3

Predicates and Quantifiers

THE SENTENCE $x^2 - 2 = 0$ IS NOT a proposition. It cannot be assigned a truth value unless some more information is supplied about the variable x. Such a statement is called a **predicate** or a **propositional function**.

3.1 Predicates

Instead of using a single letter to denote a predicate, a symbol such as $S(x)$ will be used to indicate the dependence of the sentence on a variable. Here are two more examples of predicates.

(1) $A(c)$: *Al drives a* c, and

(2) $B(x, y)$: x *is the brother of* y.

The second example is an instance of a **two-place predicate**.

With a given predicate, there is an associated set of objects which can be used in place of the variables. For example, in the predicate $S(x) : x^2 - 2 = 0$, it is understood that the x can be replaced by a number. Replacing x by, say, the word *blue* does not yield a meaningful sentence. For the predicate $A(c)$ above, c can be replaced by, say, makes of cars (or maybe types of nails!). For $B(x, y)$, the x can be replaced by any human male, and the y by any human. The collection of possible replacements for a variable in a predicate is called the **domain of discourse** for that variable.

Usually the domain of discourse is left for the reader to guess, but if the domain of discourse is something other than an obvious choice, the writer will mention the domain to be used.

3.2 *Instantiation and Quantification*

A predicate is not a proposition, but it can be converted into a proposition. There are three ways to modify a predicate to change it into a proposition. Let's use $S(x) : x^2 - 2 = 0$ as an example.

The first way to change $S(x)$ to make it into a proposition is to assign a specific value from the variable's domain of discourse to the variable. For example, setting $x = 3$, gives the (false) proposition $S(3) : 3^2 - 2 = 0$. On the other hand, setting $x = \sqrt{2}$ gives the (true) proposition $S(\sqrt{2}) : (\sqrt{2})^2 - 2 = 0$. The process of setting a variable equal to a specific object in its domain of discourse is called **instantiation**. Looking at the two-place predicate $B(x, y) : x$ is the brother of y, we can instantiate both variables to get the (true) proposition $B($*Donny, Marie*$)$: *Donny is the brother of Marie*. Notice that the sentence $B($*Donny*, $y)$: *Donny is the brother of y* has not been converted into a proposition since it cannot be assigned a truth value without some information about y. But it has been converted from a two-place predicate to a one-place predicate.

A second way to convert a predicate to a proposition is to precede the predicate with the phrase *There is an x such that*. For example, *There is an x such that* $S(x)$ would become *There is an x such that* $x^2 - 2 = 0$. This proposition is true if there is at least one choice of x in its domain of discourse for which the predicate becomes a true statement. The phrase *There is an x such that* is denoted in symbols by $\exists x$, so the proposition above would be written as $\exists x\, S(x)$ or $\exists x\, (x^2 - 2 = 0)$. When trying to determine the truth value of the proposition $\exists x\, P(x)$, it is important to keep the domain of discourse for the variable in mind. For example, if the domain for x in $\exists x\, (x^2 - 2 = 0)$ is all integers, the proposition is false. But if its domain is all real numbers, the proposition is true. The phrase *There is an x such that* (or, in symbols, $\exists x$) is called **existential quantification**[1].

The third and final way to convert a predicate into a proposition is by **universal quantification**[2] . The universal quantification of a predicate, $P(x)$, is obtained by preceding the predicate with the phrase

[1] In English it can also be read as *There exists x* or *For some x*.

[2] The phrase *For all x* is also rendered in English as *For each x* or *For every x*.

For all x, producing the proposition *For all* $x, P(x)$, or, in symbols, $\forall x\, P(x)$. This proposition is true provided the predicate becomes a true proposition for every object in the variable's domain of discourse. Again, it is important to know the domain of discourse for the variable since the domain will have an effect on the truth value of the quantified proposition in general.

For multi-placed predicates, these three conversions can be mixed and matched. For example, using the obvious domains for the predicate $B(x, y)$: *x is the brother of y* here are some conversions into propositions:

(1) $B(Donny, Marie)$ has both variables instantiated. The proposition is true.

(2) $\exists y\, B(Donny, y)$ is also a true proposition. It says *Donny* is somebody's brother. The first variable was instantiated, the second was existentially quantified.

(3) $\forall y\, B(Donny, y)$ says everyone has Donny for a brother, and that is false.

