38
Graphs

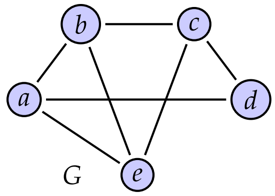
In chapter 8 we represented a relation with a graph. In this chapter we discuss a more general notion of a graph.

38.1 Some Graph Terminology

There is a lot of new vocabulary to absorb concerning graphs! For this chapter, a **graph** will consist of a number of points (called **vertices**) (singular: **vertex**) together with lines (called **edges**) joining some (possibly none, possibly all) pairs of vertices. Unlike the graphs of earlier chapters, we will not allow an edge from a vertex back to itself (so no loops allowed), we will not allow multiple edges between vertices, and the edges will not be directed (there will be no edges with arrowheads on one or both ends). All of our graphs will have a finite vertex sets, and consequently a finite number of edges. Graphs are typically denoted by an uppercase letter such as G or H.

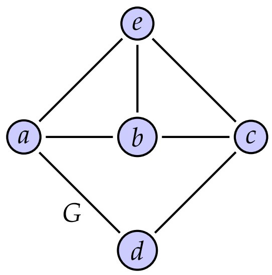
If you would like a formal definition: a **graph**, G consists of a set of vertices V and a set E of edges, where an edge $t \in E$ is written as an unordered pair of vertices $\{u, v\}$, (in other words, a set consisting of two <u>different</u> vertices). We say that the edge $t = \{u, v\}$ has **endpoints** u and v, and that the edge t is **incident** to both u and v. The vertices u and v are **adjacent** when there is an edge with endpoints u and v; otherwise they are not adjacent. Such a formal definition is necessary, but a more helpful way to think of a graph is as a diagram.

Here is an example of a graph G with vertex set $\{a, b, c, d, e\}$ illustrating these concepts.



G

The placement of the vertices in a diagram representing a graph is (within reason!) not important. Here is another diagram of that same graph G.



G

In this diagram, we again have vertex set a, b, c, d, e, and edges $\{a, b\}, \{b, c\}, \{c, d\}, \{a, d\}, \{a, e\}, \{b, e\}, \{c, e\}$, and that is all that matters. It is a good idea to draw a diagram that is easy to understand! In particular, while any curve can be used to represent an edge between two vertices, whenever it is reasonable, edges are normally drawn as straight lines. The vertices b and e are adjacent and the vertices b and d are not adjacent. The vertices a and c are not adjacent since there is no edge $\{a, c\}$. If we use s to denote the edge joining b to c, then s has endpoints b and c, and s is incident to b and c.

Applying the *a-picture-is-worth-a-thousand-words* principle, for the small graphs we will be working with, a graph diagram is generally the easiest way to represent a graph.

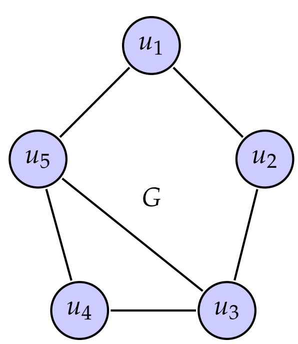
38.1.1 Representing a graph in a computer

There are two standard ways to represent a graph in computer memory, both involving matrices (in other words, tables of numbers). The matrices are of a special type called $0, 1$-matrices since the table entries will all be either 0 or 1.

Adjacency matrix: If there are n vertices in the graph G, the adjacency matrix is an n by n square table of numbers. The rows and columns of the table are labeled with the symbols used to name the vertices. The names are used in the same order for the rows and columns, so there are $n!$ possible labelings. Often there will be some *natural* choice of the order of the labels, such as alphabetic or numeric order. The entries in the table are determined as follows: the matrix entry with row label x and column label y is 1 if x and y are adjacent, and 0 otherwise.

Incidence matrix: Suppose the graph G has n vertices and m edges. The table will have n rows, labeled with the names of the vertices, and m columns labeled with the edges. Which of the $n!m!$ possible orderings of these labelings has to be specified in some way. The entry in the row labeled with vertex u and column labeled with edge e is 1 if e is incident with u, and 0 otherwise. Since every edge is incident to exactly two vertices, every column of the incidence matrix will have exactly two 1's.

Example 38.1. *Let G have vertex set $\{u_1, u_2, u_3, u_4, u_5\}$ and edges $\{u_1, u_2\}, \{u_2, u_3\}, \{u_3, u_4\}, \{u_4, u_5\}, \{u_5, u_1\}, \{u_5, u_3\}$. A graphical representation of G is*



Here are the adjacency matrix A_G, and the incidence matrix M_G of G using the vertices and edges in the orders given above.

$$A_G = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \qquad M_G = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

38.2 An Historical Interlude: The origin of graph theory

Unlike most areas of mathematics, it is possible to point the a specific person as the creator of graph theory and a specific problem that led to its creation. On the following pages the *Seven Bridges of Königsberg* problem and the graph theoretic approach to a solution provided by Leonard Euler in 1736 is described.

The notion of a graph discussed in the article is a little more general that the graphs we will be working with in the chapter. To model the bridge problem as a graph, Euler allowed multiple edges between vertices. In modern terminology, graphs with multiple edges are called **multigraphs**.

While we are on the topic of extensions of the definition of a graph, let's also mention the case of graphs with *loops*. Here we allow an edge to connect a vertex to itself, forming a loop. Multigraphs with loops allowed are called **pseudographs**. Another generalization of the basic concept of a graph is **hypergraph**: in a hypergraph, a single edge is allowed to connect not just two, but any number of vertices.

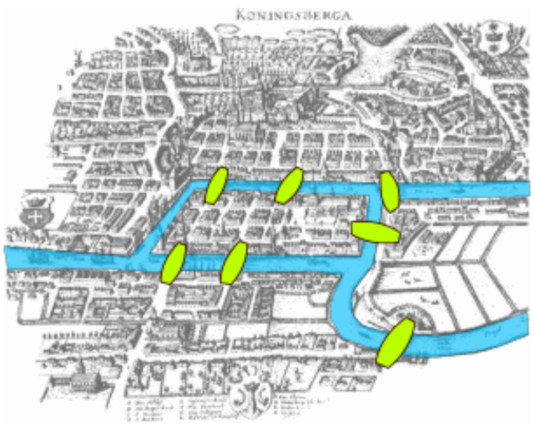
Finally, for all these various types of graphs, we can consider the **directed** versions in which the edges are given arrowheads on one or both ends to indicate the permitted direction of travel along that edge.

In the following article, multigraphs are employed. *But after the article we will again refer only to graphs with no multiple edges and and no loops.*

Seven Bridges of Königsberg

This article is about an abstract problem. For the historical group of bridges in the city once known as Königsberg, and those of them that still exist, see § Present state of the bridges.

The **Seven Bridges of Königsberg** is a historically notable problem in mathematics. Its negative resolution by Leonhard Euler in 1736 laid the foundations of graph theory and prefigured the idea of topology.[1]



Map of Königsberg in Euler's time showing the actual layout of the seven bridges, highlighting the river Pregel and the bridges

The city of Königsberg in Prussia (now Kaliningrad, Russia) was set on both sides of the Pregel River, and included two large islands which were connected to each other, or to the two mainland portions of the city, by seven bridges. The problem was to devise a walk through the city that would cross each of those bridges once and only once.

By way of specifying the logical task unambiguously, solutions involving either

1. reaching an island or mainland bank other than via one of the bridges, or

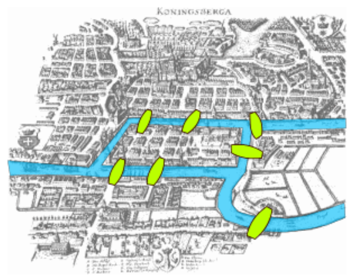
2. accessing any bridge without crossing to its other end

are explicitly unacceptable.

Euler proved that the problem has no solution. The difficulty he faced was the development of a suitable technique of analysis, and of subsequent tests that established this assertion with mathematical rigor.

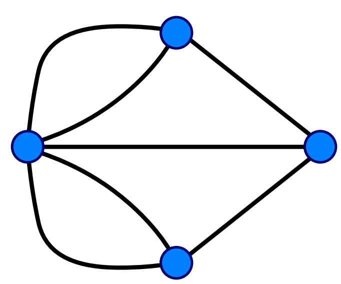
1 Euler's analysis

First, Euler pointed out that the choice of route inside each land mass is irrelevant. The only important feature of a route is the sequence of bridges crossed. This allowed him to reformulate the problem in abstract terms (laying the foundations of graph theory), eliminating all features except the list of land masses and the bridges connecting them. In modern terms, one replaces each land mass with an abstract "vertex" or node, and each bridge with an abstract connection, an "edge", which only serves to record which pair of vertices (land masses) is connected by that bridge. The resulting mathematical structure is called a graph.



