Math 208: Discrete Mathematics
Lesson 1: Lecture Video Notes

Topics

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2. Logical equivalences
    (a) logical equivalences
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Readings: Chapters 1-2

0. Motivations

- Euclid used logical arguments to prove geometric results in *The Elements*.
- Geometers attempted to prove the Parallel Postulate (i.e. Fifth Postulate)
  Given a line ℓ in $\mathbb{R}^2$ and a point P not on ℓ, there is exactly one line which is parallel to ℓ and passes though P.
- General problem: How does one reason deductively to make inferences? That is, how can one logically proceed from known statements or assumptions/axioms/postulates (i.e. premises) to conclusions?
- Paradoxes: Barber of Seville, Hilbert's Hotel, Zeno's Paradox, Liar Paradox, etc.
- Careful development of rules of logic and inference in early 1900's.
- Discovery of non-Euclidean geometry (i.e. elliptical and hyperbolic geometries) and resolution of paradoxes.
- Gödel's Incompleteness Theorem (1931): Roughly, under any given set of axioms, there are always statements which can neither be proved or disproved.

§1. Logical connectives and compound propositions.

1a. Propositions

Defn. A proposition is a statement which is either true (T) or false (F) but not both. We often use $p, q, r, s, \ldots$ to denote propositions.

Remarks. Binary notation uses 1 for T and 0 for F. Fuzzy logic explores logic using more than two possible truth values (e.g. any real number in [0,1]). Logic with two possible truth values is sometimes called Boolean logic in honor of George Boole.

Example and nonexamples.

	Proposition?	Truth Value
$1 + 2 = 3$		
There are 3,129,417 planets in the Milky Way galaxy.		
Give me that book!		
Have you ever visited Washington, DC?		
$x + 1 = 9$		
Today is Monday.		
This sentence is false.		
That person is more than 5 feet tall.		
It is raining.		

Observation. We can combine simple propositions using connectives to form *compound propositions.*

Example. It is sunny, and I studied today.

Important. Compound propositions are propositions, so they are either T or F but not both.

1b. Connectives

Logical connectives are used to combine propositions. Common logical connectives include:

- negation: *not* ($\neg$ or $\sim$)
- conjunction: *and* or *meet* ($\wedge$)
- disjunction: *or* or *join* ($\vee$)
- implication: *if..., then...* ($\rightarrow$ or $\Rightarrow$)
- biconditional: *...if and only if...* or *iff* ($\leftrightarrow$ or $\Leftrightarrow$)
- exclusive or: *xor* ($\oplus$)

Case 1: Logical connective involving one proposition.

Negation: To negate a statement p, either (1) preceed the statement with *it is not the case that...*, (2) add the word *not* in the appropriate location, or (3) add a slanted slash for mathematical statements.

Ex: Negate the following.

(i) $3 + 5 = 7$

(ii) $3 + 5 \neq 7$

(iii) There are no mosquitoes in North Dakota.

Case 2: Logical connective involving two propositions.

Two given propositions, say p and q, can be combined using the logical connectives conjunction, disjunction, implication, bicondional, and 'exclusive or' in the natural ways.

Ex. Consider the propositions:

p : it is below freezing
q : it is sunny

Write the following in ordinary language or symbolic language:

(i) It is below freezing but not sunny.

(ii) It is either not below freezing or it is sunny (or both).

(iii) It is either not below freezing or it is sunny (but not both).

(iv) $(\neg p) \rightarrow q$

(v) $\neg(p \rightarrow q)$

(vi) $q \leftrightarrow (\neg p)$

1c. Truth Tables

Ex. Compute the truth value of $\neg p$ given the truth value of p.

A few natural questions...

Q1. How many logical connectives are there on one proposition, say p?

Q2. How many logical connectives are there on two propositions, say p and q?

Q3. How many logical connectives are there on n propositions, say $p_1, \ldots, p_n$? (HW problem)

Next, let's discuss the truth tables for the common "binary" logical connectives.

Summary.

p	$\neg p$	$\mathbb{T}$	$\mathbb{F}$
T			
F			

p	q	$p \wedge q$	$p \vee q$	$p \oplus q$	$p \to q$	$p \leftrightarrow q$	$\mathbb{T}$	$\mathbb{F}$
T	T							
T	F							
F	T							
F	F							

Ex. Construct a truth table for $(p \to q) \vee (\neg p \to q)$.

Ex. Construct a truth table for $(q \wedge (\neg r)) \to (\neg p \oplus r)$.

1d. Translating between symbolic and ordinary English statements

Ex. Convert the following implications to symbolic statements.

(i) The apple trees bloom if it stays warm for a week.

(ii) It is necessary to walk 8 miles to get to Long's Peak.

(iii) That you get the job implies that you had the best credentials.

(iv) The beach erodes whenever there is a storm.

Ex. Translate to symbolic statements.

(i) To take discrete mathematics, you must have taken calculus or a course in computer science.

(ii) School is closed if more than 2 feet of snow falls or if the wind chill is below -100$°F$.

(iii) An integer n is odd if and only if n is not divisible by 2.

(iv) Dinner for includes soup or salad.

1e. Connections with operations on bit strings

Defn. A *bit string* of length n is any sequence of n bits (i.e. 0's or 1's).

Identifying T with 1 and F with 0, we can apply logical connectives to bit strings. When $n > 1$, apply the logical connectives to the corresponding bits from each bit string.

Ex. Simplify each of the following expressions.

(i) $1 \vee 0$

(ii) $1 \oplus (0 \to 1)$

(iii) $011 \leftrightarrow 110$

(iv) $0111 \to (1010 \wedge 0011)$

§2. Logical equivalences

2a. Logical equivalences

Ex. Compute the truth tables for the following expressions.

