8

Relations

TWO-PLACE PREDICATES, such as $B(x, y)$: *x is the brother of y*, play a central role in mathematics. Such predicates can be used to describe many basic concepts. As examples, consider the predicates given verbally:

(1) $G(x, y)$: *x is greater than or equal to y* which compares the magnitudes of two values.

(2) $P(x, y)$: *x has the same parity as y* which compares the parity of two integers.

(3) $S(x, y)$: *x has square equal to y* which relates a value to its square.

8.1 Relations

Two-place predicates are called **relations**, probably because of examples such as the *brother of* given above. To be a little more complete about it, if $P(x, y)$ is a two-place predicate, and the domain of discourse for x is the set A, and the domain of discourse for y is the set B, then P is called a **relation from** A **to** B. When working with relations, some new vocabulary is used. The set A (the domain of discourse for the first variable) is called the **domain** of the relation, and the set B (the domain of discourse for the second variable) is called the **codomain** of the relation.

8.2 *Specifying a relation*

There are several different ways to specify a relation. One way is to give a verbal description as in the examples above. As one more example of a verbal description of a relation, consider

$E(x, y)$: *The word x ends with the letter y*. Here the domain will be words in English, and the codomain will the the twenty-six letters of the alphabet. We say the ordered pair (cat, t) **satisfies** the relation E, but that (dog, w) does not.

8.2.1 *By ordered pairs*

When dealing with abstract relations, a verbal description is not always convenient. An alternate method is to tell what the domain and codomain are to be, and then simply list the ordered pairs which will satisfy the relation. For example, if $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d\}$, then one of many possible relations from A to B would be $\{(1, b), (2, c), (4, c)\}$. If we name this relation R, we will write $R = \{(1, b), (2, c), (4, c)\}$. It would be tough to think of a natural verbal description of R.

When thinking of a relation, R, as a set of ordered pairs, it is common to write aRb in place of $(a, b) \in R$. For example, using the relation G defined above, we can convey the fact that the pair $(3, 2)$ satisfies the relation by writing any one of the following: (1) $G(3, 2)$ is true, (2) $(3, 2) \in G$, or (3) $3G2$. The third choice is the preferred one when discussing relations abstractly.

Sometimes the ordered pair representation of a relation can be a bit cumbersome compared to the verbal description. Think about the ordered pair form of the relation E given above: $E = \{\,(cat, t), (dog, g), (antidisestablishmentarianism, m), \cdots\}$.

8.2.2 *By graph*

Another way represent a relation is with a **graph**[1]. Here, a graph is a diagram made up of dots, called **vertices**, some of which are joined by lines, called **edges**. To draw a graph of a relation R from A to B, make a column of dots, one for each element of A, and label the dots

[1] Here graph does **not** mean the sorts of graphs of lines, curves and such discussed in an algebra course.

with the names of those elements. Then, to the right of A's column make a column of dots for the elements of B. Then connect the vertex labelled $a \in A$ to a vertex $b \in B$ with an edge provided $(a, b) \in R$. The diagram is called the **bipartite graph representation** of R.

Example 8.1. *Let* $A = \{1, 2, 3, 4\}$ *and* $B = \{a, b, c, d\}$, *and let* $R = \{(1, a), (2, b), (3, c), (3, d), (4, d)\}$. *Then the bipartite graph which represents R is given in figure 8.1.*

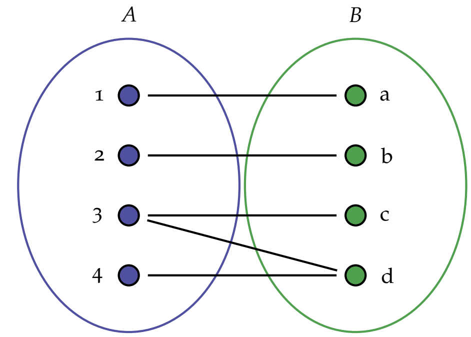

Figure 8.1: Example bipartite graph

The choices made about the ordering and the placement of the vertices for the elements of A and B may make a difference in the appearance of the graph, but all such graphs are considered equivalent. Also, edges can be curved lines. All that matters is that such diagrams convey graphically the same information as R given as a set of ordered pairs.

8.2.3 *By digraph: domain=codomain*

It is common to have the domain and the codomain of a relation be the same set. If R is a relation from A to A, then we will say R is a **relation on** A. In this case there is a shorthand way of representing the relation by using a **digraph**. The word digraph is shorthand for *directed graph* meaning the edges have a direction indicated by an arrowhead. Each element of A is used to label a single point. An arrow connects the vertex labelled s to the one labelled t provided $(s, t) \in R$. An edge of the form (s, s) is called a **loop**.

Example 8.2. *Let* $A = \{1, 2, 3, 4, 5\}$ *and*
$R = \{(1, 1), (1, 2), (2, 2), (2, 3), (3, 3), (3, 4), (4, 4)\}$. *Then a digraph for R is shown in figure 8.2*

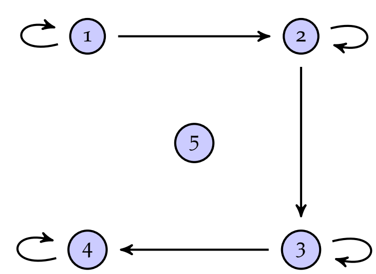

Figure 8.2: Example digraph

Again it is true that a different placement of the vertices may yield a different-looking, but equivalent, digraph.