→

→



Since only the connection information is relevant, the shape of pictorial representations of a graph may be distorted in any way, without changing the graph itself. Only the existence (or absence) of an edge between each pair of nodes is significant. For example, it does not matter whether the edges drawn are straight or curved, or whether one node is to the left or right of another.

Next, Euler observed that (except at the endpoints of the walk), whenever one enters a vertex by a bridge, one leaves the vertex by a bridge. In other words, during any walk in the graph, the number of times one enters a non-terminal vertex equals the number of times one leaves it. Now, if every bridge has been traversed exactly once, it follows that, for each land mass (except for the ones chosen for the start and finish), the number of bridges touching that land mass must be *even* (half of them, in the particular traversal, will be traversed "toward" the landmass; the other half, "away" from it). However, all four of the land masses in the original problem are touched by an *odd* number of bridges (one is touched by 5 bridges, and each of the other three is touched by 3). Since, at most, two land masses can serve as the endpoints of a walk, the proposition of a walk traversing each bridge once leads to a contradiction.

In modern language, Euler shows that the possibility of a walk through a graph, traversing each edge exactly once, depends on the degrees of the nodes. The degree of a node is the number of edges touching it. Euler's argument shows that a necessary condition for the walk of the desired form is that the graph be connected and have exactly zero or two nodes of odd degree. This condition turns out also to be sufficient—a result stated by Euler and later proven by Carl Hierholzer. Such a walk is now called an *Eulerian path* or *Euler walk* in his honor. Further, if there are nodes of odd degree, then any Eulerian path will start at one of them and end at the other. Since the graph corresponding to historical Königsberg has four nodes of odd degree, it cannot have an Eulerian path.

An alternative form of the problem asks for a path that traverses all bridges and also has the same starting and ending point. Such a walk is called an *Eulerian circuit* or an *Euler tour*. Such a circuit exists if, and only if, the graph is connected, and there are no nodes of odd degree at all. All Eulerian circuits are also Eulerian paths, but not all Eulerian paths are Eulerian circuits.

Euler's work was presented to the St. Petersburg Academy on 26 August 1735, and published as *Solutio problematis ad geometriam situs pertinentis* (The solution of a problem relating to the geometry of position) in the journal *Commentarii academiae scientiarum Petropolitanae* in 1741.[2] It is available in English in The World of Mathematics.

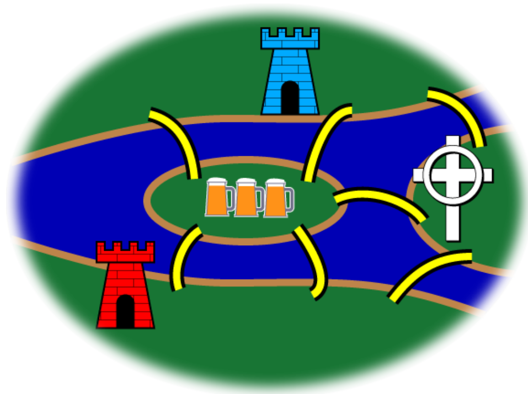
2 Significance in the history of mathematics

In the history of mathematics, Euler's solution of the Königsberg bridge problem is considered to be the first theorem of graph theory and the first true proof in the theory of networks,[3] a subject now generally regarded as a branch of combinatorics. Combinatorial problems of other types had been considered since antiquity.

In addition, Euler's recognition that the key information was the number of bridges and the list of their endpoints (rather than their exact positions) presaged the development of topology. The difference between the actual layout and the graph schematic is a good example of the idea that topology is not concerned with the rigid shape of objects.

3 Variations

The classic statement of the problem, given above, uses **unidentified** nodes—that is, they are all alike except for the way in which they are connected. There is a variation in which the nodes are **identified**—each node is given a unique name or color.



A variant with red and blue castles, a church and an inn.

The northern bank of the river is occupied by the *Schloß*, or castle, of the Blue Prince; the southern by that of the Red Prince. The east bank is home to the Bishop's *Kirche*, or church; and on the small island in the center is a *Gasthaus*, or inn.

It is understood that the problems to follow should be taken in order, and begin with a statement of the original problem:

It being customary among the townsmen, after some hours in the *Gasthaus*, to attempt to **walk the bridges**, many have returned for more refreshment claiming success. However, none have been able to repeat the feat by the light of day.

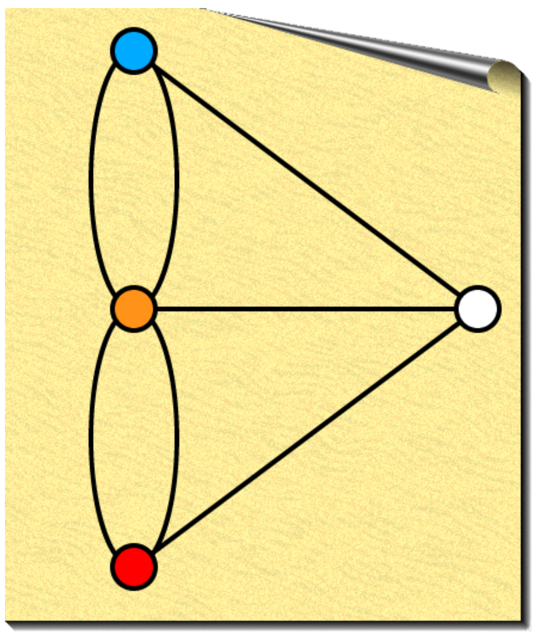
Bridge 8: The Blue Prince, having analyzed the town's bridge system by means of graph theory, concludes that the bridges cannot be walked. He contrives a stealthy plan to build an eighth bridge so that he can begin in the evening at his *Schloß*, walk the bridges, and end at the *Gasthaus* to brag of his victory. Of course, he wants the Red Prince to be unable to duplicate the feat from the Red Castle. *Where does the Blue Prince build the eighth bridge?*

Bridge 9: The Red Prince, infuriated by his brother's

Gordian solution to the problem, wants to build a ninth bridge, enabling *him* to begin at his *Schloß*, walk the bridges, and end at the *Gasthaus* to rub dirt in his brother's face. As an extra bit of revenge, his brother should then no longer be able to walk the bridges starting at his *Schloß* and ending at the *Gasthaus* as before. *Where does the Red Prince build the ninth bridge?*

Bridge 10: The Bishop has watched this furious bridge-building with dismay. It upsets the town's *Weltanschauung* and, worse, contributes to excessive drunkenness. He wants to build a tenth bridge that allows *all* the inhabitants to walk the bridges and return to their own beds. *Where does the Bishop build the tenth bridge?*

3.1 Solutions

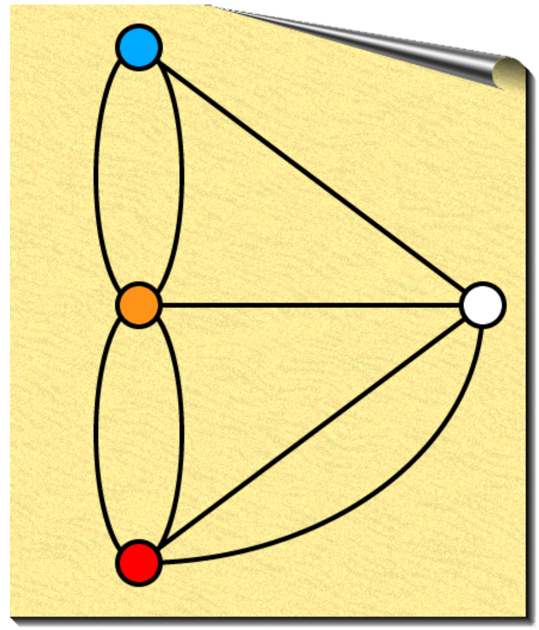


The colored graph

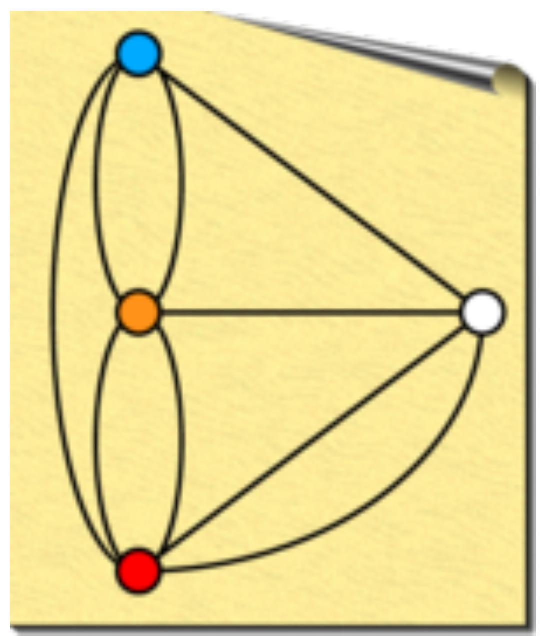
Reduce the city, as before, to a graph. Color each node. As in the classic problem, no Euler walk is possible; coloring does not affect this. All four nodes have an odd number of edges.