(4) $\forall x\, \exists y\, B(x, y)$ says every male is somebody's brother, and that is false.

(5) $\exists y\, \forall x\, B(x, y)$ says there is a person for whom every male is a brother, and that is false.

(6) $\forall x\, B(x, x)$ says every male is his own brother, and that is false.

3.3 Translating to symbolic form

Translation between ordinary language and symbolic language can get a little tricky when quantified statements are involved. Here are a few more examples.

Example 3.1. *Let $P(x)$ be the predicate x **owns a Porsche**, and let $S(x)$ be the predicate x **speeds**. The domain of discourse for the variable in each predicate will be the collection of all drivers. The proposition $\exists x P(x)$ says **Someone owns a Porsche**. It could also be translated as **There is a person** x **such that** x **owns a Porsche**, but that sounds too stilted for* *ordinary conversation. A smooth translation is better. The proposition $\forall x(P(x) \to S(x))$ says* ***All Porsche owners speed.***

Translating in the other direction, the proposition ***No speeder owns a Porsche*** *could be expressed as $\forall x(S(x) \to \neg P(x))$.*

Example 3.2. *Here's a more complicated example: translate the proposition* ***Al knows only Bill*** *into symbolic form. Let's use $K(x, y)$ for the predicate x* ***knows*** *y. The translation would be $K(Al, Bill) \wedge \forall x\,(K(Al, x) \to (x = Bill))$.*

Example 3.3. *For one last example, let's translate* ***The sum of two even integers is even*** *into symbolic form. Let $E(x)$ be the predicate x* ***is even****. As with many statements in ordinary language, the proposition is phrased in a shorthand code that the reader is expected to unravel. As given, the statement doesn't seem to have any quantifiers, but they are implied. Before converting it to symbolic form, it might help to expand it to its more long winded version:* ***For every choice of two integers, if they are both even, then their sum is even****. Expressed this way, the translation to symbolic form is duck soup: $\forall x\,\forall y\,((E(x) \wedge E(y)) \to E(x+y))$.*

3.4 Quantification and basic laws of logic

Notice that if the domain of discourse consists of finitely many entries $a_1, ..., a_n$, then $\forall x\, p(x) \equiv p(a_1) \wedge p(a_2) \wedge ... \wedge p(a_n)$. So the quantifier $\forall$ can be expressed in terms of the logical connective $\wedge$. The existential quantifier and $\vee$ are similarly linked: $\exists x\, p(x) \equiv p(a_1) \vee p(a_2) \vee ... \vee p(a_n)$.

From the associative and commutative laws of logic we see that we can rearrange any system of propositions which are linked only by $\wedge$'s or linked only by $\vee$'s.[3] Consequently any more generally quantified proposition of the form $\forall x \forall y\, p(x, y)$ is logically equivalent to $\forall y \forall x\, p(x, y)$. Similarly for statements which contain only existential quantifiers. But the distributive laws come into play when $\wedge$'s and $\vee$'s are mixed. So care must be taken with predicates which contain both existential and universal quantifiers, as the following example shows.

[3] For instance, consider examples 3.1 – 3.3 with finite domains of discourse.

Example 3.4. *Let $p(x,y) : \mathbf{x}+\mathbf{y}=\mathbf{0}$ and let the domain of discourse be all real numbers for both x and y. The proposition $\forall y\,\exists x\, p(x,y)$ is true, since, for any given y, by setting (instantiating) $x=-y$ we convert $\mathbf{x}+\mathbf{y}=\mathbf{0}$ to the true statement $(-\mathbf{y})+\mathbf{y}=\mathbf{0}$[4]. However the proposition $\exists x\,\forall y\, p(x,y)$ is false. If we set (instantiate) $y=1$, then $\mathbf{x}+\mathbf{y}=\mathbf{0}$ implies that $x=-1$. When we set $y=0$, we get $x=0$. Since $0\neq -1$ there is no x which will work for all y, since it would have to work for the specific values of $y=0$ and $y=1$.*

[4] $(\forall y\in\mathbb{R})[(-y)+y=0]$ is a tautology.

