Math 208: Discrete Mathematics
Lesson 3: Lecture Video Notes

Topics

5. Sets: Basic definitions
   (a) specifying sets: roster method and set-builder notation
   (b) special standard sets
   (c) empty and universal sets
   (d) subset and equality relations
   (e) cardinality
   (f) power set
6. Set operations
   (a) intersection
   (b) Venn diagrams
   (c) union
   (d) symmetric difference
   (e) complement
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   (g) Cartesian product
   (h) laws of set theory
   (i) proving set identities
   (j) bit string operations

Readings: Chapters 5-6

§5. Sets: Basic definitions

Defn. A *set* is a collection of objects and usually denoted using capital letters such as $A, B, C, \ldots$. The objects of the set are called *elements* or *members* of the set.

Notation. We write $x \in A$ to indicate that the object x is an element of the set A.
On the other hand, $x \notin A$ indicates that the object x is not an element of the set A.

Q. When are two sets considered the same?

The sets A and B are *equal* if A and B are comprised of exactly the same elements. That is, $x \in A$ if and only if $x \in B$. We write $A = B$ when A and B are equal.

Ex. Let $A = \{1, 2, 3\}$ and $B = \{2, 3, 1\}$. Determine whether or not the sets A and B are equal.

5a. Specifying sets

We consider two common ways of specifying a given set.

Roster Method: A set can be specified by listing all its elements. Braces are used to signify the beginning and ending of the list. Commas are used to separate elements.

Ex. Here are some examples of the roster method.

(i) $A = \{11, 7, 5, 3, 2\}$

(ii) $B = \{5, 2, 2, 5, 6, 3, 2, 5, 6\}$

(iii) $C = \{2, 4, 6, 8, \ldots, 100\}$

(iv) $D = \{J\clubsuit, Q\clubsuit, K\clubsuit, J\spadesuit, Q\spadesuit, K\spadesuit, J\diamondsuit, Q\diamondsuit, K\diamondsuit, J\heartsuit, Q\heartsuit, K\heartsuit\}$

Remark. A few observations to note...

- The order that the elements appear in the list is not important.
- Any repetitions in the list can be ignored.
- Ellipsis $(\ldots)$ can be used *provided* the reader can follow the intended pattern and correctly fill in any missing elements.

- Individual playing cards from a standard 52-card deck can be identified by two features:
  - 4 suits: clubs ($\clubsuit$), spades ($\spadesuit$), diamonds ($\diamondsuit$), hearts ($\heartsuit$)
    - red cards consists of diamonds and hearts
    - black cards consist of clubs and spades
  - 13 ranks: $A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K$
    - face cards consists of J, Q, and K (Jacks, Queens, and Kings)
    - A indicates an Ace

<u>Set-builder notation</u>: A set can be specified using the form:

$$A = \{x : p(x)\} \qquad \text{or} \qquad A = \{x \mid p(x)\}.$$

Here the set A is defined as all objects x for which the predicate $p(x)$ is true.

Ex. We can express the sets from the roster method example using set-builder notation.

(i) $A = \{x : x \text{ is a prime number and } 2 \leq x \leq 12\}$

(ii) $B = \{x \mid x \text{ is an integer and } 1 < x < 7 \text{ and } x \neq 4\}$

(iii) $C = \{x \mid x \text{ is an even integer and } 1 \leq x \leq 100\}$

(iv) $D = \{x : x \text{ is a face card from a standard deck}\}$

5b. Special standard sets

Several commonly used sets of numbers have special notation for them.

Name	Notation
natural numbers	$\mathbb{N} = \{x : x \text{ is a non-negative whole number}\} = \{0, 1, 2, 3, \ldots\}$
integers	$\mathbb{Z} = \{x : x \text{ is a whole number}\} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
rational numbers	$\mathbb{Q} = \{x : x = \frac{p}{q},\ p \text{ and } q \text{ are integers with } q \neq 0\}$
irrational numbers	$\mathbb{I} = \{x : x \text{ is a real number and not a rational number}\}$
real numbers	$\mathbb{R} = \{x : x \text{ is a real number}\}$
complex numbers	$\mathbb{C} = \{x : x = a + bi \text{ where } a, b \in \mathbb{R} \text{ and } i^2 = -1\}$

Remarks.