Bridge 8: Euler walks are possible if exactly zero or two nodes have an odd number of edges. If we have 2 nodes with an odd number of edges, the walk must begin at one such node and end at the other. Since there are only 4 nodes in the puzzle, the solution is simple. The walk desired must begin at the blue node and end at the orange node. Thus, a new edge is drawn between the other two nodes. Since they each formerly had an odd number of edges, they must now have an even number of edges, fulfilling all conditions. This is a change in parity from an odd to even degree.

Bridge 9: The 9th bridge is easy once the 8th is solved.



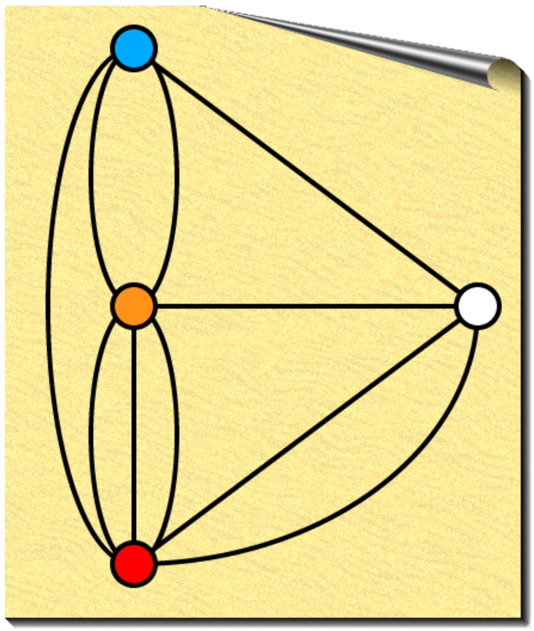
The 8th edge



The 9th edge

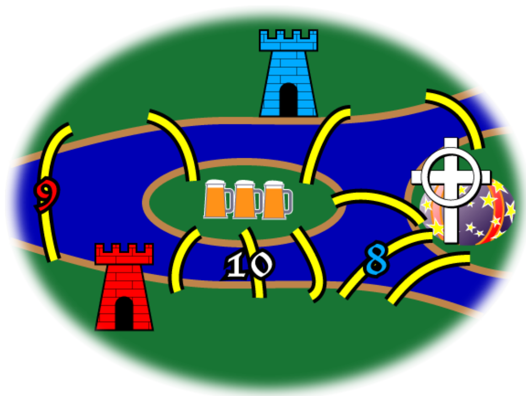
The desire is to enable the red castle and forbid the blue castle as a starting point; the orange node remains the end of the walk and the white node is unaffected. To change the parity of both red and blue nodes, draw a new edge between them.

Bridge 10: The 10th bridge takes us in a slightly different direction. The Bishop wishes every citizen to return to his starting point. This is an Euler circuit and requires that all nodes be of even degree. After the solution of the 9th bridge, the red and the orange nodes have odd degree,



The 10th edge

so their parity must be changed by adding a new edge between them.

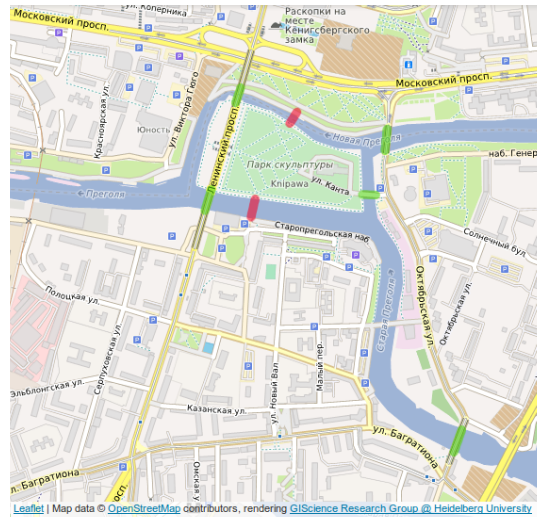


8th, 9th, and 10th bridges

4 Present state of the bridges

Two of the seven original bridges did not survive the bombing of Königsberg in World War II. Two others were later demolished and replaced by a modern highway. The three other bridges remain, although only two of them are from Euler's time (one was rebuilt in 1935).[4] Thus, as of 2000, there are five bridges in Kaliningrad that were a part of the Euler's problem.

In terms of graph theory, two of the nodes now have degree 2, and the other two have degree 3. Therefore, an Eulerian path is now possible, but it must begin on one island and end on the other.[5]



Modern map of Kaliningrad. Locations of the remaining bridges are highlighted in green, while those destroyed are highlighted in red.

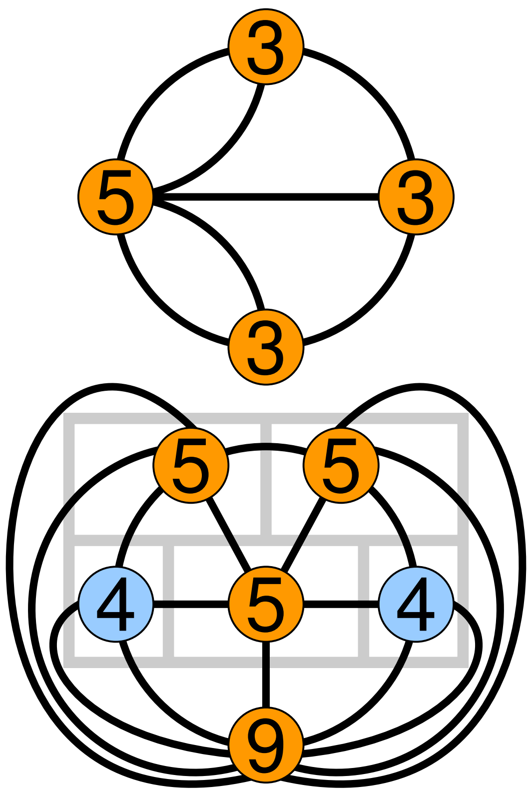
Canterbury University in Christchurch, New Zealand, has incorporated a model of the bridges into a grass area between the old Physical Sciences Library and the Erskine Building, housing the Departments of Mathematics, Statistics and Computer Science.[6] The rivers are replaced with short bushes and the central island sports a stone tōrō. Rochester Institute of Technology has incorporated the puzzle into the pavement in front of the Gene Polisseni Center, an ice hockey arena that opened in 2014.[7]

5 See also

- Eulerian path
- Five room puzzle
- Glossary of graph theory
- Hamiltonian path
- Icosian game
- Water, gas, and electricity

6 References

[1] See Shields, Rob (December 2012). 'Cultural Topology: The Seven Bridges of Königsburg 1736' in Theory Culture and Society 29. pp.43-57 and in versions online for a discussion of the social significance of Euler's engagement with this popular problem and its significance as an example of (proto-)topological understanding applied to everyday life.



Comparison of the graphs of the Seven bridges of Konigsberg (top) and Five-room puzzles (bottom). The numbers denote the number of edges connected to each node. Nodes with an odd number of edges are shaded orange.

[2] The Euler Archive, commentary on publication, and original text, in Latin.

[3] Newman, M. E. J. *The structure and function of complex networks* (PDF). Department of Physics, University of Michigan.

[4] Taylor, Peter (December 2000). "What *Ever* Happened to Those Bridges?". Australian Mathematics Trust. Archived from the original on 19 March 2012. Retrieved 11 November 2006.

[5] Stallmann, Matthias (July 2006). "The 7/5 Bridges of Koenigsberg/Kaliningrad". Retrieved 11 November 2006.

[6] "About – Mathematics and Statistics – University of Canterbury". *math.canterbury.ac.nz*. Retrieved 4 November 2010.

