11

Functions and Their Properties

In algebra, functions are thought of as formulas such as $f(x) = x^2$ where x is any real number. This formula gives a rule that describes how to determine one number if we are handed some number x. So, for example, if we are handed $x = 2$, the function f says that determines the value 4, and if we are handed 0, f says that determines 0. There is one condition that, by mutual agreement, such a function rule must obey to earn the title function: the rule must always determine *exactly one value* for each (reasonable) value it is handed. Of course, for the example above, $x =$ blue isn't a reasonable choice for x, so f doesn't determine a value associated with *blue*. The **domain** of this function is all real numbers.

Instead of thinking of a function as a formula, we could think of a function as any rule which determines exactly one value for every element of a set A. For example, suppose W is the set of all words in English, and consider the rule, I, which associates with each word, w, the first letter of w. Then $I(cat) = c$, $I(dog) = d$, $I(a) = a$, and so on. Notice that for each word w, I always determines exactly one value, so it meets the requirement of a function mentioned above. Notice that for the same set of all English words, the rule $T(w)$ *is the third letter of the word* w is not a function since, for example, $T(be)$ has no value.

11.1 Definition of function

Here is the semi-formal definition of a function: A **function** from the set A to the set B is any rule which describes how to determine exactly one element of B for each element of A. The set A is called the **domain** of f, and the set B is called the **codomain** of f. The notation $f : A \to B$ means f is a function from A to B.

There are cases where it is not convenient to describe a function with words or formulas. In such cases, it is often possible to simply make a table listing the members of the domain along with the associated member of the codomain.

x	$f(x)$
1	a
2	a
3	c
4	b
5	d
6	e

Table 11.1: A simple function

Example 11.1. *Let $A = \{1, 2, 3, 4, 5, 6\}, B = \{a, b, c, d, e\}$ and let $f : A \to B$ be specified by table 11.1*

It is hard to imagine a verbal description that would act like f, but the table says it all. It is traditional to write such tables in a more compact form as

$$f = \{ (1, a), (2, a), (3, c), (4, b), (5, d), (6, e) \}.$$

The last result in example 11.1 looks like a relation, and that leads to the modern definition of a function:

Definition 11.2. A function, f, with domain A and codomain B is a relation from A to B (hence $f \subseteq A \times B$) such that each element of A is the first coordinate of exactly one ordered pair in f.

That completes the evolution of the concept of function from formula, through rule, to set of ordered pairs. When dealing with functions, it is traditional to write $b = f(a)$ instead of $(a, b) \in f$.

11.2 Functions with discrete domain and codomain

In algebra and calculus, the functions of interest have a domain and a codomain consisting of sets of real numbers, $A, B \subseteq \mathbb{R}$. The *graph* of f is the set of ordered pairs in the Cartesian plane of the form $(x, f(x))$. Normally in this case, the output of the function f is determined by some formula. For example, $f(x) = x^2$.

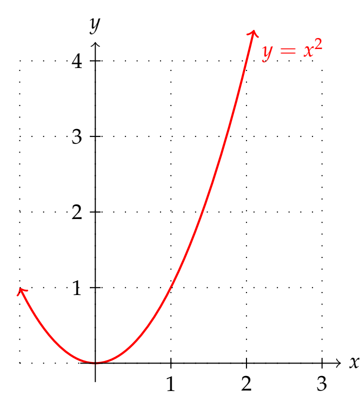

Figure 11.1: Graph of $y = x^2$

We can spot a function in this case by the **vertical line test**. A relation from a subset A of $\mathbb{R}$ to another subset of $\mathbb{R}$ is a function if every vertical line of the form $x = a$, where $a \in A$ intersects the graph of f exactly once.

In discrete mathematics, most functions of interest have a domain and codomain some finite sets, or, perhaps a domain or codomain consisting of integers. Such domains and codomains are said to be **discrete**.

11.2.1 Represention*s by* 0-1 *matrix or bipartite graph*

When $f : A \to B$ is a function and both A and B are finite, then since f is a relation, we can represent f either as a $0 - 1$ matrix or a bipartite graph. If M is a $0 - 1$ matrix which represents a function, then since every element of A occurs as the first entry in exactly one ordered pair in f, it must be that every row of M has exactly one 1 in it. So it is easy to distinguish which relations are functions, and which are not from the matrix for the relation. This is the discrete analog of the vertical line test, (but notice that rows are horizontal).