- Some authors do not include 0 in the set $\mathbb{N}$ and define $\mathbb{N} = \{1, 2, 3, \dots\}$. It's good practice to check which convention is being used when $\mathbb{N}$ is used. Recently it has become more common for authors to use the alternatives $\mathbb{Z}_{\geq 0}$ and $\mathbb{Z}_{>0}$ to avoid any confusion.
- The choice of $\mathbb{Z}$ originates from Zahlen, the German word for integers.
- The choice of $\mathbb{Q}$ likely originates from rational numbers being the quotients of integers (and avoiding division by zero).

Ex. Determine whether the following are true or false.

(i) $\frac{2}{3} \in \mathbb{Q}$

(ii) $3 \in \mathbb{Q}$

(iii) $\frac{2\pi}{7\pi} \in \mathbb{Q}$

(iv) $\sqrt{2} \in \mathbb{Q}$

(v) $\pi \in \mathbb{Q}$

Remark. The Swiss mathematician Johan Heinrich Lambert first proved $\pi \notin \mathbb{Q}$ in the 1760s using continued fractions. He is also famous for introducing hyperbolic functions into trigonometry and making conjectures about non-Euclidean spaces.

5c. Empty and universal sets

Defn. The set which contains no elements is called the *empty set* and denoted $\emptyset$ or $\{\}$.

Remarks.

- $\emptyset \neq 0$ since $\emptyset$ is the set with no elements and 0 is a number. In fact, 0 is not even a set!
- $\emptyset \neq \{\emptyset\}$ since $\emptyset$ is the set with no elements and $\{\emptyset\}$ is the set containing the empty set. That is, $\{\emptyset\}$ is not the empty set since it contains the element $\emptyset$. Note that sets can be elements of sets.

Ex. $\{x \in \mathbb{R} \mid x^2 = -1\} = \emptyset$

Defn. The set which contains all elements under consideration is called the *universal set* and denoted $\mathcal{U}$. Often times, the universal set is left to the reader to figure out, but usually pointed out when no the obvious one.

Ex. In the topic of interest discusses prime numbers or divisibility, the universal set is most likely the integers. That is, most likely $\mathcal{U} = \mathbb{Z}$.

Often times, the universal set $\mathcal{U}$ is indicated in set-builder notation as:

$$\{x \in \mathcal{U} : p(x)\} \qquad \text{or} \qquad \{x \in \mathcal{U} \mid p(x)\}.$$

Ex. List the elements for each of the following sets.

(i) $\{x \in \mathbb{Z} : -1 \leq x \leq 1\}$

(ii) $\{x \in \mathbb{N} : -1 \leq x \leq 1\}$

(iii) $\{x \in \mathbb{R} : -1 \leq x \leq 1\}$

5d. Subset and equality relations

Defn. The two sets are *equal* if A and B have exactly the same elements. That is, $x \in A$ iff $x \in B$. We write $A = B$ when A and B are equal.

Ex. Determine whether the following propositions are true or false.

(i) $\{1, 2\} = \{2, 1\}$

(ii) $\{1, 2, 1, 2, 2, 2\} = \{2, 1, 1, 1, 1\}$

(iii) $\{2, \mathrm{II}, \bullet\bullet\} = \{2\}$

(iv) $\left\{\frac{1}{2}, \frac{3}{6}, \frac{-5}{-10}\right\} = \left\{\frac{1}{2}\right\}$

(v) $\mathbb{Q} = \{x : x = \frac{p}{q},\ p$ and q are integers with $q \neq 0\}$

Remark. An elements may be represented in different ways when defining a set. Any repeats may be ignored. Only the membership status is important.