[7] https://twitter.com/ritwhky/status/501529429185945600

7 External links

- Kaliningrad and the Konigsberg Bridge Problem at Convergence
- Euler's original publication (in Latin)
- The Bridges of Königsberg
- How the bridges of Königsberg help to understand the brain
- Euler's Königsberg's Bridges Problem at Math Dept. Contra Costa College
- Pregel – A Google graphing tool named after this problem

Coordinates: 54°42′12″N 20°30′56″E / 54.70333°N 20.51556°E

8 Text and image sources, contributors, and licenses

8.1 Text

- **Seven Bridges of Königsberg** *Source:* https://en.wikipedia.org/wiki/Seven_Bridges_of_K%C3%B6nigsberg?oldid=759230125 *Contributors:* Zundark, Eclecticology, Deb, D, Michael Hardy, Chris-martin, Gabbe, Seav, Den fjättrade ankan~enwiki, Mark Foskey, Bogdangiusca, Berteun, Ed Cormany, Reddi, Dysprosia, Wik, Shizhao, AnonMoos, Jerzy, Robbot, Murray Langton, Fredrik, Donreed, Altenmann, Kneiphof, Bkell, Matt Gies, Giftlite, JamesMLane, Harp, MSGJ, Dratman, Finn-Zoltan, Macrakis, Matthead, Gadfium, Antandrus, DRE, Icairns, Ukexpat, Clubjuggle, Deadlock, Wikiacc, Ascánder, Bender235, Shanes, C S, Blotwell, La goutte de pluie, AllTom, Arthena, Keenan Pepper, Cdc, Americanadian, Oghmoir, Gene Nygaard, Alai, Ghirlandajo, Hq3473, Jftsang, Apokrif, Tabletop, Cbdorsett, Audiovideo, Xiong, Marudubshinki, StefanFuhrmann~enwiki, Graham87, BD2412, Qwertyus, Salix alba, Mkehrt, FlaBot, Gurch, Bgwhite, YurikBot, Wavelength, Hairy Dude, Snillet, Michael Slone, Sikon, Stallions2010, CptnMisc, Arthur Rubin, Cmglee, DVD R W, SmackBot, McGeddon, Stegano~enwiki, Wzhao553, Betacommand, Anachronist, Bird of paradox, Thumperward, DHN-bot~enwiki, Scray, John Reid, LtPowers, John, JLeander, NongBot~enwiki, DaBjork, TheFarix, BranStark, Dilip rajeev, Eyefragment, Courcelles, Stuart Wimbush, CmdrObot, Ivan Pozdeev, Phauly, Nalpdii~enwiki, Nczempin, WLior, Iempleh, Thijs!bot, Headbomb, Lethargy, Gswitz, Seaphoto, Smith2006, Hurmari, JAnDbot, MER-C, CheMechanical, The Anomebot2, David Eppstein, WPaulB, DerHexer, Gwern, LapisQuem, R'n'B, CommonsDelinker, Nev1, Maproom, Smitty, Independentdependent, TXiKiBoT, David Condrey, LFStokols, Falcon8765, Dusti, YonaBot, Hertz1888, Triwbe, Smsarmad, Foljiny, Ctxppc, Ken123BOT, Nic bor, Mikeharris111, Pnijssen, ClueBot, Kl4m, CounterVandalismBot, Piledhigheranddeeper, Excirial, Steveheric, Manu-ve Pro Ski, Jth1994, Addbot, Andunie, Godwin100, Fottry55i6, LinkFABot, Numbo3-bot, Komischn, PV=nRT, Teles, Zorrobot, Luckas-bot, Yobot, AnakngAraw, Ciphers, Materialscientist, Smmlit, ArthurBot, Obersachsebot, Xqbot, Zevyefa, RibotBOT, Tmgreen, Zmorell, Chenopodiaceous, AstaBOTh15, Winterst, MarcelB612, Bmclaughlin9, Serols, Christopher1968, MFrawn, Nascar1996, Deadlyops, WikitanvirBot, TuHan-Bot, Thecheesykid, Cobaltcigs, Donaldm314, Donner60, Scientific29, Orange Suede Sofa, Haythamdouaihy, ClueBot NG, Lord Chamberlain, the Renowned, Vacation9, Santacloud, Ianr790, Athos, MusikAnimal, Cyberbot II, Dexbot, Hmainsbot1, Christallkeks, RockvilleRideOn, Ynaamad, MasterTriangle12, Monkbot, Acekqj, Blois2014, Imdifferentyo123456789, Poppy sheppard and Anonymous: 170

8.2 Images

- **File:7_bridges.svg** *Source:* https://upload.wikimedia.org/wikipedia/commons/9/91/7_bridges.svg *License:* CC-BY-SA-3.0 *Contributors:* ? *Original artist:* ?
- **File:7_bridgesID.png** *Source:* https://upload.wikimedia.org/wikipedia/commons/b/b3/7_bridgesID.png *License:* CC-BY-SA-3.0 *Contributors:* Transferred from en.wikipedia to Commons. *Original artist:* The original uploader was Xiong at English Wikipedia
- **File:7b-graph09.png** *Source:* https://upload.wikimedia.org/wikipedia/commons/6/69/7b-graph09.png *License:* CC-BY-SA-3.0 *Contributors:* Transferred from en.wikipedia to Commons. *Original artist:* Xiong at English Wikipedia
- **File:Commons-logo.svg** *Source:* https://upload.wikimedia.org/wikipedia/en/4/4a/Commons-logo.svg *License:* PD *Contributors:* ? *Original artist:* ?
- **File:Comparison_7_bridges_of_Konigsberg_5_room_puzzle_graphs.svg** *Source:* https://upload.wikimedia.org/wikipedia/commons/f/fb/Comparison_7_bridges_of_Konigsberg_5_room_puzzle_graphs.svg *License:* CC BY-SA 3.0 *Contributors:* Own work *Original artist:* Cmglee
- **File:Koenigsberg_Bridges_Variations_Graph10.png** *Source:* https://upload.wikimedia.org/wikipedia/commons/b/b7/Koenigsberg_Bridges_Variations_Graph10.png *License:* CC-BY-SA-3.0 *Contributors:* ? *Original artist:* ?
- **File:Koenigsberg_Bridges_Variations_Graph7.png** *Source:* https://upload.wikimedia.org/wikipedia/commons/a/af/Koenigsberg_Bridges_Variations_Graph7.png *License:* CC-BY-SA-3.0 *Contributors:* ? *Original artist:* ?
- **File:Koenigsberg_Bridges_Variations_Graph8.png** *Source:* https://upload.wikimedia.org/wikipedia/commons/9/90/Koenigsberg_Bridges_Variations_Graph8.png *License:* CC-BY-SA-3.0 *Contributors:* ? *Original artist:* ?
- **File:Koenigsberg_Bridges_Variations_Problem.png** *Source:* https://upload.wikimedia.org/wikipedia/commons/6/66/Koenigsberg_Bridges_Variations_Problem.png *License:* CC-BY-SA-3.0 *Contributors:* Transferred from en.wikipedia to Commons by Legoktm using CommonsHelper. *Original artist:* Xiong at English Wikipedia
- **File:Konigsberg_bridges.png** *Source:* https://upload.wikimedia.org/wikipedia/commons/5/5d/Konigsberg_bridges.png *License:* CC-BY-SA-3.0 *Contributors:* Public domain (PD), based on the image
  -

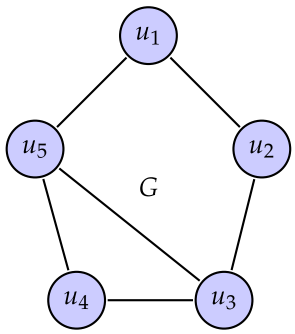
  Original artist: Bogdan Giuşcă
- **File:Königsberg_graph.svg** *Source:* https://upload.wikimedia.org/wikipedia/commons/9/96/K%C3%B6nigsberg_graph.svg *License:* CC-BY-SA-3.0 *Contributors:* ? *Original artist:* ?
- **File:Present_state_of_the_Seven_Bridges_of_Königsberg.png** *Source:* https://upload.wikimedia.org/wikipedia/commons/e/e8/Present_state_of_the_Seven_Bridges_of_K%C3%B6nigsberg.png *License:* CC BY-SA 2.5 *Contributors:* http://openstreetmap.ru/#map=15/54.7044/20.5175&layer=S *Original artist:* Map data by OpenStreetMap contributors; rendering by GIScience Research Group @ Heidelberg University; produced work by Santacloud

- **File:Question_book-new.svg** *Source:* https://upload.wikimedia.org/wikipedia/en/9/99/Question_book-new.svg *License:* Cc-by-sa-3.0 *Contributors:*
Created from scratch in Adobe Illustrator. Based on Image:Question book.png created by User:Equazcion *Original artist:*
Tkgd2007

8.3 Content license

38.3 The First Theorem of Graph Theory

For a vertex v in a graph we denote the number of edges incident to v as the **degree** of v, written as $deg(v)$. For example, consider the graph



Vertices u_1, u_2, u_4 each have degree 2, while $deg(u_3)$ and $deg(u_5)$ are each 3. The list of the degrees of the vertices of a graph is called the **degree sequence** of the graph. The degrees are traditionally listed in increasing order. So the degree sequence of the graph G above is $2, 2, 2, 3, 3$.

The following theorem is usually referred to as the *First Theorem of Graph Theory*

Theorem 38.2. *The sum of the degrees of the vertices of a graph equals twice the number of edges. In particular, the sum of the degrees is even.*

Proof. *Notice that, when adding the degrees for the vertices, each edge will contribute two to the total, once for each end. So the sum of the degrees is twice the number of edges.* ♣

For example, in the graph G above, there are 6 edges, and the sum of the degrees of the vertices is $2 + 2 + 2 + 3 + 3 = 12 = 2(6)$.

