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Logical Connectives and Compound Propositions

Logic is concerned with forms of reasoning. Since reasoning is involved in most intellectual activities, logic is relevant to a broad range of pursuits. The study of logic is essential for students of computer science. It is also very valuable for mathematics students, and others who make use of mathematical proofs, for instance, linguistics students. In the process of reasoning one makes inferences. In an inference one uses a collection of statements, the premises, in order to justify another statement, the conclusion. The most reliable types of inferences are deductive inferences, in which the conclusion must be true if the premises are. Recall elementary geometry: Assuming that the postulates are true, we prove that other statements, such as the Pythagorean Theorem, must also be true. Geometric proofs, and other mathematical proofs, typically use many deductive inferences. (Robert L. Causey)[1]

[1] www.cs.utexas.edu/~rlc/whylog.htm

1.1 Propositions

The basic objects in logic are **propositions**. A proposition is a statement which is either true (T) or false (F) but not both. For example in the language of mathematics $p : 3 + 3 = 6$ is a true proposition while $q : 2 + 3 = 6$ is a false proposition. *What do you want for lunch?* is a question, not a proposition. Likewise *Get lost!* is a command, not a proposition. The sentence *There are exactly* $10^{87} + 3$ *stars in the universe* is a proposition, despite the fact that no one knows its truth value. Here are two, more subtle, examples:

(1) *He is more than three feet tall* is not a proposition since, until we are told to whom *he* refers, the statement cannot be assigned a truth value. The mathematical sentence $x + 3 = 7$ is not a proposition for the same reason. In general, sentences containing variables are not propositions unless some information is supplied about the variables. More about that later however.

(2) *This sentence is false* is not a proposition. It seems to be both true and false. In fact if is T then it says it is F and if it is F then it says it is T. It can be dangerous using sentences that refer to themselves. If course, using a knife can also be dangerous, but we do use knives safely when we are careful. Likewise, using self-referential sentences can be done safely if care is taken.

Sometimes a little common sense is required. For example *It is raining* is a proposition, but its truth value is not constant, and may be arguable. That is, someone might say *It is not raining, it is just drizzling*, or *Do you mean on Venus?* Feel free to ignore this sort of quibbling.

Simple propositions, such as *It is raining*, and *The streets are wet*, can be combined to create more complicated propositions such as *It is raining and the streets are not wet*. These sorts of involved propositions are called **compound propositions**. Compound propositions are built up from simple propositions using a number of **connectives** to join or modify the simple propositions. In the last example, the connectives are **and** which joins the two clauses, and **not**, which modifies the second clause.

It is important to keep in mind that since a compound proposition is, after all, a proposition, it must be classifiable as either true or false. That is, it must be possible to assign a truth value to any compound proposition. There are mutually agreed upon rules to allow the determination of exactly when a compound proposition is true and when it is false. Luckily these rules jive nicely with common sense (with one small exception), so they are easy to remember and understand.

1.2 *Negation: **not***

The simplest logical connective is **negation**. In normal English sentences, this connective is indicated by appropriately inserting *not* in the statement, by preceding the statement with *it is not the case that*, or for mathematical statements, by using a slanted slash. For example, if p is the proposition $2 + 3 = 4$, then the negation of p is denoted by the symbol $\neg p$ and it is the proposition $2 + 3 \neq 4$. In this case, p is false and $\neg p$ is true. If p is *It is raining*, then $\neg p$ is *It is not raining* or even the stilted sounding *It is not the case that it is raining*. The negation of a proposition p is the proposition whose truth value is the opposite of p in all cases. The behavior of $\neg p$ can be exhibited in a **truth table**. In each row of the truth table 1.1 we list a possible truth value of p and the corresponding truth value of $\neg p$.

p	$\sim$	p
T	F	T
F	T	F

Table 1.1: Logical Negation

1.3 *Conjunction:* ***and***

The connective that corresponds to the word *and* is called **conjunction**. The conjunction of p with q is denoted by $p \wedge q$ and read as *p and q*. The conjunction of p with q is declared to be true exactly when both of p, q are true. It is false otherwise. This behavior is exhibited in the truth table 1.2.

p	q	p	$\wedge$	q
T	T	T	T	T
T	F	T	F	F
F	T	F	F	T
F	F	F	F	F

Table 1.2: Logical Conjunction

Four rows are required in this table since when p is true, q may be either true or false and when p is false it is possible for q to be either true or false. Since a truth value must be assigned to $p \wedge q$ in every possible case, one row in the truth table is needed for each of the four possibilities.