Defn. Let A be a set. We say B is a *subset* of A if every element of B is also an element of A and write $B \subseteq A$. That is, $B \subseteq A$ iff $\forall x(x \in B \rightarrow x \in A)$.

Ex. $\{2\} \subseteq \{1, 2, 3\}$ and $\{2, 4\} \not\subseteq \{1, 2, 3\}$ and $\emptyset \subseteq \{1, 2, 3\}$

Fact. The empty set is a subset of every set A.

Why? Every element of the empty set is an element of A. Thus $\forall x(x \in \emptyset \rightarrow x \in A)$ holds vacuously.

Ex. True or false:

(i) $\emptyset \subseteq \emptyset$

(ii) $\{\} \subseteq \emptyset$

(iii) Every element of the empty set weights more than one ounce.

Ex. List all elements and subsets of $A = \{a, b, c\}$.

Theorem. If $A \subseteq B$ and $B \subseteq A$, then $A = B$.

Defn. If $A \subset B$ and $A \neq B$, we say A is a *proper subset* of B and write $A \subset B$. Thus $A \subset B$ means every element of A is also an element of B *and* there is some element of B which is not an element of A.

Ex. $\emptyset \subset \{1, 2, 3\}$ and $\{2, 4\} \subset \{2, 4, 6\}$.

Memory aid. It is helpful to think of subsets ($\subseteq$) and proper subsets ($\subset$) as analogous to inequalities ($\leq$) and strict inequalities ($<$).

Cautionary remark. Some authors use $\subset$ to indicate both subsets and proper subsets. It is good practice to verify which convention is being used just like for the definition of $\mathbb{N}$.

Ex. Identify any subset relations between $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{I}$, $\mathbb{R}$, and $\mathbb{C}$.

5e. Cardinality

A useful property of sets is its size or *cardinality*.

Defn. If A is a finite set, the cardinality of A is the number of distinct elements in the set A. If there are n elements, we write $|A| = n$. If A is infinite, then the cardinality of A is infinite.

Ex. Determine the cardinality of the following.

(i) $|\{\}|$

(ii) $|\{\{\}\}|$

(iii) $|\{\{\{\}\}\}|$

(iv) $|\{2, 3, 5, 7, 11\}|$

(v) $|\{\emptyset, \{a, b, \{c\}\}, 2, \{a\}, \{\emptyset\}\}|$

(vi) $|\mathbb{Z}|$

(vii) $|\mathbb{R}|$

Remarks.

- There are different sizes of cardinality for infinite sets: countable and uncountable.
- There are different sizes of uncountable cardinality.
- Somewhat related to cardinal numbers are the ordinal numbers. Ordinal numbers extend the idea of order of natural numbers whereas cardinality captures the size of the set. Arithmetic on ordinal numbers have several surprising results. For example, $\omega + 1 \neq \omega$ as cardinal numbers even though each has infinite cardinality.

Ex. List all elements and subsets of $X = \{a, b, c\}$

5f. Power set

Defn. Let A be a set. The *power set* of A is the set of all subsets of A and denoted $\mathcal{P}(A)$.

Ex. If $A = \{2, 3\}$, then $\mathcal{P}(A) = \{\emptyset, \{2\}, \{3\}, \{2, 3\}\}$.

Ex. Let $B = \emptyset$. Compute $\mathcal{P}(B)$.

Ex. Let $C = \{\emptyset\}$. Compute $\mathcal{P}(C)$.

Ex. Let $X = \{a, b, c\}$. Compute $\mathcal{P}(X)$.

Theorem, If $|A| = n$, then $|\mathcal{P}(A)| = 2^n$.

Sketch of Proof. Let $a_1, a_2, a_3, \ldots, a_n$ be the n elements of A. Consider

$$\underline{\qquad} \quad \underline{\qquad} \quad \underline{\qquad} \quad \cdots \quad \underline{\qquad}$$

Each subset can be encoded using a sequence of Y's and N's. There are 2^n such sequences.