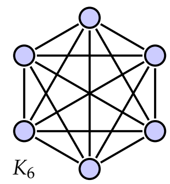
Corollary 38.3. *A graph must have an even number of vertices of odd degree.*

Proof. *Split the vertices into two groups: the vertices with even degree and the vertices with odd degree. The sum of all the degrees is even, and the sum of all the even degrees is also even. That implies that the sum of all the odd degrees must also be even. Since an odd number of odd integers adds up to an odd integer, it must be that there is an even number of odd degrees.* ♣

38.4 A Brief Catalog of Special Graphs

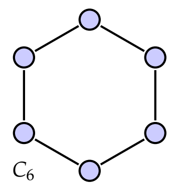
It is convenient to have names for some particular types of graphs that occur frequently.

For $n \geq 1$, K_n denotes the graph with n vertices where every pair of vertices is adjacent. K_n is the **complete graph** on n vertices. So K_n is the largest possible graph with n vertices in the sense that it has the maximum possible number of edges.



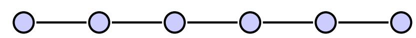
K_6

For $n \geq 3$, C_n denotes the graph with n vertices, $v_1, ..., v_n$, where each vertex in that list is adjacent to the vertex that follows it and v_n is adjacent to v_1. The graph C_n is called the n-**cycle**. The graph C_3 is called a **triangle**.



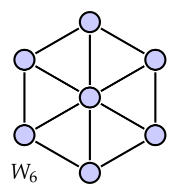
C_6

For $n \geq 2$, L_n denotes the n-**link**. An n-link is a row of n vertices with each vertex adjacent to the following vertex. Alternatively, for $n \geq 3$, an n-link is produced by erasing one edge from an n-cycle.



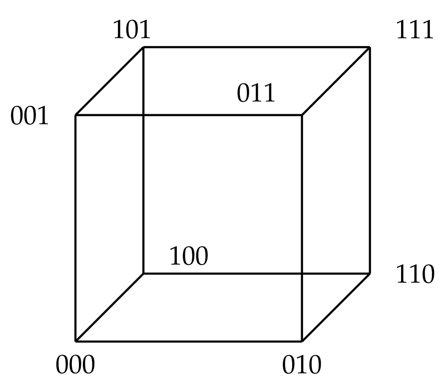
L_6

For $n \geq 3$, W_n denotes the n-**wheel**. To form W_n add one vertex to C_n and make it adjacent to every other vertex. Notice that the n-wheel has $n + 1$ vertices.



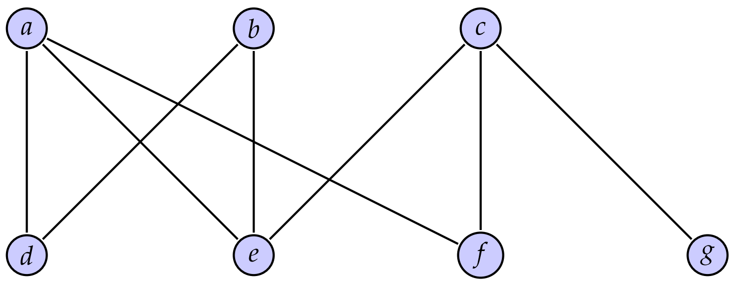
W_6

For $n \geq 1$, the n-**cube**, Q_n, is the graph whose vertices are labeled with the 2^n bit strings of length n. The unusual choice of names for the vertices is made so it will be easy to describe the edges in the graph: two vertices are adjacent provided their labels differ in exactly one bit. Except for $n = 1, 2, 3$ it is not easy to draw a convincing diagram of Q_n. The graph Q_3 can be drawn so it looks like what you would probably draw if you wanted a picture of a 3-dimensional cube. In the graph below, there is a vertex placed at each of the eight corners of the 3-cube labeled with the name of the vertex.

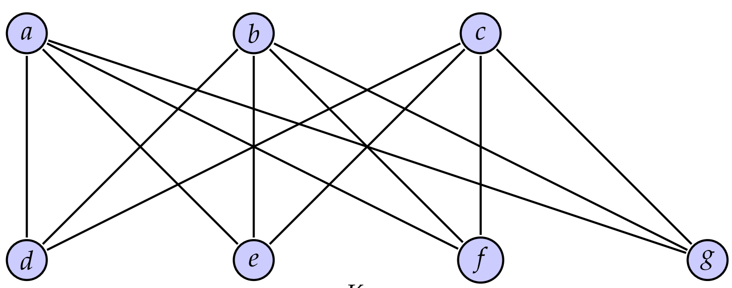


A graph is **bipartite** if it is possible to split the vertices into two subsets, let's call them T and B for top and bottom, so that all the edges go from a vertex in one of the subsets to a vertex in the other subset.

For example, the graph below is a bipartite graph with $T = \{a, b, c\}$ and $B = \{d, e, f, g\}$.

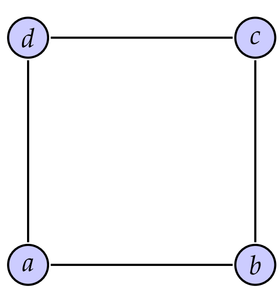


If T has m vertices and B has n vertices, and every vertex in T is adjacent to every vertex in B, the graph is called the **complete bipartite graph**, and it is denoted by $K_{m,n}$. Here is the graph $K_{3,4}$:

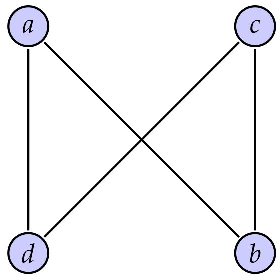


$K_{3,4}$

It is not always obvious if a graph is bipartite or not when looking at a diagram. For example the square



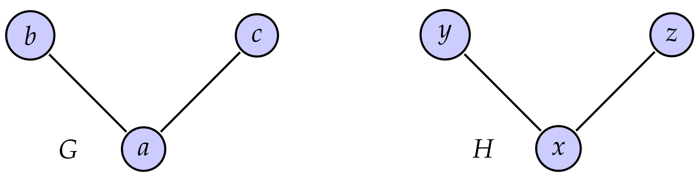
is bipartite since the graph can be redrawn as



so we can see the graph is actually $K_{2,2}$ in disguise.

38.5 Graph isomorphisms

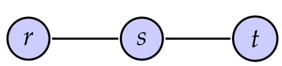
The graphs G and H are obviously really the same except for the labels used for the vertices.



This idea of *sameness* (the official phrase is the graphs G and H are **isomorphic**) for graphs is defined as follows: Two graphs G and H are isomorphic provided we can relabel the vertices of one of the graphs using the labels of the other graph in such a way that the two graphs will have exactly the same edges. As you can probably guess, the notion of isomorphic graphs is an equivalence relation on the collection of all graphs.

In the example above, if the vertices of H are relabeled as $a \to x$ (meaning replace x with a), and $b \to y$, $c \to z$, then the graph H will have edges $\{a, b\}$ and $\{a, c\}$ just like the graph G. So we have proved G and H are isomorphic graphs. The set of replacement rules, $a \to x$, $b \to y$, $c \to z$, is called an **isomorphism**.

The graph G is also isomorphic to the 3-link L_3:



L_3

In this case, an isomorphism is $a \to s$, $b \to r$, $c \to t$.

On the other hand, G is certainly not isomorphic to the 4-cycle, C_4 since that graph does not even have the same number of vertices as G. Also G is not isomorphic to the 3-cycles, C_3. In this case, the two graphs do have the same number of vertices, but not the same number of edges. For two graphs have a chance of being isomorphic, the two graphs must have the same number of vertices and the same number of edges. But **warning**: even if two graphs have the same number of vertices and the same number of edges, they need not be isomorphic. For example L_4 and $K_{1,3}$ are both graphs with 4 vertices and 3 edges, but they are not isomorphic. This is so since L_4 does not have a vertex of degree 3, but $K_{1,3}$ does.

Extending that idea: to have a chance of being isomorphic, two graphs will have to have the same degree sequences since they will end up with the same edges after relabeling. But even having the same degree sequences is not enough to conclude two graphs are isomorphic as the margin example shows. We can see those two graphs are not isomorphic since G has three vertices that form a triangle, but there are no triangles in H.

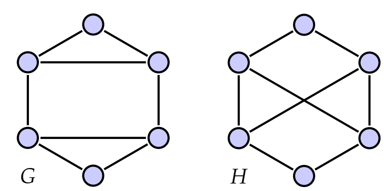


Figure 38.1: Nonisomorphic grades with the same degree sequences.

For graphs with a few vertices and a few edges, a little trial and error is typically enough to determine if the graphs are isomorphic. For more complicated graphs, it can be very difficult to determine if they are isomorphic or not. One of the big goals in theoretical computer science is the design of efficient algorithms to determine if two graphs are isomorphic.

Example 38.4. *Let G be a 5-cycle on a, b, c, d, e drawn as a regular pentagon with vertices arranged clockwise, in order, at the corners. Let H have vertex set v, w, x, y, z and graphical presentation as a pentagram (five-pointed star), where the vertices of the graph are the ends of the points of the star, and are arranged clockwise, (see figure 38.2).*

An isomorphism is $a \to v,\ b \to x,\ c \to z,\ d \to w,\ e \to y$.

