Jusuk Lee
Department of Mechanical and Aerospace Engineering
Seoul National University, Seoul

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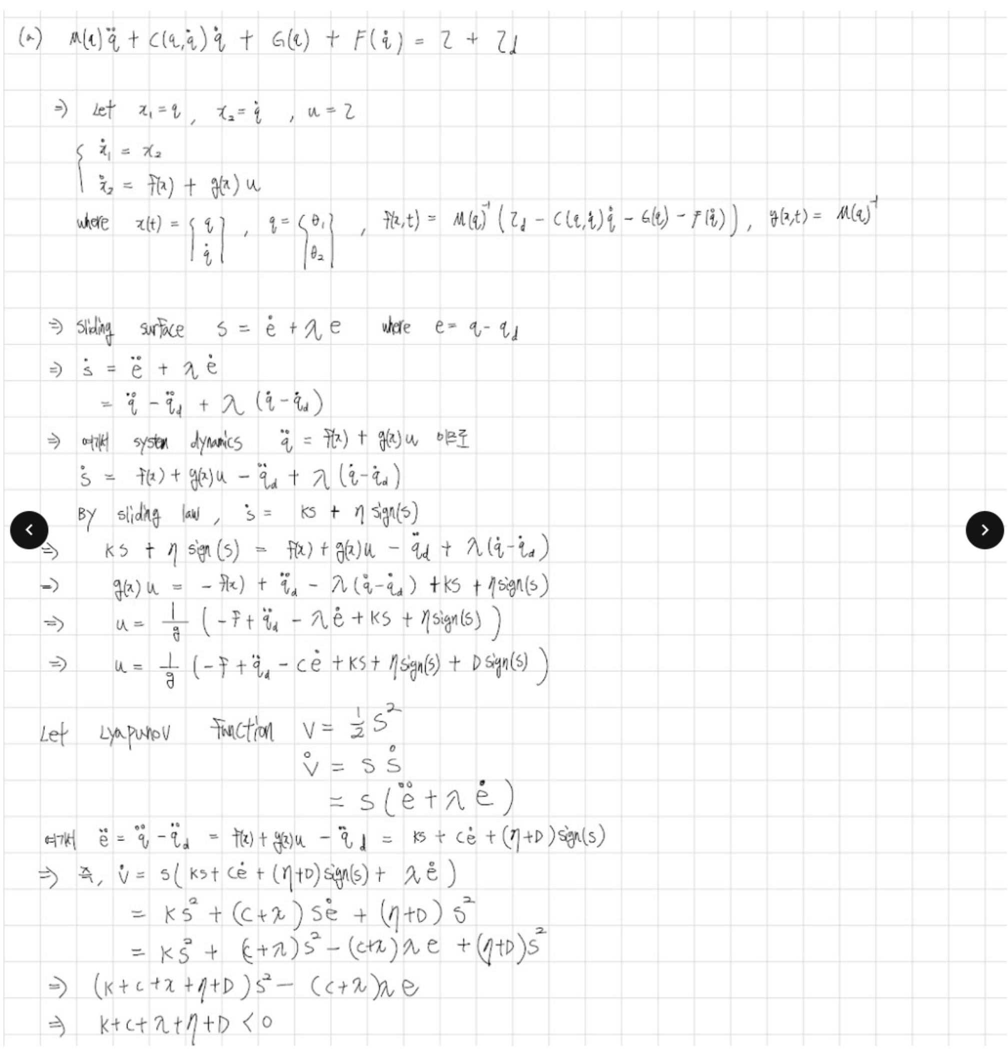
HOMEWORK — 2

You can check the code below.
`https://github.com/JU-SUK/motion_planning-24spring`

1 Sliding Mode control

(a)



(a) $M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) + F(\dot{q}) = \tau + \tau_d$

⇒ Let $x_1 = q$, $x_2 = \dot{q}$, $u = \tau$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f(x) + g(x)u \end{cases}$$

where $x(t) = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$, $q = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$, $f(x,t) = M(q)^{-1}\left(\tau_d - C(q,\dot{q})\dot{q} - G(q) - F(\dot{q})\right)$, $g(x,t) = M(q)^{-1}$

⇒ Sliding surface $s = \dot{e} + \lambda e$ where $e = q - q_d$

⇒ $\dot{s} = \ddot{e} + \lambda\dot{e}$

$= \ddot{q} - \ddot{q}_d + \lambda(\dot{q} - \dot{q}_d)$

⇒ 여기서 system dynamics $\ddot{q} = f(x) + g(x)u$ 이므로

$\dot{s} = f(x) + g(x)u - \ddot{q}_d + \lambda(\dot{q} - \dot{q}_d)$

By sliding law, $\dot{s} = ks + \eta\,\mathrm{sign}(s)$

⇒ $ks + \eta\,\mathrm{sign}(s) = f(x) + g(x)u - \ddot{q}_d + \lambda(\dot{q} - \dot{q}_d)$