8.2.4 *By* 0-1 *matrix*

The last method for representing a relation is by using a 0-1 matrix. This method is particularly handy for encoding a relation in computer memory. An $m \times n$ **matrix** is a rectangular array with m rows and n columns. Matrices are usually denoted by capital English letters. The entries of a matrix, usually denoted by lowercase English letters, are indexed by row and column. Either $a_{i,j}$ or a_{ij} stands for the entry in a matrix in the ith row and jth column. A 0-1 **matrix** is one all of whose entries are 0 or 1. Given two finite sets A and B with m and n elements respectively, we may use the elements of A (in some fixed order) to index the rows of an $m \times n$ 0-1 matrix, and use the elements of B to index the columns. So for a relation R from A to B, there is a matrix of R, M_R with respect to the orderings of A and B which represents R. The entry of M_R in the row labelled by a and column labelled by b is 1 if aRb and 0 otherwise. This is exactly like using characteristic vectors to represent subsets of $A \times B$, except that the vectors are cut into n chunks of size m.

Example 8.3. *Let* $A = \{1, 2, 3, 4\}$ *and* $B = \{a, b, c, d\}$ *as before, and consider the relation* $R = \{(1, a), (1, b), (2, c), (4, c), (4, a)\}$. *Then a* 0-1 *matrix which represents* R *using the natural orderings of* A *and* B *is*

Note: This matrix may change appearance if A or B is listed in a different order.

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

8.3 Set operations with relations

Since relations can be thought of as sets of ordered pairs, it makes sense to ask if one relation is a subset of another. Also, set operations such as union and intersection can be carried out with relations.

8.3.1 Subset relation using matrices

These notions can be expressed in terms of the matrices that represent the relations. Bit-wise operations on 0-1 matrices are defined in the obvious way. Then $M_{R \cup S} = M_R \vee M_S$, and $M_{R \cap S} = M_R \wedge M_S$. Also, for two 0-1 matrices of the same size $M \leq N$ means that wherever N has a 0 entry, the corresponding entry in M is also 0. Then $R \subseteq S$ means the same as $M_R \leq M_S$.

8.4 Special relation operations

There are two new operations possible with relations.

8.4.1 Inverse of a relation

First, if R is a relation from A to B, then by reversing all the ordered pairs in R, we get a new relation, denoted R^{-1}, called the **inverse** of R. In other words, R^{-1} is the relation from B to A given by $R^{-1} = \{(b, a)|(a, b) \in R\}$. A bipartite graph for R^{-1} can be obtained from a bipartite graph for R simply by interchanging the two columns of vertices with their attached edges (or, by rotating the diagram 180°).

If the matrix for R is M_R, then the matrix for R^{-1} is produced by taking the columns of $M_{R^{-1}}$ to be the rows of M_R. A matrix obtained by changing the rows of M into columns is called the **transpose** of M, and written as M^T. So, in symbols, if M is a matrix for R, then M^T is a matrix for R^{-1}.

8.4.2 Composition of relations

The second operation with relations concerns the situation when S is a relation from A to B and R is a relation from B to C. In such a case, we can form the **composition of** S **by** R which is denoted $R \circ S$. The composition is defined as

$$R \circ S = \{(a, c)|a \in A, c \in C \text{ and } \exists b \in B, \text{ such that } (a, b) \in S \text{ and } (b, c) \in R\}.$$

Example 8.4. *Let* $A = \{1, 2, 3, 4\}, B = \{\alpha, \beta\}$ *and* $C = \{a, b, c\}$. *Further let* $S = \{(1, \alpha), (1, \beta), (2, \alpha), (3, \beta), (4, \alpha)\}$ *and* $R = \{(\alpha, a), (\alpha, c), (\beta, b)\}$.

Since $(1, \alpha) \in S$ *and* $(\alpha, a) \in R$, *it follows that* $(1, a) \in R \circ S$. *Likewise, since* $(2, \alpha) \in S$ *and* $(\alpha, c) \in R$, *it follows that* $(2, c) \in R \circ S$. *Continuing in that fashion shows that*

$$R \circ S = \{(1, a), (1, b), (1, c), (2, a), (2, c), (3, b), (4, a), (4, c)\}.$$

The composition can also be determined by looking at the bipartite graphs. Make a column of vertices for A labelled $1, 2, 3, 4$, *then to the right a column of points for B labelled* α, β, *then again to the right a column of points for C labelled* a, b, c. *Draw in the edges as usual for R and S. Then a pair* (x, y) *will be in* $R \circ S$ *provided there is a two edge path from* x *to* y. *(See figure 8.3 at right.)*

From the picture it is instantly clear that, for example, $(1, c) \in R \circ S$.

In terms of 0-1 *matrices if* M_S *is the* $m \times k$ *matrix of S with respect to the given orderings of A and B, and if* M_R *is the* $k \times n$ *matrix of R with respect to the given orderings of B and C, then whenever the* i, l *entry of S and* l, j *entry of R are both* 1, *then* $(a_i, c_j) \in R \circ S$.