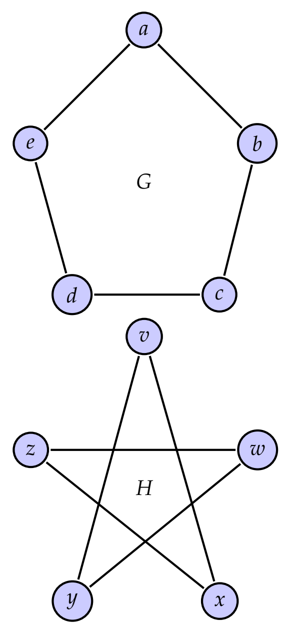


Figure 38.2: Isomorphic graphs

Example 38.5. *The two graphs in figure 38.3 are isomorphic as shown by using the relabeling*

$$u_1 \to v_1,\ u_2 \to v_2,\ u_3 \to v_3,\ u_4 \to v_4,\ u_5 \to v_9,$$

$$u_6 \to v_{10},\ u_7 \to v_5,\ u_8 \to v_7,\ u_9 \to v_8,\ u_{10} \to v_6.$$

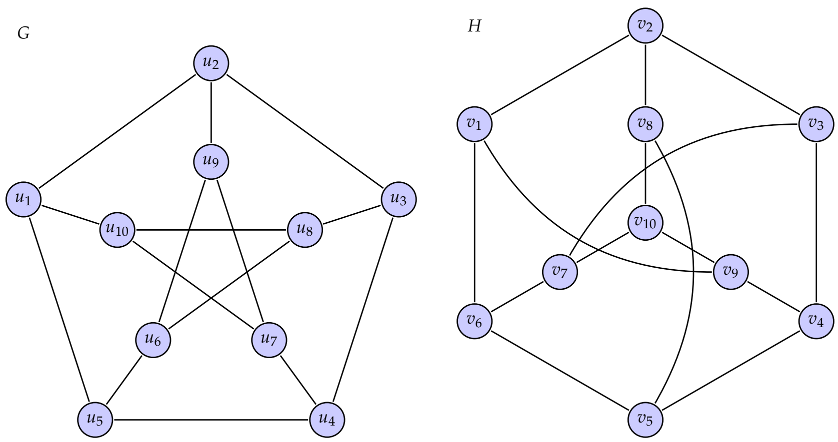


Figure 38.3: More Isomorphic graphs

The graph G is the traditional presentation of the **Petersen Graph**. It could be described as the graph whose vertex set is labeled with all the two element subsets of a five element set, with an edge joining two vertices if their labels have exactly one element in common.

38.6 Paths

The origins of graph theory had to do with bridges, and possible routes crossing the bridges. In this section we will consider that sort of question in graphs in general. We will think of walking along edges, from one vertex in the list to the next, and visiting vertices. Remember that we do not allow multiple edges or loops in our graphs.

We begin with a collection of definitions. Warning: These terms are used differently in different texts. If you look at another graph theory text, be sure to see how the terms are used there.

A **path** of **length** n in a graph is a sequence of $n + 1$ vertices

$v_0, v_1, v_2, \ldots, v_n$, where each vertex in the list is adjacent to the following vertex. Repeated vertices and repeated edges in a path are allowed. The vertices v_0 and v_n are the **endpoints** of the path. Think of starting at v_0, walking along the edges, and ending up at v_n. The length n of the walk is the number of edges transversed in the path. A path of length three or more for which the endpoints are the same (so $v_0 = v_n$) is called a **circuit**. A **simple** path (or circuit) is one that does not repeat any edges. A single vertex v will be considered to be a path (but not a circuit!) of length 0.

Here is an example illustrating these definitions.

Example 38.6. *In the graph shown in figure 38.4, a, b, e, c, f, c is a path of length 5. That is an example of an a, c-path, meaning it starts at vertex a and ends at vertex c. That path is not simple since the edge c, f is repeated. Note that direction does not matter. The vertex sequence a, b, c, f, e is an a, e-path. Here are two simple circuits in that graph: a, b, e, d, a and a, b, c, e, d, a. Notice that the circuit a, e, b, c, f, e, d, a is also simple even though it repeats the vertex e. It does not repeat any edges.*

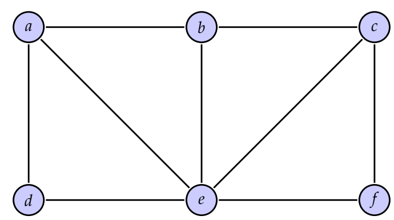


Figure 38.4: paths and circuits

A graph is **connected** if there is a path between any two vertices. In plain English, a connected graph consists of a single piece. The individual connected pieces of a graph are called its **connected components**. The length of the shortest path between two vertices in a connect component of a graph is called the **distance** between the vertices. In figure 38.4, the distance between a and f is 2.

Theorem 38.7. *In a connected graph there is a simple path between any two vertices. In other words, if there is a way to get from one vertex to another vertex along edges, then there is a way to get between those two vertices without repeating any edges.*

Proof. *Problem 38.8. The idea is simple: in a path with a repeated edge, just eliminate the side trip made between the two occurrences of that edge from the path. Do that until all the repeated edges are eliminated. For example, in the graph shown in figure 38.4, The a, c-path a, e, b, e, c can be reduced to the path a, e, c, eliminating the side trip to b.* ♣

A vertex in a graph is a **cut vertex**, if removal of the vertex and edges incident to it results in a graph with more connected components. Similarly a **bridge** is an edge whose removal (keeping the vertices it is incident to) yields a graph with more connected components.

We close this section with a discussion of two special types of paths.

38.6.1 Eulerian paths and circuits

An **eulerian path** in a graph is a simple path which transverses every edge of the graph. In other words, an eulerian path in a graph is a path that transverses every edge of the graph exactly once. An interesting property of a graph with an eulerian path is that it can be drawn completely without lifting pencil from paper and without retracing any edges.

An **eulerian circuit** is a simple circuit in a graph that transverses every edge of the graph. So an eulerian circuit is a path of length three or more that transverses every edge of the graph and ends up at its initial vertex. A graph is called **eulerian** if it has an eulerian circuit.

Example 38.8. *The graph C_5 is an eulerian graph. In fact, the graph itself is an eulerian circuit.*

Example 38.9. *The graph K_5 is an eulerian graph.*

Example 38.10. *The graph L_n is itself an eulerian path, but does not have an eulerian circuit.*

Example 38.11. *The graph K_4 is not an eulerian graph.* [1].

[1] Try it!

38.6.2 Hamiltonian paths and circuits

A **hamiltonian path** in a graph is a simple path that visits every vertex in the graph exactly once. A **hamiltonian circuit** in a graph is a simple circuit that, except for the last vertex of the circuit, visits every vertex in the graph exactly once. A graph is **hamiltonian** if it has a hamiltonian circuit.

Example 38.12. *K_n is hamiltonian for $n \geq 3$.*

Example 38.13. *W_n has a hamiltonian circuit for $n \geq 3$.*

Example 38.14. *L_n has no hamiltonian circuit for $n \geq 2$*

38.6.3 Some facts about eulerian and hamiltonian graphs

A few easy observation: if G is a graph with either an eulerian circuit or hamiltonian circuit, then

(1) G is connected.

(2) every vertex has degree at least 2.

(3) G has no bridges.

If G has a hamiltonian circuit, then G has no cut vertices.

Leonhard Euler gave a simple way to determine exactly when a graph is eulerian. On the other hand, despite considerable effort, no one has been able to devise a test to distinguish between hamiltonian and nonhamiltonian graphs that is much better than a brute force trial-and-error search for a hamiltonian circuit.

Theorem 38.15. *A connected graph is eulerian if and only if every vertex has even degree.*

Proof. *Let G be an eulerian graph, and suppose that v is a vertex in G with odd degree, say $2m + 1$. Let i denote the number of times an eulerian circuit passes through v. Since every edge is used exactly once in the circuit, and each time v is visited two different edges are used, we have $2i = 2m + 1$, which is impossible. $\rightarrow\!\leftarrow$. So G cannot have any vertices of odd degree.*

Conversely, let G be a connected graph where every vertex has even degree. Select a vertex u and build a simple path starting at u as long as possible: each time we visit a vertex we select an unused edge leaving that vertex to extend the simple path. For any vertex $v \neq u$ we visit, its even degree guarantees there will be an unused edge out, since each time v is visited used two edges incident to v and one more edge to arrive at v, for a total of an odd number of edges incident to v, and the vertex has even degree, so there must be at least one unused edge leading out of v. Since the process of extending the simple path must eventually come to an end, that shows the end must be at u when the simple path cannot be extended, and so we have constructed an eulerian circuit.