Example 11.3. *Again, let's consider the function defined, as in example 11.1, by f is from $A = \{1, 2, 3, 4, 5, 6\}$ to $B = \{a, b, c, d, e\}$ given by the relation $f = \{\, (1, a), (2, a), (3, c), (4, b), (5, d), (6, e)\, \}$.*

If we take the given orderings of A and B, then the 0-1 *matrix representing the function f appears in figure 11.2.*

Notice that in matrix form the number of 1*'s in a column coincides with the number of occurrences of the column label as output of the function. So the sum of all entries in a given column equals the number of times the element labeling that column is an output of the function.*

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Figure 11.2: A function in 0-1 matrix form

When a function from A to B is represented as a bipartite graph, every vertex of A is connected to exactly one element of B.

11.3 *Special properties*

In the case of a function whose domain is a subset of $\mathbb{R}$, the number of times that the horizontal line $y = b$ intersects the graph of f, is the number of inputs from A for which the function value is b. Notice that these criteria are twisted again. In the finite case we are now considering vertical information, and in the other case we are considering horizontal information. In either case, these criteria will help us determine which of several special properties a function either has or lacks.

11.3.1 One-to-one (injective)

We say that a function $f : A \to B$ is **one-to-one** provided $f(s) = f(t)$ implies $s = t$. The two dollar word for one-to-one is **injective**. The definition can also be expressed in the contrapositive as: f is one- -to-one provided $s \neq t$ implies $f(s) \neq f(t)$. But the definition is even easier to understand in words: a function is one-to-one provided different inputs always result in different outputs. As an example, consider the function $f : \mathbb{R} \to \mathbb{R}$ given by the formula $f(x) = x^2$. This function is not one-to-one since both inputs 2 and -2 are associated with the same output: $f(2) = 2^2 = 4$ and $f(-2) = (-2)^2 = 4$.

Example 11.4. *Proving a function is one-to-one can be a chore. Let* $f : \mathbb{R} \to\to \mathbb{R}$ *be given by* $f(x) = x^3 - 2$. *Let's prove* f *is one-to-one.*

Proof. *Suppose* $f(s) = f(t)$, *then* $s^3 - 2 = t^3 - 2$. *Thus* $s^3 = t^3$. *So* $s^3 - t^3 = 0$. *Now* $s^3 - t^3 = (s-t)(s^2 + st + t^2) = 0$ *implies* $s - t = 0$ *or* $s^2 + st + t^2 = 0$. *The first case leads to* $s = t$. *Using the quadratic formula, the second case leads to* $s = -t \pm \dfrac{\sqrt{t^2 - 4t^2}}{2}$. *Since* s *has to be a real number, the expression under the radical cannot be negative. The only other option is that it is* 0, *and that means* $t = 0$. *Of course if* $t = 0$ *this leads to* $s = 0 = t$. *So, in any case,* $s = t$. ♣

The one-to-one property is very easy to spot from either the matrix or the bipartite graph of a function. When $f : A \to B$ is one-to-one, and $|A| = m$ and $|B| = n$ for some $m, n \in \mathbb{N} - \{0\}$, then when f is represented by a 0-1 matrix M, there can be no more than one 1 in any column. So the column sums of any $0 - 1$ matrix representing a one-to-one function are all less than or equal to 1. Since every row sum of M is 1 and there are m rows, we must have $m \leq n$. The bipartite graph of a one-to-one function can be recognized by the feature that no vertex of the codomain has more than one edge leading to it.

11.3.2 Onto (surjective)

We say that a function $f : A \to B$ is **onto**, or **surjective**, if every element of B equals $f(a)$ for some $a \in A$. Consequently any matrix representing an onto function has each column sum at least one, and thus $m \geq n$. In terms of bipartite graphs, for an onto function, every element of the codomain has at least one edge leading to it.