Ex. Consider $X = \{a, b, c\}$. Then

subset	sequence	subset	sequence
$\emptyset$		$\{a, b\}$	
$\{a\}$		$\{a, c\}$	
$\{b\}$		$\{b, c\}$	
$\{c\}$		$\{a, b, c\}$	

§6. Set operations

There are many ways to combine sets to produce new sets.

6a. Intersection

Defn. The *intesection* of the sets A and B is the set

$$\begin{aligned} A \cap B &= \{x \mid x \in A \text{ and } x \in B\} \\ &= \{x \mid (x \in A) \wedge (x \in B)\}. \end{aligned}$$

Ex. Let $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 4, 6, 8, 10\}$ and $C = \{1, 3, 5, 7, 9\}$. Compute:

(i) $A \cap B$

(ii) $A \cap C$

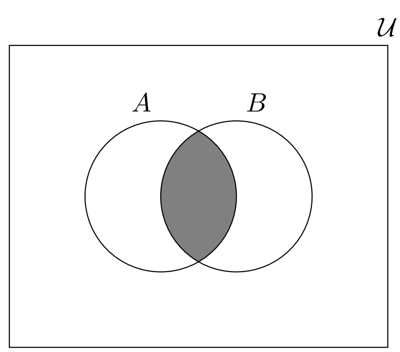
(iii) $B \cap C$

Defn. If $A \cap B = \emptyset$, then we say the sets A and B are *disjoint*.

6b. Venn diagrams

A useful way to visualize set operations is with *Venn diagrams*. The outer rectangle represents the universal set, i.e. all elements under consideration. A circle is used to represent each set. Points inside the circle represent elements of the set; points outside the circle represent elements not in the set. Some elements may appear in the overlap of several circles which indicates it is a member of each of the corresponding sets.

<u>Venn diagram for the intersection $A \cap B$</u>:



Remarks.

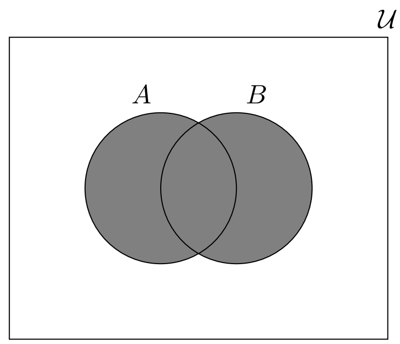
- Venn diagrams work best for visualizing set operations involving up to 3 sets.
- The English mathematician John Venn introduced these diagrams in a paper in 1880.
- Edwards-Venn diagrams have been developed for visualizing set operations with more than 3 sets. These diagrams allow a set to be represented by a simple region (less restrictive) instead of a circle.

6c. Union

Defn. The *union* of the sets A and B is the set

$$\begin{aligned} A \cup B &= \{x \mid x \in A \text{ or } x \in B\} \\ &= \{x \mid (x \in A) \vee (x \in B)\}. \end{aligned}$$

Venn diagram for the union $A \cup B$:



Memory aid:

$\cap$ goes with $\wedge$

$\cup$ goes with $\vee$

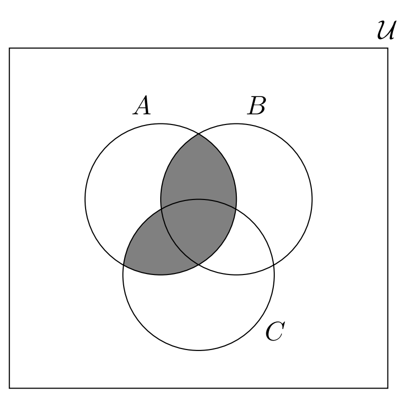
Ex. Let $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 4, 6, 8, 10\}$ and $C = \{1, 3, 5, 7, 9\}$. Compute:

(i) $A \cup B$

(ii) $A \cup C$

(iii) $A \cap (B \cup C)$

Below is the Venn diagram depicting $A \cap (B \cup C)$.