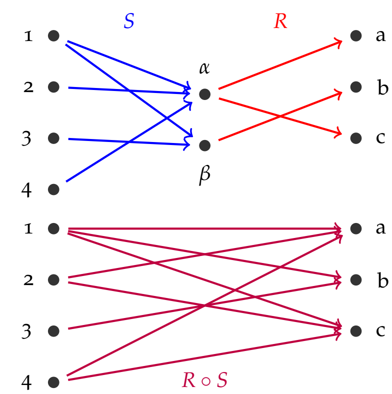

Figure 8.3: Composing relations: $R \circ S$

8.4.3 *Composition with matrices: Boolean product*

This example motivates the definition of the **Boolean product** of M_S and M_R as the corresponding matrix $M_{R \circ S}$ of the composition. More rigorously when M is an $m \times k$ 0-1 matrix and N is an $k \times n$ 0-1 matrix, $M \odot N$ is the $m \times n$ 0-1 matrix whose i, j entry is $(m_{i,1} \wedge n_{1,j}) \vee (m_{i,2} \wedge n_{2,j}) \vee, ..., (m_{i,k} \wedge n_{k,j})$. This looks worse than it is. It achieves the desired result[2].

For the relations in the example above example

$$M_{R \circ S} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = M_S \odot M_R$$

[2] The boolean product is computed the same way as the ordinary matrix product where multiplication and addition have been replaced with `and` and `or`, respectively.

8.5 Exercises

Exercise 8.1. *Let* $A = \{a, b, c, d\}$ *and*
$R = \{(a,a),(a,c),(b,b),(b,d),(c,a),(c,c),(d,b),(d,d)\}$ *be a relation on* A*. Draw a digraph which represents* R*. Find the matrix which represents* R ***with respect to the ordering*** (d,c,a,b)**.**

Exercise 8.2. *The matrix of a relation* S *from* $\{1,2,3,4,5\}$ *to* $\{a,b,c,d\}$ *with respect to the given orderings is displayed below. Represent* S *as a bipartite graph, and as a set of ordered pairs.*

$$M_S = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Exercise 8.3. *Find the composition of* S *by* R *(as given in exercises 8.1 and 8.2) as a set of ordered pairs. Use the Boolean product to find* $M_{R\circ S}$ *with respect to the natural orderings.* [3]

[3] The natural ordering for R is **not** the ordering used in exercise 8.1.

Exercise 8.4. *Let* $B = \{1,2,3,4,5,6\}$ *and let*

$$\begin{aligned} R_1 = \{&(1,2),(1,3),(1,5),(2,1),(2,2),(2,4),(3,3),(3,4),\\ &(4,1),(4,5),(5,5),(6,6)\} \text{ and} \\ R_2 = \{&(1,2),(1,6),(2,1),(2,2),(2,3),(2,5),(3,1),(3,3),(3,6),\\ &(4,2),(4,3),(4,4),(5,1),(5,5),(5,6),(6,2),(6,3),(6,6)\}. \end{aligned}$$

(a) *Find* $R_1 \cup R_2$*,* $R_1 \cap R_2$*, and* $R_1 \oplus R_2$*.*

(b) *With respect to the given ordering of* B *find the matrix of each relation in part (a)*

8.6 Problems

Problem 8.1. *Let* $A = \{a, b, c, d\}$ *and* $R = \{(a, a), (a, c), (b, d), (c, a), (c, c), (d, b)\}$ *be a relation on* A*. Draw a digraph which represents* R*. Draw the bipartite graph which represents* R*.*

Problem 8.2. *Let* $A = \{a, b, c, d\}$ *and* $R = \{(a, a), (a, c), (b, d), (c, a), (c, c), (d, b)\}$ *be a relation on* A*. What is the inverse of* R*?*

Problem 8.3. *Find the composition,* $R \circ S$*, where* $S = \{\ (1, a), (4, a), (5, b), (2, c), (5, c), (3, d)\ \}$ *with* $R = \{(a, x), (a, y), (b, x), (c, z), (d, z)\}$ *as a set of ordered pairs.*

Problem 8.4. *Let* $R_1 = \{(1, 2), (1, 3), (1, 5), (2, 1), (6, 6)\}$ *and* $R_2 = \{(1, 2), (1, 6), (3, 6), (4, 2), (5, 6), (6, 2), (6, 3)\}$*. Find* $R_1 \cup R_2$ *and* $R_1 \cap R_2$*.*

Problem 8.5. *Let* L *be the relation less than on the set of integers. Examples* $3\,L\,7$ *and* $-8\,L\,0$ *are true, but* $5\,L\,2$ *and* $6\,L\,6$ *are false. How would describe the relation* L^{-1}*?*

Problem 8.6. *True or False: For any relation* R*,* $(R^{-1})^{-1} = R$*. Explain your answer.*

Problem 8.7. *Are there relations* R *for which* $R = R^{-1}$*? If not, explain why it is not possible. If so, give an example of such a relation.*

Problem 8.8. *Let* R *be a relation on a set* A*, and let* R^{-1} *be its inverse. Prove that if* $(a, b) \in R \circ R^{-1}$*, then* $(b, a) \in R \circ R^{-1}$*.*

Problem 8.9. *Let* A *and* B *be two sets. Explain why the empty set,* $\emptyset$*, is a relation from* A *to* B*.*

Problem 8.10. *Let* S *be a relation from* A *to* B*, and let* R *be a relation from* B *to* C*. Prove* $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$*.*

9

Properties of Relations

There are several conditions that can be imposed on a relation R on a set A that make it useful. These requirements distinguish those relations which are interesting for some reason from the garden variety junk, which is, let's face it, what most relations are.

9.1 *Reflexive*

Reflexive: $(\forall a \in A)[aRa]$

A relation R on A is **reflexive** provided $\forall a \in A, aRa$. In plain English, a relation is reflexive if every element of its domain is related to itself. The relation $B(x, y)$: x **is the brother of** y is not reflexive since no person is his own brother. On the other hand, the relation $S(m, n)$: $m + n$ **is even.** is a reflexive relation on the set of integers since, for any integer m, $m + m = 2m$ is even.