1.4 *Disjunction:* ***or***

The logical connective **disjunction** corresponds to the word *or* of ordinary language. The disjunction of p with q is denoted by $p \vee q$, and read as *p or q*. The disjunction $p \vee q$ is true if at least one of p, q is true.

Disjunction is also called **inclusive-or**, since it includes the possibility that both component statements are true. In everyday language, there is a second use of *or* with a different meaning. For example, in the proposition *Your ticket wins a prize if its serial number contains a* 3 *or a* 5, the *or* would normally be interpreted in the inclusive sense (tickets that have both a 3 and 5 are still winners), but in the proposition *With dinner you get mashed potatoes or french fries*, the *or* is being used in the **exclusive-or** sense.

The rarely used (at least in mathematics)[2] exclusive-or is also

[2] In a mathematical setting, always assume the inclusive-or is intended unless the exclusive sense is explicitly indicated.

called the **disjoint disjunction** of p with q and is denoted by $p \oplus q$. Read that as *p xor q* if it is necessary to say it in words. The value of $p \oplus q$ is true if exactly one of p, q is true. The exclusion of both being true is the difference between inclusive-or and exclusive-or. The truth table shown officially defines these two connectives.

p q	p ∨ q	p ⊕ q
T T	T T T	T F T
T F	T T F	T T F
F T	F T T	F T T
F F	F F F	F F F

Table 1.3: Logical or and xor

1.5 *Logical Implication and Biconditional*

The next two logical connectives correspond to the ordinary language phrases *If* $\cdots$, *then* $\cdots$ and the (rarely used in real life but common in mathematics) $\cdots$ *if and only if* $\cdots$.

1.5.1 *Implication:* ***If . . . , then . . .***

In mathematical discussions, ordinary English words are used in ways that usually correspond to the way we use words in normal conversation. The connectives **not**, **and**, **or** mean pretty much what would be expected. But the **implication**, denoted $p \to q$ and read as *If p, then q* can be a little mysterious at first. This is partly because when the *If p, then q* construction is used in everyday speech, there is an implied connection between the proposition p (called the **hypothesis**) and the proposition q (called the **conclusion**). For example, in the statement *If I study, then I will pass the test*, there is an assumed connection between studying and passing the test. However, in logic, the connective is going to be used to join any two propositions, with no relation necessary between the hypothesis and conclusion. What truth value should be assigned to such bizarre sentences as *If I study, then the moon is* $238,000$ *miles from earth*?

Is it true or false? Or maybe it is neither one? Well, that last option isn't too pleasant because that sentence is supposed to be a proposition, and to be a proposition it has to have truth value either T or F. So it is going to have to be classified as one or the other. In everyday conversation, the choice isn't likely to be too important whether it is classified it as either true or false in the case described. But an important part of mathematics is knowing when propositions are true and when they are false. The official choices are given in the truth table for $p \to q$. We can make sense of this with an example.

p	q	p	$\to$	q
T	T	T	T	T
T	F	T	F	F
F	T	F	T	T
F	F	F	T	F

Table 1.4: Logical Implication

Example 1.1. *First consider the statement which Bill's dad makes to Bill: If you get an A in math, then I will buy you a new car. If Bill gets an A and his dad buys him a car, then dad's statement is true, and everyone is happy (that is the first row in the table). In the second row, Bill gets an A, and his dad doesn't come through. Then Bill's going to be rightfully upset since his father lied to him (dad made a false statement). In the last row of the table he can't complain if he doesn't get an A, and his dad doesn't buy him the car (so again dad made a true statement). Most people feel comfortable with those three rows. In the third row of the table, Bill doesn't get an A, and his dad buys him a car anyhow. This is the funny case. It seems that calling dad a liar in this case would be a little harsh on the old man. So it is declared that dad told the truth. Remember it this way: an implication is true unless the hypothesis is true and the conclusion is false.*

