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The Tensor Product, Demystified

November 18, 2018 • Algebra

Previously on the blog, we've discussed a recurring th
mathematics: making new things from old things. Mat

hen you have two integers, you can find their g
ast common multiple.

hen you have some sets, you can form their Ca
nion.

hen you have two groups, you can construct th
roduct.

hen you have a topological space, you can look
pace.

hen you have some vector spaces, you can ask
ntersection.

Home

he list goes on!

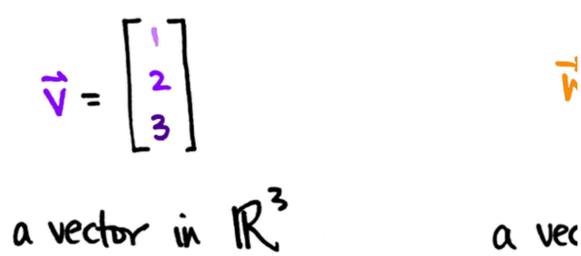
Ps. 148

I'd like to focus on a particular way to build a n
tor spaces: *the tensor product*. This constructio
sterious, but I hope to help shine a little light a
particular, we won't talk about axioms, univers
ms. Instead, we'll take an elementary, concrete l

wo vectors **v** and **w**, we can build a new vector
But what is that vector, *really*? Likewise, given
build a new vector space, also called their *tens*
vector space, *really*?

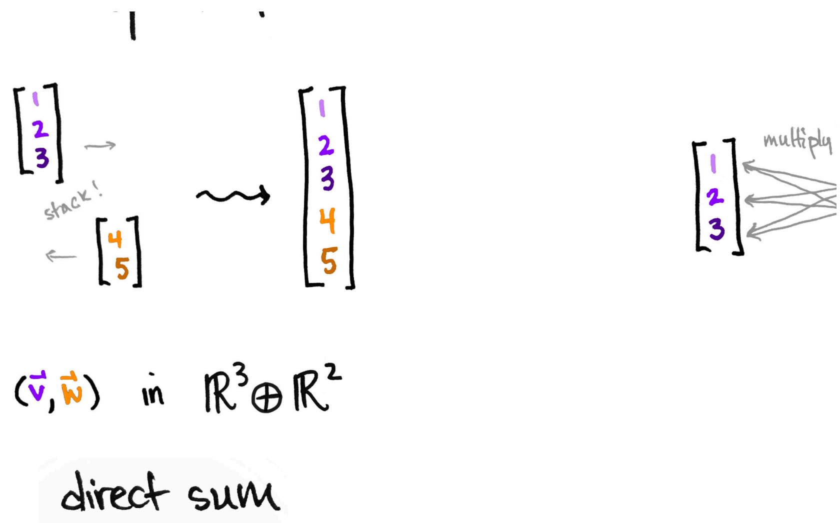
ing new vectors from old

discussion, we'll assume V and W are finite din
eans we can think of V as $\mathbb{R}^n$ and W as $\mathbb{R}^m$ for
So a vector **v** in $\mathbb{R}^n$ is really just a list of n num
a list of m numbers.



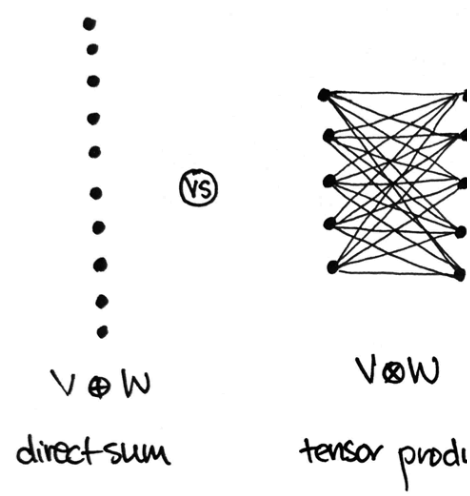
Let's try to make new, third vector out of **v** and **w**. Bu
We can stack them on top of each other, or we can fir
together and *then* stack them on top of each other.