It is easy to spot a reflexive relation from its digraph: there is a loop at every vertex. Also, a reflexive relation can be spotted quickly from its matrix. First, let's agree that when the matrix of a relation on a set A is written down, the same ordering of the elements of A is used for both the row and column designators. For a reflexive relation, the entries on the **main diagonal** of its matrix will all be 1's. The main diagonal of a square matrix runs from the upper left corner to the lower right corner.

9.2 *Irreflexive*

Irreflexive: $(\forall a \in A)[a \not R a]$
$\equiv \neg(\exists a \in A)[aRa]$

The flip side of the coin from reflexive is irreflexive. A relation R on A is **irreflexive** in case $a \not R a$ for all $a \in A$. In other words, no element of A is related to itself. The *brother of* relation is irreflexive. The digraph of an irreflexive relation contains no loops, and its matrix has all 0's on the main diagonal.

Actually, that discussion was a little careless. To see why, consider the relation $S(x, y)$: **the square of** x is bigger than or equal to y. Is this relation reflexive? The answer is: we can't tell. The answer depends on the domain of the relation, and we haven't been told what that is to be. For example, if the domain is the set $\mathbb{N}$ of natural numbers, then the relation is reflexive, since $n^2 \geq n$ for all $n \in \mathbb{N}$. However, if the domain is the set $\mathbb{R}$ of all real numbers, the relation is not reflexive. In fact, a counterexample to the claim that S is reflexive on $\mathbb{R}$ is the number $\frac{1}{2}$ since $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$, and $\frac{1}{4} < \frac{1}{}$, so $\frac{1}{2} \not S \frac{1}{2}$. The lesson to be learned from this example is that the question of whether a relation is reflexive cannot be answered until the domain has been specified. The same is true for the irreflexive condition and the other conditions defined below. Always be sure you know the domain before trying to determine which properties a relation satisfies.

9.3 *Symmetric*

Symmetric: $(\forall a, b \in A)[aRb \rightarrow bRa]$

A relation R on A is **symmetric** provided $(a, b) \in R \rightarrow (b, a) \in R$. Another way to say the same thing: R is symmetric provided $R = R^{-1}$. In words, R is symmetric provided that whenever a is related to b, then b is related to a. Any digraph representing a symmetric relation R will have a return edge for every non-loop. Think of this as saying the graph has no one-way streets. The matrix M of a symmetric relation satisfies $M = M^T$. In this case M is symmetric about its main diagonal in the usual geometric sense of symmetry. The $B(x, y)$: x **is the brother of** y relation mentioned before is not symmetric if the domain is taken to be all people since, for example, *Donny* B *Marie*, but *Marie* $\not B$ *Donny*. On the other hand, if we take the domain to be all (human) males, then B is symmetric.

9.4 Antisymmetric

Antisymmetric:

$$(\forall a, b \in A)[(aRb) \wedge (bRa) \to (a = b)]$$

A relation R on A is **antisymmetric** if whenever $(a, b) \in R$ and $(b, a) \in R$, then $a = b$. In other words, the only objects that are each related to the other are objects that are the same. For example, the usual $\leq$ relation for the integers is antisymmetric since if $m \leq n$ and $n \leq m$, then $n = m$. A digraph representing an antisymmetric relation will have all streets one-way except loops. If M is a matrix for R, then whenever $a_{i,j} = 1$ and $i \neq j$, $a_{j,i} = 0$.

9.5 Transitive

Transitive:

$$(\forall a, b, c \in A)[(aRb) \wedge (bRc) \to (aRc)]$$

A relation R on A is **transitive** if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$. This can also be expressed by saying $R \circ R \subseteq R$. In a digraph for a transitive relation whenever we have a directed path of length two from a to c through b, we must also have a direct link from a to c. This means that any digraph of a transitive relation has lots of triangles. This includes degenerate triangles where a, b and c are not distinct. A matrix M of a transitive relation satisfies $M \odot M \leq M$. The relation $\leq$ on $\mathbb{N}$ is transitive, since from $k \leq m$ and $m \leq n$, we can conclude $k \leq n$.

9.6 Examples

Example 9.1.

Define a relation, N on the set of all living people by the rule a N b if and only if a, b live within one mile of each other. This relation is reflexive since every person lives within a mile of himself. It is not irreflexive since I live within a mile of myself. It is symmetric since if a lives within a mile of b, then b lives within a mile of a. It is not antisymmetric since Mr. and Mrs. Smith live within a mile of each other, but they are not the same person. It is not transitive: to see why, think of the following situation (which surely exists somewhere in the world!): there is a straight road of length 1.5 *miles. Say Al lives at one end of the road, Cal lives at the other end, and Sal lives half way between Al and Cal. Then Al N Sal and Sal N Cal, but not Al N Cal.*

Example 9.2. *Let $A = \mathbb{R}$ and define aRb iff $a \leq b$, then R is a reflexive, transitive, antisymmetric relation. Because of this example, any relation on a set that is reflexive, antisymmetic, and transitive is called an **ordering** relation. The subset relation on any collection of sets is another ordering relation.*

Example 9.3. *Let $A = \mathbb{R}$ and define aRb iff $a < b$. Then R is irreflexive, and transitive.*

Example 9.4. *If $A = \{1, 2, 3, 4, 5, 6\}$ then*

$$\begin{aligned} R = \{&(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(1,3),(3,1),(1,5),\\ &(5,1),(2,4),(4,2),(2,6),(6,2),(3,5),(5,3),(4,6),(6,4)\} \end{aligned}$$

is reflexive, symmetric, and transitive. In artificial examples such as this one, it can be a tedious chore checking that the relation is transitive.

