Math 208: Discrete Mathematics
Lesson 6: Lecture Video Notes

Topics

11. Functions and their properties
    (a) definition of function
    (b) functions with discrete domain and codomain
    (c) special properties: one-to-one, onto, bijective
    (d) composition of functions
    (e) invertible discrete functions
    (f) characteristic functions
12. Special functions
    (a) floor and ceiling functions
    (b) fractional and integral parts
    (c) power functions
    (d) exponential functions
    (e) logarithmic functions and laws of logarithms

Readings: Chapters 11-12

§11. Functions and their properties

11a. Definition of function

Defn. (Informal) A *function* f from the set A to the set B is any rule which assigns exactly one element of B to each element of A. The set A is called the *domain* of f. The set B is called the *codomain* of f.

Notation. We write $f : A \to B$ to indicate f is a function from A to B. The element of B which is assigned to a given $x \in A$ is called the *value of x under f* and denoted as $f(x)$.

Ex. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{a, b, c\}$. Specify $f : A \to B$ via the table

x	$f(x)$
1	c
2	b
3	c
4	a
5	b

Alternatively, the table can be written as the set of ordered pairs

$$f = \{(1, c), (2, b), (3, c), (4, a), (5, b)\}.$$

Observation. Every function $f : A \to B$ is a relation from A to B. However, not all relations are functions.

Defn. (Formal) A function $f : A \to B$ is a relation from A to B (so $f \subseteq A \times B$) such that each element of A is the first coordinate of exactly one ordered pair of f.

11b. Functions with discrete domain and codomain

Functions are often classified as continuous or discrete.

Continuous functions are functions where the domain and codomain are usually $\mathbb{R}$, $\mathbb{C}$, or connected subsets or subinvervals. These type of functions are studied in introductory algebra, calculus, and analysis. The properties of interest include continuity, differentiability, optimization, etc.

In discrete mathematics, the focus is more on functions on a discrete set. Examples of discrete sets include any finite set, $\mathbb{N}$, $\mathbb{Z}$, or any subset of $\mathbb{Z}$. Usually $\mathbb{Q}$ is not considered a discrete set. The formal definition of a discrete set comes from topology, an area of math which looks at collection of subsets and their connections to the shape of spaces.

Let's make some observations about the 0-1 matrix and bipartite graph of functions with discrete domain and codomains. After all, a function is just a certain type of relation.

Ex. Consider the function $f : \{1, 2, 3, 4, 5\} \to \{a, b, c\}$ given by $f = \{(1, c), (2, b), (3, c), (4, a), (5, b)\}$. Then the 0-1 matrix M_f and bipartite graph are:

Fact 1. Let $f : A \to B$ be a function with discrete domain A and codomain B. Then the 0-1 matrix M_f of f has exactly one 1 in each row of M_f. Conversely, if the 0-1 matrix M_f of a relation f from A to B has exactly one 1 in each row, then f is a function from A to B.

Fact 2. Let $f : A \to B$ be a function with discrete domain A and codomain B. Then the bipartite graph of f has exactly one edge for each vertex of A. Conversely, if the bipartite graph of the relation f from A to B has exactly one edge for each vertex of A, then f is a function from A to B.

Remark. These two facts are the discrete version of the vertical line test from introductory algebra.

11c. Special properties: one-to-one, onto, bijective

We next consider three special properties of functions: one-to-one, onto, bijective.

<u>One-to-one (injective)</u>:

Defn. A function $f : A \to B$ is *one-to-one* or *injective* provided $f(s) = f(t)$ implies $s = t$. Equivalently, a function $f : A \to B$ is *one-to-one* or *injective* provided $s \neq t$ implies $f(s) \neq f(t)$.

Other characterizations a function $f : A \to B$ having the one-to-one property

- The 0-1 matrix M_f of f has at most one 1 in each column.
- The bipartite graph of f has at most one edge for each vertex of B.
- For continuous functions on $\mathbb{R}$, the graph of f passes the horizontal line test.

Remark. If $|B| < |A|$, then $f : A \to B$ is not one-to-one. Equivalently, if $f : A \to B$ is one-to-one, then $|B| \geq |A|$.

Ex. Consider the function $f : \{1, 2, 3, 4, 5\} \to \{a, b, c\}$ given by $f = \{(1, c), (2, b), (3, c), (4, a), (5, b)\}$. Is f one-to-one?