[redacted]



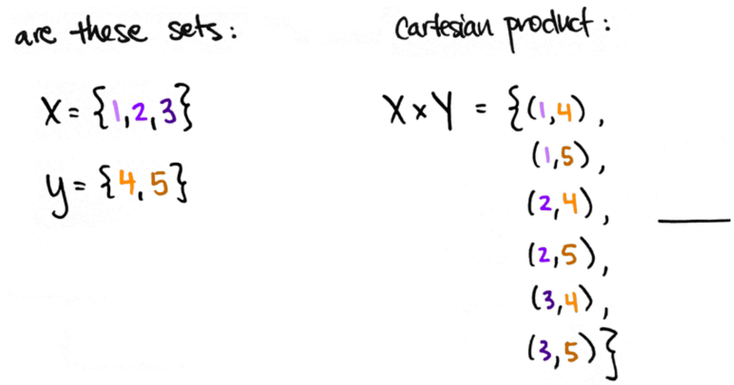
The first option gives a new list of $n+m$ numbers, wh
a new list of nm numbers. The first gives a way to buil
dimensions *add*; the second gives a way to build a new
dimensions *multiply*. The first is a vector $(\mathbf{v}, \mathbf{w})$ in the
the same as their direct product $V \times W$); the second i
tensor product $V \otimes W$.

And that's it!

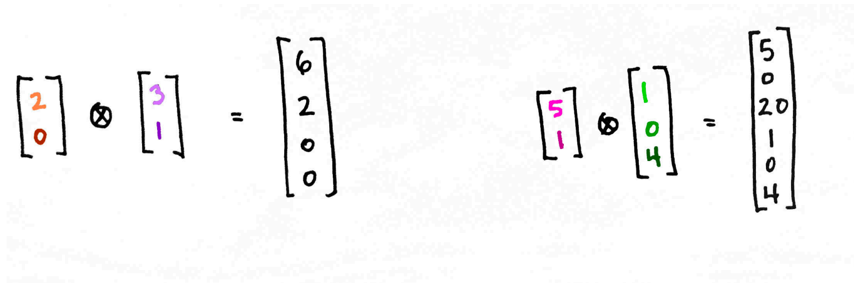


Forming the tensor product $\mathbf{v} \otimes \mathbf{w}$ of two vectors is *a*
Cartesian product of two sets $X \times Y$. In fact, that's ex
think of X as the set whose elements are the entries c

[redacted]

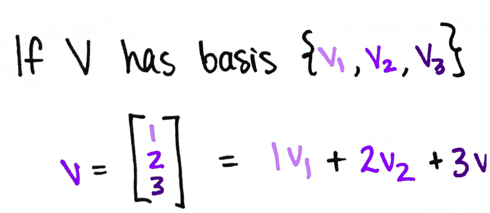


So a tensor product is like a grown-up version of mult
when you systematically multiply a bunch of numbers
results into a list. It's multi-multiplication, if you will.



There's a little more to the stor

Does *every* vector in $V \otimes W$ look like $\mathbf{v} \otimes \mathbf{w}$ for some
quite. Remember, a vector in a vector space can be wr
basis vectors, which are like the space's building block
making new things from existing ones: we get a new v
sum of some special vectors!



So a typical vector in $V \otimes W$ is a weighted sum of bas
basis vectors? Well, there must be exactly nm of them
$V \otimes W$ is nm. Moreover, we'd expect them to be built

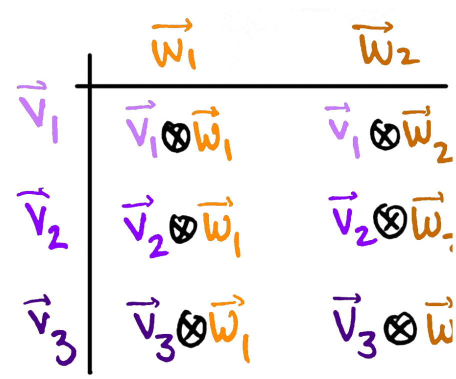
the basis of W. This brings us again to the "How can w

[redacted]

old things?" question. Asked explicitly: If we have n ba we have m bases $\mathbf{w}_1, \ldots, \mathbf{w}_m$ for W then how can we set of nm vectors?