1.5.2 *Biconditional:* ***...if and only if ...***

The **biconditional** is the logical connective corresponding to the phrase $\cdots$ *if and only if* $\cdots$. It is denoted by $p \longleftrightarrow q$, (read *p if and only if q*), and often more tersely written as *p iff q*. The biconditional is true when the two component propositions have the same truth value, and it is false when their truth values are different. Examine the truth table to see how this works.

p	q	p	$\longleftrightarrow$	q
T	T	T	T	T
T	F	T	F	F
F	T	F	F	T
F	F	F	T	F

Table 1.5: Logical biconditional

1.6 *Truth table construction*

The connectives described above combine at most two simple propositions. More complicated propositions can be formed by joining compound propositions with those connectives. For example, $p \wedge (\neg q)$, $(p \vee q) \to (q \wedge (\neg r))$, and $(p \to q) \longleftrightarrow ((\neg p) \vee q)$ are compound propositions, where parentheses have been used, just as in ordinary algebra, to avoid ambiguity. Such extended compound propositions really are propositions. That is, if the truth value of each component is known, it is possible to determine the truth value of the entire proposition. The necessary computations can be exhibited in a truth table.

Example 1.2. *Suppose that p, q and r are propositions. To construct a truth table for $(p \wedge q) \rightarrow r$, first notice that eight rows will be needed in the table to account for all the possible combinations of truth values of the simple component statements p, q and r. This is so since there are, as noted above, four rows needed to account for the choices for p and q, so there will be those four rows paired with r having truth value T, and four more with r having truth value F, for a total of $4 + 4 = 8$. In general, if there are n simple propositions in a compound statement, the truth table for the compound statement will have 2^n rows. Here is the truth table for $(p \wedge q) \rightarrow r$, with an auxiliary column for $p \wedge q$ to serve as an aid for filling in the last column.*

p	q	r	(p	∧	q)	→	r
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	F
T	F	T	T	F	F	T	T
T	F	F	T	F	F	T	F
F	T	T	F	F	T	T	T
F	T	F	F	F	T	T	F
F	F	T	F	F	F	T	T
F	F	F	F	F	F	T	F

Table 1.6: Truth table for $(p \wedge q) \rightarrow r$

Be careful about how propositions are grouped. For example, if truth tables for $p \wedge (q \rightarrow r)$ and $(p \wedge q) \rightarrow r$ are constructed, they turn out not to be the same in every row. Specifically if p is false, then $p \wedge q$ is false, and $(p \wedge q) \rightarrow r$ is true. Whereas when p is false $p \wedge (q \rightarrow r)$ is false. So writing $p \wedge q \rightarrow r$ is ambiguous.

1.7 Translating to propositional forms

Here are a few examples of translating between propositions expressed in ordinary language and propositions expressed in the language of logic.

Example 1.3. *Let c be the proposition It is cold and s : It is snowing, and h : I'm staying home. Then $(c \wedge s) \rightarrow h$ is the proposition If it is cold and snowing, then I'm staying home. While $(c \vee s) \rightarrow h$ is If it is either cold or snowing, then I'm staying home. Messier is $\neg(h \rightarrow c)$ which could be expressed as It is not the case that if I stay home, then it is cold, which is a little too convoluted for our minds to grasp quickly. Translating in the other direction, the proposition It is snowing and it is either cold or I'm staying home would be symbolized as $s \wedge (c \vee h)$.* [3]

[3] Notice the parentheses are needed in this last proposition since $(s \wedge c) \vee h$ does not capture the meaning of the ordinary language sentence, and $s \wedge c \vee h$ is ambiguous.