As an example, consider again the function L from all English words to the set of letters of the alphabet defined by the rule $L(w)$ is the last letter of the word w. This function is not one-to-one since, for example, $L(cat) = L(mutt)$, so two different members of the domain of L are associated with the same member (namely t) of the codomain. However, L is onto. We could prove that by making a list of twenty-six words, one ending with a, one ending with b, $\cdots$, one ending with z. (Only the letters j and q might take more than a moment's thought.)

11.3.3 Bijective

A function $f : A \to B$ which is both one-to-one and onto is called **bijective**. In the matrix of a bijection, every column has exactly one 1 and every row has exactly one 1. So the number of rows must equal the number of columns. In other words, if there is a bijection $f : A \to B$, where A is a finite set, then A and B have the same number of elements. In such a case we will say the sets have the same **cardinality** or that they are **equinumerous**, and write that as $|A| = |B|$. The general definition (whether A and B are finite or not) is:

Definition 11.5. A and B are *equinumerous* provided there exists a bijection from A to B.

Notice that for finite sets with the same number of elements, A, B, any one-to-one function must be onto and vice versa. This is not true for infinite sets. For example the function $f : \mathbb{Z} \to \mathbb{Z}$ by $f(m) = 2m$ is one-to-one, but not onto, since $f(n) = 1$ is impossible for any n. On the other hand, the function $g : \mathbb{Z} \to \mathbb{Z}$ given by the rule $g(n)$ is the smallest integer that is greater than or equal to $\frac{n}{2}$ is onto, but not one-to-one. As examples, $g(6) = 3$ and $g(-5) = -2$. This function is onto since clearly $g(2n) = n$ for any integer n, so every element of the codomain has at least one edge leading to it. But $g(1) = g(2) = 1$, so g is not one-to-one.

11.4 Composition of functions

Since functions are relations, the **composition** of a function $g : A \to B$ by a function $f : B \to C$, makes sense. As usual, this is written as $f \circ g : A \to C$, but that's a little presumptive since it seems to assume that $f \circ g$ really is a function.

Theorem 11.6. *If $g : A \to B$ and $f : B \to C$, then $f \circ g$ is a function.*

Proof. *We need to show that for each $a \in A$ there is exactly one $c \in C$ such that $(a, c) \in f \circ g$. So suppose $a \in A$. since $g : A \to B$, there is some $b \in B$ with $(a, b) \in g$. Since $f : B \to C$, there is a $c \in C$ such that $(b, c) \in f$. So, by the definition of composition, $(a, c) \in f \circ g$. That proves there is at least one $c \in C$ with $(a, c) \in f \circ g$. To complete the proof, we need to show that there is only one element of C that $f \circ g$ pairs up with a. So, suppose that (a, c) and (a, d) are both in $f \circ g$. We need to show $c = d$. Since (a, c) and (a, d) are both in $f \circ g$, there must be elements $s, t \in B$ such that $(a, s) \in g$ and $(s, c) \in f$, and also $(a, t) \in g$ and $(t, d) \in f$. Now, since g is a function, and both (a, s) and (a, t) are in g, we can conclude $s = t$. So when we write $(t, d) \in f$, we might as well write $(s, d) \in f$. So we know (s, c) and (s, d) are both in f. As f is a function, we can conclude $c = d$.* ♣

If $g : A \to B$ and $f : B \to C$, and $(a, b) \in g$ and $(b, c) \in f$, then $(a, c) \in f \circ g$. Another way to write that is $g(a) = b$ and $f(b) = c$. So $c = f(b) = f(g(a))$. That last expression look like the familiar formula for the composition of functions found in algebra texts: $(f \circ g)(x) = f(g(x))$.

11.5 Invertible discrete functions

When $f : A \to B$ is a function, we can form the relation f^{-1} from B to A. But f^{-1} might not be a function. For example, suppose f :

$\{a,b\} \to \{1,2\}$ is $f = \{(a,1),(b,1)\}$. Then $f^{-1} = \{(1,a),(1,b)\}$, definitely not a function.

