Math 208: Discrete Mathematics
Lesson 8: Lecture Video Notes

Topics

13. Sequences and summation
    (a) specifying sequences
    (b) arithmetic sequences
    (c) geometric sequences
    (d) summation notation
    (e) formulas for arithmetic and geometric summations
14. Recursively defined sequences
    (a) closed form formulas
    (b) arithmetic and geometric sequences by recursion

Readings: Chapters 13-14

§13. Sequences and summation

Defn. (Informal) A *sequence* is a list of items in a specific order. A *term* of a sequence is one of the items appearing in the sequence.

Ex. Some example of sequences are:

(i) $3, 1, 4, 1, 5, 9, \ldots$

(ii) $a, b, c, d, \ldots, z$

(iii) $\alpha, \beta, \gamma, \delta, \ldots, \omega$

(iv) red, orange, yellow, green, blue, indigo, violet

(v) $2, 3, 5, 7, 11, \ldots$

(vi) $1, 2, 3, 4, 5, \ldots, 100$

Observations.

- sequences can be infinite or finite
- terms of the sequence can be numbers or members from an arbitrary set

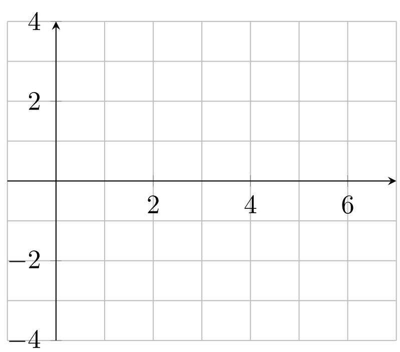
Remark. We are most interested in sequences of numbers, so we can discuss the summation of a sequence (i.e. adding all terms of the sequence together). This is straightforward for finite sequences. More care is needed when dealing with infinite sequences.

13a. Specifying sequences

Sequences can be thought of as a special type of function.

Defn. A sequence is a function whose domain is $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$, $\{1, 2, 3, \ldots\}$, or $\{1, 2, 3, \ldots, n\}$ and codomain is some arbitrary set A.

Ex. Consider the sequence s given by $1, 2, 4, 1, 2, 4, \ldots$.



Remark. When the domain of the sequence is $\{1, 2, 3, \ldots\}$, we say $s(1)$ is the *first term* of the sequence; $s(2)$ is the *second term*; and so on. With sequences, we often use subscripts to indicate the term and write s_1 for $s(1)$, s_2 for $s(2)$, etc.

Caution. If the domain of the sequence s is $\{0, 1, 2, \ldots\}$, then s_0 is the first term, s_1 is the second term, etc.

Ways to specify a sequence

- describe the ith term of the sequence using words
- use a formula. E.g. $a_n = n^2$ for $n = 1, 2, 3, \ldots$
- by suggesting a pattern (although this can lead to confusion at times)

Ex. Let p be the sequence of prime numbers. What is the p_7?

Ex. Let $s_n = \frac{1}{2n+1}$ for $n = 0, 1, 2, \ldots$. What is the 21st term of the sequence? *Hint:* Be careful!

Ex. What are the next terms in the sequence: $36, 12, 4, \frac{4}{3}, \ldots$?

Remark. Defining a sequence by pattern assumes that the reader can guess the intended pattern. Sometimes this can lead to miscommunication or confusion!

Ex. What are the next terms in the sequence: M, T, W, T, F, …?

Ex. What are the next terms in the sequence: $2, 1, 3, 4, 7, \ldots$?

Ex. What are the next terms in the sequence: O, T, T, F, F, S, S, E, N, …?

13b. Arithmetic sequences

Ex. Determine the next terms in the following sequences.

(i) $10, 6, 2, -2, \ldots$

(ii) $4, 4.6, 5.2, 5.8, \ldots$

Observatons. Both sequences in the last example started with some fixed number a. Furthermore, the next term in the sequence was obtains by adding a (possibly negative) constant d.