1.8 Bit strings

There is a connection between logical connectives and certain operations on bit strings. There are two **binary digits** (or **bits**): 0 and

1. A **bit string of length** n is any sequence of n bits. For example, 0010 is a bit sting of length four. Computers use bit strings to encode and manipulate information. Some bit string operations are really just disguised truth tables. Here is the connection: Since a bit can be one of two values, bits can be used to represent truth values. Let T correspond to 1, and F to 0. Then given two bits, logical connectives can be used to produce a new bit. For example $\neg 1 = 0$, and $1 \vee 1 = 1$. This can be extended to strings of bits of the same length by combining corresponding bit in the two strings. For example, $01011 \wedge 11010 = (0 \wedge 1)(1 \wedge 1)(0 \wedge 0)(1 \wedge 1)(1 \wedge 0) = 01010$.

1.9 Exercises

Exercise 1.1. *Determine which of the following sentences are propositions. Assume you are speaking the sentence.*

a) There are seven days in a week.

b) Get lost!

c) Pistachio is the best ice cream flavor.

d) If $x = 2$*, then* $x^2 - 2x + 1 = 0$.

e) If $x > 1$*, then* $x^2 + 2x + 1 > 5$.

f) All unicorns have four legs.

Exercise 1.2. *Construct truth tables for each of the following.*

a) $p \oplus \neg q$

b) $\neg(q \to p)$

c) $q \wedge \neg p$

d) $\neg q \vee p$

e) $p \to (\neg q \wedge r)$

Exercise 1.3. *Perform the indicated bit string operations. The bit strings are given in groups of four bits each for ease of reading.*

a) $(1101\ 0111 \oplus 1110\ 0010) \wedge 1100\ 1000$

b) $(1111\ 1010 \wedge 0111\ 0010) \vee 0101\ 0001$

c) $(1001\ 0010 \vee 0101\ 1101) \wedge (0110\ 0010 \vee 0111\ 0101)$

Exercise 1.4. *Let* s *be the proposition It is snowing and* f *be the proposition It is below freezing. Convert the following English sentences into statements using the symbols* s*,* f *and logical connectives.*

a) It is snowing and it is not below freezing.

b) It is below freezing and it is not snowing.

c) If it is not snowing, then it is not below freezing.

Exercise 1.5. *Let j be the proposition Jordan played and w be the proposition The Wizards won. Write the following propositions as English sentences.*

a) $\neg j \wedge w$ *b)* $j \to \neg w$ *c)* $w \vee j$

d) $w \to \neg j$

Exercise 1.6. *Let c be the proposition Sam plays chess, let b be Sam has the black pieces, and let w be Sam wins.*

a) Translate into English: $(c \wedge \neg b) \to w$.

b) Translate into symbols: If Sam didn't win his chess game, then he played black.

1.10 Problems

Problem 1.1. *Determine which of the following sentences are propositions*

Assume you are speaking the sentence.

a) Today is Tuesday.

b) Why are you whining?

c) The Vikings are the worst team in professional sports.

d) This sentence has five words.

e) There is a black hole at the center of every galaxy.

Problem 1.2. *Construct truth tables for each of the following.*

a) $\neg q \longrightarrow \neg p$.

b) $p \longrightarrow (q \wedge r)$.
(You will need eight rows for this one.)

Problem 1.3. *Perform the indicated bit string operations. The bit strings are given in groups of four bits each for ease of reading.*

a) $(1001\ 0101 \oplus 1010\ 0110) \wedge 1100\ 1000$

b) $(1110\ 1010 \wedge 0101\ 0010) \vee 0111\ 1001$

c) $(1111\ 0011 \vee 0111\ 0101) \wedge (0010\ 0010 \vee 0110\ 0100)$

Problem 1.4. *Let s be the proposition It is snowing and f be the proposition It is below freezing. Convert the following English sentences into statements using the symbols s, f and logical connectives.*

a) It is snowing and it is below freezing.

b) If it is snowing, then it is below freezing.

