Lesson 4

For these exercises, you will need to know the definitions of even and odd integers. An integer n is *even* if $n = 2k$ for some integer k. An integer n is *odd* if $n = 2k + 1$ for some integer k. There are no integers that are both even and odd! Examples: 6 is even since $6 = (2)(3)$, -8 is even since $-8 = (2)(-4)$, 0 is even since $0 = (2)(0)$, 3 is odd since $3 = 2(1) + 1$, and -9 is odd since $-9 = (2)(-5) + 1$.

(1) Give a direct proof that the sum of an odd integer and an even integer is odd.
Hint: Start by letting m be an odd integer and letting n be an even integer. That means $m = 2k + 1$ for some integer k and $n = 2j$ for some integer j. Notice that if we let the odd and even integers be $2k + 1$ and $2k$, the proof will only account for the cases in which n is one less than m. That is why we need to have $m = 2k + 1$ and $n = 2j$ for integers k, j, so that the sum of any odd and any even will be considered. You are interested in $m + n$, so add them up and see what you get. Why is the thing you get an odd integer (think about the definition of *odd*)?

(2) Give an indirect proof that if n^3 is even, then n is even. Hint: Study the solution of a similar statement in the sample solutions for this lesson.

(3) Give a proof by contradiction that if $5n - 4$ is odd, then n is odd.
Hint: This is the problem in this set that gives the most grief. Study the section in the notes where the mechanics of proving a statement of the form If P, then Q by contradiction is discussed. Be sure you understand why the first line of the proof should be something like *Suppose* $5n - 4$ *is odd* ***and*** n *is even.*

(4) Give an example of a predicate $P(n)$ about positive integers n, such that $P(n)$ is true for every positive integer from 1 to one billion, but which is never-the-less not true for all positive integers. (Hints: (1) There is a really simple choice possible for the predicate $P(n)$, (2) Make sure you write down a **predicate** with variable n, and not a **proposition**!) The purpose of this problem is to convince you that when checking a *for all* type proposition, it is not good enough to just check the truth for a few sample cases, or, for that matter, even a few billion sample cases. A general proof that covers all possible cases is necessary.

(5) Give a counterexample to the proposition *Every positive integer that ends with the digits* 13 *is a prime.*

(6) (bonus) The **maximum** of two numbers, a and b, is a provided $a \geq b$. Notation: $\max(a, b) = a$. The **minimum** of a and b is a provided $a \leq b$. Notation: $\min(a, b) = a$. Examples: $\max(2, 3) = 3$, $\max(5, 0) = 5$, $\min(2, 3) = 2$, $\min(5, 0) = 0$, $\max(4, 4) = \min(4, 4) = 4$.
Give a proof by cases (two cases is the natural choice for this problem) that for any numbers s, t,

$$\min(s, t) + \max(s, t) = s + t.$$