Ex. Consider the function $g : \{a, b, c\} \to \{1, 2, 3, 4\}$ given by $g = \{(a, 4), (b, 2), (c, 1)\}$. Is g injective?

Onto (surjective):

Defn. A function $f : A \to B$ is *onto* or *surjective* provided every $b \in B$ equals $f(a)$ for some $a \in A$.

Other characterizations a function $f : A \to B$ having the onto property

- The 0-1 matrix M_f of f has at least one 1 in each column.
- The bipartite graph of $f : A \to B$ has at least one edge for each vertex of B.

Remark. If $|B| > |A|$, then $f : A \to B$ is not onto. Equivalently, if $f : A \to B$ is onto, then $|B| \leq |A|$.

Ex. Consider the function $f : \{1, 2, 3, 4, 5\} \to \{a, b, c\}$ given by $f = \{(1, c), (2, b), (3, c), (4, 1), (5, b)\}$. Is f onto?

Ex. Consider the function $g : \{a, b, c\} \to \{1, 2, 3, 4\}$ given by $g = \{(a, 4), (b, 2), (c, 1)\}$. Is g surjective?

Bijective:

Defn. A function $f : A \to B$ is *bijective* or an *one-to-one correspondence* provided it is both one-to-one and onto (i.e. injective and surjective).

Other characterizations a function $f : A \to B$ having the bijective property

- The 0-1 matrix M_f of f has exactly one 1 in each column.
- The bipartite graph of $f : A \to B$ has exactly one edge for each vertex of B.

Remark. Suppose A and B are finite sets. Then, $f : A \to B$ is bijective implies $|A| = |B|$. That is, A and B have the same cardinality, i.e. number of elements.

Defn. Two finite sets A and B with the same cardinality are said to be *equinumerous* and we write $|A| = |B|$. More generally, the sets A and B are *equinumerous* if there exists a bijection from A to B.

Ex. Let $\mathbb{E}$ be the set of even integers, so $\mathbb{E} = \{x \in \mathbb{Z} : x = 2k \text{ for some } k \in \mathbb{Z}\} = \{\dots, -2, 0, 2, \dots\}$. Show $\mathbb{E}$ and $\mathbb{Z}$ are equinumerous.

11d. Composition of functions

Since functions are relations, we can discuss the composition of functions. A priori, we only know that the composition of two functions produces a relation. In fact, the composition of functions is a function!

Theorem. If $g : A \to B$ and $f : B \to C$ are functions, then $f \circ g : A \to C$ is also a function.

Proof. It needs to be shown that for each $a \in A$ there is exactly one $c \in C$ such that $(a, c) \in f \circ g$. See textbook for details.

Ex. Let $g = \{(1, a), (2, c), (3, a), (4, c)\}$ and $f = \{(a, y), (b, w), (c, z), (d, w)\}$. Compute $f \circ g$.

Remarks. If $g : A \to B$ and $f : B \to C$, and $(a, b) \in g$ and $(b, c) \in f$, then $(a, c) \in f \circ g$. Sometimes this is written as $b = g(a)$ and $c = f(b)$, so $c = f(b) = f(g(a))$. This leads to the general notation $(f \circ g)(x) = f(g(x))$ found in introductory algebra and calculus textbooks.

11e. Invertible discrete functions

Since a function $f : A \to B$ is a relation, we can form a relation f^{-1} from B to A.

Q. If $f : A \to B$ is a function, when is $f^{-1} : B \to A$ also a function?

Ex. Let $f : \{1, 2, 3\} \to \{a, b\}$ be defined by $f = \{(1, a), (2, b), (3, b)\}$. Compute f^{-1}. Is f^{-1} a function?

Defn. Let $f : A \to B$ be a function. If $f^{-1} : B \to A$ is a function, we say f is *invertible.*

Alternative characterization of invertibility: Let $1_S : S \to S$ be a function defined by $1_S(x) = x$ for all $x \in S$. We say 1_S is the identity function on the set S. Then the function $f : A \to B$ is invertible iff $f \circ f^{-1} = 1_B$ and $f^{-1} \circ f = 1_A$.

Theorem. A function $f : A \to B$ is invertible if and only if f is bijective.

Proof. See textbook.

Ex. Consider $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = n + 2$. Is f invertible? If so, find f^{-1}.