Problem 1.5. *Like any natural language, spoken English can be ambiguous. It is sometimes necessary to rely on context, voice inflection or other clues, to correctly interpret a sentence. Propositional logic, correctly written, is never ambiguous. Express the following two sentence pairs in symbolic form that correctly conveys the intended meaning.*

English ambiguity:

(1) A pedestrian is hit by a New York taxi every three minutes. He is getting sick and tired of it.

(2) One morning I shot an elephant in my pajamas. How he got in my pajamas, I don't know.

(Groucho Marx as Captain Spaulding)

a) *Dinner comes with peas and carrots or french fries.*
Intended: You get either the peas/carrots combination, or else you get french fries.
Intended: You get peas together with your choice of one of carrots or french fries.

b) *A similar scenario with algebraic operations: Two plus three times four.*
Intended: add two and three, and multiply the total by four.
Intended: add two to the product of three and four.

2

Logical Equivalence

IT IS CLEAR THAT the propositions *It is sunny and it is warm* and *It is warm and it is sunny* mean the same thing. More generally, for any propositions p, q, we see that $p \wedge q$ and $q \wedge p$ have the same meaning. To say it a little differently, for any choice of truth values for p and q, the propositions $p \wedge q$ and $q \wedge p$ have the same truth value. One more time: $p \wedge q$ and $q \wedge p$ have identical truth tables.

2.1 *Logical Equvalence*

Two propositions with identical truth values are called **logically equivalent**. The expression $p \equiv q$ means p, q are logically equivalent.

Some logical equivalences are not as transparent as the example above. With a little thought it should be clear that *I am not taking math or I am not taking physics* means the same as *It's not the case that I taking math and physics*. In symbols, $(\neg m) \vee (\neg p)$ means the same as $\neg(m \wedge p)$.

To be convinced these two proposition really have the same content, look at the truth table, (2.1), for the two propositions, and notice that the final truth values are identical.

Example 2.1 (De Morgan)**.** *Prove that* $\neg(p \wedge q) \equiv (\neg p \vee \neg q)$ *using a truth table. We construct the truth table 2.1 in the order or precedence:* $\neg$ *before* $\wedge$ *or* $\vee$*, but the expresion in parentheses has highest precedence.*

p	q	¬	(p	∧	q)	¬	p	∨	¬	q
T	T	F	T	T	T	F	T	F	F	T
T	F	T	T	F	F	F	T	T	T	F
F	T	T	F	F	T	T	F	T	F	T
F	F	T	F	F	F	T	F	T	T	F

Table 2.1: De Morgan's Law

It is probably a little harder to believe $(p \to q) \equiv (\neg p \vee q)$, but checking a truth table shows they are in fact equivalent. Saying *If it is Monday, then I am tired* is identical to saying *It isn't Monday or I am tired.* Complete table 2.2 to demonstrate their equivalence.

p	q	p	→	q	∼	p	∨	q
T	T	T		T		T		T
T	F	T		F		T		F
F	T	F		T		F		T
F	F	F		F		F		F

Table 2.2: Prove $p \to q \equiv \neg p \vee q$

2.2 *Tautologies and Contradictions*

A proposition , $\mathbb{T}$, which is always true is called a **tautology**. A **contradiction** is a proposition, $\mathbb{F}$, which is always false. The prototype example of a tautology is $p \vee \neg p$, and for a contradiction, $p \wedge \neg p$. Notice that since $p \longleftrightarrow q$ is T exactly when p and q have the same truth value, two propositions p and q will be logically equivalent provided $p \longleftrightarrow q$ is a tautology.

2.3 *Related **If . . . , then . . .** propositions*

There are three propositions related to the basic **If . . . , then . . .** implication: $p \rightarrow q$. First $\neg q \rightarrow \neg p$ is called the **contrapositive** of the implication. The **converse** of the implication is the proposition $q \rightarrow p$. Finally, the **inverse** of the implication is $\neg p \rightarrow \neg q$. Using a truth table, it is easy to check that an implication and its contrapositive are logically equivalent, as are the converse and the inverse. A common slip is to think the implication and its converse are logically equivalent. Checking a truth table shows that isn't so. The implication *If an integer ends with a* 2*, then it is even* is T, but its converse, *If an integer is even, then it ends with a* 2, is certainly F.

