Math 208: Discrete Mathematics
Lesson 10: Lecture Video Notes

Topics

16. Mathematical Induction
    (a) mathematical induction
    (b) principle of mathematical induction
    (c) proofs by inductions
    (d) examples
    (e) second principle of mathematical induction

Readings: Chapter 16

§16. Mathematical induction

We have discussed several methods of proof so far:

- direct
- indirect
- proof by contradiction
- proof by cases

16a. Mathematical induction

Another method of proof is mathematical induction. It is used to prove theorems of the form $\forall x\, p(x)$ where the universe of discourse is a well-ordered set (usually $\mathbb{N}$).

Ex. Statements of the form $\forall x\, p(x)$

- Every classroom on campus has a trash can.
- The square of an odd integer is odd.
- For every positive integer n, we have

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

- For every integer $n \geq 0$, the expression $11^n - 6$ is divisible by 5.
- Using only 5¢ and 6¢ stamps, any postage amount 20¢ or greater can be formed.

Ways to prove a theorem of the form $\forall x\, p(x)$:

- Check $p(c)$ is true for every $c \in \mathcal{U}$. This approach is feasible if $\mathcal{U}$ is finite.
- Let $c \in \mathcal{U}$ be arbitrary and show $p(c)$ is true. Then the theorem follows by universal generalization.
- mathematical induction (typically $\mathcal{U} = \mathbb{N}$)

Idea of mathematical induction. Suppose we have a sequence of theorems that we would like to be true. Let's say we want to prove the theorem $\forall x\, p(x)$ where the universe of discourse is $\mathbb{N}$. That is, we want to prove

$$p(0) \wedge p(1) \wedge p(2) \wedge p(3) \wedge \dots$$

One approach for a proof uses ideas similar to recursive definitions for sequences and sets. There are two parts for mathematical induction:

(i) basis step: $p(0)$ is true

(ii) inductive step: for any $n \in \mathbb{N}$, $p(n) \to p(n+1)$.

By showing these two parts, it can be logically concluded that $\forall x\, p(x)$ where $x \in \mathbb{N}$.

Remark. Mathematical induction can be used when the domain of discourse is any well ordered set. The natural numbers $\mathbb{N}$ is an example of a well ordered set which appears frequently.

Ex. Use mathematical induction to prove: for all $n \in \mathbb{N}$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

16b. The principle of mathematical induction

Defn. A set S is *well ordered* if every not empty subset has a least member.

Examples and non-examples.

(i) $\mathbb{N}$ is well ordered

(ii) any subset of a well ordered set is well ordered

(ii) $\mathbb{Z}$ is not well ordered

Theorem. (Principle of Mathematical Induction) Suppose we have a list of statements: $p(k), p(k+1), p(k+2), \ldots, p(n), p(n+1), \ldots$.

If

(1) $p(k)$ is true, and

(2) $p(n) \to p(n+1)$ for every $n \geq k$

then all the statements in the list are true.

Note. This theorem is established in the textbook using a proof by contradiction.

16c. Proofs by induction

Mathematical induction is a useful method of proof to establish theorems of the form:

$$p(k) \wedge p(k+1) \wedge p(k+2) \wedge \cdots \wedge p(n) \wedge p(n+1) \wedge \ldots$$

By the Principle of Mathematical induction, to give a proof we must show

(1) $p(k)$ is true, and

(2) $p(n) \to p(n+1)$ for every $n \geq k$.

Memory aid: Think of induction as knocking over dominos.

Ex. Use induction to prove $n < 2^n$ for every $n \geq 1$.

16d. Examples

Ex. Assume $r \neq 1$. Use induction to prove that for all $n \geq 0$

$$\sum_{j=0}^{n} r^j = \frac{r^{n+1} - 1}{r - 1}.$$

Ex. Use induction to show that using only 5¢ and 6¢ stamps any postage amount 20¢ or greater can be formed.

Ex. Use induction to prove that for every integer $n \geq 0$, the expression $11^n - 6$ is divisible by 5.

16e. Second principle of mathematical induction

There is a variation of mathematical induction that arises from time to time. So far, we've discussed:

Theorem. (Principle of Mathematical Induction) Suppose we have a list of statements: $p(k), p(k+1), p(k+2), \ldots, p(n), p(n+1), \ldots$.

If

(1) $p(k)$ is true, and

(2) $p(n) \to p(n+1)$ for every $n \geq k$

then all the statements in the list are true.

Note. The version above is sometimes called *weak induction* or simply *induction.* The following alternative is sometimes called *strong induction.*

Theorem. (Second Principle of Mathematical Induction) Suppose we have a list of statements: $p(k), p(k+1), p(k+2), \ldots, p(n), p(n+1), \ldots$.

If

(1) $p(k)$ is true, and

(2) $[p(k) \wedge p(k+1) \wedge \cdots \wedge p(n-1) \wedge p(n)] \to p(n+1)$ for every $n \geq k$

then all the statements in the list are true.

Ex. Using strong induction to prove the Fundamental Theorem of Arithmetic That is, prove that every positive integer $n \geq 2$ can be uniquely factored into positive prime numbers.

Ex. Use strong induction to prove that using only 5¢ and 6¢ stamps any postage amount 20¢ or greater can be formed.

Note. We also gave proof of this theorem using weak induction.

Ex. A sequence is defined recursively by $a_0 = 1$, $a_1 = 4$, and for $n \geq 2$, $a_n = 5a_{n-1} - 6a_{n-2}$. Use induction to prove that the closed form formula for a_n is $a_n = 2 \cdot 3^n - 2^n$ for $n \geq 0$.

Remark. Often times, authors refer to either method of induction as simply induction when writing proofs. It is up to the reader to decipher which version is being used.