3.5 *Negating quantified statements*

To form the negation of quantified statements, we apply De Morgan's laws. This can be seen in case of a finite domain of discourse as follows:

$$\begin{aligned}\neg(\forall x\, p(x)) &\equiv \neg(p(a_1)\wedge p(a_2)\wedge \ldots \wedge p(a_n))\\ &\equiv \neg p(a_1)\vee\neg p(a_2)\vee\cdots\vee\neg p(a_n)\\ &\equiv \exists x\,\neg p(x)\end{aligned}$$

In the same way, we have $\neg(\exists x\, p(x))\equiv(\forall x\,\neg p(x))$. [5]

[5] Use De Morgan's laws to find a similar expression for $\neg(\forall x p(x))$.

3.6 Exercises

Exercise 3.1. *Let $p(x) : 2x \geq 4$, for $x \in \mathbb{R}$. Determine the truth values of the following propositions.*

a) $p(2)$

b) $p(-3)$

c) $\forall x\, ((x \leq 10) \rightarrow p(x))$

d) $\exists x\, \neg p(x)$

Exercise 3.2. *Let $p(x, y)$ be x has read y, where the domain of discourse for x is all students in this class, and the domain of discourse for y is all novels. Express the following propositions in English.*

a) $\forall x\, p(x, \textit{War and Peace})$

b) $\exists x\, \neg p(x, \textit{The Great Gatsby})$

c) $\exists x\, \forall y\, p(x, y)$

d) $\forall y\, \exists x\, p(x, y)$

Exercise 3.3. *Let $F(x, y)$ be the statement x can fool y, where the domain of discourse for both x and y is all people. Use quantifiers to express each of the following statements.*

a) I can fool everyone.

b) George can't fool anybody.

c) No one can fool himself.

d) There is someone who can fool everybody.

e) There is someone everyone can fool.

f) Ralph can fool two different people.

Exercise 3.4. *Negate each of the statements from exercise 3.2 in English.*

Exercise 3.5. *Negate each statement from exercise 3.3 in logical symbols. Of course, the easy answer would be to simply put $\neg$ in front of each statement. But use the principle given at the end of this chapter to move the negation across the quantifiers.*

Exercise 3.6. *Express symbolically: The product of an even integer and an odd integer is even.*

Exercise 3.7. *Express in words the meaning of*

$$\exists x\, P(x) \wedge \forall x\, \forall y\, ((P(x) \wedge P(y)) \rightarrow (x = y)) .$$

3.7 Problems

Problem 3.1. *Let h be Ben is healthy., w: Ben is wealthy., and s: Ben is wise.*

Express the following in English:

a) $h \wedge w$

b) $w \vee s$

c) $h \rightarrow (w \wedge s)$

d) $(h \rightarrow w) \wedge s$

Problem 3.2. *Let $S(x, y)$ be the predicate x has seem y where the domain of discourse for x is all students in this class and the domain of discourse for y is all movies. Express the following in logical symbols using quantifiers.*

a) *Every student in this class has seen Gone With The Wind.*

b) *No student in this class has seen Jaws.*

c) *The Shape of Water has been seen by someone in this class.*

d) *Some students have seen every movie.*

e) *For each movie, there is at least one student in the class who has seen that movie.*

Problem 3.3. *Negate the propositions in 3.1 in English.*

Problem 3.4. *Negate the propositions in 3.2 in symbols. Note: An easy way to do this is to simply write $\neg$ in front of the answers in 3.2. Don't do that! Give the negation with no quantifiers coming after a negation symbol.*

Problem 3.5. *Negate the propositions in 3.2 in English. Note: An easy way to do this is to simply write It is not the case that .... in front of each proposition. Don't do that! Give the negation as a reasonably natural English sentence.*

4
Rules of Inference

THE HEART OF MATHEMATICS is proof. In this chapter, we give a careful description of what exactly constitutes a proof in the realm of propositional logic. Throughout the course various methods of proof will be demonstrated, including the particularly important style of proof called *induction*. It's important to keep in mind that all proofs, no matter what the subject matter might be, are based on the notion of a valid argument as described in this chapter, so the ideas presented here are fundamental to all of mathematics.

Imagine trying carefully to define what a proof is, and it quickly becomes clear just how difficult a task that is. So it shouldn't come as a surprise that the description takes on a somewhat technical looking aspect. But don't let all the symbols and abstract-looking notation be misleading. All these rules really boil down to plain old common sense when looked at correctly.

