Math 208: Discrete Mathematics
Lesson 20: Lecture Video Notes

Topics

38. Graphs
    (a) Some graph terminology
    (b) A Historical Interlude: The origins of graph theory
    (c) The First Theorem of Graph Theory
    (d) Brief Catalog of Special Graphs
    (e) Graph isomorphisms
    (f) Paths
        (i) Some path terminology
        (ii) Eulerian and Hamiltonian paths
        (iii) Some facts on Eulerian and Hamiltonian graphs
    (g) Trees

Readings: Chapter 38

§38. Graphs

38a. Some graph terminology

Defn. A *graph* consists of a number of points called *vertices* together with lines called *edges* joining some (possibly none, some, or all) pairs of vertices. A *loop* is an edge from a vertex to itself. A *simple* graph is a graph with no loops and no multiple edges.

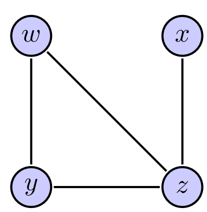
Ex. Draw a simple graph G with three vertices. Then draw a graph H with four vertices and a loop. List the vertices and edges.

We can use two types of 0-1 matrices to represent a given graph G in computers, namely the adjacency matrix A_G and incidence matrix M_G.

Adjacency matrix A_G: The rows and columns are indexed by the vertex set. The (v_1, v_2) entry has a value of 1 if there is an edge between v_1 and v_2 (i.e. v_1 and v_2 are adjacent) and value of 0 otherwise.

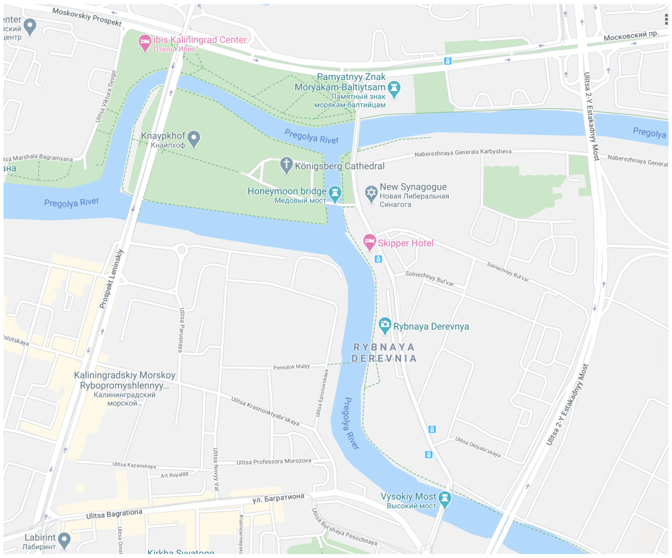
Incidence matrix M_G: The rows are indexed by the vertex set and columns by the edges in some order. The (v, e) entry has a value of 1 if there is the edge e contains v as an endpoint (i.e. e is incident with v) and value of 0 otherwise.

Ex. Compute the adjacency matrix and incidence matrix for the graph given below.



38b. A Historical Interlude: The origins of graph theory

The origins of graph theory lie in Leonhard Euler's solution to the Königsberg Bridge Problem in 1736. Today Königsberg is known as Kalingrad, Russia.

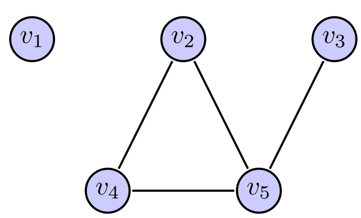


Euler's Conclusion: There does not exist a walking path (i.e. Eulerian path) which crosses each of the seven bridges exactly once.

38c. The First Theorem of Graph Theory

Defn. The *degree* of a vertex v of a graph, written $deg(v)$, is the number of edges incident to v. The list of degrees of the vertices in a graph is called the *degree sequence*. A degree sequence is traditionally written in increasing order.

Ex. Compute the degree of each vertex and the degree sequence for the graph given below.



General Problem. Construct a graph which has a given a degree sequence.

First Theorem of Graph Theory. The sum of the degrees of the vertices of a graph equals twice the number of edges. In particular, the sum of the degrees is even.

Remark. The last theorem is also known as the Handshaking Lemma or Handshaking Theorem.

Ex. Draw any graph and verify the First Theorem of Graph Theory.

Corollary. A graph must have an even number of vertices of odd degree.

38d. Brief Catalog of Special Graphs

There are certain families of graphs which occur frequently in graph theory. We mention some of them here.

