16

Mathematical Induction

As mentioned earlier, to show that a proposition of the form $\forall x\, P(x)$ is true, it is necessary to check that $P(c)$ is true for every possible choice of c in the domain of discourse. If that domain is not too big, it is feasible to check the truth of each $P(c)$ one by one. For instance, consider the proposition *For every page in these notes, the letter **e** appears at least once on the page*. To express the proposition in symbolic form we would let the domain of discourse be the set of pages in these notes, and we would let the predicate E be *has an occurrence of the letter **e***, so the proposition becomes $\forall p\, E(p)$. The truth value of this proposition can be determined by the tedious but feasible task of checking every page of the notes for an **e**. If a single page is found with no **e**'s, that page would constitute a counterexample to the proposition, and the proposition would be false. Otherwise it is true.

When the domain of discourse is a finite set, it is, in principle, always possible to check the truth of a proposition of the form $\forall x\, P(x)$ by checking the members of the domain of discourse one by one. But that option is no longer available if the domain of discourse is an infinite set since no matter how quickly the checks are made there is no practical way to complete the checks in a finite amount of time. For example, consider the proposition *For every natural number n, $n^5 - n$ ends with a* 0.The truth of the proposition could be established by

Here the domain of discourse is the set $\mathbb{N} = \{0, 1, 2, 3, \cdots\}$.

checking:

$$\begin{array}{llll} 0^5 - 0 = 0 & 1^5 - 1 = 0 & 2^5 - 2 = 30 & 3^5 - 3 = 240 \\ 4^5 - 4 = 1020 & 5^5 - 5 = 3120 & 6^5 - 6 = 7770 & 7^5 - 7 = 16800 \\ 8^5 - 8 = 32760 & 9^5 - 9 = 59040 & 10^5 - 10 = 99990 & 11^5 - 11 = 161040 \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

(and so on forever.)

Checking these facts one by one is obviously a hopeless task, and, of course, just checking a few of them (or even a few billion of them) will never suffice to prove they are all true. And it is not sufficient to check a few and say that the facts are all clear. That's not a proof, it's only a suspicion. So verifying the truth of $\forall\, n\, (n^5 - n)$ *ends with a* 0 for domain of discourse $\mathbb{N}$ seems tough.

16.1 Mathematical induction

In general, proving a universally quantified statement when the domain of discourse is an infinite set is a tough nut to crack. But, in the special case when the domain of discourse is the set $\mathbb{N} = \{0, 1, 2, 3, \cdots\}$, there is a technique called **mathematical induction** that comes to the rescue.

The method of proof by induction provides a way of checking that all the statements in the list are true without actually verifying them one at a time. The process is carried out in two steps. First (the **basis step**) we check that the first statement in the list is correct. Next (the **inductive step**), we show that if any statement in the list is known to be correct, then the one following must also be correct. Putting these two facts together, it ought to appear reasonable that all the statements in the list are correct. In a way, it's pretty amazing: we learn infinitely many statements are true just by checking two facts. It's like killing infinitely many birds with two stones.

So, suppose a list of statements, $p(0), p(1), p(2), \cdots, p(k), p(k+1) \cdots$ is presented and we want to show they are all true. The plan is to show two facts:

(1) $p(0)$ is true, and

(2) for any $n \in \mathbb{N}$, $p(n) \longrightarrow p(n+1)$.

We then conclude all the statements in the list are true.

16.2 The principle of mathematical induction

The **well ordering property** of the positive integers provides the justification for proof by induction. This property asserts that every non-empty subset of the natural numbers contains a smallest number. In fact, given any nonempty set of natural numbers, we can determine the smallest number in the set by the process of checking to see, in turn, if 0 is in the set, and, if the answer is *no*, checking for 1, then for 2, and so on. Since the set is nonempty, eventually the answer will be *yes, that number is in the set*, and in that way, the smallest natural number in the set will have been found. Now let's look at the proof that induction is a valid form of proof. The statement of the theorem is a little more general than described above. Instead of beginning with a statement $p(0)$, we allow the list to begin with a statement $p(k)$ for some integer k (almost always, $k = 0$ or $k = 1$ in practice). This does not have any effect of the concept of induction. In all cases, we have a list of statements, and we show the first statement is true, and then we show that if any statement is true, so is the next one. The particular name for the starting point of the list doesn't really matter. It only matters that there is a starting point.

Theorem 16.1 (Principle of Mathematical Induction)**.** *Suppose we have a list of statements* $p(k), p(k+1), p(k+2), \cdots, p(n), p(n+1) \cdots$.