If in fact f^{-1} is a function, then for all $a \in A$ with $b = f(a)$, we have $f^{-1}(b) = a$ so $(f^{-1} \circ f)(a) = f^{-1}(f(a)) = f^{-1}(b) = a, \forall a \in A$. Similarly $(f \circ f^{-1})(b) = b, \forall b \in B$. In this case we say f is **invertible**. Another way to say the same thing: the inverse of a function $f : A \to B$ is a function $g : B \to A$ which undoes the operation of f. As a particular example, consider the function $f : \mathbb{Z} \to \mathbb{Z}$ given by the formula $f(n) = n+3$. In words, f is the **add** 3 function. The operation which undoes the effect of f is clearly the **subtract** 3 function. That is, $f^{-1}(n) = n-3$.

For any set, S, define $1_S : S \to S$ by $1_S(x) = x$ for every $x \in S$. In other words, $1_S = \{(x,x) \mid x \in S\}$. The function 1_S is called the **identity function on** S. So the computations above show $f^{-1} \circ f = 1_A$ and $f \circ f^{-1} = 1_B$.

Theorem 11.7. *A function $f : A \to B$ is invertible iff f is bijective.*

Proof. *First suppose that $f : A \to B$ is invertible. Then $f^{-1} : B \to A$ exists. If $f(a_1) = f(a_2)$, then since f^{-1} is a function, $a_1 = f^{-1}(f(a_1)) = f^{-1}(f(a_2)) = a_2$. Thus f is one-to-one. Also if $b \in B$ with $f^{-1}(b) = a$, then $f(a) = f(f^{-1}(b)) = b$. So f is onto. Since f is one-to-one and onto, f is bijective.*

Now suppose that f is bijective, and let $b \in B$. Since f is onto, we have some $a \in A$ with $f(a) = b$. If $e \in A$ with $f(e) = b$, then $e = a$ since f is one-to-one. Thus b is the first entry in exactly one ordered pair in the inverse relation f^{-1}. Whence, f^{-1} is a function. ♣

Do not make the error[1] of confusing inverses and reciprocals when dealing with functions. The **reciprocal** of $f : \mathbb{Z} \to \mathbb{Z}$ given by the formula $f(n) = n+3$ is $\dfrac{1}{f(n)} = \dfrac{1}{n+3}$ which is not the inverse function for f. For example $f(0) = 3$, but the reciprocal of f does not convert 3 back into 0, instead the reciprocal associates $\frac{1}{6}$ with 3. In fact, there are other problems with the reciprocal: it doesn't even make sense when $n = -3$ since that would give a division by 0, which is undefined. So, be very careful when working with functions not to confuse the words reciprocal and inverse.

[1] (Danger ahead!) They are entirely different things.

11.6 Characteristic functions

The characteristic vector (see section 6.10) of a set may be used to define a special 0-1 function representing the given set.

Example 11.8. *Let $\mathcal{U}$ be a finite universal set with n elements ordered $u_1, ..., u_n$. Let B_n denote all binary strings of length n. The characteristic function $\chi : \mathcal{P}(\mathcal{U}) \to B_n$, which takes a subset A to its characteristic vector is bijective. Thus there is no danger of miscomputation. We can either manipulate subsets of $\mathcal{U}$ using set operations and then represent the result as a binary vector or we can represent the subsets as binary vectors and manipulate the vectors with appropriate bit string operations. We'll get exactly the same answer either way.*

The process in example 11.8 allows us therefore to translate any set theory problem with finite sets into the world of 0's and 1's. This is the essence of computer science.

11.7 Exercises

Exercise 11.1. *Recall that $\mathbb{R}$ is the set of all real numbers. In each case, give an example of a function $f : \mathbb{R} \to \mathbb{R}$ with the indicated properties, or explain why no such function exists.*

(a) f *is bijective, but* f *is not the identity function* $f(x) = x$.