If this simple path contains every edge we are done. Otherwise when these edges are removed from G we obtain a set of connected components $H_1, ..., H_m$ *which are subgraphs of G and which each satisfy that all vertices have even degree. Since their sizes are smaller, we may inductively construct an eulerian circuit for each* H_i. *Since each G is connected, each* H_i *contains a vertex of the initial circuit, say* v_j. *If we call the eulerian circuit of* H_i, C_i, *then* $v_0, ...v_j, C_i, v_j, ..., v_n, v_0$ *is a circuit in G. Since the* H_i *are disjoint, we may insert each eulerian partial circuit thus obtaining an eulerian circuit for G.* ♣

As a corollary we have

Theorem 38.16. *A connected graph has an eulerian path, but not an eulerian circuit, if and only if it has exactly two vertices of odd degree.*

The following theorem is an example of a sufficient (but not necessary) condition for a graph to have a hamiltonian circuit.

Theorem 38.17. *Let G be a connected graph with* $n \geq 3$ *vertices. If* $deg(v) \geq n/2$ *for every vertex v, then G is hamiltonian.*

Proof. *Suppose that the theorem is false. Let G be a connected graph with* $deg(v) \geq n/2$ *for every vertex v. Moreover suppose that of all counterexamples on n vertices, G is a graph with the largest possible number of edges.*

G is not complete, since K_n *has a hamiltonian circuit, for* $n \geq 3$. *Therefore G has two nonadjacent vertices* v_1 *and* v_n. *By maximality the graph* G_1 *formed by adding the edge* $\{v_1, v_n\}$ *to G has a hamiltonian circuit. Moreover this circuit uses the edge* $\{v_1, v_n\}$, *since otherwise G has a hamiltonian circuit. So we may suppose that the hamiltonian circuit in* G_1 *is of the form* $v_1, v_2, ..., v_n, v_1$. *Thus* $v_1, ..., v_n$ *is a path in G.*

Let $k = deg(v_1)$. *If* v_{i+1} *is adjacent to* v_1, *then* v_i *cannot be adjacent to* v_n, *since otherwise* $v_1, ..., v_i, v_n, v_{n-1}, ..., v_{i+1}, v_1$ *is a hamiltonian circuit in G. Therefore, we have the contradiction*

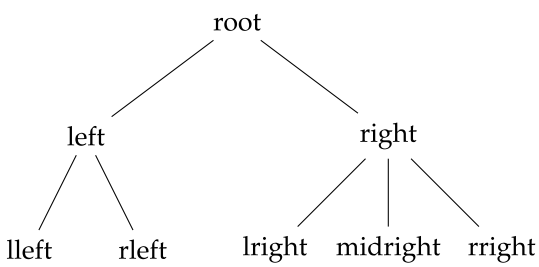
$$deg(v_n) \leq (n-1) - k \leq n - 1 - n/2 = n/2 - 1. \rightarrow\leftarrow$$

♣

WARNING: Do not read too much into this theorem. The condition is not a necessary condition. The 5-cycle, C_5, is obviously hamiltonian, but the vertices all have degree 2 which is less than $\frac{5}{2}$.

38.7 Trees

Trees form an important class of graphs. A **tree** is a connected graph with no circuits. Trees are traditionally drawn *upside down*, with the tree growing down rather than up, starting at a root vertex.



Theorem 38.18. *A graph G is a tree if and only if there is a unique path between any two vertices.*

Proof. *Suppose that G is a tree, and let u and v be two vertices of G. Since G is connected, there is a path of the form $u = v_0, v_1, ..., v_n = v$. If there is a different path from u to v, say $u = w_0, w_1, ..., w_n = v$ let i be the smallest subscript so that $w_i = v_i$, but $v_{i+1} \neq w_{i+1}$. Also let j be the next smallest subscript where $v_j = w_j$. By construction $v_i, v_{i+1}, ..., v_j, w_{j-1}, w_{j-2}, ..., w_i$ is a circuit in G $\rightarrow\!\leftarrow$.*

Conversely, if G is a graph where there is a unique path between any pair of vertices, then by definition G is connected. If G contained a circuit, C, then any two vertices of C would be joined by two distinct paths.$\rightarrow\!\leftarrow$ Therefore G contains no circuits, and is a tree. ♣

A consequence of theorem 38.18 is that given any vertex r in a tree, we can draw the tree with r at the top, as the root vertex, and the other vertices in levels below. [2] The neighbors of r that appear at the first level below r are called r's **children**. The children of r's children are put in the second level below r, and are r's **grandchildren**. In general the ith level consists of those vertices in the tree which are at distance i from r. The result is called a **rooted tree**. The **height** of a rooted tree is the maximum level number.

[2] Redraw the tree diagram above with vertex midright as the root vertex.

Naturally, besides child and parent, many genealogical terms apply to rooted trees, and are suggestive of the structure. For example if a rooted tree has root r, and $v \neq r$, the **ancestors** of v are all vertices on the path from r to v, including r, but excluding v. The **descendants** of a vertex, w consist of all vertices which have w as one of their ancestors. The **subtree rooted at** w is the rooted tree consisting of w, its descendants, and all the required edges. A vertex with no children is a **leaf**, and a vertex with at least one child is called an **internal vertex**.

To distinguish rooted trees by breadth, we use the term m**-ary** to mean that any internal vertex has at most m children. An m-ary tree is **full** if every internal vertex has exactly m children. When $m = 2$, we use the term **binary**.

Theorem 38.19. *A tree on n vertices has $n - 1$ edges.*

Proof. *(by induction on n.)*

Basis: Let $n = 1$, this is the trivial tree with 0 edges. So true the theorem is true for $n = 1$.

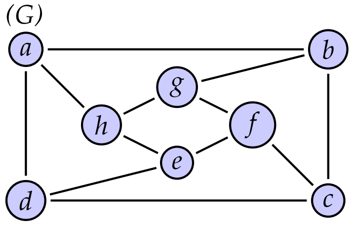
Inductive Step: Suppose that for some $n \geq 1$ every tree with n vertices has $n - 1$ edges. Now suppose T is a tree with $n + 1$ vertices. Let v be a leaf of T. If we erase v and the edge leading to it, we are left with a tree with n vertices. By the inductive hypothesis, this new tree will have $n - 1$ edges. Since it has one less edge than the original tree, we conclude T has n edges.

♣

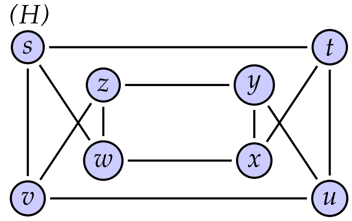
38.8 Exercises

Exercise 38.1. *Find a graph isomorphism $\phi : G \to H$. Verify the adjacency preserving property by showing the adjacency matrices satisfy $A_G = A_G^{\phi}$.*

(G)

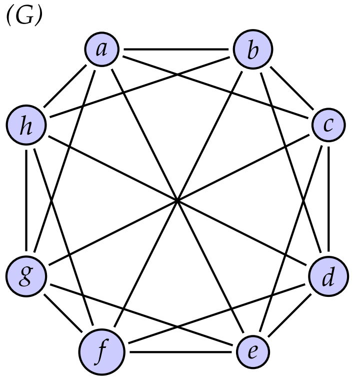


(H)

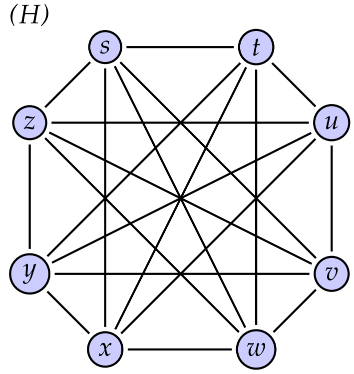


Exercise 38.2. *Prove that G and H are not isomorphic.*

(G)

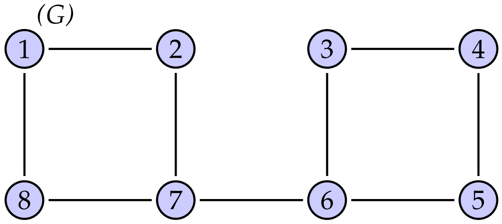


(H)



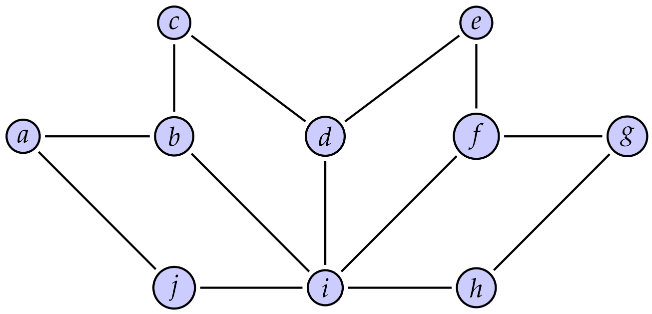
Exercise 38.3. *Redraw the graph G as a bipartite graph.*

(G)

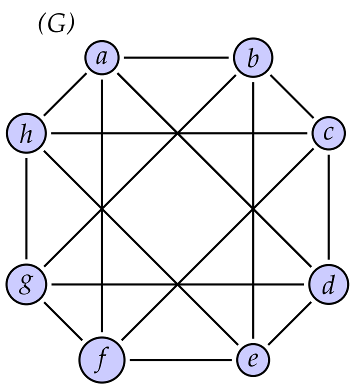


Exercise 38.4. *Explain why the graph G is not eulerian, but is hamiltonian.*

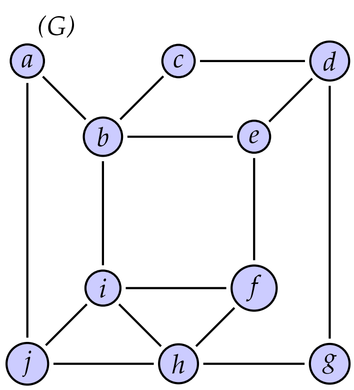
(G)



Exercise 38.5. *Find an eulerian circuit for the graph G as a list of vertices.*



Exercise 38.6. *Prove that the graph G has no hamiltonian circuit.*



38.9 Problems

Problem 38.1.