2.4 *Fundamental equivalences*

Table 2.3 contains the most often used equivalences. These are well worth learning by sight and by name.

2.5 *Disjunctive normal form*

Five basic connectives have been given: $\neg, \wedge, \vee, \rightarrow, \longleftrightarrow$, but that is really just for convenience. It is possible to eliminate some of them using logical equivalences. For example, $p \longleftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$ so there really is no need to explicitly use the biconditional. Likewise, $p \rightarrow q \equiv \neg p \vee q$, so the use of the implication can also be avoided. Finally, $p \wedge q \equiv \neg(\neg p \vee \neg q)$ so that there really is no need ever to use the connective $\wedge$. Every proposition made up of the five

Table 2.3: Logical Equivalences

Equivalence	Name
$\neg(\neg p) \equiv p$	Double Negation
$p \wedge \mathbb{T} \equiv p$ $p \vee \mathbb{F} \equiv p$	Identity laws
$p \vee \mathbb{T} \equiv \mathbb{T}$ $p \wedge \mathbb{F} \equiv \mathbb{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv (\neg p \vee \neg q)$ $\neg(p \vee q) \equiv (\neg p \wedge \neg q)$	De Morgan's laws
$p \vee \neg p \equiv \mathbb{T}$ $p \wedge \neg p \equiv \mathbb{F}$	Law of Excluded Middle Law of Contradiction
$p \to q \equiv \neg p \vee q$	Disjunctive form
$p \to q \equiv \neg q \to \neg p$ $\neg p \to \neg q \equiv q \to p$	Implication $\equiv$ Contrapositive Inverse $\equiv$ Converse

basic connectives can be rewritten using only $\neg$ and $\vee$ (probably with a great loss of clarity however).

The most often used standardization, or normalization, of logical propositions is the **disjunctive normal form (DNF)**, using only $\neg$ (negation), $\wedge$ (conjunction), and $\vee$ (disjunction). A propositional form is considered to be in DNF if and only if it is a disjunction of one or more conjunctions of one or more *literals* (a *literal* is a letter or a letter preceded by the negation symbol). For example, the following are all in disjunctive normal form:

- $p \wedge q$
- p
- $(a \wedge q) \vee r$
- $(p \wedge \neg q \wedge \neg r) \vee (\neg s \wedge t \wedge u)$

While, these are **not** in DNF: [1]

- $\neg(p \vee q)$ *this is **not** the disjunction of literals.*

[1] Use the fundamental equivalences to find DNF versions of each.

- $p \wedge (q \wedge (r \vee s))$ an or is embedded in a conjunction.

2.6 *Proving equivalences*

It is always possible the verify a logical equivalence via a truth table. But it also possible to verify equivalences by stringing together previously known equivalences. Here are two examples of this process.

Example 2.2. *Show* $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$. [2]

[2] The plan is to start with the expression $\neg(p \vee (\neg p \wedge q))$, work through a sequence of equivalences ending up with $\neg p \wedge \neg q$. It's pretty much like proving identities in algebra or trigonometry.

Proof.

$$
\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \textit{De Morgan's Law} \\
&\equiv \neg p \wedge (\neg(\neg p) \vee \neg q) && \textit{De Morgan's Law} \\
&\equiv \neg p \wedge (p \vee \neg q) && \textit{Double Negation Law} \\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \textit{Distributive Law} \\
&\equiv (p \wedge \neg p) \vee (\neg p \wedge \neg q) && \textit{Commutative Law} \\
&\equiv \mathbb{F} \vee (\neg p \wedge \neg q) && \textit{Law of Contradiction} \\
&\equiv (\neg p \wedge \neg q) \vee \mathbb{F} && \textit{Commutative Law} \\
&\equiv \neg p \wedge \neg q && \textit{Identity Law}
\end{aligned}
$$

♣

Example 2.3. *Show* $(p \wedge q) \to (p \vee q) \equiv \mathbb{T}$.