The usual form of a theorem in mathematics is: If a is true and b is true and c is true, etc., then s is true. The $a, b, c, \cdots$ are called the **hypotheses**, and the statement s is called the **conclusion**. For example, a mathematical theorem might be: if m is an even integer and n is an odd integer, then mn is an even integer. Here the hypotheses are *m is an even integer* and *n is and odd integer*, and the conclusion is *mn is an even integer*.

4.1 *Valid propositional arguments*

In this section we are going to be concerned with proofs from the realm of propositional logic rather than the sort of theorem from mathematics mentioned above. We will be interested in arguments in which the **form** of the argument is the item of interest rather than the **content** of the statements in the argument.

For example, consider the simple argument: (1) *My car is either red or blue* and (2) *My car is not red*, and so (3) *My car is blue*. Here the hypotheses are (1) and (2), and the conclusion is (3). It should be clear that this is a **valid argument**. That means that if you agree that (1) and (2) are true, then you *must* accept that (3) is true as well.

Definition 4.1. An argument is called **valid** provided that if you agree that all the hypotheses are true, then you must accept the truth of the conclusion.

Now the content of that argument (in other words, the stuff about *my* and *cars* and *colors*) really have nothing to do with the validity of the argument. It is the **form** of the argument that makes it valid. The form of this argument is (1) $p \vee q$ and (2) $\neg p$, therefore (3) q. Any argument that has this form is valid, whether it talks about cars and colors or any other notions. For example, here is another argument of the very same form: (1) *I either read the book or just looked at the pictures* and (2) *I didn't read the book*, therefore (3) *I just looked at the pictures*.

Some arguments involve quantifiers. For instance, consider the classic example of a logical argument: (1) *All men are mortal* and (2) *Socrates is a man*, and so (3) *Socrates is mortal*. Here the hypotheses are the statements (1) and (2), and the conclusion is statement (3). If we let $M(x)$ be *x is a man* and $D(x)$ be *x is mortal* (with domain for x being everything!), then this argument could be symbolized as shown.

$$\begin{array}{l} \forall x(M(x) \to D(x)) \\ M(\text{Socrates}) \\ \hline \therefore D(\text{Socrates}) \end{array}$$

The general form of a proof that a logical argument is valid consists in assuming all the hypotheses have truth value T, and showing, by applying valid rules of logic, that the conclusion must also have truth value T.

Just what are the valid rules of logic that can be used in the course of the proof? They are called the Rules of Inference, and there are seven of them listed in the table below. Each rule of inference arises from a tautology, and actually there is no end to the rules of inference, since each new tautology can be used to provide a new rule of inference. But, in real life, people rely on only a few basic rules of inference, and the list provided in the table is plenty for all normal purposes.

Table 4.1: Basic rules of inference

Name	**Rule of Inference**	
Modus Ponens	p and $p \to q$	$\therefore q$
Modus Tollens	$\neg q$ and $p \to q$	$\therefore \neg p$
Hypothetical Syllogism	$p \to q$ and $q \to r$	$\therefore p \to r$
Addition	p	$\therefore p \vee q$
Simplification	$p \wedge q$	$\therefore p$
Conjunction	p and q	$\therefore p \wedge q$
Disjunctive Syllogism	$p \vee q$ and $\neg p$	$\therefore q$

It is important not to merely look on these rules as marks on the page, but rather to understand what each one says in words. For example, Modus Ponens corresponds to the common sense rule: if we are told *p is true*, and also *If p is true, then so is q*, then we would leap to the reasonable conclusion that *q is true*. That is all Modus Ponens says. Similarly, for the rule of proof of Disjunctive Syllogism: knowing *Either p or q is true*, and *p is not true*, we would immediately conclude *q is true*. That's the rule we applied in the *car* example above. Translate the remaining six rules of inference into such common sense statements. Some may sound a little awkward, but they ought to all elicit an *of course that's right* feeling once understood. Without such an understanding, the rules seem like a jumble of mystical symbols, and building logical arguments will be pretty difficult.

