CSCI-PHYS 3090 – Quantum Computing – Spring 2023
Homework #0

Due Wednesday, January 25, at 2:30 pm

Homework is graded for clarity of explanation as much as for mere "correctness" of the final answer. You will earn partial credit much easier if your writing is legible and organized and all of your steps are shown. Submit to gradescope, making sure that your scan can be easily read!

Problem 1: Complex numbers (10 points)

Recall that a complex number z can be represented in one of two different ways:

1. In Cartesian form: $z = a + bi$ (for $a, b \in \mathbb{R}$)
2. In polar form: $z = r \cdot e^{i\theta}$ for $r \geq 0$ and $0 \leq \theta < 2\pi$

Consider the complex number $z = 2 + 4i$.

(a) What is the value of the modulus of z? What is the geometric interpretation of the modulus?

(b) Plot this number in the complex plane (x-axis for the real part, and y-axis for the imaginary part).

(c) Write z in the polar form form $z = r \cdot e^{i\theta}$. Hint: You have already calculated r.

(d) Using the same r and θ from part (c), write the number $w = r^2 \cdot e^{\frac{i\theta}{2}}$ in Cartesian form.

Problem 2: Linear equations (15 points)

Solve the following linear system and write the solution set in vector form.

$$\begin{aligned} 3x + \ \ y \quad\ \ &= 4 \\ -2x + \ \ y + z &= 1 \\ x + 2y + z &= 5 \end{aligned}$$

What does the solution set look like geometrically?

Problem 3: Matrix manipulation (15 points)

Consider matrices $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$.

(a) Find the transposes A^T and B^T.

(b) What does it mean for a matrix M to be *symmetric*? Are A or B symmetric?

(c) Compute the products AB and BA. Are these matrix products equal to each other? Explain why or why not.

(d) Compute the transpose of AB and the matrix $B^T A^T$. What do you notice?

(e) Given two matrices with variable entries, write the product in terms of those variables:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

Problem 4: Determinant and characteristic polynomial (20 points)

Consider the matrix

$$M = \begin{bmatrix} 2 & 3 & 2 \\ 1 & 1 & 1 \\ 4 & 6 & 5 \end{bmatrix}$$

(a) Compute $\det(M)$. What does this imply about the invertibility of M?

(b) For $|x\rangle \in \mathbb{R}^3$, find and describe all solutions to $M|x\rangle = 0$.

(c) For $|x\rangle \in \mathbb{R}^3$, find and describe all solutions to $M|x\rangle = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

(d) Find the characteristic polynomial of M (you do not need to factor it). What is the meaning of the roots of the characteristic polynomial of M?

(e) What is the purpose of the characteristic polynomial and why is it important? You may use online resources but please write 2-3 sentences in your own words.

Problem 5: Diagonalization (20 points)

If a matrix A is diagonalizable, it can be decomposed into the product of special matrices as

$$A = SDS^{-1}$$

(a) What are S and D in this decomposition? Describe both their relationship with A as well as any matrix properties they satisfy.

Determine whether the given matrix is diagonalizable and, if so, find the decomposition.

(b) $A = \begin{bmatrix} 3 & 2 \\ 3 & 4 \end{bmatrix}$

(c) $B = \begin{bmatrix} -5 & 3 \\ -3 & 1 \end{bmatrix}$

(d) $Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

(e) $H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Problem 6: Orthonormality and independence (20 points)

(a) What does it mean for two vectors $|x\rangle$ and $|y\rangle$ to be *orthogonal*? Give an example of two such orthogonal vectors in $\mathbb{R}^3$ not including the zero vector.

(b) What does it mean for a set of vectors $\{|x_1\rangle, \ldots, |x_m\rangle\}$ to be *linearly independent*? Give a set of three linearly independent vectors in $\mathbb{R}^3$ not including $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, or $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Can you come up with a set of four linearly independent vectors in $\mathbb{R}^3$?

(c) What does it mean for a set of vectors $\{|x_1\rangle, \ldots, |x_m\rangle\}$ to be *orthonormal*? Give an orthonormal set of three vectors in $\mathbb{C}^3$ in which all vectors have non-zero complex components.

(d) Can you come up with a set of four orthonormal vectors in $\mathbb{C}^3$? How about four orthogonal vectors?

(e) True or false: if $|v\rangle$ and $|w\rangle$ are orthogonal vectors, then they are linearly independent. If true, argue why it is true. If false, provide a counterexample. Is the converse of this statement true?