(1) $p(k)$ *is true, and*

(2) $p(n) \longrightarrow p(n+1)$ *for every* $n \geq k$,

then all the statements in the list are true.

Proof. *The proof will be by contradiction.*

Suppose that 1 and 2 are true, but that it is not the case that $p(n)$ *is true for all* $n \geq k$. *Let* $S = \{n | n \geq k \text{ and } p(n) \text{ is false}\}$, *so that* $S \neq \emptyset$. *Since* S *is a non-empty set of integers* $\geq k$ *it has a least element, say* t. *So* t *is the* *smallest positive integer for which $p(n)$ is false. In the ever colorful jargon of mathematics, t is usually called the minimal criminal.*

Since $p(k)$ is true, $k \notin S$. Therefore $t > k$. So $t - 1 \geq k$. Since t is the smallest integer $\geq k$ for which p is false, it must be that $p(t-1)$ is true. Now, by part 2, we also know $p(t-1) \to p(t)$ is true. So it must be that $p(t)$ is true, and that is a contradiction. ♣

16.3 Proofs by induction

Many people find proofs by induction a little bit black-magical at first, but just keep the goals in mind (namely check [1] the first statement in the list is true, and [2] that if any statement in the list is true, so is the one that follows it) and the process won't seem so confusing.

A handy way of viewing mathematical induction is to compare proving the sequence $p(k) \wedge p(k+1) \wedge p(k+2) \wedge ... \wedge p(m) \wedge ...$ to knocking down a set of dominos set on edge and numbered consecutively $k, k+1, ...$ If we want to knock all of the dominos down, which are numbered k and greater, then we must knock the kth domino down, and ensure that the spacing of the dominos is such that every domino will knock down its successor. If either the spacing is off ($\exists m \geq k$ with $p(m)$ not implying $p(m+1)$), or if we fail to knock down the kth domino (we do not demonstrate that $p(k)$ is true), then there may be dominos left standing.

When checking the inductive step, $p(n) \to p(n+1)$, the statement $p(n)$, is called the **inductive hypothesis**.

To discover how to prove the inductive step most people start by explicitly listing several of the first instances of the inductive hypothesis $p(n)$. Then, look for how to make, in a general way, an argument from one, or more, instances to the next instance of the hypothesis. Once an argument is discovered that allows us to advance from the truth of previous one, or more, instances, that argument, in general form, becomes the pattern for the proof on the inductive hypothesis. Let's examine an example.

Example 16.2. *Let's prove that, for each positive integer n, the sum of the first n positive integers is $\dfrac{n(n+1)}{2}$. Here is the list of statements we want to verify:*

$$
\begin{aligned}
p(1):&\quad 1=\frac{1(1+1)}{2} && \textit{add 2 to both sides, can you make } p(2) \textit{ appear?}\\
p(2):&\quad 1+2=\frac{2(2+1)}{2} && \textit{add 3 to both sides, can you make } p(3) \textit{ appear?}\\
p(3):&\quad 1+2+3=\frac{3(3+1)}{2}\\
&\quad\vdots\\
p(n):&\quad 1+2+\cdots+n=\frac{n(n+1)}{2}\\
p(n+1):&\quad 1+2+\cdots+(n+1)=\frac{(n+1)((n+1)+1)}{2}\\
&\quad\vdots
\end{aligned}
$$

Once you figure out the general form of the argument[1] that takes us from one instance of $p(\cdot)$ to the next, you have form of the inductive argument.

[1] For this example it will be some calculation

Proof. *Basis: Let's check the first statement in the list,*
$p(1)\ :\ \ 1 = \dfrac{1(1+1)}{2}$, *is correct. The left-hand side is* 1, *and the right-hand side is* $\dfrac{1(1+1)}{2} = \dfrac{2}{2} = 1$, *so the two sides are equal as claimed.*

Inductive Step: Suppose $p(n)$ is true for some integer $n \geq 1$. In other words, suppose $1+2+\cdots+n = \frac{n(n+1)}{2}$. We need to show $p(n+1)$ is true. In other words, we need to verify $1+2+\cdots+(n+1) = \frac{(n+1)((n+2)}{2}$. Here are the computations:

To prove an equality, the usual strategy is to start on one side of the equation, $p(n+1)$ in this case, obtain the other side. We do this through a series of algebraic manipulations and using the general induction hypothesis, $p(n)$, along the way.

$$
\begin{aligned}
1+2+\cdots+(n+1) &= 1+2+\cdots+n+(n+1)\\
&= \frac{n(n+1)}{2}+(n+1) \textit{ using the inductive hypothesis}\\
&= \frac{n(n+1)}{2}+\frac{2(n+1)}{2}\\
&= \frac{n(n+1)+2(n+1)}{2}\\
&= \frac{(n+1)(n+2)}{2}
\end{aligned}
$$

as we needed to show. So we conclude all the statements in the list are true.