What exactly goes into a logical argument? Suppose we want to prove (or show valid) an argument of the form *If a and b and c are true, then so is s*. One way that will always do the trick is to construct a truth table as in examples earlier in the course. We check the rows in the table where all the hypotheses are true, and make sure the conclusion is also true in those rows. That would complete the proof. In fact that is exactly the method used to justify the seven rules of inference given in the table. But building truth tables is certainly tedious business, and it certainly doesn't seem too much like the way we learned to do proofs in geometry, for example. An alternative is the construction of a logical argument which begins by assuming the hypotheses are all true and applies the basic rules of inferences from the table until the desired conclusion is shown to be true.

Here is an example of such a proof. Let's show that the argument displayed in figure 4.1 is valid.

$$\begin{array}{l} p \\ p \to q \\ s \lor r \\ r \to \neg q \\ \hline \therefore s \lor t \end{array}$$

Figure 4.1: A logical argument

Each step in the argument will be justified in some way, either (1) as a hypothesis (and hence assumed to have truth value T), or (2) as a consequence of previous steps and some rule of inference from the table, or (3) as a statement logically equivalent to a previous statement in the proof. Finally the last statement in the proof will be the desired conclusion. Of course, we could prove the argument valid by constructing a 32 row truth table instead! Well, actually we wouldn't need all 32 rows, but it would be pretty tedious in any case.

Such proofs can be viewed as games in which the hypotheses serve as the starting position in a game, the goal is to reach the conclusion as the final position in the game, and the rules of inference (and logical equivalences) specify the legal moves. Following this outline, we can be sure every step in the proof is a true statement, and, in particular, the desired conclusion is true, as we hoped to show.

Table 4.2: Proof of the validity of an argument

Argument:		**Proof:**			
	p		(1)	p	hypothesis
	$p \to q$		(2)	$p \to q$	hypothesis
	$s \lor r$		(3)	q	Modus Ponens (1) and (2)
	$r \to \neg q$		(4)	$r \to \neg q$	hypothesis
	$\therefore s \lor t$		(5)	$q \to \neg r$	logical equivalent of (4)
			(6)	$\neg r$	Modus Ponens (3) and (5)
			(7)	$s \lor r$	hypothesis
			(8)	$r \lor s$	logical equivalence of (7)
			(9)	s	Disjunctive Syllogism (6) and (8)
			(10)	$s \lor t$	Addition

One step more complicated than the last example are arguments that are presented in words rather than symbols. In such a case, it is necessary to first convert from a verbal argument to a symbolic argument, and then check the argument to see if it is valid. For example, consider the argument: *Tom is a cat. If Tom is a cat, then Tom likes fish. Either Tweety is a bird or Fido is a dog. If Fido is a dog, then Tom does not like fish. So, either Tweety is a bird or I'm a monkey's uncle.* Just reading this argument, it is difficult to decide if it is valid or not. It's just a little too confusing to process. But it is valid, and in fact it is the very same argument as given above. Let p be *Tom is a cat*, let q be *Tom likes fish*, let s be *Tweety is a bird*, let r be *Fido is a dog*, and let t be *I'm a monkey's uncle*. Expressing the statements in the argument in terms of p, q, r, s, t produces exactly the symbolic argument proved above.

4.2 *Fallacies*

Some logical arguments have a convincing ring to them but are nevertheless invalid. The classic example is an argument of the form *If it is snowing, then it is winter. It is winter. So it must be snowing.* A moment's thought is all that is needed to be convinced the conclusion does not follow from the two hypotheses. Indeed, there are many winter days when it does not snow. The error being made is called the **fallacy of affirming the conclusion**. In symbols, the argument is claiming that $[(p \to q) \land q] \to p$ is a tautology, but in fact, checking a truth table shows that it is not a tautology. Fallacies arise when statements that are not tautologies are treated as if they were tautologies.

4.3 *Arguments with quantifiers*

Logical arguments involving propositions using quantifiers require a few more rules of inference. As before, these rules really amount to no more than a formal way to express common sense. For instance, if the proposition $\forall x\, P(x)$ is true, then certainly for every object c in the universe of discourse, $P(c)$ is true. After all, if the statement $P(x)$ is true for every possible choice of x, then, in particular, it is true when $x = c$. The other three rules of inference for quantified statements are just as obvious. All four quantification rules appear in table 4.3.