Defn. An *arithmetic sequence* is a sequence of the form

$$a, a+d, a+2d, a+3d, \ldots$$

where a is the *initial term* and constant d is called the *common difference.*

Note. Most often a and d are taken to be real numbers. However, arithmetic sequences can be defined in any set of numbers where addition exists (e.g. $\mathbb{C}$).

Q. What is the nth term of an arithmetic sequence?

Fact. The nth term of an arithmetic sequence s is $s_n = a + (n-1)d$.

Ex. What is the 200th term of the arithmetic sequence which begins $10, 6, 2, -2, \ldots$?

Ex. Suppose an arithmetic sequence has terms $s_6 = 24$ and $s_{11} = 79$. Find a formula for the nth term.

13c. Geometric sequences

To get the next term in an arithmetic sequence, we added a common difference to the previous term. If instead we multiply by a fixed number, say r, we produce a geometric sequence.

Defn. A *geometric sequence* is a sequence of the form

$$a, ar, ar^2, ar^3, \ldots$$

where a is the *initial term* and constant r is called the *common ratio.*

Note. Most often a and r are taken to be real numbers. However, geometric sequences can be defined in any set of numbers where multiplication exists (e.g. $\mathbb{C}$).

Q. What is the nth term of a geometric sequence?

Fact. The nth term of a geometric sequence s is $s_n = ar^{n-1}$.

Ex. What is the 6th term of the geometric sequence which begins $24, 12, 6, \ldots$?

Ex. Suppose an geometric sequence has terms $s_2 = 18$ and $s_5 = -\frac{16}{3}$. Find a formula for the nth term.

Remarks. We mention a few special geometric series. A geometric series with $r = 1$ is just a constant sequence, namely

$$a, a, a, a, a, \ldots.$$

When $r = 0$, the geometric series has the form:

$$a, 0, 0, 0, \ldots.$$

Lastly, a geometric series with $r = -1$ has an alternating pattern:

$$a, -a, a, -a, a, \ldots.$$

13d. Summation notation

Defn. A *summation* or *sum* is a sequence of numbers added up.

Notation. A sum of n terms of a sequence is often denoted S_n.

There can be a lot of terms to add in a sequence, so mathematicians have developed some notation to write sums more compactly.

Ex.

$$\sum_{j=1}^{100} j = 1 + 2 + 3 + \cdots + 100$$

$$\sum_{i=0}^{4} i^2 = 0^2 + 1^2 + 2^2 + 3^2 + 4^2$$

$$\sum_{k=-2}^{3} (2k+1) = (2(-2)+1) + (2(-1)+1) + (2(0)+1) + (2(1)+1) + (2(2)+1) + (2(3)+1)$$

$$\sum_{n=1}^{\infty} 5\left(\frac{1}{3}\right)^{n-1} = 5\left(\frac{1}{3}\right)^{0} + 5\left(\frac{1}{3}\right)^{1} + 5\left(\frac{1}{3}\right)^{2} + \dots$$

Observations.

- The capital Greek letter sigma, $\sum$, indications summation.
- Terms in the summation are determined using the index of summation which appear below the $\sum$, namely i, j, k, n. Another common choice for index is m.
- The lower value indicates the value of the index for the initial term of the sum and upper value indicates the value of the index for the final term (possibly infinite). The step size of the index is $+1$.

Ex. Expand the following sums.

(i) $\displaystyle\sum_{i=1}^{4}(3i-5)$

(ii) $\displaystyle\sum_{m=-1}^{2}(3m+1)$

Remark. The last example illustrates that the same sum can be represented in many different summation expressions.

Note. It is also possible to consider multiplying the terms of the sequence together to form a product. The capital Greek letter pi, $\prod$, indications is used for products. As an example,

$$\prod_{j=1}^{4} j! = 1! \cdot 2! \cdot 3! \cdot 4! = 1 \cdot 2 \cdot 6 \cdot 24 = 288.$$

13e. Formulas for arithmetic and geometric summations

We are interested in summation formulas for three types of summations.