(a) *How many edges are there in K_n, the complete graph with n vertices?*

(b) *How many edges are there in C_n, the n-cycle with n vertices?*

(c) *How many edges are there in L_n, the n-link with n vertices?*

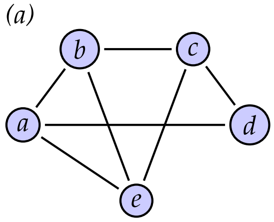
(d) *How many edges are there in W_n, the n-wheel with $n+1$ vertices?*

(e) *How many edges are there in Q_n, the n-cube with 2^n vertices?*

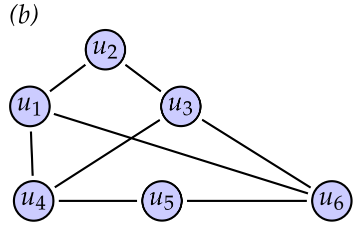
(f) *How many edges are there in $K_{m,n}$, the complete bipartite graph with m top and n bottom vertices?*

Problem 38.2. *Determine whether each graph is bipartite. If it is, redraw it as a bipartite graph.*

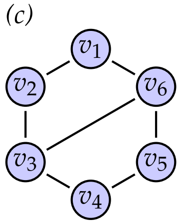
(a)



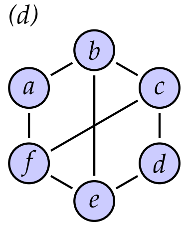
(b)



(c)



(d)



Problem 38.3.

(a) *For which values of n is C_n bipartite?*

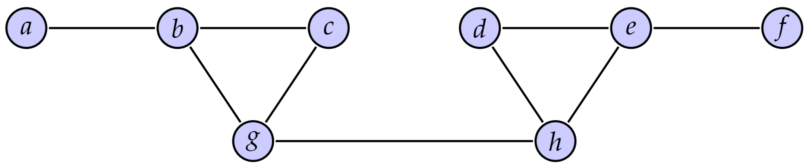
(b) *For which values of n is Q_n bipartite?*

Problem 38.4. *Draw the Petersen graph with vertices labeled with the ten different subsets of the five element set $\{a, b, c, d, e\}$ as suggested in example38.5.*

Problem 38.5. *For the graph below*

(1) *Determine all the bridges.*

(2) *Determine all the cut vertices.*



Problem 38.6. *For each candidate degree sequence below, either draw a graph with that degree sequence or explain why that list cannot be the degree sequence of a graph.*

(1) $4, 4, 4, 4, 4$

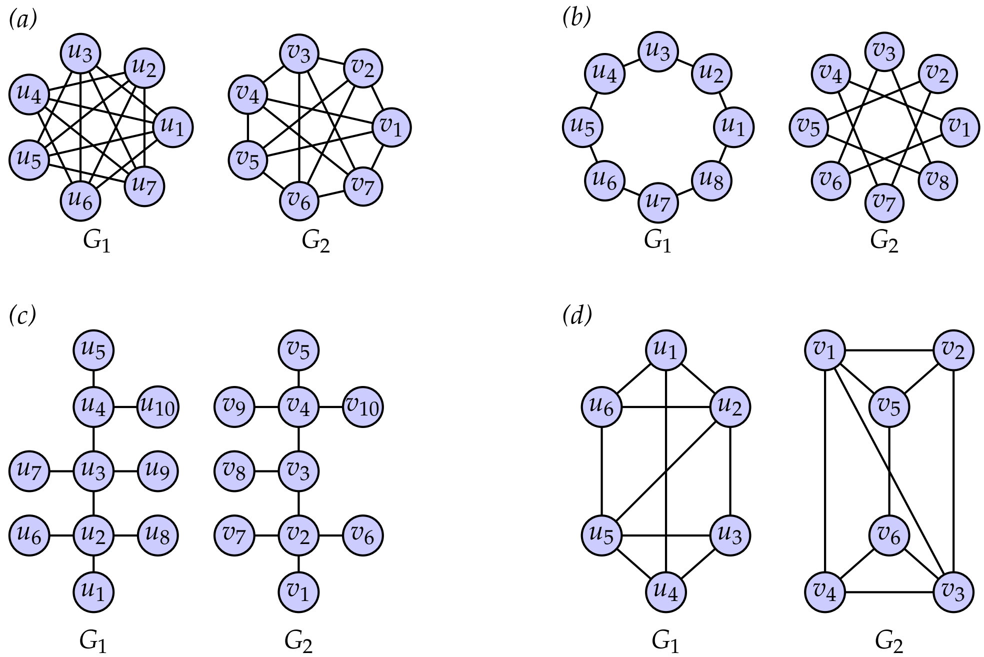
(2) $6, 4, 4, 4, 4$

(3) $0, 0, 0, 0, 0$

(4) $3, 2, 1, 1, 1$

(5) $3, 3, 2, 2, 1$

Problem 38.7. *For each pair of graphs either prove that G_1 and G_2 are not isomorphic, or else show they are isomorphic by exhibiting a graph isomorphism.*

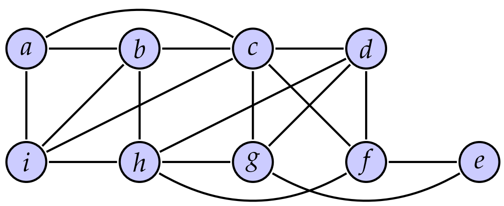


Problem 38.8. *Prove theorem 38.7 from section 38.6 on paths:* If G is a connected graph, then there is a simple path between any two different vertices.

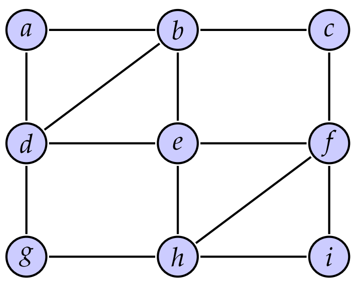
Problem 38.9. *For each graph below (i) find an eulerian circuit, or prove that none exists, and (ii) find a hamiltonian circuit or prove that none exists.*

(a) *The* 3*-cube* Q_3*.*

(b)



(c)



(d) *The Petersen Graph. (See figure* 38.3*.)*

Problem 38.10. *A forest is a graph consisting of one or more (separate) trees. If the total number of vertices in a forest is* f*, and the number of trees in the forest is* t*, what is the total number of edges in the forest?*

Problem 38.11. *A tree is called* **star-like** *if there is exactly one vertex with degree greater than* 2*. How many different (that it, nonisomorphic) star-like trees are there with six vertices? (Note: If you draw the graph with the vertex of of degree greater than* 2 *having the arms of the tree radiating out from it like spokes on a wheel, the name star-like will make sense.)*

Problem 38.12. *Answer the following questions about the rooted tree shown in figure 38.5.*

(a) Which vertex is the root?

(b) Which vertices are internal?

(c) Which vertices are leaves?

(d) Which vertices are children of b?

(e) Which vertices are grandchildren of b?

(f) Which vertex is the parent of m?

(g) Which vertices are siblings of q?

(h) Which vertices are ancestors of p?

(i) Which vertices are descendants of d?

(j) What level is i at?

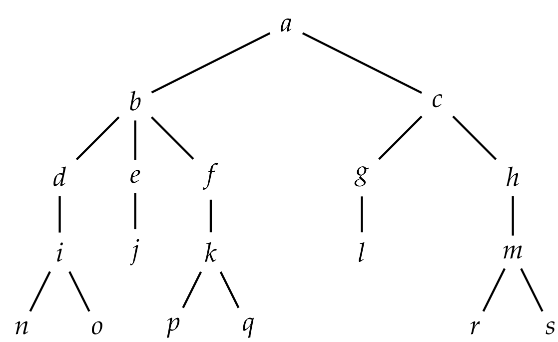


Figure 38.5: Tree for problem 38.12