Example 9.5. *If $A = \{1, 2, 3, 4\}$ and $R = \{(1,1),(1,2),(2,3),(1,3),(3,4),(2,4),(4,1)\}$ then R is not reflexive, not irreflexive, not symmetric, and not transitive but it is antisymmetric.*

9.7 Exercises

Exercise 9.1. *Define a relation on* $\{1, 2, 3\}$ *which is both symmetric and antisymmetric.*

Exercise 9.2. *Define a relation on* $\{1, 2, 3, 4\}$ *by*

$$R = \{(1,2), (2,1), (2,3), (3,2), (3,4), (4,3)\}.$$

For each of the five properties of a relation defined in this chapter (reflexive, irreflexive, symmetric, antisymmetric, and transitive) either show R satisfies the property, or explain why it does not.

Exercise 9.3. *Each matrix below specifies a relation R on* $\{1, 2, 3, 4, 5, 6\}$ *with respect to the given ordering* $1, 2, 3, 4, 5, 6$.

For each of the five properties of a relation defined in this chapter (reflexive, irreflexive, symmetric, antisymmetric, and transitive) either show R satisfies the property, or explain why it does not.

$$a)\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \qquad b)\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$c)\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad d)\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Exercise 9.4. *Define the relation* $C(A, B) : |A| \leq |B|$, *where the domains for* A *and* B *are all subsets of* $\mathbb{Z}$.

For each of the five properties of a relation defined in this chapter (reflexive, irreflexive, symmetric, antisymmetric, and transitive) either show S satisfies the property, or explain why it does not.

Exercise 9.5. *Explain why* $\emptyset$ *is a relation on any set.*

Exercise 9.6. *Define the relation* $M(A, B) : |A \cap B| = 1$ *(or, in plain English, A and B have exactly one element in common), where the domains for A and B are all subsets of* $\mathbb{Z}$. *A few examples:*

- $\{5, 10\}\, M\, \{1, 2, 3, 4, 5, 6\}$ *is true since the sets* $\{5, 10\}$ *and* $\{1, 2, 3, 4, 5, 6\}$ *have exactly one element in common (namely* 5*).*
- $\{1, 2, 3\}\, M\, \{6, 7, 8, 9\}$ *is false since* $\{1, 2, 3\}$ *and* $\{6, 7, 8, 9\}$ *have no elements in common.*
- $\{1, 2, 3, 4\}\, M\, \{2, 4, 6, 8\}$ *is false since* $\{1, 2, 3, 4\}$ *and* $\{2, 4, 6, 8\}$ *have more than one element in common.*
- $\{n | n \in Z \text{ and } n \leq 0\}\, M\, \{n | n \in Z \text{ and } n \geq 0\}$ *is true since* $\{n | n \in Z \text{ and } n \leq 0\}$ *and* $\{n | n \in Z \text{ and } n \geq 0\}$ *have exactly one element in common (namely* 0*).*

For each of the five properties of a relation defined in this chapter (reflexive, irreflexive, symmetric, antisymmetric, and transitive) either show M satisfies the property, or explain why it does not.

9.8 Problems

Problem 9.1. *Let R be the relation $\{(1,1)\}$ on the set $A = \{1,2\}$. For each of the five properties of a relation defined in this chapter (reflexive, irreflexive, symmetric, antisymmetric, and transitive) either show S satisfies the property, or explain why it does not.*

Problem 9.2. *Let R be the relation $\{(1,1),(1,2),(1,3),(2,3)\}$ on the set $A = \{1,2,3\}$. For each of the five properties of a relation defined in this chapter (reflexive, irreflexive, symmetric, antisymmetric, and transitive) either show S satisfies the property, or explain why it does not.*

Problem 9.3. *Let A be the relation on the set $\mathbb{Z}$ of all integer defined by $s\,A\,t$ if and only if $|s| \leq |t|$. For each of the five properties of a relation defined in this chapter (reflexive, irreflexive, symmetric, antisymmetric, and transitive) either show S satisfies the property, or explain why it does not.*

Problem 9.4. *Let D be the relation on the natural numbers defined by the rule mDn if and only if m does not equal n. Examples: $5D7$ is true and $4D4$ is false. For each of the five properties of a relation defined in this chapter (reflexive, irreflexive, symmetric, antisymmetric, and transitive) either show S satisfies the property, or explain why it does not.*

Problem 9.5. *Let R be the relation $\{(1,2),(2,3),(3,4)\}$ on the set $A = \{1,2,3\}$. The relation R is not transitive on A. What is the fewest number of ordered pairs that need to be added to R so it becomes a transitive relation on A?*

Problem 9.6. *Give a counterexample to the claim that a relation R on a set A that is both symmetric and transitive must be reflexive. Hint: There is a very simple example!*

Problem 9.7. *Define the relation $M(A,B) : A \cap B = \emptyset$, where the domains for A and B are all subsets of $\mathbb{Z}$.*

For each of the five properties of a relation defined in this chapter (reflexive, irreflexive, symmetric, antisymmetric, and transitive) either show M satisfies the property, or explain why it does not.