- sum of a finite arithmetic sequence
- sum of a finite geometric sequence
- sum of an infinite geometric sequence (with certain restrictions)

<u>Sum of a finite arithmetic sequence</u>

Let's find a formula for

$$S_n = a + (a+d) + (a+2d) + \cdots + (a+(n-1)d) = \sum_{j=1}^{n}[a+(j-1)d].$$

Ex. (Guass' example) Compute

$$\sum_{i=1}^{100} i$$

<u>Sum of a finite geometric sequence</u>

Let's find a formula for

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1} = \sum_{j=1}^{n} ar^{j-1}.$$

Ex. Compute (*Hint:* Be careful here!)

$$\sum_{i=0}^{6} 9\left(\frac{2}{5}\right)^{i-1}$$

Summation formulas.

$$\sum_{j=1}^{n}(a+(j-1)d) = a+(a+d)+(a+2d)+\cdots+(a+(n-1)d) = \frac{n[a+(a+(n-1)d)]}{2}$$

$$\sum_{j=1}^{n} ar^{j-1} = a+ar+ar^2+\cdots+ar^{n-1} = a\left(\frac{1-r^n}{1-r}\right)$$

$$\sum_{j=1}^{\infty} ar^{j-1} = a+ar+ar^2+\cdots = \frac{a}{1-r} \qquad \text{is } |r|<1$$

<u>Sum of a finite geometric sequence</u>

Let's find a formula for

$$\sum_{j=1}^{\infty} ar^{j-1} = a+ar+ar^2+\dots$$

Ex. Compute

$$\sum_{k=2}^{\infty} \frac{2}{3}\left(\frac{-1}{2}\right)^{k}$$

§14. Recursively defined sequences

We've studied several ways to specify sequences. Additionally, a sequence can be defined recursively.

Ways to specify a sequence

- describe the ith term of the sequence using words
- use a formula. E.g. $a_n = n^2$ for $n = 1, 2, 3, \ldots$
- by suggesting a pattern (although this can lead to confusion at times)
- give a recursive definition

Ex. List the first few terms for each of the recursively defined sequences below.

(a) $a_1 = 4$ and $a_n = 3 \cdot a_{n-1}$ for $n > 1$.

(b) $b_1 = 0$ and $b_{k+1} = 2b_k + 1$ for $k \geq 1$.

(c) (Fibonacci) $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$.

Remarks. A recursive definition for a sequence has two parts: initial conditions and a recursive formula. If we can find a formula to directly compute the nth term of the sequence, such a formula is called a *closed form formula.*

Ex. The closed form formula for the Fibonacci sequence is:

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Note. We'll study how to derive this formula later in the course.

Ex. Identify the closed for formula for the following sequences:

(a) $a_1 = 4$ and $a_n = 3 \cdot a_{n-1}$ for $n > 1$

(b) $b_1 = 0$ and $b_{k+1} = 2b_k + 1$ for $k \geq 1$

14a. Closed form formula

Many sequences naturally have recursive definitions. Computing the nth term in a recursively defined sequence requires computing all previous terms. This is very inefficient for large n.

Give any sequence, it would be ideal to be able to find a closed form formula. Then we could always compute the nth term directly. However, not all sequences have nice closed form formulas that can be derived using known methods. Recursive definitions are better than nothing!

Defn. For $n \in \mathbb{N} = \{0, 1, 2, \dots\}$, define $0! = 1$ and $n! = n \cdot (n-1) \cdot \dots \cdot 1$ for $n \geq 1$. We read $n!$ as n *factorial*.

Ex. Compute the terms of the factorial sequence. Then find a recursive definition for $n!$.

14b. Arithmetic and geometric sequences by recursion

Arithmetic and geometric sequences can be defied recursively.

Ex. Give a recursive definition of the following sequences.

(a) $34, 40, 46, 52, \dots$

(b) $2, -6, 18, -54, \dots$

Facts. (a) The arithmetic sequence

$$a, a+d, a+2d, a+3d, \ldots$$

can be defined recursively as $a_1 = a$ and $a_n = a_{n-1} + d$ for $n > 1$.

(b) The geometric sequence

$$a, ar, ar^2, ar^3, \ldots$$

can be defined recursively as $a_1 = a$ and $a_n = r \cdot a_{n-1}$ for $n > 1$.