♣

16.4 Examples

The next example reproves the useful formula for the sum of the terms in a geometric sequence. Recall that to form a geometric sequence, fix a real number $r \neq 1$, and list the integer powers of r starting with $r^0 = 1$: $1, r, r^2, r^3, \cdots, r^n, \cdots$. The formula given in the next example shows the result of adding $1 + r + r^2 + \cdots r^n$.

Example 16.3. *For all $n \geq 0$, we have $\sum_{k=0}^{n} r^k = \frac{r^{n+1} - 1}{r - 1}$, (if $r \neq 1$).*

In this example, $p(n)$ is the statement:

$$p(n) : \sum_{k=0}^{n} r^k = \frac{r^{n+1} - 1}{r - 1}$$

.

Proof (by induction on n:). *(We assume $r \neq 1$.)*

Basis: *When $n = 0$ we have $\sum_{k=0}^{0} r^k = r^0 = 1$. We also have* $\frac{r^{n+1} - 1}{r - 1} = \frac{r - 1}{r - 1} = 1$.

Inductive Step: *Now suppose that $\sum_{k=0}^{n} r^k = \frac{r^{n+1} - 1}{r - 1}$ is true for some $n \geq 0$. Then, we see that*

$$\begin{aligned}
\sum_{k=0}^{n+1} r^k &= \left(\sum_{k=0}^{n} r^k\right) + r^{n+1} && \textit{by the recursive definition of a sum} \\
&= \frac{r^{n+1} - 1}{r - 1} + r^{n+1} && \textit{by inductionhypothesis,} \\
&= \frac{r^{n+1} - 1}{r - 1} + \frac{r^{n+2} - r^{n+1}}{r - 1} \\
&= \left[\frac{r^{n+1} - 1 + r^{n+2} - r^{n+1}}{r - 1}\right] \\
&= \frac{r^{n+2} - 1}{r - 1}.
\end{aligned}$$

♣

Example 16.4. *For every integer $n \geq 2$, $2^n > n + 1$.*

Proof. *Basis:* *When $n = 2$, the inequality to check is $2^2 > 2 + 1$, and that is correct.*

Inductive Step: *Now suppose that $2^n > n + 1$ for some integer $n \geq 2$. Then $2^{n+1} = 2 \cdot 2^n > 2(n + 1) = 2n + 2 > n + 2$, as we needed to show.*

♣

Example 16.5. *Show that using only* $5¢$ *stamps and* $9¢$ *stamps, any postage amount* $32¢$ *or greater can be formed.*

Proof. Basis: *$32¢$ can be formed by using one $5¢$ stamp and three $9¢$ stamps.*

Inductive Step: *Now suppose we can form $n¢$ postage for some $n \geq 32$. We need to show we can form $(n+1)¢$ postage. Since $n \geq 32$, when we form $n¢$ postage, we must use either (1) at least seven $5¢$ stamps, or (2) at least one $9¢$ stamps. For if both of those possibilities are wrong, we will have at most $30¢$ postage.*

case 1: *If there are seven (or more) $5¢$ stamps in the $n¢$ postage, remove seven $5¢$, and put in four $9¢$ stamps. Since we removed $35¢$ and put back $36¢$, we now have $(n+1)¢$ postage.*

case 2: *If there is one (or more) $9¢$ stamps in the $n¢$ postage, remove one $9¢$, and put back two $5¢$. Since we removed $9¢$ and put back $10¢$, we now have $(n+1)¢$ postage.*

So, in any case, if we can make $n¢$ postage for some $n \geq 32$, we can form $(n+1)¢$ postage. Thus, by induction, we can make any postage amount $32¢$ or greater. ♣

Example 16.6. *Let's now look at an example of an induction proof with a geometric flavor. Suppose we have a 4×5 chess board:*

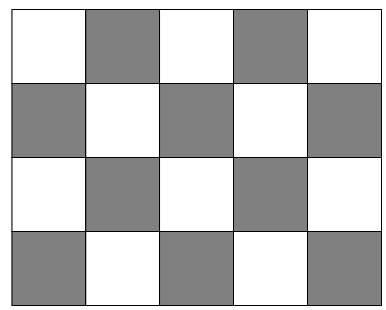
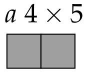