This is totally analogous to the construction we saw al and a list of m things, we can obtain a list of nm thing together. So we'll do the same thing here! We'll simply with the $\mathbf{w}_j$ in all possible combinations, *except* "multi "take the tensor product of $\mathbf{v}_i$ and $\mathbf{w}_j$."

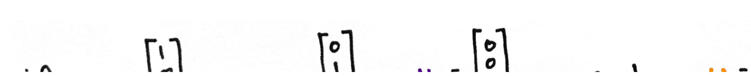
Concretely, a basis for $V \otimes W$ is the set of all vectors ranges from 1 to n and j ranges from 1 to m. As an exa $m = 2$ as before. Then we can find the six basis vector a 'multiplication chart.' (The sophisticated way to say t vector space on $A \times B$, where A is a set of generators generators for W.")



So $V \otimes W$ is the six-dimensional space with basis

$$\{\mathbf{v}_1 \otimes \mathbf{w}_1,\ \mathbf{v}_1 \otimes \mathbf{w}_2,\ \mathbf{v}_2 \otimes \mathbf{w}_1,\ \mathbf{v}_2 \otimes \mathbf{w}_2,\ \mathbf{v}$$

This might feel a little abstract with all the $\otimes$ symbols don't forget—we know exactly what each $\mathbf{v}_i \otimes \mathbf{w}_j$ look numbers! *Which list of numbers*? Well,



if $v_1 = \begin{bmatrix}\vdots\\0\end{bmatrix}$ $v_2 = \begin{bmatrix}\vdots\\0\end{bmatrix}$ $v_3 = \begin{bmatrix}\vdots\\1\end{bmatrix}$ and $w_1 =$

$$v_1 \otimes w_1 = \begin{bmatrix}1\\0\\0\\0\\0\\0\end{bmatrix} \quad v_1 \otimes w_2 = \begin{bmatrix}0\\1\\0\\0\\0\\0\end{bmatrix} \quad v_2 \otimes w_1 = \begin{bmatrix}0\\0\\1\\0\\0\\0\end{bmatrix} \quad v_2 \otimes w_2 = \begin{bmatrix}0\\0\\0\\1\\0\\0\end{bmatrix}$$

So what is $V \otimes W$? It's the vector space whose vector the $\mathbf{v}_i \otimes \mathbf{w}_j$. For example, here are a couple of vectors

$$7v_1 \otimes w_1 + 3v_3 \otimes w_2 = \begin{bmatrix}7\\0\\0\\0\\0\\3\end{bmatrix} \qquad -v_1 \otimes w_2 +$$

Well, *technically*...

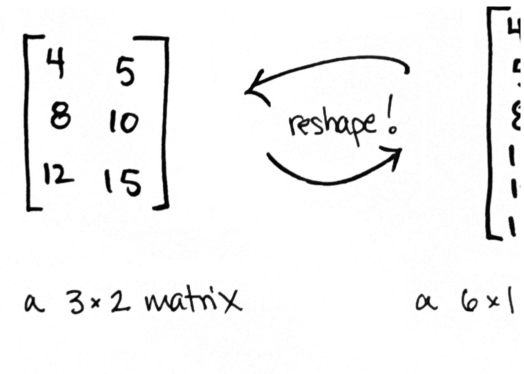
Technically, $\mathbf{v} \otimes \mathbf{w}$ is called the **outer product** of $\mathbf{v}$ an

$$\mathbf{v} \otimes \mathbf{w} := \mathbf{v}\mathbf{w}^\top$$

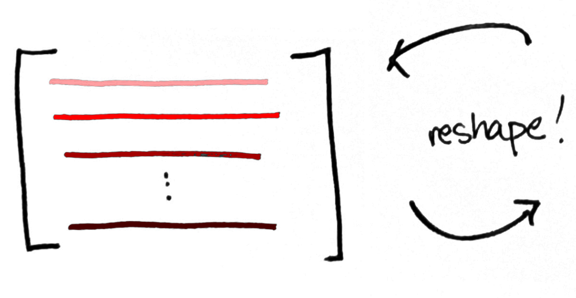
where $\mathbf{w}^\top$ is the same as $\mathbf{w}$ but written as a row vecto are complex numbers, then we also replace each entry So *technically* the tensor product of vectors is matrix:

$$\vec{v} \otimes \vec{w} = \begin{bmatrix}1\\2\\3\end{bmatrix}\begin{bmatrix}4 & 5\end{bmatrix} = \begin{bmatrix}1\cdot 4 & 1\cdot 5\\2\cdot 4 & 2\cdot 5\\3\cdot 4 & 3\cdot 5\end{bmatrix}$$