(b) f *is neither one-to-one nor onto.*

(c) f *is one-to-one, but not onto.*

(d) f *is onto, but not one-to-one.*

Exercise 11.2. *Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{a, b, c, d, e, f\}$. In each case, give an example of a function $f : A \to B$ with the indicated properties, or explain why no such function exists.*

(a) $f : A \to B$, f *is one-to-one.*

(b) $g : B \to A$, g *is one-to-one.*

(c) $f : A \to B$, f *is onto.*

(d) $g : B \to A$, g *is onto.*

Exercise 11.3. *Prove or give a counterexample: If E is an equivalence relations on a set A, then $E \circ E$ is an equivalence relation on A.*

Exercise 11.4. *Suppose $g : A \to B$ and $f : B \to C$ are both one-to-one. Prove $f \circ g$ is one-to-one.*

11.8 Problems

Problem 11.1. *Let $A = \{1, 2, 3, 4, 5, 6\}$. In each case, give an example of a function $f : A \to A$ with the indicated properties, or explain why no such function exists.*

(a) *f is bijective, but is not the identity function $f(x) = x$.*

(b) *f is neither one-to-one nor onto.*

(c) *f is one-to-one, but not onto.*

(d) *f is onto, but not one-to-one.*

Problem 11.2. *Repeat problem 11.1 with the set $A = \mathbb{N}$.*

Problem 11.3. *Repeat problem 11.1 with the set $A = \mathbb{Z}$.*

12

Special Functions

CERTAIN FUNCTIONS arise frequently in discrete mathematics. Here is a catalog of some important ones.

12.1 Floor and ceiling functions

To begin with, the **floor function** is a function from $\mathbb{R}$ to $\mathbb{Z}$ which assigns to each real number x, the largest integer which is less than or equal to x. We denote the floor function by $\lfloor x \rfloor$. So $\lfloor x \rfloor = n$ means $n \in \mathbb{Z}$ and $n \leq x < n+1$. For example, $\lfloor 4.2 \rfloor = 4$, and $\lfloor 7 \rfloor = 7$. Notice that for any integer n, $\lfloor n \rfloor = n$. Be a little careful with negatives: $\lfloor \pi \rfloor = 3$, but $\lfloor -\pi \rfloor = -4$. A dual function is denoted $\lceil x \rceil$, where $\lceil x \rceil = n$ means $n \in \mathbb{Z}$ and $n \geq x > n-1$. This is the **ceiling function**. For example, $\lceil 4.2 \rceil = 5$ and $\lceil -4.2 \rceil = -4$.

The graph (in the college algebra sense!) of the floor function appears in figure 12.1.

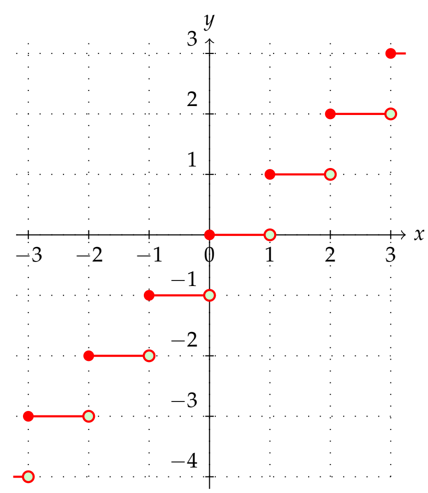

Figure 12.1: Floor function

12.2 Fractional part

The **fractional part**[1] of a number $x \geq 0$ is denoted $frac(x)$ and equals $x - \lfloor x \rfloor$. For numbers $x \geq 0$, the fractional part of x is just what would be expected: the stuff following the decimal point. For example, $frac(5.2) = 5.2 - \lfloor 5.2 \rfloor = 5.2 - 5 = 0.2$. When x is negative

[1] This is the Mathematica and Wolfram/Alpha definition. Often, the Graham definition is used:

$$frac(x) = x - \lfloor x \rfloor, \text{ for all } x.$$

its fractional part is defined to be $frac(x) = x - \lceil x \rceil$. Hence, we have

$$frac(x) = \begin{cases} x - \lfloor x \rfloor, & x \geq 0, \\ x - \lceil x \rceil, & x < 0. \end{cases}$$

For example, $frac(-5.2) = -5.2 - \lceil -5.2 \rceil = -5.2 - (-5) = -0.2$ In plain English, to determine the fractional part of a number x, take the stuff after the decimal point and keep the sign of the number. The graph of the fractional part function is shown in figure 12.2.