Figure 16.1: 4×5 chessboard

and a supply of 1×2 dominos: 

Each domino covers exactly two squares on the board. A ***perfect cover*** *of the board consists of a placement of dominos on the board so that each domino covers two squares on the board (dominos can be either vertically or horizontally orientated), no dominos overlap, no dominos extend beyond the edge of the board, and all the squares on the board are covered by a domino. It's easy to see that the 4×5 board above has a perfect cover. More generally, it is not hard to prove:*

Theorem 16.7. *An $m \times n$ board has a perfect cover with 1×2 dominos if and only if at least one of m and n is even.*

Example 16.8. *Now consider a $2^n \times 2^n$ board for n a positive integer. Suppose somewhere on the board there is one free square which does not have to be covered by a domino. For $n = 3$ the picture could appear as in figure 16.2, where the shaded square is the free square.*

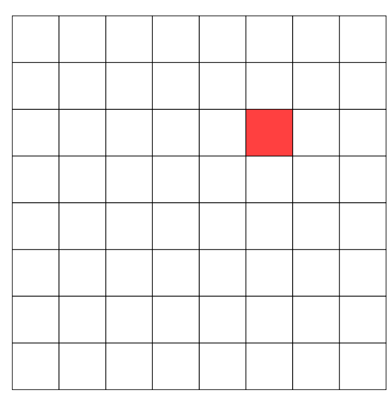
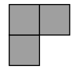


Figure 16.2: $2^3 \times 2^3$ chessboard

This time we have a supply of L-shaped dominos:  *These dominos (which can be rotated) each cover exactly three squares on the board. We will prove by induction that every such board has a perfect cover using L-shaped dominos.*

Proof. Basis: *For $n = 1$, the board to cover is an L-shaped domino, so it certainly has a perfect cover.*

Inductive Step: *Assume now that for some integer $n \geq 1$, any $2^n \times 2^n$ with one free square can be perfectly covered by L-shaped dominos. Consider a $2^{n+1} \times 2^{n+1}$ board with one free square. Divide the board in half horizontally and vertically. Each quarter of the board will be a $2^n \times 2^n$ board, and one of those quarters will have a free square in it (see figure 16.2).*

We now add one L-shaped domino as shown in figure 16.4.

This leaves us with essentially four $2^n \times 2^n$ boards, each with one free square. So, by the inductive assumption, they can each be perfectly covered by the L-shaped dominos, and so the entire board can be perfectly covered. ♣

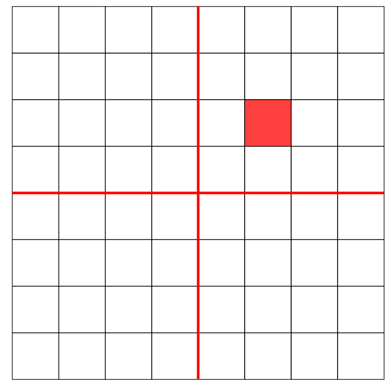


Figure 16.3: Divided $2^3 \times 2^3$ chessboard

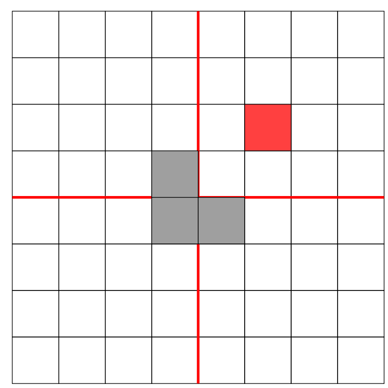


Figure 16.4: $2^3 \times 2^3$ board with domino

16.5 *Second principle of mathematical induction*

There is a second version of mathematical induction. Anything that can be proved with this second version can be proved with the method described above, and vice versa, but this second version is often easier to use. The change occurs in the induction assumption made in the inductive step of the proof. The inductive step of the method described above ($p(n) \to p(n+1)$ for all $n \geq k$) is replaced with $[p(k) \wedge p(k+1) \wedge \cdots \wedge p(n)] \to p(n+1)$ for all $n > k$. The effect is that we now have a lot more hypotheses to help us derive $p(n+1)$. In more detail, the second form of mathematical induction is described in the following theorem.