Name	symbol	Examples
complete graph	K_n	
n-cycle	C_n	
n-link	L_n	
n-wheel	W_n	
n-cube	Q_n	

Def. A graph is *bipartite* if it is possible to split the vertices into two subsets, say T and B, where any edge has one endpoint in T and the other endpoint in B.

Ex. Draw an example of a bipartite graph with five vertices.

Name	symbol	Examples
complete bipartite graph	$K_{m,n}$	

Remark. There are many overlaps between these families for certain choices of n. For instance, Q_2 is the "same" as $K_{2,2}$.

38e. Graph isomorphisms

Consider the two graphs G and H below

These graphs are really the "same" up to relabeling the vertices. To make the idea of *sameness* more precise, the notion of graph isomorphism was introduced.

Def. We say the graphs G and H are isomorphic provided we can relabel the vertices of one of the graphs using the labels of the other in such a way that both graphs have exactly the same edges.

Ex. Give a graph isomorphism for the example above. Can you find a second isomorphism?

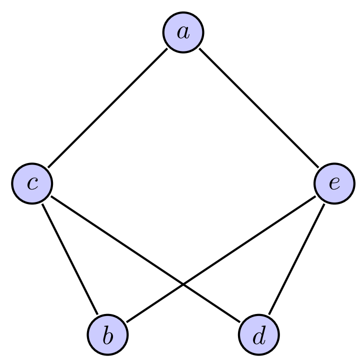
General problem. Find efficient ways to determine if two graphs G and H are isomorphic or not.

Facts. If two graphs G and H are isomorphic, then:

- G and H have the same number of vertices and edges
- G and H have the same degree sequences

Note that if two graphs have the same number of vertices and edges as well as the same degree sequence, the graphs may or may not be isomorphic. That is, these conditions are *necessary* but not *sufficient* for a pair of graphs to be isomorphic.

Ex. Determine whether or not the following graph is isomorphic to $K_{2,3}$.



38f. Paths

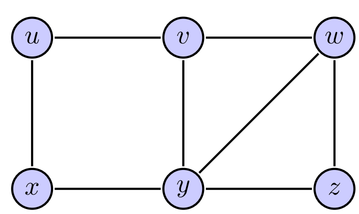
Graph theory originated by considering routes crossing bridges. We next consider this problem more carefully by thinking of walking along edges of a graph and visiting vertices. Note that we only consider simple graphs – no multiple edges and no loops are allowed in our graphs.

(i) Some path terminology

We begin by introducing some path terminology. This terminology is frequently used differently in different texts. It's a good idea to check the definitions used in each reference.

Defn. A *path of length* n in a graph is a sequence of $n+1$ vertices $v_0, v_1, \ldots, v_n$. The endpoints of the path are the vertices v_0 and v_n. Repeated vertices and edges are allowed.

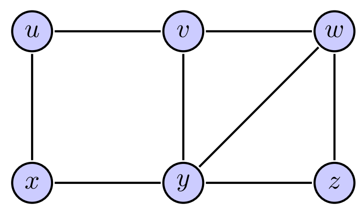
Ex. Identify a path of length 5 in the graph below. Identify its endpoints.



Defn. A *circuit* is a path of length three or more for which the endpoints are the same. A *simple* graph does not repeat any edges.

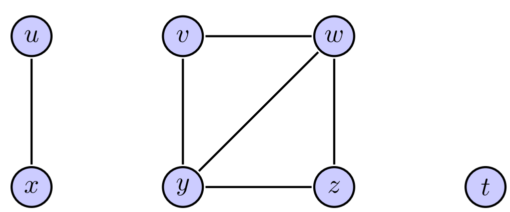
Note. A single vertex v is considered a path of length 0 but not a circuit (since circuits require at least 3 edges).

Ex. Identify a circuit and a simple path in the graph below, if possible.



Defn. A graph is *connected* if there is a path between any two vertices. That is, a connected graph consists of a single piece. The individual connected pieces of a graph are called its *connected components*. The length of the shortest path between two vertices in a connected component of a graph is call the *distance* between the vertices.

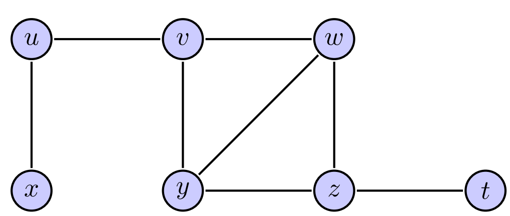
Ex. How many connected components are there? Compute the distance between w and y.