This may seem to be in conflict with what we did abov hand-in-hand. Any $m \times n$ matrix can be reshaped into and vice versa. (So thus far, we've exploiting the fact th $\mathbb{R}^6$.) You might refer to this as *matrix-vector duality.*



It's a little like a **process-state duality**. On the one har *process*—it's a concrete representation of a (linear) *tra* hand, $\mathbf{v} \otimes \mathbf{w}$ is, abstractly speaking, a vector. And a *ve* gadget that physicists use to describe the *state* of a qu encode processes; vectors encode states. The upshot product $V \otimes W$ can be viewed in *either way* simply by list or as a rectangle.



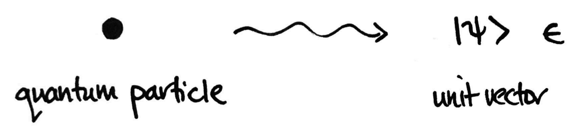
By the way, this idea of viewing a matrix as a process c *higher dimensional arrays*, too. These arrays are calle do a bunch of *these* processes together, the resulting r **tensor network**. But manipulating high-dimensional a very messy very quickly: there are lots of numbers tha together. This is like multi-multi-multi-multi...plicatio networks come with lovely pictures that make these c goes back to Roger Penrose's graphical calculus.) This have here, but it'll have to wait for another day!

In quantum physics

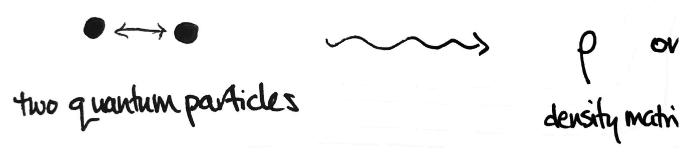
[redacted]

in quantum physics

One application of tensor products is related to the br "A *vector* is the mathematical gadget that physicists us quantum system." To elaborate: if you have a little qua like to know what it's doing. Or what it's capable of do it'll be doing something. In essence, you're asking: Wh its *state*? The answer to this question— provided by a mechanics—is given by a unit vector in a vector space $\mathbb{C}^n$.) That unit vector encodes information about that



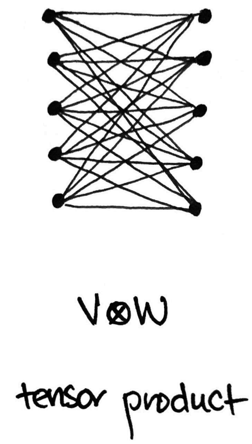
The dimension n is, loosely speaking, the number of d observe after making a measurement on the particle. quantum particles? The state of that two-particle syst something called a *density matrix* ρ on the tensor pro spaces $\mathbb{C}^n \otimes \mathbb{C}^n$. A density matrix is a generalization o for interactions between the two particles.



The same story holds for N particles—the state of an described by a density matrix on an N-fold tensor pro

But why the tensor product? Why is it that this constru describes the interactions within a quantum system s know the answer, but perhaps the appropriateness of

too surprising. The tensor product itself captures all w "interact" with each other!



Of course, there's lots more to be said about tensor pr
snippet of basic arithmetic. For a deeper look into the
reading through Jeremy Kun's wonderfully lucid How
Tensorphobia and Tensorphobia and the Outer Produ

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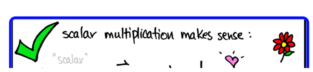
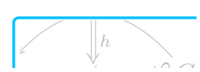
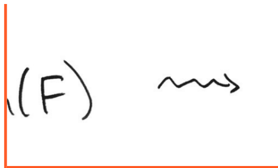


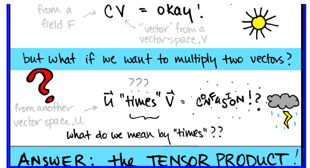




A Quotient of the General Linear Group, Intuitively

December 15, 2016

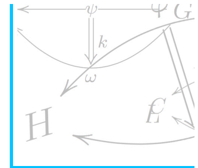
in Algebra



Motivation for the Tensor Product

August 16, 2015

in The Back Pocket



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February 16,

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G gbd_…

4 years ago

Great article!