Theorem 16.9 (Second Principle of Mathematical Induction)**.**

For integers k and n, if

(1) $p(k)$ *is true, and*

(2) $[p(k) \wedge p(k+1) \wedge ... \wedge p(n)] \to p(n+1)$ *for an arbitrary $n \geq k$,*

then $p(n)$ is true for all $n \geq k$.

This principle is shown to be valid in the same way the first form of induction was justified. The utility lies in dealing with cases where we want to use inductive reasoning, but cannot deduce the $(n+1)$st case form the nth case directly. Let's do a few examples of proofs using this second form of induction. One more comment before doing the examples. In many induction proofs, it is convenient to check several initial cases in the basis step to avoid having to include special cases in the inductive step. The examples below illustrate this idea.

Example 16.10. *Show that using only* 5¢ *stamps and* 9¢ *stamps, any postage amount* 32¢ *or greater can be formed.*

Proof.

Basis: We can certainly make

$$\begin{aligned}
32¢ &= (1)5¢ + (3)9¢ \\
33¢ &= (3)5¢ + (2)9¢ \\
34¢ &= (5)5¢ + (1)9¢ \\
35¢ &= (7)5¢ + (0)9¢ \\
36¢ &= (0)5¢ + (4)9¢
\end{aligned}$$

Inductive Step: Suppose we can make all postage amounts from 32¢ *up to some amount* k¢ *where* $k \geq 36$. *Now consider the problem of making* $(k+1)$¢. *We can make* $(k+1-5)$¢ $= (k-4)$¢ *postage since* $k-4$ *is between* 32 *and* k. *Adding a* 5¢ *stamp to that gives the needed* $k+1$¢ *postage.* ♣

In that example, the basis step was a little messier than our first solution to the problem, but to make up for that, the inductive step required much less cleverness.

Example 16.11. *Induction can be used to verify a guessed closed from formula for a recursively defined sequence. Consider the sequence defined recursively by the initial conditions* $a_0 = 2$, $a_1 = 5$ *and the recursive rule, for* $n \geq 2$, $a_n = 5a_{n-1} - 6a_{n-2}$. *The first few terms of this sequence are* $2, 5, 13, 35, 97, \cdots$. *A little experimentation leads to the guess* $a_n = 2^n + 3^n$.

Let's verify that guess using induction. For the basis of the induction we check our guess gives the correct value of a_n *for* $n = 0$ *and* $n = 1$. *That's easy. For the inductive step, let's suppose our guess is correct up to* n *where* $n \geq 2$. *Then, we have*

$$\begin{aligned} a_{n+1} &= 5a_n - 6a_{n-1} \\ &= 5(2^n + 3^n) - 6(2^{n-1} + 3^{n-1}) \\ &= (5 \cdot 2 - 6)2^{n-1} - (6 - 5 \cdot 3)3^{n-1} \\ &= 4 \cdot 2^{n-1} - (-9) \cdot 3^{n-1} \\ &= 2^{n+1} + 3^{n+1} \text{ as we needed to show.} \end{aligned}$$

Example 16.12. *In the game of* ***Nim****, two players are presented with a pile of matches. The players take turns removing one, two, or three matches at a time. The player forced to take the last match is the loser. For example, if the pile initially contains* 8 *matches, then first player can, with correct play, be sure to win. Here's how: player 1: take* 3 *matches leaving* 5*; player 2's options will leave* 4, 3, *or* 2 *matches, and so player 1 can reduce the pile to* 1 *match on her turn, thus winning the game. Notice that if player 1 takes only* 1 *or* 2 *matches on her first turn, she is bound to lose to good play since player 2 can then reduce the pile to* 5 *matches.*

Let's prove that if the number of matches in the pile is 1 *more than a multiple of* 4*, the second player can force a win; otherwise, the first player can force a win.*

Proof. *For the basis, we note that obviously the second player wins if there is* 1 *match in the pile, and for* 2, 3, *or* 4 *matches the first player wins by taking* 1, 2, *or* 3 *matches in each case, leaving* 1 *match.*

For the inductive step, suppose the statement we are to prove is correct for the number of matches anywhere from 1 *up to* k *for some* $k \geq 4$. *Now consider a pile of* $k + 1$ *matches.*

case 1: *If* $k + 1$ *is* 1 *more than a multiple of* 4*, then when player 1 takes her matches, the pile will* <u>not</u> *contain* 1 *more than a multiple of* 4 *matches, and so the next player can force a win by the inductive assumption. So player 2 can force a win.*

case 2: *If* $k + 1$ *is not* 1 *more than a multiple of* 4*, then player 1 can select matches to make it* 1 *more than a multiple of* 4*, and so the next player* *is bound to lose (with best play) by the inductive assumption. So player 1 can force a win.* ♣

So, to win at Nim, when it is your turn, make sure you leave 1 more than a multiple of 4 matches in the pile (which is easy to do unless your opponent knows the secret as well, in which case you can just count the number of matches in the pile to see who will win, and skip playing the game altogether!).