Caution. The inverse and reciprocal of functions are generally different. For example, the inverse of $f(n) = n + 2$ is $f^{-1}(n) = n - 2$. This is different than the reciprocal of $f(n) = n + 2$ which is

$$\frac{1}{f(n)} = \frac{1}{n+2}.$$

11f. Characteristic functions

There is a special 0-1 function on the power set of a finite universal set $\mathcal{U}$. Let $\mathcal{U} = \{u_1, u_2, \ldots, u_n\}$ be a finite universal set with n ordered elements. Then define the *characteristic function* $\mathcal{X} : \mathcal{P}(\mathcal{U}) \to B_n$ which assigns $A \in \mathcal{P}(\mathcal{U})$ to $\mathcal{X}(A)$. Here B_n denotes the binary strings of length n.

Ex. Compute the characteristic function for the ordered set $\mathcal{U} = \{a, b, c\}$.

Remark. Characteristic functions allows any set theory problem with finite sets to be translated into 0's and 1's. This is the essence of computer science.

§12. Special functions

Next we survey some functions which appear frequently in discrete mathematics.

12a. Floor and ceiling functions

Defn. The *floor function* is a function from $\mathbb{R}$ to $\mathbb{Z}$ which assigns to each real number x the largest integer which is less than or equal to x. We denote the value of x under the floor function by $\lfloor x \rfloor$ and refer to $\lfloor x \rfloor$ as the *floor of* x.

Ex. Compute the following:

(a) $\lfloor 2.08 \rfloor$

(b) $\lfloor \pi \rfloor$

(c) $\lfloor 1883 \rfloor$

(d) $\lfloor -10 \rfloor$

(e) $\lfloor -\pi \rfloor$

Defn. The *ceiling function* is a function from $\mathbb{R}$ to $\mathbb{Z}$ which assigns to each real number x the smallest integer which is greater than or equal to x. We denote the value of x under the ceiling function by $\lceil x \rceil$ and refer to $\lceil x \rceil$ as the *ceiling of* x.

Ex. Compute the following:

(a) $\lceil \sqrt{5} \, \rceil$

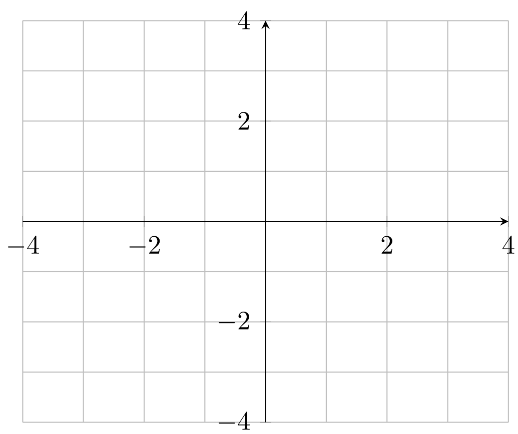
(b) $\lceil \pi \rceil$

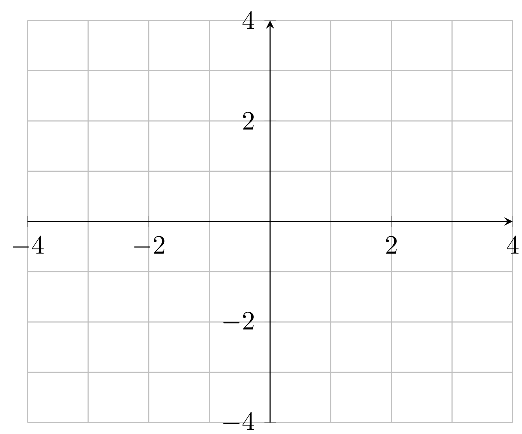
(c) $\lceil 1889 \rceil$

(d) $\lceil -2.08 \rceil$

(e) $\lceil -\pi \rceil$

The graphs of the floor and ceiling functions are stepwise functions.





12b. Fractional and integral parts

Defn. The *fractional part* of a real number x is defined as

$$\operatorname{frac}(x) = \begin{cases} x - \lfloor x \rfloor & \text{if } x \geq 0 \\ x - \lceil x \rceil & \text{if } x < 0 \end{cases}$$

Informally, $\operatorname{frac}(x)$ is the part of the decimal representation of x after the decimal point while keeping the sign of the number.