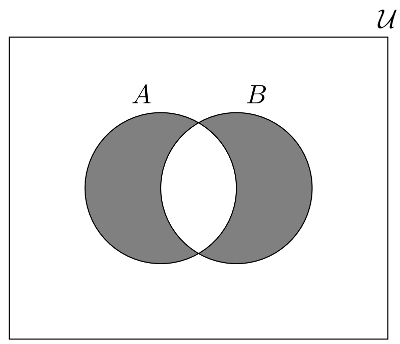
6d. Symmetric difference

Defn. The *symmetric difference* of the sets A and B is the set

$$\begin{aligned} A \oplus B &= \{x \mid x \in A \text{ xor } x \in B\} \\ &= \{x \mid (x \in A) \oplus (x \in B)\}. \end{aligned}$$

Note. Another common notation for symmetric difference is $A \Delta B$.

Venn diagram for the symmetric difference $A \oplus B$:



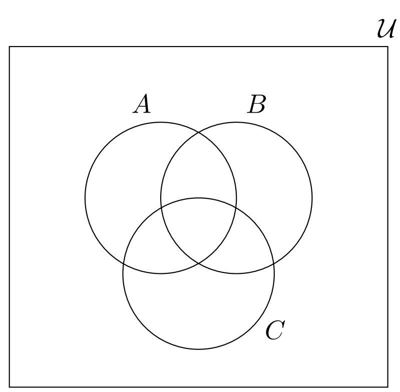
Ex. Let $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 4, 6, 8, 10\}$ and $C = \{1, 3, 5, 7, 9\}$. Compute:

(i) $A \oplus B$

(ii) $A \oplus C$

(iii) $A \oplus (B \oplus C)$

(iv) Sketch the region $A \oplus (B \oplus C)$ in the Venn diagram below.

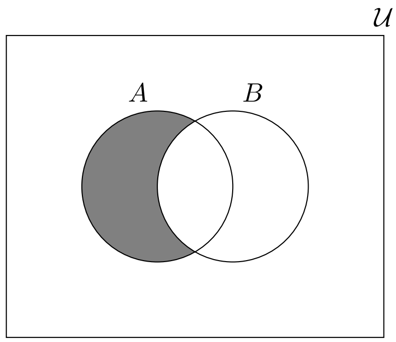


6e. Complement

Defn. The *complement of the set B relative to the set A* is the set

$$\begin{aligned} A - B &= \{x \mid x \in A \text{ and } x \notin B\} \\ &= \{x \mid (x \in A) \wedge (x \notin B)\}. \end{aligned}$$

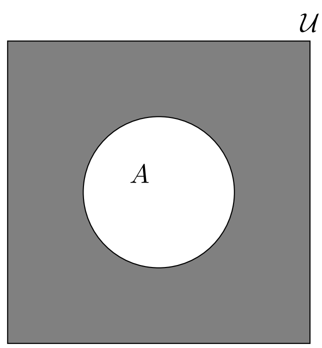
<u>Venn diagram for the complement of A relative to B, i.e. $A - B$:</u>



Observations.

- Order matters! In general, $A - B \neq B - A$.
- It is common to consider the complement of A relative to the universal set $\mathcal{U}$. In this situation, we write $\overline{A}$ for $\mathcal{U} - A$. Note that another common notation used is A^c for $\overline{A} = \mathcal{U} - A$

<u>Venn diagram for the $\overline{A}$:</u>



Ex. Let $A = \{1, 2, 3, 4, 5\}$, $C = \{1, 3, 5, 7, 9\}$, and $\mathcal{U} = \{1, 2, 3, \ldots, 10\}$. Compute:

(i) $A - C$

(ii) $C - A$

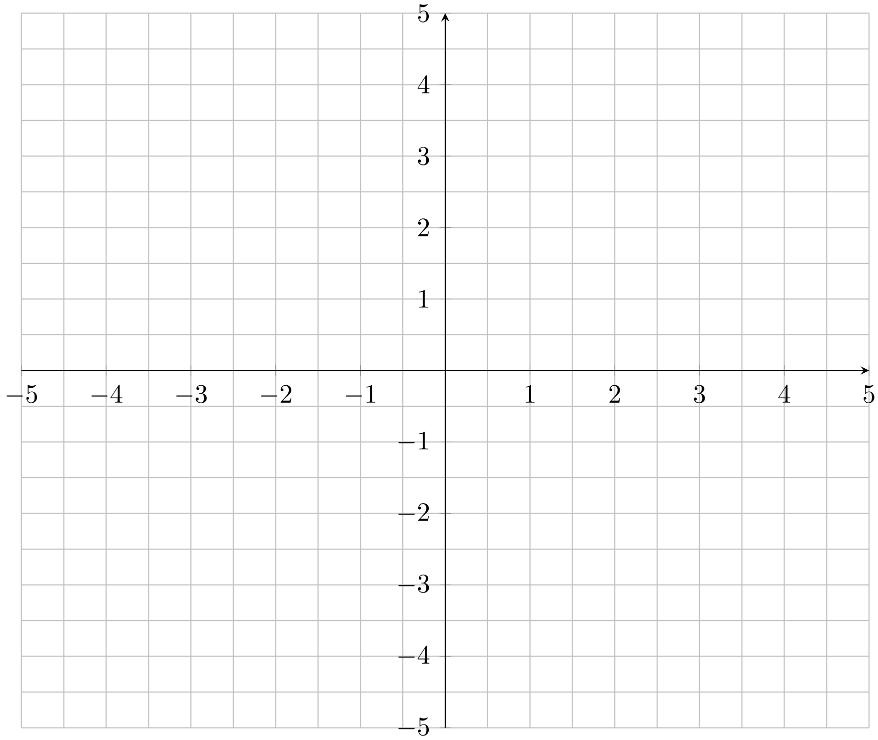
(iii) $\overline{C}$

(iv) $\overline{A \cap C}$

6f. Ordered lists

When using the roster method to specify a set, the order that elements appear is not important. However, order is important when specifying coordinates (x, y) in the usual Cartesian plane $\mathbb{R}^2$.

Ex. Plot the points: $P = (1, 4)$ and $Q = (4, 1)$.



In $\mathbb{R}^3$, coordinates of points are specified as (x, y, z). More generally, points in $\mathbb{R}^n$ are specified via $(x_1, x_2, \ldots, x_n)$. This leads to the following notion.

Defn. An *ordered pair* is a collection of two objects with one specified as the first (i.e. the first coordinate) and the other as the second (the second coordinate). We write (a, b) for the ordered pair with a specified as first and b as second. Two ordered pairs (a, b) and (c, d) are equal iff $a = c$ and $b = d$. That is, $(a, b) = (c, d)$ iff $a = c$ and $b = d$.

The idea of ordered pair generalized to an *ordered n-tuple* $(a_1, a_2, \ldots, a_n)$ where a_i is the ith coordinate. Two n-tuples are the same if they match in every coordinate.

6g. Cartesian product

Ordered pairs and ordered n-tuples are central to the set operation of Cartesian product.

Defn. The *Cartesian product* of the sets A and B is the set:

$$A \times B \quad = \quad \{(a, b) \mid a \in A,\ b \in B\}.$$

More generally, the Cartesian product of the sets $A_1, A_2, \ldots, A_n$ is the set

$$\prod_{k=1}^{n} A_k = A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } 1 \leq i \leq n\}.$$

Note. $\emptyset \times B = A \times \emptyset = \emptyset$

Special Case. If $A_1 = A_2 = \cdots = A_n = A$, then we write A^n for $\prod_{k=1}^n A_k$.

Ex. $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ and $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

Ex. Suppose $A = \{1, 2, 3\}$ and $B = \{p, q\}$. True or false:

(i) $\{q, 2\} \in B \times A$.