Problem 9.8.

(a) *Let $A = \{1\}$, and consider the empty relation, $\emptyset$, on A. For each of the five properties of a relation defined in this chapter (reflexive, irreflexive, symmetric, antisymmetric, and transitive) either show $\emptyset$ satisfies the property, or explain why it does not.*

(b) *Same question as (a), but now with $A = \emptyset$.*

10

Equivalence Relations

Relations capture the essence of many different mathematical concepts. In this chapter, we will show how to put the idea of *are the same kind* in terms of a special type of relation.

Before considering the formal concept of *same kind* let's look at a few simple examples. Consider the question, posed about an ordinary deck of 52 cards: *How many different kinds of cards are there?* One possible answer is: *There are* 52 *kinds of cards*, since all the cards are different. But another possible answer in certain circumstances is: *There are four kinds of cards* (namely clubs, diamonds, hearts, and spades). Another possible answer is: *There are two kinds of cards, red and black*. Still another answer is: *There are* 13 *kinds of cards: aces, twos, threes,* $\cdots$*, jacks, queens, and kings*. Another answer, for the purpose of many card games is: *There are ten kinds of cards, aces, twos, threes, up to nines, while tens, jacks, queens, and kings are all considered to be the same value (usually called 10)*. You can certainly think of many other ways to split the deck into a number of different kinds.

Whenever the idea of *same kind* is used, some properties of the objects being considered are deemed important and others are ignored. For instance, when we think of the the deck of cards made of the 13 different ranks, ace through king, we are agreeing the the suit of the card is irrelevant. So the jack of hearts and the jack of clubs are taken to be the same for what ever purposes we have in mind.

10.1 Equvialence relation

The mathematical term for *same kind* is **equivalent**. There are three basic properties always associated with the idea of equivalence.

(1) *Reflexive*: Every object is equivalent to itself.

(2) *Symmetric*: If object a is equivalent to object b, then b is also equivalent to a.

(3) *Transitive*: If a is equivalent to b and b is equivalent to c, then a is equivalent to c.

To put the idea of equivalence in the context of a relation, suppose we have a set A of objects, and a rule for deciding when two objects in A are the same kind (equivalent) for some purpose. Then we can define a relation E on the set A by the rule that the pair (s, t) of elements of A is in the relation E if and only if s and t are the same kind. For example, consider again the deck of cards, with two cards considered to be the same if they have the same rank. Then a few of the pairs in the relation E would be (ace hearts, ace spades), (three diamonds, three clubs), (three clubs, three diamonds), (three diamonds, three diamonds), (king diamonds, king clubs), and so on.

Using the terminology of the previous chapter, this relation E, and in fact any relation that corresponds to notion of equivalence, will be reflexive, symmetric, and transitive. For that reason, any reflexive, symmetric, transitive relation on a set A is called an **equivalence relation** on A.

10.2 Equivalence class of a relation

Suppose E is an equivalence relation on a set A and that x is one particular element of A. The **equivalence class of** x is the set of all the things in A that are equivalent to x. The symbol used for the equivalence class of x is $[x]$, so the definition can be written in symbols as $[x] = \{y \in A | y\, E\, x\}$.

For instance, think once more about the deck of cards with the equivalence relation *having the same rank*. The equivalence class of the two of spades would be the set $[2\spadesuit] = \{2\clubsuit, 2\diamondsuit, 2\heartsuit, 2\spadesuit\}$. That would also be the equivalence class of the two of diamonds. On the other hand, if the equivalence relation we are using for the deck is *having the same suit*, then the equivalence class of the two of spades would be

$[2\spadesuit] = \{A\spadesuit, 2\spadesuit, 3\spadesuit, 4\spadesuit, 5\spadesuit, 6\spadesuit, 7\spadesuit, 8\spadesuit, 9\spadesuit, 10\spadesuit, J\spadesuit, Q\spadesuit, K\spadesuit\}$.

The most important fact about the collection of different equivalence classes for an equivalence relation on a set A is that they split the set A into separate pieces. In fancier words, they **partition** the set A. For example, the equivalence relation of having the same rank splits a deck of cards into 13 different equivalence classes. In a sense, when using this equivalence relation, there are only 13 different objects, four of each kind.

10.3 Examples

Here are a few more examples of equivalence relations.

Example 10.1. *Define R on $\mathbb{N}$ by aRb iff $a = b$. In other words, equality is an equivalence relation. If fact, this example explains the choice of name for such relations.*

Example 10.2. *Let A be the set of logical propositions and define R on A by pRq iff $p \equiv q$.*

Example 10.3. *Let A be the set of people in the world and define R on A by aRb iff a and b are the same age in years.*

Example 10.4. *Let $A = \{1, 2, 3, 4, 5, 6\}$ and R be the relation on A with the matrix from exercise 3. part a) of chapter 9.*

Example 10.5. *Define P on $\mathbb{Z}$ by $a\,P\,b$ iff a and b are both even, or both odd. We say a and b have the same parity.*

For the equivalence relation *has the same rank* on a set of cards in a 52 card deck, there are 13 different equivalence classes. One of the classes contains all the aces, another contains all the 2's, and so on.