Ex. Compute the following:

(a) $\mathrm{frac}(2.08)$

(b) $\mathrm{frac}(7/3)$

(c) $\mathrm{frac}(41)$

(d) $\mathrm{frac}(-2.08)$

Remark. There are some subtleties since certain rational numbers have two decimal representations. For example, $1 = 0.9999\ldots$ and $1.25 = 1.249999\ldots$. In these examples,

$$\mathrm{frac}(0.9999\ldots) = \mathrm{frac}(1) = 0$$

and

$$\mathrm{frac}(1.249999\ldots) = \mathrm{frac}(1.25) = 0.25.$$

Defn. The *integral part* of a real number x is defined as

$$\mathrm{int}(x) = \begin{cases} \lfloor x \rfloor & \text{if } x \geq 0 \\ \lceil x \rceil & \text{if } x < 0 \end{cases}$$

Informally, $\mathrm{int}(x)$ is the part of the decimal representation of x before the decimal point while keeping the sign of the number.

Remark. Sometimes the integral part of x is written as $[x]$.

Fact. For all real numbers x, we have $x = \mathrm{int}(x) + \mathrm{frac}(x)$.

Ex. Compute the following:

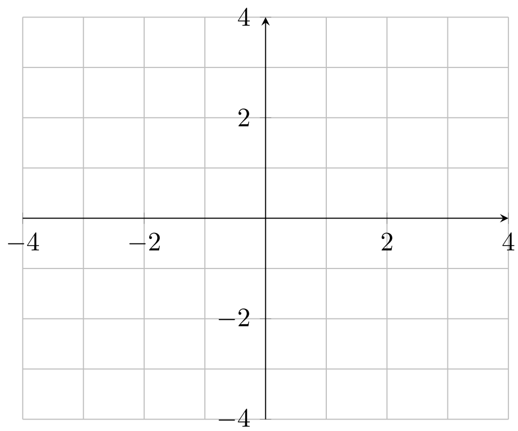
(a) $\mathrm{int}(2.08)$

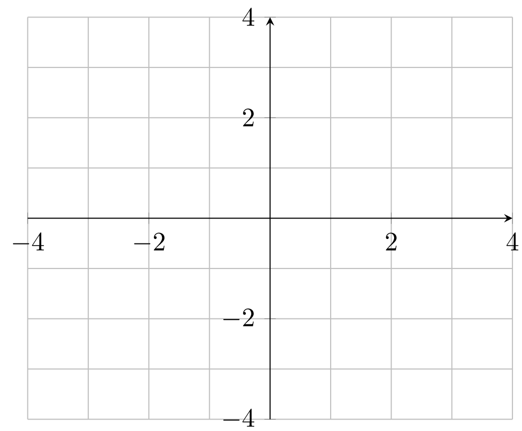
(b) $\mathrm{int}(7/3)$

(c) $\mathrm{int}(41)$

(d) $\mathrm{int}(-2.08)$

The graphs of the fractional and integral part functions are piecewise functions.





12c. Power functions

Defn. The *power functions* are functions of the form $f(x) = x^a$ for some constant $a \geq 1$.

Note. Other textbooks define power functions as functions of the form $f(x) = cx^a$ where $c \neq 0$ and $a \in \mathbb{R}$.

Ex. $f(x) = x$, $f(x) = x^2$, $f(x) = x^3$, $f(x) = x^\pi$.

Remark. The power function $f(x) = x$ is called the *identity function*.

Note. When a is irrational, the values of the power functions are defined in terms of limits from calculus. For example, consider $f(x) = x^\pi$. Then $f(2) = 2^\pi$ where

$$2^\pi := \lim_{\substack{r \to \pi \\ r \in \mathbb{Q}}} 2^r.$$

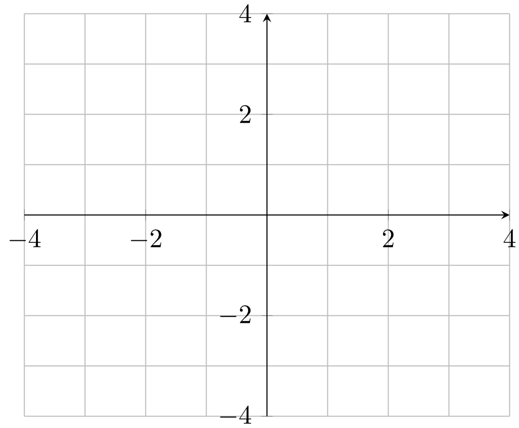
12d. Exponential functions

Interchanging the variable and constant of power functions $f(x) = x^a$ produces the exponential functions $f(x) = a^x$.