(ii) $\{q, 2\} \subseteq B \times A$

(iii) $|A \times B| = 5$

6h. Laws of set theory

There are several identities for sets and set operations which are analogous to logical equivalences for statements and logical connectives. In particular, we have

$\cap$ goes with $\wedge$

$\cup$ goes with $\vee$

complement goes with $\neg$

$\mathcal{U}$ goes with $\mathbb{T}$

$\emptyset$ goes with $\mathbb{F}$

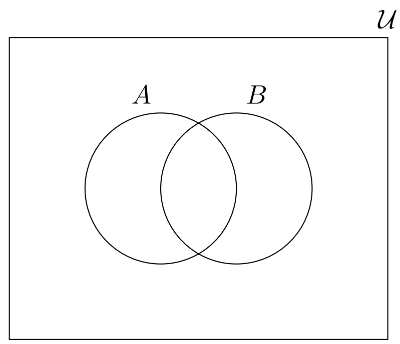
The following laws of set theory are taken from Table 6.1:

Name	Set Identity
Double negation	$\overline{(\overline{A})} = A$
Identity laws	$A \cap \mathcal{U} = A$ $A \cup \emptyset = A$
Domination laws	$A \cup \mathcal{U} = \mathcal{U}$ $A \cap \emptyset = \emptyset$
Idempotent laws	$A \cup A = A$ $A \cap A = A$
Commutative laws	$A \cup B = B \cup A$ $A \cap B = B \cap A$
Associative laws	$(A \cup B) \cup C = A \cup (B \cup C)$ $(A \cap B) \cap C = A \cap (B \cap C)$
Distributive laws	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
De Morgan's laws	$\overline{(A \cap B)} = \overline{A} \cup \overline{B}$ $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$
Law of Excluded Middle Law of Contradiction	$A \cup \overline{A} = \mathcal{U}$ $A \cap \overline{A} = \emptyset$

Q. How can we prove/establish set identities?

One approach is to use a so called *membership table*. This is analogue of a truth table. Here we use a "1" if $x \in A$ and "0" if $x \notin A$.

Ex. Use a membership table to confirm the identity $A \oplus B = (A \cup B) - (A \cap B)$.



Set membership table:

A	B	$A \oplus B$	$A \cup B$	$A \cap B$	$(A \cup B) - (A \cap B)$
1	1				
1	0				
0	1				
0	0				

6i. Proving set identities

Using truth tables and membership tables is often inefficient when there are many propositions or sets involved. To establish logical or set identities, it is considered more elegant approach to use basic identities.

Ex. Prove $A \cup (A \cap B) = A$ using set identities.

6j. Bit string operations

Programmers use a correspondence between set operations of finite sets and bit string operations to manipulate sets in computer memory. Here we briefly illustrate the correspondence through an example.

Let $\mathcal{U} = \{m, a, t, h\}$ be a finite universal set with distinct elements listed in a specific order. Alternative, we can view $\mathcal{U}$ as the 4-tuple $\mathcal{U} = (m, a, t, h)$. For each set $A \subseteq \mathcal{U}$, define the binary string of length $|\mathcal{U}| = n$ by assigning to the ith coordinate either 1 if $u_i \in A$ or 0 if $u_i \notin A$ for each element $u_i \in \mathcal{U}$.

Ex. Let $\mathcal{U} = \{m, a, t, h\}$. Compute:

(i) $\chi(A)$ where $A = \{a, t\}$

(ii) $\chi(B)$ where $B = \{m, a\}$

Properties.

- $\chi(A \cap B) = \chi(A) \wedge \chi(B)$
- $\chi(A \cup B) = \chi(A) \vee \chi(B)$
- $\chi(\overline{A}) = \neg\chi(A)$

Ex. Verify the properties above hold for the previous example.
Recall, $\mathcal{U} = \{m, a, t, h\}$; $A = \{a, t\}$; and $B = \{m, a\}$.