Example 10.6. *For the equivalence relation from example 10.5, the equivalence class of* 2 *is the set of all even integers.*

$$\begin{aligned}[2] &= \{n \mid 2\,P\,n\} = \{n \mid 2 \text{ has the same parity as } n\} \\ &= \{n \mid n \text{ is even}\} = \{\cdots, -4, -2, 0, 2, 4, \cdots\}\end{aligned}$$

In this example, there are two different equivalence classes, the one comprising all the even integers, and the other comprising all the odd integers. As far as parity is concerned, -1232215 *and* 171717 *are the same.*

Suppose E is an equivalence relation on A. The most important fact about equivalence classes is that every element of A belongs to exactly one equivalence class. Let's prove that.

Theorem 10.7. *Let E be an equivalence relation on a set A, and let $a \in A$. Then there is exactly one equivalence class to which a belongs.*

Proof. *Let E be an equivalence relation on a set A, and suppose $a \in A$. Since E is reflexive, $a\,E\,a$, and so $a \in [a]$ is true. That proves that a is in at least one equivalence class. To complete the proof, we need to show that if $a \in [b]$ then $[b] = [a]$.*

Now, stop and think: Here is what we know:

(1) E is an equivalence relation on A,

(2) $a \in [b]$, and

(3) the definition of equivalence class.

Using those three pieces of information, we need to show the two sets $[a]$ and $[b]$ are equal. Now, to show two sets are equal, we show they have the same elements. In other words, we want to prove

(1) If $c \in [a]$, then $c \in [b]$, and

(2) If $c \in [b]$, then $c \in [a]$.

For homework, you will complete the proof of this theorem by doing part (1).

Let's give a direct proof of (2).

Suppose $c \in [b]$. Then, according to the definition of $[b]$, $c\,E\,b$. The goal is to end up with So $c \in [a]$. Now, we know $a \in [b]$, and that means $a\,E\,b$. Since E is symmetric and $a\,E\,b$, it follows that $b\,E\,a$. Now we have $c\,E\,b$ and $b\,E\,a$. Since E is transitive, we can conclude $c\,E\,a$, which means $c \in [a]$ as we hoped to show. That proves (2). ♣

10.4 Partitions

Definition 10.8. A **partition** of a set A is a collection of nonempty, pairwise disjoint subsets of A, so that A is the union of the subsets in the collection. So for example $\{\{1,2,3\}\ \{4,5,6\}\}$ is a partition of $\{1,2,3,4,5,6\}$. The subsets forming a partition are called the **parts of the partition**.

So to express the meaning of theorem 10.7 above in different words: The different equivalence classes of an equivalence relation on a set partition the set into nonempty disjoint pieces. More briefly: the equivalence classes of E **partition** A.

10.5 Digraph of an equivalence relation

The fact that an equivalence relation partitions the underlying set is reflected in the digraph of an equivalence relation. If we pick an equivalence class $[a]$ of an equivalence relation E on a finite set A and we pick $b \in [a]$, then $b\,E\,c$ for all $c \in [a]$. This is true since $a\,E\,b$ implies $b\,E\,a$ and if $a\,E\,c$, then transitivity fills in $b\,E\,c$. So in any digraph for E every vertex of $[a]$ is connected to every other vertex in $[a]$ (including itself) by a directed edge. Also no vertex in $[a]$ is connected to any vertex in $A - [a]$. So the digraph of E consists of separate components, one for each distinct equivalence class, where each component contains every possible directed edge.

10.6 Matrix representation of an equivalence relation

In terms of a matrix representation of an equivalence relation E on a finite set A of size n, let the distinct equivalence classes have size $k_1, k_2, ...k_r$, where $k_1 + k_2 + ... + k_r = n$. Next list the elements of A as $a_{1,1}, ..., a_{k_1,1}, a_{1,2}, ..., a_{k_2,2}, ..., a_{1,r}, ..., a_{k_r,r}$ where the ith equivalence class is $\{a_{1,i}, ..., a_{k_i,i}\}$. Then the matrix for R with respect to this ordering is of the form

$$\begin{bmatrix} J_{k_1} & 0 & 0 & ... & 0 \\ 0 & J_{k_2} & 0 & ... & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & ... & 0 & J_{k_{r-1}} & 0 \\ 0 & ... & 0 & 0 & J_{k_r} \end{bmatrix}$$

where J_m is the all 1's matrix of size $k_m \times k_m$. Conversely if the digraph of a relation can be drawn to take the above form, or if it has a matrix representation of the above form, then it is an equivalence relation and therefore reflexive, symmetric, and transitive.

10.7 Exercises

Exercise 10.1. *Let* $A = \{0, 1, 2\}$*. Let* $R = \{(0, 0), (1, 1), (2, 2), (0, 1), (1, 0)\}$*. Is R and equivalence relation on A? If it is, what are the equivalence classes?*

Exercise 10.2. *Let* $A = \{0, 1, 2, 3\}$*. Let* $R = \{(0, 0), (1, 1), (2, 2), (0, 1), (1, 0)\}$*. Is R and equivalence relation on A? If it is, what are the equivalence classes?*

Exercise 10.3. *Let* $A = \{0, 1, 2\}$*. Let* $R = \{(0, 0), (1, 1), (2, 2), (0, 1)\}$*. Is R and equivalence relation on A? If it is, what are the equivalence classes?*

Exercise 10.4. *Let* $A = \{0, 1, 2\}$*. Let* $R = \{(0, 0), (1, 1), (2, 2), (0, 1), (1, 0), (1, 2), (2, 1)\}$*. Is R and equivalence relation on A? If it is, what are the equivalence classes?*

Exercise 10.5. *True or False: The relation* $R = \{(1, 1), (2, 2)\}$ *on* $A = \{1, 2\}$ *is both symmetric and antisymmetric.*