Defn. The *exponential function with base* b are functions of the form $f(x) = b^x$ for some constant $b > 0$ and $b \neq 1$.

Remark. The most common bases are $b = 2$, e, and 10. When the base is $e \approx 2.71828$, $f(x) = e^x$ is called the *natural exponential function* and e is called the *natural base*. The choice of "natural" with base e has its origins in calculus. There one sees that $f(x) = b^x = f'(x)$ iff $b = e$.

Ex. Graph $y = (0.5)^x$, $y = 2^x$, and $y = e^x$.



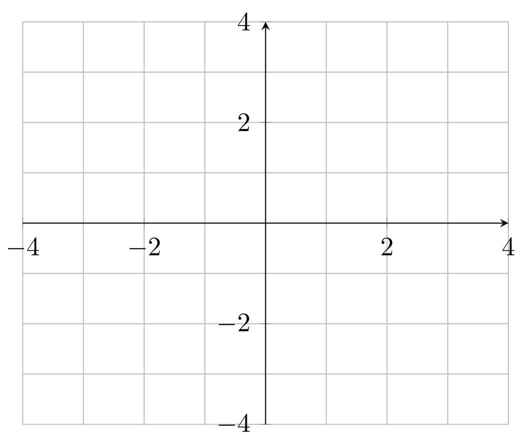
12e. Logarithmic functions and laws of logarithms

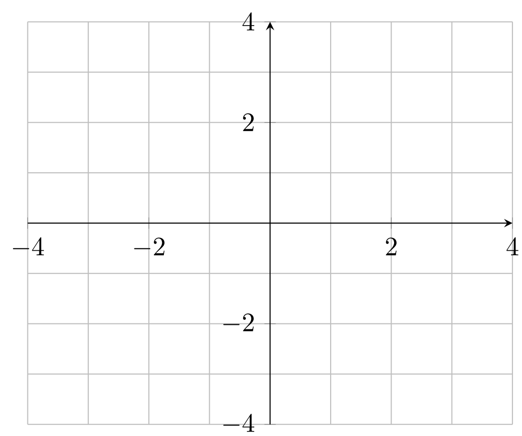
Exponential functions $f : \mathbb{R} \to (0, \infty)$ with $f(x) = b^x$ where $b > 0$ is a constant and $b \neq 1$ are bijective. Their inverse functions $f^{-1} : (0, \infty) \to \mathbb{R}$ are the logarithic functions.

Defn. The inverse function $f^{-1} : (0, \infty) \to \mathbb{R}$ of the exponential function $f(x) = b^x$ with base b is the *logarithmic function with base* b and denoted $\log_b(x)$.

Notation. If $b = 10$, we write $\log(x)$ for $\log_{10}(x)$ and refer to this as the *common logarithmic function*. If $b = e$, we write $\ln(x)$ for $\log_e(x)$ and refer to this as the *natural logarithmic function*.

Ex. Graph $f(x) = e^x$ and $f^{-1}(x) = \ln(x)$. Also graph $g(x) = (0.5)^x$ and $g^{-1}(x) = \log_{0.5}(x)$.





There are several useful (and hopefully familiar) properties of exponents and logarithms which we summarize below.

Laws of Exponents. For $a, b, c \in \mathbb{R}$, the following holds:

(i) $a^{b+c} = a^b \cdot a^c$

(ii) $a^{bc} = (a^b)^c$

(iii) $a^c \cdot b^c = (ab)^c$

Changing between exponential and logarithmic statements

$$y = b^x \quad \text{iff} \quad \log_b(y) = x$$

The law of exponents can be used to derive the laws of logarithms. See the textbook for the derivations of (i) and (iii). Note the change on conditions for a, b, and c from Law of Exponents to Laws of Logarithms.

Laws of Logarithms. For $a, b, c > 0$, the following holds:

(i) $\log_a(bc) = \log_a(b) + \log_a(c)$

(ii) $\log_a\left(\frac{b}{c}\right) = \log_a(b) - \log_a(c)$

(iii) $\log_a\left(b^c\right) = c \cdot \log_a(b)$

Note. (iii) holds for $c \in \mathbb{R}$

Change of base formula

$$\log_a(b) = \frac{\log(b)}{\log(a)} = \frac{\ln(b)}{\ln(a)}$$

Ex. Approximate $\log_2(\pi)$ using operations found on a common calculator.

Ex. Assume the y^x key on a common calculator is broken. If all the other buttons work, how can we approximate 2^π?