CSCI-PHYS 3090 – Quantum Computing – Spring 2023
Homework #4

Due Wednesday, March 1, at 2:30 pm

Homework is graded for clarity of explanation as much as for mere "correctness" of the final answer. You will earn partial credit much easier if your writing is legible and organized and all of your steps are shown. Submit to gradescope, making sure that your scan can be easily read!

Problem 1: Counting (and looking ahead to Deutsch-Josza) (30 points)

An n-bit to 1-bit function f is said to be constant if either $f(x) = 0$ for all x or $f(x) = 1$ for all x. The function f is said to be balanced if $f(x) = 0$ for 2^{n-1} values of x and $f(x) = 1$ for the other 2^{n-1} values.

(a) For $n = 2$, how many functions from 2 bits to 1 bit are constant? How many are balanced?

(b) Repeat (a), but for $n = 4$.

(c) How many functions $f : \{0, 1, \ldots, q\}^n \to \{0, 1, \cdots, q\}$ are there? How many of them are constant?

(d) How many one-to-one functions are there from a set with m elements to a set with n elements? (A function f is one-to-one if no two elements in the domain of f correspond to the same element in the range of f.)

Problem 2: Quantum teleportation of entanglement (40 points)

Suppose that Alice and Bob share an entangled pair $|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}\big(|00\rangle_{AB} + |11\rangle_{AB}\big)$ and Alice and Carol share an entangled pair $|\psi\rangle_{AC} = \frac{1}{\sqrt{2}}\big(|00\rangle_{AC} + |11\rangle_{AC}\big)$. Now Alice creates a third entangled pair $|\phi\rangle_{\alpha\beta} = \frac{1}{\sqrt{2}}\big(|00\rangle_{\alpha\beta} + |11\rangle_{\alpha\beta}\big)$ and she uses the quantum teleportation protocol to teleport one half of $|\phi\rangle_{\alpha\beta}$ to Bob and one half of $|\phi\rangle_{\alpha\beta}$ to Carol. Show that Bob and Carol now share an entangled pair.

[Hint: Start by writing out the full quantum state $|\psi\rangle$ as a product state of the various entangled pairs. Then operate with your control-NOTs and Hadamards as part of the two quantum teleportation protocols, apply Alice's local measurements on her qubits, communicate the results to Bob and Carol on a classical channel, and finally see what quantum state you are left with after Bob and Carol make their local operations.]

Problem 3: Entanglement swapping and the GHZ basis (30 points)

Consider three local qubits A, B, C and three very distant qubits A', B', C', where the distant qubits have been spread far apart to the corners of the galaxy. Suppose we have previously prepared three maximally entangled pairs between AA', BB', and CC', i.e.,

$$\frac{1}{2^{3/2}}\left(|00\rangle_{AA'} + |11\rangle_{AA'}\right) \otimes \left(|00\rangle_{BB'} + |11\rangle_{BB'}\right) \otimes \left(|00\rangle_{CC'} + |11\rangle_{CC'}\right)$$

(a) Consider what happens if we do a local measurement (i.e., on the ABC qubits) in the computational basis, and we realize the specific measurement result $|000\rangle_{ABC}$. Write down what the resulting state of the distant qubits (i.e., $A'B'C'$)? Is this state entangled?

(b) Now consider what happens if we do a local measurement (i.e., on the ABC qubits) in the GHZ basis? The GHZ basis is a generalization of the Bell basis for 3 qubits. It consists of the 8 states

$$\begin{gathered}
\tfrac{1}{\sqrt{2}}\left(|000\rangle_{ABC} + |111\rangle_{ABC}\right) \\
\tfrac{1}{\sqrt{2}}\left(|000\rangle_{ABC} - |111\rangle_{ABC}\right) \\
\tfrac{1}{\sqrt{2}}\left(|001\rangle_{ABC} + |110\rangle_{ABC}\right) \\
\tfrac{1}{\sqrt{2}}\left(|001\rangle_{ABC} - |110\rangle_{ABC}\right) \\
\tfrac{1}{\sqrt{2}}\left(|010\rangle_{ABC} + |101\rangle_{ABC}\right) \\
\tfrac{1}{\sqrt{2}}\left(|010\rangle_{ABC} - |101\rangle_{ABC}\right) \\
\tfrac{1}{\sqrt{2}}\left(|100\rangle_{ABC} + |011\rangle_{ABC}\right) \\
\tfrac{1}{\sqrt{2}}\left(|100\rangle_{ABC} - |011\rangle_{ABC}\right)
\end{gathered}$$

Again determine what is the resulting state is of the distant qubits (i.e., $A'B'C'$)? Is this state entangled?