(i) $\neg(p \vee q)$

(ii) $\neg p \wedge \neg q$

Defn. Two propositions with identical truth values are called *logically equivalent*. If p_1 and p_2 are logically equivalent, this can be denoted as $p_1 \equiv p_2$.

Ex. From our last example, we see that $\neg(p \vee q) \equiv \neg p \wedge \neg q$. This is one of De Morgan's laws.

Ex. Show $\neg(\neg p) \equiv p$.

Ex. Determine whether or not the propositions $p \to (q \to r)$ and $(p \to q) \to r$ are equivalent.

Ex. Determine whether or not the propositions $p \to q$ and $\neg p \vee q$ are equivalent.

2b. Tautologies and contradictions

Defn. A *tautology* is a proposition whose truth value is always true and denoted $\mathbb{T}$.
A *contradiction* is a proposition whose truth value is always false and denoted $\mathbb{F}$.

Ex. Show $p \vee \neg p \equiv \mathbb{T}$ and $p \wedge \neg p \equiv \mathbb{F}$. That is, show $p \vee \neg p$ is a tautology, and $p \wedge \neg p$ is a contradiction.

2c. Relatives of implications

The implication $p \to q$ is read as "if p, then q". There are three related propositions of the form "if...,then..."

implication	$p \to q$
converse	$q \to p$
contrapositive	$\neg q \to \neg p$
inverse	$\neg p \to \neg q$

Ex. Consider the implication: If it is a warm day, I will eat ice cream.

(i) Write the contrapositive of the implication.

(ii) Write the inverse of the implication.

(iii) Write the converse of the contrapositive of the implications.

Proposition. An implication is logically equivalent to its contrapositive. Furthermore, the converse of an implication is logically equivalent to its inverse. That is, $p \to q \equiv \neg q \to \neg p$ and $q \to p \equiv \neg p \to \neg q$.

Remark. The logical equivalence between an implication and its contrapositive is often useful when writing proofs.

2d. Fundamental equivalences

Commonly used logical equivalences (taken from Table 2.3):

Name	Equivalence
Double negation	$\neg(\neg p) \equiv p$
Identity laws	$p \wedge \mathbb{T} \equiv p$
	$p \vee \mathbb{F} \equiv p$
Domination laws	$p \vee \mathbb{T} \equiv \mathbb{T}$
	$p \wedge \mathbb{F} \equiv \mathbb{F}$
Idempotent laws	$p \vee p \equiv p$
	$p \wedge p \equiv p$
Commutative laws	$p \vee q \equiv q \vee p$
	$p \wedge q \equiv q \wedge p$
Associative laws	$p \vee (q \vee r) \equiv (p \vee q) \vee r$
	$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$
Distributive laws	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
De Morgan's laws	$\neg(p \wedge q) \equiv \neg p \vee \neg q$
	$\neg(p \vee q) \equiv \neg p \wedge \neg q$
Law of Excluded Middle	$p \vee \neg p \equiv \mathbb{T}$
Law of Contradiction	$p \wedge \neg p \equiv \mathbb{F}$
Disjunctive form	$p \to q \equiv \neg p \vee q$
Implication $\equiv$ Contrapositive	$p \to q \equiv \neg q \to \neg p$
Inverse $\equiv$ Converse	$\neg p \to \neg q \equiv q \to p$

Remark. For homework problems, be sure to cite the name of any logical equivalences used. However, it will not be required to know all these names for the exams. Nonetheless, it is well worthwhile to recognize these identities by sight and name.

2e. Disjunctive normal form

Observe that we have studied six basic connectives: $\neg$, $\wedge$, $\vee$, $\to$, $\leftrightarrow$, and $\oplus$.

Fact. Using logical equivalences, we can take any given proposition involving the six connectives above and create a logically equivalence proposition involving only $\neg$, $\wedge$, and $\vee$.

Q. How can we eliminate any $\to$, $\leftrightarrow$, and $\oplus$'s?

Reduction identities.

$$\begin{aligned}
p \leftrightarrow q &\equiv (p \to q) \wedge (q \to p) \\
p \to q &\equiv \neg p \vee q \\
p \oplus q &\equiv (p \wedge \neg q) \vee (\neg p \wedge q) \stackrel{HW}{\equiv} (p \vee q) \wedge \neg(p \wedge q)
\end{aligned}$$

Ex. Write using only negations, conjunctions, and (inclusive) disjunctions: $p \leftrightarrow \neg q$.

Defn. A proposition is in *disjunctive normal form* (DNF) if it is written as a disjunction of conjunctions of literals where literals are a simple proposition with possibly a negation sign (e.g. p or $\neg p$).

Examples and nonexamples.

$$p \wedge q \qquad\qquad (p \wedge \neg q \wedge \neg r) \vee (\neg s \wedge t \wedge u)$$

$$\neg(p \vee q) \qquad\qquad p \wedge (q \wedge (r \vee s))$$

2f. Proving equivalences

A two-column style of proof is commonly used to prove logical equivalences. Note that an alternative is to show the truth values for each are always same. Using logical equivalences is considered more elegant.

General strategies.

1. Start with more complicated looking side.
2. Use identities to eliminate any $\leftrightarrow$, $\to$, $\oplus$'s.
3. Expand and simplify whenever possibly helpful.

Ex. Prove using logical equivalences: $\neg(p \vee (\neg p \wedge q)) \equiv \neg(p \vee q)$.

Ex. Prove using logical equivalences: $(p \wedge q) \to (p \vee q) \equiv \mathbb{T}$.