One thing that helped me conquer tensorphobia was the idea of bilinearity (or multilinearity more generally, I guess). I don't think it's the best way of first introducing the topic---this article's approach is probably best for that---but it helped make it clearer to me why tensors matter. I'll lay it out here in case it helps anyone.

First off, I think "bilinearity" is a misnomer. It makes it sound like it's sort of ultra-linear. Like, you have normal old linear maps, but then you have maps, but then you have *bilinear* maps, which are *twice* as linear, somehow! But no, bilinearity is a special case of *non*-linearity, not of linearity. Namely, if you have a product space V×W, then a bilinear map `f : V×W -> U` out of it satisfies:

f(v1 + v2, w) = f(v1,w) + f(v2,w)
f(v, w1 + w2) = f(v, w1) + f(v, w2)
f(a*v, w) = f(v, a*w) = a*f(v,w)

As you can see, it's *like* the conditions for linearity, but not quite the same. If you hold the right argument constant, you have a linear map V -> U, and if you hold the left argument constant, you have a linear map W -> U. But if neither argument is held constant, the map *isn't* linear. A linear map out of the same space would satisfy:

f(v1+v2, w1+w2) = f(v1, w1) + f(v2, w2)
f(a*v, a*w) = a*f(v, w)

As you can see, a linear map acts more uniformly; the conditions have to hold for both conditions at once. f(ax,ay) is a*f(x,y) for linear f, but it has to be a^2*f(x,y) for bilinear maps; you need to "pull out" the "a" twice.

Okay, so anyway, bilinear maps are a useful type of nonlinear maps out of the product of two vector spaces. For now, you kinda have to take my word for it that this class of maps is useful and interesting, but hopefully it shouldn't be too hard to believe. After all, linear maps are extremely useful, and bilinearity is linear in each of its arguments. It's like you're "encoding" two different classes of interrelated linear maps.

But, since bilinear maps fail to *actually* be linear as a whole, linear algebra isn't really equipped to deal with them directly! That's where tensor products come in. There is a one-to-one correspondence between BILINEAR maps f : V×W -> U and LINEAR maps V⊗W -> U! (In fact, that's actually one way to *define* a tensor product in a more category-theoretic way; the "universal property" of the tensor product, at least in the category of vector spaces, is based on the preceding fact.) So, tensor products gives you a way to use the tools of linear algebra on bilinear maps, even though bilinear maps are nonlinear. Generalizing to multilinear maps, with higher-order products and tensor products, is pretty easy from here.

5 0 Reply

Tai

4 years ago

Thanks for sharing!

0 0 Reply

Leon...

4 years ago

Thank you!

0 0 Reply

Lisan...

LO

4 years ago

Awesome !!! Thanks a bunch !!!

0 0 Reply

Dinus...

DA

3 years ago

great article. when i searched what is the tensor product for one of my undergraduate project there were lots of explanation which i cant understand. this is very good explanation and very easy to understand.

0 0 Reply

Jan R...

3 years ago

Thanks for putting all this together :-)

0 0 Reply

DF **Danie...**

3 years ago

Love the article - it's really helping me familiarize myself with tensors ! Your content is beautifully put and very insightful. One question though, when you write the example of the tensor product of the vector spaces V and W, you write "V (x) W is the free vector space on A x B, where A is a set of generators for V and B is a set of generators for Y". Should this Y be a W ? Thanks !

0 0 Reply

> **Tai...**
>
> 3 years ago
>
> Whoops yes, fixed. Thanks!
>
> 0 0 Reply

Abrah...

2 years ago

Awesome article! Thanks in advance for your easy explanation.

2 0 Reply

HR **hami...**

2 years ago

[redacted]

Amazing article. Thanks a lot