16.6 Exercises

Exercise 16.1. *Prove: For every integer $n \geq 1$,*

$$1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n+2) = \frac{n(n+1)(2n+7)}{6}.$$

Exercise 16.2. *Prove: For every integer $n \geq 1$,*

$$1 \cdot 2^1 + 2 \cdot 2^2 + 3 \cdot 2^3 + ... + n \cdot 2^n = (n-1)2^{n+1} + 2.$$

Exercise 16.3. *The Fibonacci sequence is defined recursively by $f_0 = 0$, $f_1 = 1$, and, for $n \geq 2$, $f_n = f_{n-1} + f_{n-2}$. Use induction to prove that for all $n \geq 0$, $f_0 + f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$.*

Exercise 16.4. *Prove by induction: For every integer $n > 4$, we have $2^n > n^2$.*

Exercise 16.5. *Prove by induction: For every integer $n \geq 0$, $11^n - 6$ is divisible by 5.*

Exercise 16.6. *A pizza is cut into pieces (maybe some pretty oddly shaped) by making some integer $n \geq 0$ number of straight line cuts. Prove: The maximum number of pieces is $\frac{n^2+n+2}{2}$.*

Exercise 16.7. *A sequence is defined recursively by $a_0 = 0$, and, for $n \geq 1$, $a_n = 5a_{n-1} + 1$. Use induction to prove the closed form formula for a_n is*

$$a_n = \frac{5^n - 1}{4}.$$

Exercise 16.8. *A sequence is defined recursively by $a_0 = 1, a_1 = 4$, and for $n \geq 2$, $a_n = 5a_{n-1} - 6a_{n-2}$. Use induction to prove that the closed form formula for a_n is $a_n = 2 \cdot 3^n - 2^n, n \geq 0$.*

16.7 Problems

Problem 16.1. *Prove: For every integer $n \geq 1$,*

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}.$$

Problem 16.2. *Prove by induction: For $n \geq 2$,*

$$\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right)\left(1 - \frac{1}{16}\right) \cdots \left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}.$$

Problem 16.3. *Show that using only 3¢ stamps and 5¢ stamps, any postage amount 8¢ or greater can be formed. Do this twice, using both styles of induction.*

Problem 16.4. *Prove by induction: For every integer $n \geq 1$,*
$\displaystyle\sum_{k=1}^{n} (-1)^k k^2 = (-1)^n \frac{n(n+1)}{2}$.

Problem 16.5. *Prove by induction: For every integer $n \geq 1$, the number $n^5 - n$ is divisible by 5.*

Problem 16.6. *Prove by induction: For the Fibonacci sequence, for all $n \geq 0$,*
$f_0^2 + f_1^2 + f_3^2 + \cdots + f_n^2 = f_n f_{n+1}$.

Problem 16.7. *Prove by induction: For the Fibonacci sequence, for all $n \geq 1$,*
$f_{n-1} f_{n+1} = f_n^2 + (-1)^n$.

Problem 16.8. *Here is a proof that for $n \geq 0$,*
$1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1}$.

Proof. *Suppose $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1}$ for some $n \geq 0$. Then*

$$\begin{aligned} 1 + 2 + 2^2 + \cdots + 2^n + 2^{n+1} &= 2^{n+1} + 2^{n+1} && \textit{using the inductive hypothesis} \\ &= 2(2^{n+1}) = 2^{n+2} = 2^{(n+1)+1} \end{aligned}$$

as we needed to show. ♣

Now, obviously there is something wrong with this proof by induction since, for example, $1 + 2 + 2^2 = 7$, but $2^{2+1} = 2^3 = 8$. Where does the proof good bad?

Problem 16.9. *Prove by induction: Suppose that for some* $n \geq 1$, $2n$ *dots are placed around the outside of the circle, with* n *dots colored red and the remaining* n *colored blue. Going around the circle clockwise, you keep a count of how many red and blue dots you have passed. If at all times the number of red dots you have passed is at least the number of blue dots, you consider it a successful trip around the circle. Prove that no matter how the dots are colored red and blue, it is possible to have a successful trip around the circle if you start at the correct point.*