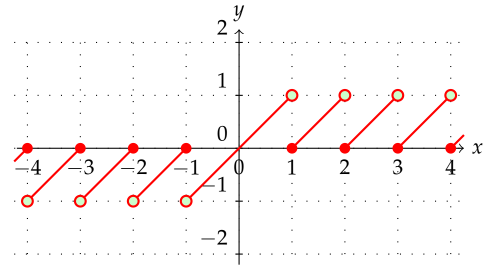

Figure 12.2: Fractional part function

12.3 *Integral part*

For any real number x its **integral part** is defined to be $x - frac(x)$.

The integral part can equivalently be defined by

$$[x] = \begin{cases} \lfloor x \rfloor, & x \geq 0, \\ \lceil x \rceil, & x < 0. \end{cases}$$

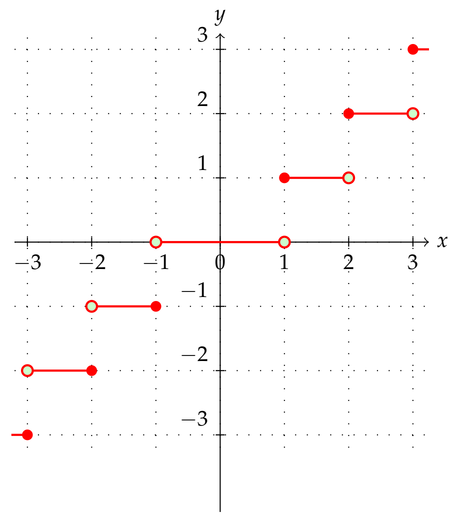

Figure 12.3: Integral part function

The integral part of x is denoted by $[x]$, or, sometimes, by $int(x)$. In words, the integral part of x is found by discarding everything following the decimal (at least if we agree not to end decimals with an infinite string of 9's such as $2.9999\cdots$). The graph of the integral part function is displayed in figure 12.3.

12.4 *Power functions*

The **power functions** are familiar from college algebra. They are functions of the form $f(x) = x^2$, $f(x) = x^3$, $f(x) = x^4$, and so on. By extension, $f(x) = x^a$, where a is any constant greater than or equal to 1 will be called a power function.

For any set X, the unit power function $1_X(x) = x$ for all $x \in X$ is called the **identity** function.

12.5 *Exponential functions*

Exchanging the roles of the variable and the constant in the power functions leads to a whole class of interesting functions, those of the form $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = a^x$, and $0 < a$. Such a function f is called the **base a exponential function**. The function is not very interesting when $a = 1$. Also if $0 < b < 1$, then the function $g(x) = b^x = \dfrac{1}{f(x)}$, where $f(x) = a^x$, and $a = \dfrac{1}{b} > 1$. So we may focus on $a > 1$. In fact the most important values for a are $2, e$ and 10. The number $e \approx 2.718281828459...$ is called the **natural base**, but that story belongs to calculus. Base 2 is the usual base for computer science. Engineers are most interested in base 10, while mathematicians often use the natural exponential function, e^x.

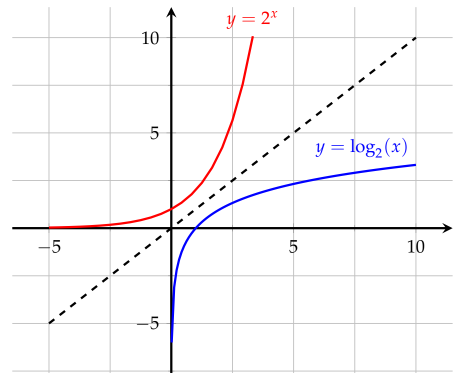

Figure 12.4: 2^x and $\log_2(x)$ functions

12.6 *Logarithmic functions*

By graphing the function $f : \mathbb{R} \to (0, \infty)$ defined by $y = f(x) = e^x$ we can see that it is bijective. We denote the inverse function $f^{-1}(x)$ by $\ln x$ and call it the **natural log function**. Since these are inverse functions we have

$$e^{\ln a} = a, \forall a > 0 \text{ and } \ln(e^b) = b, \forall b \in \mathbb{R}.$$

As a consequence $a^x = (e^{\ln a})^x = e^{(\ln a)\cdot x} = e^{x \ln a}$ is determined as the composition of $y = (\ln a)x$ by the natural exponential function. So every exponential function is invertible with inverse denoted as $\log_a x$, the **base a logarithmic function**. Besides the natural log, $\ln x$, we often write $\lg x$ for the base 2 logarithmic function, and $\log x$ with no subscript to denote the base 10 logarithmic function.