Exercise 10.6. *The relation S is defined on the set* $\mathbb{Z}$ *of all integers by the rule* $m\,S\,n$ *if and only if* $m^2 = n^2$*. Is S an equivalence relation on* $\mathbb{Z}$*? If it is, what are the equivalence classes of S?*

Exercise 10.7. *Let L be the collection of all straight lines in the plane. Four examples of elements in L:* $x + y = 0$*, 2x-y - 5,* $x = 7, y = 0$*. A relation C on L is defined by the rule* $l_1\,C\,l_2$ *provided the lines* l_1 *and* l_2 *have at least one point in common. (The letter C should remind us of cross, and, loosely speaking, two lines are related if they cross each other. We will have to agree that a line crosses itself.) Is C an equivalence relation on L? If it is, what are the equivalence classes of C?*

Exercise 10.8. *Let R be a relation on a non-empty set A that is both symmetric, transitive. And, suppose that for each* $a \in A$*, aRb for at least one* $b \in A$*. Prove that R is reflexive, hence, an equivalence relation.*

Exercise 10.9. *Let E be an equivalence relation on a set A, and let* $a, b \in A$*. Prove that either* $[a] \cap [b] = \emptyset$ *or else* $[a] = [b]$*.*

Exercise 10.10. *Let* $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$*. Form a partition of A using* $\{1, 2, 4\}, \{3, 5, 7\}$*, and* $\{6, 8\}$*. These are the equivalence classes for an equivalence relation, E, on A.*

a) *Draw a digraph of E.* b) *Determine a* 0-1 *matrix of E.*

Exercise 10.11. *Determine if each matrix represents an equivalence relation on* $\{a, b, c, d, e, f, g, h\}$*. If the matrix represents an equivalence relation find the equivalence classes. The natural order of the elements,* $[a, b, c, d, e, f, g, h]$*, defines the matrices.*

$$\text{(a)} \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Exercise 10.12. *Complete the proof of theorem 10.7 on page 110 by proving part (1).*

10.8 Problems

Problem 10.1. *Let A be the set of people alive on earth. For each relation defined below, determine if it is an equivalence relation on A. If it is, describe the equivalence classes. If it is not, determine which properties of an equivalence relation fail.*

a) *$a\,H\,b \longleftrightarrow a$ and b are the same height.*

b) *$a\,G\,b \longleftrightarrow a$ and b have a common grandparent.*

c) *$a\,L\,b \longleftrightarrow a$ and b have the same last name.*

d) *$a\,N\,b \longleftrightarrow a$ and b have a name (first name or last name) in common.*

e) *$a\,W\,b \longleftrightarrow a$ and b were born less than a day apart.*

Problem 10.2. *Let L be the collection of all straight lines in the plane. Four examples of elements in L: $x + y = 0$, 2x-y - 5, $x = 7, y = 0$. A relation P on L is defined by the rule $l_1\,P\,l_2$ provided the lines l_1 and l_2 are parallel. Is P an equivalence relation on L? If it is, what are the equivalence classes of P?*

Problem 10.3.

a) *Given an example of an equivalence relation on $\mathbb{N}$ for which there are exactly two equivalence classes.*

b) *Given an example of an equivalence relation on $\mathbb{N}$ for which every equivalence class has cardinality two.*

Problem 10.4. *Consider the relation $B(x, y)$: x **is the brother of** y on the set, M, of living human males. Is M reflexive? Is M symmetric? Is M transitive? (To be precise, brothers will mean two different males with the same two parents. Don't consider half-brothers for this problem.)*

Problem 10.5. *Let $A = \{a, b, c, d, e, f, g\}$. There are many different equivalence relations on A.*

a) *Of all the equivalence relations on A, which have the smallest number of ordered pairs?*

b) *Of all the equivalence relations on A, which have the largest number of ordered pairs?*

Problem 10.6. *The relation* $R = \{(a,a),(a,b)\}$ *is not an equivalence relation on the set* $A = \{a,b,c\}$*. What is the fewest number of ordered pairs that need to be added to R so the result is an equivalence relation on* A*?*

Problem 10.7. *Prove or give a counterexample: Suppose R is an equivalence relation on the lower case letters of the alphabet. True or False: All the equivalence classes of R have the same cardinal number.*

Problem 10.8. *Let* A *be the set of all ordered pairs of positive integers. So some members of A are* $(3,6),(7,7),(11,4),(1,2981)$*. A relation on* A *is defined by the rule* $(a,b)R(c,d)$ *if and only if* $ad = bc$*. For example* $(3,5)R(6,10)$ *is true since* $(3)(10) = (5)(6)$*.*

a) Explain why R is an equivalence relation on A*.*

b) List four ordered pairs in the equivalence class of $(2,3)$*.*

Problem 10.9. *Let* $A = \{1,2,3,4,5,6\}$*. Form a partition of A using* $\{1,2\},\{3,4,5\}$*, and* $\{6\}$*. These are the equivalence classes for an equivalence relation, E, on A. Draw the* **digraph** *of E.*

Problem 10.10. *Let* $A = \{1,2,3\}$*. The relation* $E = \{(1,1),(2,2),(3,3),(2,3),(3,2)\}$ *is an equivalence relation on* A*.* $F = \{(1,1),(2,2),(3,3),(1,2),(2,1)\}$ *is another equivalence relation on* A*. Compute the composition* $F \circ E$*. Is* $F \circ E$ *and equivalence relation on* A*?*