Thm. In a connected graph there is a simple path between any two vertices. That is, if there is a way to get from one vertex to another, then it can be done without repeating any edges.

Defn. A vertex in a graph is a *cut vertex* if its removal and edges incident to it result in a graph with more connected components. A *bridge* is an edge whose removal (keeping all vertices) produces a graph with more connected components.

Ex. Identify any cut vertices and bridges in the graph below.



(ii) Eulerian and Hamiltonian paths

There are two special types of paths that we consider: Eulerian and Hamiltonian.

Defn. An *Eulerian path* in a graph is a simple path which transverses every edge of the graph. That is, an Eulerian path is a path which uses every edge exactly once. If the starting and ending vertices are the same, we say the path is an *Eulerian circuit.* A graph is called Eulerian if it has an Eulerian circuit.

Ex. Consider the graphs L_4, C_4, K_4, and K_5. Find any Eulerian paths and/or circuits.

Defn. An *Hamiltonian path* in a graph is a simple path that visits every vertex of the graph exactly once. A *Hamiltonian circuit* is a simple circuit that visits every vertex in the graph exactly once, except the beginning and ending vertices are the same. A graph is called Hamiltonian if it has an Hamiltonian circuit.

Ex. Consider the graphs L_4, W_5, K_4, and K_5. Find any Hamiltonian paths and/or circuits.

Helpful Observation: A Hamiltonian circuit uses exactly two edge at each vertex, one to enter and one to exit. This is often useful when encountering vertices of degree 2 since both edges must be part of the Hamiltonian circuit!

Remark. The Königsberg Bridge Problem involved identifying any Eulerian paths. The traveling salesman problem and its variants involve identifying Hamiltonian paths.

(iii) Some facts on Eulerian and Hamiltonian graphs

Observation. If a graph G has either an Eulerian circuit or Hamiltonian circuit, then

- G is connected
- Every vertex has degree at least 2
- G has no bridges.

Note. If G has a Hamiltonian circuit, then G has no cut vertices.

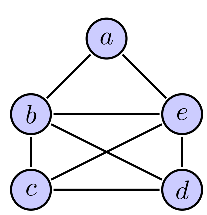
Thm. A connected graph Is Eulerian if and only if every vertex has even degree.

Ex. Determine whether or not K_4 and K_5 are Eulerian.

Ex. One more graph to check for Eulerian....

Thm. A connected graph has an Eulerian path, but not an Eulerian circuit, if and only if it has exactly two vertices of odd degrees.

Ex. Consider the graph



Thm. Let G be a connected graph with $n \geq 3$ vertices. If $\deg(v) \geq n/2$ for every vertex v, then G is Hamiltonian.

Remark. The last theorem can be limited in its use, so don't read into it too much.

Thm. Consider the graph C_5.

38g. Trees

Another important class of graphs are trees.

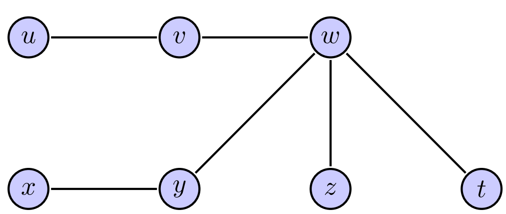
Defn. A *tree* is a connected graph with no circuits.

Ex. Family trees are example of trees.

Note. A collection of trees is called a forest.

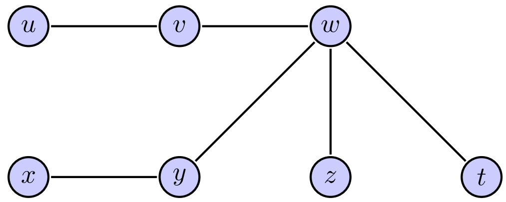
Thm. A graph G is a tree if and only if there is a unique path between any two vertices.

Ex.



One consequence of the last theorem is that trees can be drawn as *rooted trees* after specifying a vertex with the root vertex drawn on top. The next level of vertices are called children, then grandchildren, etc. Vertices can also be described as ancestors and descendants. A vertex of degree 1 is called a *leaf*.

Ex. Draw the rooted tree with vertex w. Next draw the root tree with root vertex z.



Thm. A tree on n vertices has $n-1$ edges.

Fact. Every tree has a leaf.

Pf. We proceed by induction.