12.7 *Laws of logarithms*

The basic facts needed for manipulating exponential and logarithmic functions are the laws of exponents.

Theorem 12.1 (Laws of Exponents)**.** *For $a, b, c \in \mathbb{R}$, $a^{b+c} = a^b \cdot a^c$, $a^{bc} = (a^b)^c$ and $a^c b^c = (ab)^c$.*

From the laws of exponents, we can derive the

Theorem 12.2 (Laws of Logarithms)**.** *For $a, b, c > 0, \log_a bc = \log_a b + \log_a c$, and $\log_a(b^c) = c \log_a b$.*

Proof. *We rely on the fact that all exponential and logarithmic functions are one-to-one. Hence, we have that*

$$a^{\log_a bc} = bc = a^{\log_a b} a^{\log_a c} = a^{\log_a b + \log_a c},$$

implies

$$\log_a bc = \log_a b + \log_a c.$$

Similarly, the second identity follows from

$$a^{\log_a b^c} = b^c = (a^{\log_a b})^c = a^{c \log_a b}.$$

♣

Notice that $\log_a \dfrac{1}{b} = \log_a b^{-1} = -\log_a b$.

Calculators typically have buttons for logs base e and base 10. If $\log_a b$ is needed for a base different from e and 10, it can be computed in a roundabout way. Suppose we need to find $c = \log_a b$. In other words, we need the number c such that $a^c = b$. Taking the ln of both sides of that equation we get

$$\begin{aligned} a^c &= b \\ \ln(a^c) &= \ln b \\ c \ln a &= \ln b \\ c &= \frac{\ln b}{\ln a} \end{aligned}$$

Hence, we have the general relation between logarithms as follows.

Corollary 12.3. *So, we have* $\log_a(x) = \frac{\ln x}{\ln a}$.

Example 12.4. *For example, we see that* $log_2 100 = \frac{\ln 100}{\ln 2} \approx 6.643856.$

12.8 Exercises

Exercise 12.1. *In words, $\lfloor x \rfloor$ is the largest integer less than or equal to x.* ***Complete the sentence:*** In words, $\lceil x \rceil$ is the smallest . . . .

Exercise 12.2. *Draw a (college algebra) graph of $f(x) = \lceil x \rceil$.*

Exercise 12.3. *Draw a (college algebra) graph of $f(x) = \lfloor 2x - 1 \rfloor$.*

Exercise 12.4. *Draw a (college algebra) graph of $f(x) = 2\lfloor x - 1 \rfloor$.*

Exercise 12.5. *Let $f(x) = 18x$ and let $g(x) = \frac{x^3}{2}$. Sketch the graphs of f and g for $x \geq 1$ on the same set of axes. Notice that the graph g is lower then the graph of f when $x = 1$, but it is above the graph of f when $x = 9$. Where does g cross the graph of f (in other words, where does g catch up with f)?*

Exercise 12.6. *Let $f(x) = 4x^5$ and let $g(x) = 2^x$. For values of $x \geq 1$ it appears that the graph of g is lower than the graph of f. Does g ever catch up with f, or does f always stay ahead of g?*

Exercise 12.7. *The x^y button on your calculator is broken. Show how can you approximate $2^{\sqrt{2}}$ with your calculator anyhow.*

12.9 Problems

Problem 12.1. *Write* $\frac{5}{2}\ln 5 - 4\ln 3$ *as a single logarithm.*

Problem 12.2. *Draw the college algebra style graph of* $f(x) = e^{x+3} - 1$.

Problem 12.3. *Let* $f(x) = 2x^3$ *and* $g(x) = 3^x$. *Notice that* $f(1) < g(1)$ *and* $f(2) = g(2)$. *Does* g *ever catch up with* f *again, or does* f *always stay ahead of* g*?*

Problem 12.4. *Write* $\lg 1 + \lg 2 + \lg 3 + \lg 4 + \lg 5 + \lg 6$ *as a single logarithm.*

Problem 12.5. *Write* $\sum_{k=1}^{n} \lg n$ *as a single logarithm.*