Proof.

$$
\begin{aligned}
(p \wedge q) \to (p \vee q) &\equiv \neg(p \wedge q) \vee (p \vee q) && \textit{Disjunctive form} \\
&\equiv (\neg p \vee \neg q) \vee (p \vee q) && \textit{De Morgan's Law} \\
&\equiv (p \vee \neg p) \vee (q \vee \neg q) && \textit{Associative and Commutative Laws} \\
&\equiv \mathbb{T} \vee \mathbb{T} && \textit{Commutative Law and Excluded Middle} \\
&\equiv \mathbb{T} && \textit{Domination Law}
\end{aligned}
$$

♣

2.7 Exercises

Exercise 2.1. *Use truth tables to verify each of the following equivalences:*

a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$ *b)* $p \rightarrow q \equiv \neg q \rightarrow \neg p$

c) $\neg p \wedge (p \vee q) \equiv \neg(q \rightarrow p)$ *d)* $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

Exercise 2.2. *Show that the statements are not logically equivalent.*

a) $p \rightarrow (q \rightarrow r) \not\equiv \neg o \rightarrow \neg q$

b) $p \rightarrow q \not\equiv \neg p \rightarrow \neg q$

Exercise 2.3. *Use truth tables to show that the following are tautologies.*

a) $[p \wedge (p \rightarrow q)] \rightarrow q$

b) $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$

c) $[(p \vee q) \rightarrow r] \rightarrow [(p \rightarrow r) \wedge (q \rightarrow r)]$

Exercise 2.4. *Consider the implication If it is Saturday, then I will mow the lawn.*

a) *Write the converse of the implication.*

b) *Write the inverse of the implication.*

c) *Write the contrapositive of the implication.*

Exercise 2.5. *Consider the three answers for exercise 2.4. One of the three is logically equivalent to the implication. Which one? Two of the three are not logically equivalent to the implication, but are logically equivalent to each other. Which two?*

Exercise 2.6. *The statements below are not tautologies. In each case, find an assignment of truth values to the literals, (that is, a letter or a letter preceded by the negation symbol), so the statement is false.*

a) $[(p \wedge q) \rightarrow r] \longleftrightarrow [(p \rightarrow r) \wedge (q \rightarrow r)]$

b) $[(p \wedge q) \vee r] \rightarrow [p \wedge (q \vee r)]$

Exercise 2.7. *Give proofs of the following equivalences using the Fundamental Logical Equivalences, following the pattern of examples 2.2 and 2.3.*

a) $(\neg p \wedge (p \vee q)) \to q \equiv \mathbb{T}$.

b) $(p \wedge \neg r) \to \neg q \equiv p \to (q \to r)$.

c) $p \vee (p \wedge q) \equiv p$. *(This is a tough one.)*

2.8 Problems

Problem 2.1. *Use a truth table to show* $p \to q \equiv \neg p \vee q$.

Problem 2.2. *Use a truth table to show* $(p \wedge q) \to r \equiv p \to (q \to r)$.

Problem 2.3. *Use a truth table to show* $p \to q \equiv \neg q \to \neg p$.

Problem 2.4. *Consider the implication If I work, then I get paid.*

a) Write the converse of the implication.

b) Write the inverse of the implication.

c) Write the contrapositive of the implication.

d) Write the inverse of the contrapositive of the implication.

Problem 2.5. *Consider the four answers for exercise 2.4. Which of the are logically equivalent to the implication? Which of the four are not logically equivalent to the implication, (but are logically equivalent to each other)?*

Problem 2.6. *Use a truth table to show* $[p \to (q \to r)] \not\equiv [(p \to q) \to r]$.

Problem 2.7. *Use a truth table to show that* $(p \wedge q) \to p$ *is a tautology.*

Problem 2.8. *Give a proof of* $\neg p \to (p \to q) \equiv \mathbb{T}$ *using the Fundamental Logical Equivalences, following the pattern of examples 2.2 and 2.3.*