Table 4.3: Quantification rules

Name	Instantiation Rules
Universal Instantiation	$\forall x P(x) \therefore P(c)$ if c is in the domain of x
Existential Instantiation	$\exists x P(x) \therefore P(c)$ for some c in the domain of x
Name	**Generalization Rules**
Universal Generalization	$P(c)$ for **arbitrary** c in the domain of x $\therefore \forall x\, P(x)$
Existential Generalization	$P(c)$ for some c in the domain of x $\therefore \exists x\, P(x)$

Example 4.2. *Let's analyze the following (fictitious, but obviously valid) argument to see how these rules of inference are used. All books written by Sartre are hard to understand. Sartre wrote a book about kites. So, there is a book about kites that is hard to understand. Let's use to following predicates to symbolize the argument:*

(1) $S(x)$: *x was written by Sartre.*

(2) $H(x)$: *x is hard to understand.*

(3) $K(x)$: *x is about kites.*

The domain for x in each case is all books. In symbolic form, the argument and a proof are

Argument:
$$\begin{array}{l} \forall x(S(x) \rightarrow H(x)) \\ \exists x(S(x) \land K(x)) \\ \hline \therefore \exists x(K(x) \land H(x)) \end{array}$$

Proof:

1) $\exists x(S(x) \land K(x))$	*hypothesis*
2) $S(c) \land K(c)$ *for some c*	*Existential Instantiation (1)*
3) $S(c)$	*Simplification (2)*
4) $\forall x(S(x) \rightarrow H(x))$	*hypothesis*
5) $S(c) \rightarrow H(c)$	*Universal Instantiation (4)*
6) $H(c)$	*Modus Ponens (3) and (5)*
7) $K(c) \land S(c)$	*logical equivalence (2)*
8) $K(c)$	*Simplification (7)*
9) $K(c) \land H(c)$	*Conjunction (8) and (6)*
10) $\exists x(K(x) \land H(x))$	*Existential Generalization (9)*

4.4 Exercises

Exercise 4.1. *Show $p \vee q$ and $\neg p \vee r$, $\quad \therefore q \vee r$ is a valid rule of inference. It is called **Resolution**.*

Exercise 4.2. *Prove the following argument is valid. All Porsche owners are speeders. No owners of sedans buy premium fuel. Car owners that do not buy premium fuel never speed. So Porsche owners do not own sedans. Use all car owners as the domain of discourse.*

Exercise 4.3. *Prove the following symbolic argument is valid.*

$$\begin{array}{l} \neg p \to (r \wedge \neg s) \\ t \to s \\ u \to \neg p \\ \neg w \\ u \vee w \\ \hline \therefore \neg t \vee w \end{array}$$

4.5 Problems

Problem 4.1. *Show that $p \rightarrow q$ and $\neg p$, $\quad \therefore \neg q$ is not a valid rule of inference. It is called the* ***Fallacy of denying the hypothesis****.*

Problem 4.2. *Prove the following symbolic argument is valid.*

$$\begin{array}{l} \neg p \wedge q \\ r \rightarrow p \\ \neg r \rightarrow s \\ s \rightarrow t \\ \hline \therefore t \end{array}$$

Problem 4.3. *Prove the following symbolic argument is valid.*

$$\begin{array}{l} p \vee q \\ q \rightarrow r \\ (p \wedge s) \rightarrow t \\ \neg r \\ (\neg q \rightarrow (u \wedge s) \\ \hline \therefore t \end{array}$$

Problem 4.4. *Prove the following symbolic argument is valid.*

$$\begin{array}{l} (\neg p \vee q) \rightarrow r \\ s \vee \neg q \\ \neg t \\ p \rightarrow t \\ (\neg p \wedge r) \rightarrow \neg s \\ \hline \therefore \neg q \end{array}$$

Problem 4.5. *Express the following argument is symbolic form and prove the argument is valid. If Ralph doesn't do his homework or he doesn't feel sick, then he will go to the party and he will stay up late. If he goes to the party, he will eat too much. He didn't eat too much. So Ralph did his homework.*

Problem 4.6. *In problem 4.5, show that you can logically deduce that Ralph felt sick.*

Problem 4.7. *In prob 4.5, can you logically deduce that Ralph stayed up late?*

Problem 4.8. *Prove the following symbolic argument.*

$$\begin{array}{l} \exists x(A(x) \wedge \neg B(x)) \\ \forall x(A(x) \to C(x)) \\ \hline \therefore\ \exists x(C(x) \wedge \neg B(x)) \end{array}$$