1\rangle =$

Gate C: $\left(X|0\rangle \otimes |0\rangle\right)$

$H|1\rangle$

(c) (5 points) Using your answer from part (b), write down the final operator U_{circuit} for the complete quantum circuit by combining the three consecutive gate operations, thinking carefully about the order of operations. Again use the elementary expressions and do not write out the answer in terms of the matrices.

$$\left(X|0\rangle \otimes |0\rangle\right) H|1\rangle$$

(d) (10 points) If the initial state is $|\Psi_i\rangle = |1\rangle_0 \otimes |1\rangle_1$, calculate the final result of the circuit, i.e., $|\Psi_f\rangle = U_{\text{circuit}} |\Psi_i\rangle$.

$X|1\rangle_0$

$H|1\rangle_1 \rightarrow$

$$C_{not} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Question 4: Quantum cryptography (20 points)

(a) (5 points) Consider the message $1, 0, 1, 1, 0, 1, 1, 0, 0, 0$. Encode this message as discussed in class using the one-time codepad $0, 1, 0, 1, 1, 1, 0, 1, 0, 1$. Then show that you can recover the original message using the same codepad.

$1|0\rangle, 0|1\rangle, 1|1\rangle, 1|1\rangle, 0|1\rangle, 1|0\rangle, 1|1\rangle, 0|0\rangle, 0|1\rangle$

(b) (15 points) Consider the following realization of a BB84 protocol for securely transferring a secret key from Alice to Bob

	Bit no:	1	2	3	4	5	6	7	8	9	…
Alice:	**Preparation type:**	H	1	H	1	H	1	H	1	H	…
	Message bit:	0	1	1	0	1	1	1	0	1	…
		$H\|0\rangle$	$1\|1\rangle$	$H\|1\rangle$	$1\|0\rangle$	$H\|1\rangle$	$1\|1\rangle$	$H\|1\rangle$	$1\|0\rangle$	$H\|1\rangle$	
Bob:	**Measurement type:**	1	1	H	1	H	H	1	1	H	…
	Outcome:	1	1	1	?	1	?	1	0	1	…

In this table, the "Preparation type" specifies the basis which Alice uses to prepare her bit for sending, "Message bit" is the message bit itself, "Measurement type" is the measurement basis that Bob uses, and "Outcome" is the result of Bob's measurement.

(i) In the "Outcome" row, there are three empty rectangular boxes. In those boxes complete the table by entering the result of Bob's measurement using either '0' or '1' if you know the value with *certainty*, or '?' if it is not possible to predict the value with absolute certainty.

(ii) Circle on the table each of the columns that correspond to bits that can be used as part of the secret key.

(iii) Explain how Alice and Bob mutually establish that these are the correct bits to use.

they are the bits that are known with absolute certainty and do not match

I couldn't remember most of this so I sort of had to guess.

(iv) How do Alice and Bob go about determining if the eavesdropper "Eve" has intercepted their communication?

Since an interaction with a particle no matter how far apart will change it. If there is a portion that does not match they will know someone interacted with their communication

Question 5: Entanglement (15 points)

(a) (5 points) Is the following state entangled? If it is, show that this is the case, and if not, show how the state can be separated.

$$|\Psi\rangle = \frac{1}{2}\left(|00\rangle - |01\rangle - |10\rangle + |11\rangle\right)$$

$$\frac{1}{2}\left(|0\rangle(|0\rangle - |1\rangle) - |1\rangle(|0\rangle + |1\rangle)\right)$$

it is seperable

(b) (5 points) Is the following state entangled? If it is, show that this is the case, and if not, show how the state can be separated.

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$$ Yes, it is you can not break them apart

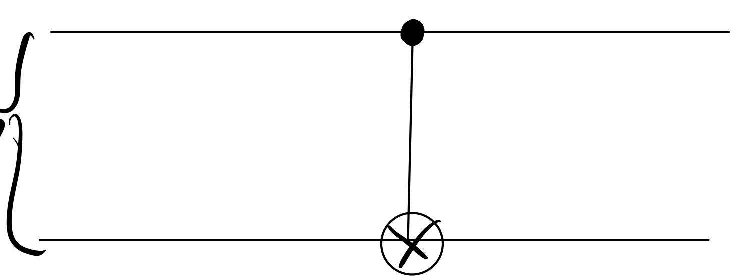
$$\frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$$

nothing can be taken from in here to seperate it to smaller blocks if it was: $(|01\rangle - |00\rangle) = |0\rangle(|1\rangle - |0\rangle)$,

but it is not

(c) (5 points) In the space below, draw a quantum circuit of two qubits that would take the input state $|00\rangle$ and generate the state given in part (b) at its output.

$|00\rangle$



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