⇒ $g(x)u = -f(x) + \ddot{q}_d - \lambda(\dot{q} - \dot{q}_d) + ks + \eta\,\mathrm{sign}(s)$

⇒ $u = \frac{1}{g}\left(-f + \ddot{q}_d - \lambda\dot{e} + ks + \eta\,\mathrm{sign}(s)\right)$

⇒ $u = \frac{1}{g}\left(-f + \ddot{q}_d - c\dot{e} + ks + \eta\,\mathrm{sign}(s) + D\,\mathrm{sign}(s)\right)$

Let Lyapunov function $V = \frac{1}{2}s^2$

$\dot{V} = s\dot{s}$

$= s(\ddot{e} + \lambda\dot{e})$

여기서 $\ddot{e} = \ddot{q} - \ddot{q}_d = f(x) + g(x)u - \ddot{q}_d = ks + c\dot{e} + (\eta + D)\,\mathrm{sign}(s)$

⇒ 즉, $\dot{V} = s\left(ks + c\dot{e} + (\eta + D)\,\mathrm{sign}(s) + \lambda\dot{e}\right)$

$= ks^2 + (c + \lambda)s\dot{e} + (\eta + D)s^2$

$= ks^2 + (c + \lambda)s^2 - (c + \lambda)\lambda e + (\eta + D)s^2$

⇒ $(k + c + \lambda + \eta + D)s^2 - (c + \lambda)\lambda e$

⇒ $k + c + \lambda + \eta + D < 0$

Figure 1: (a)

(b)

matlab 예시 코드를 이용해서 simulation을 진행해보았다. 아래는 2개의 조인트에 대한 reference trajectory이다.

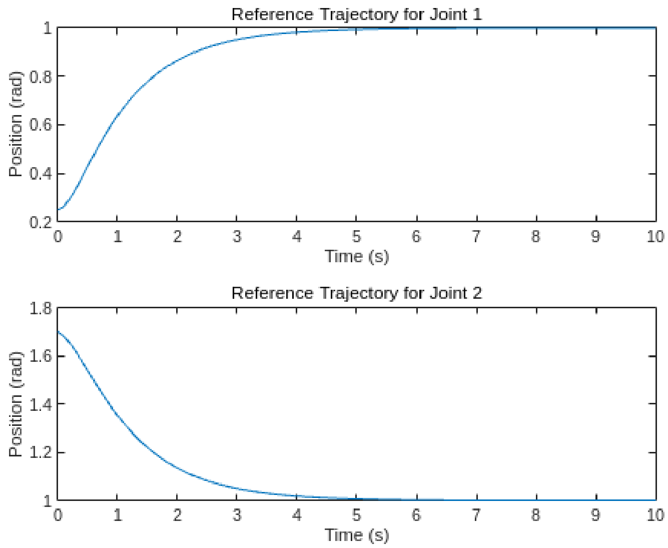


Figure 2: reference trajectory

아래의 sim1은 외부의 disturbance에 의해서 영향을 받지 않는다는 것을 보여주고, sim2은 model uncertainty에 대해서도 robust하다는 결과를 보여준다.

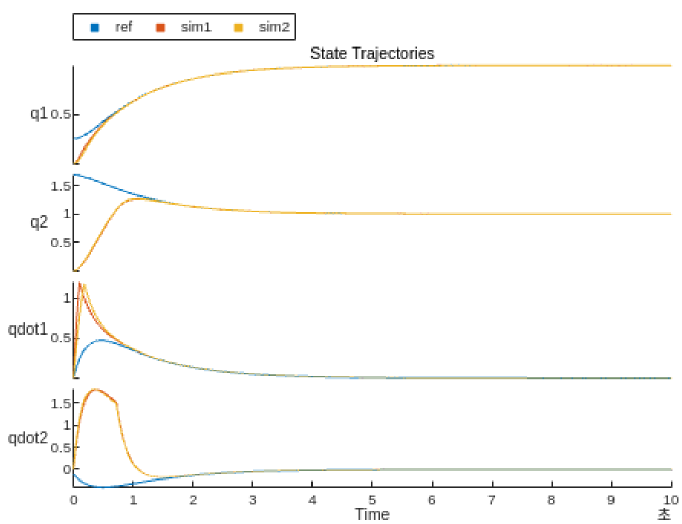


Figure 3: SMC with sign function

하지만 아래의 control input의 결과를 보면 sign function을 사용하였을 때 chattering이 발생한다는 것을 알 수 있다.

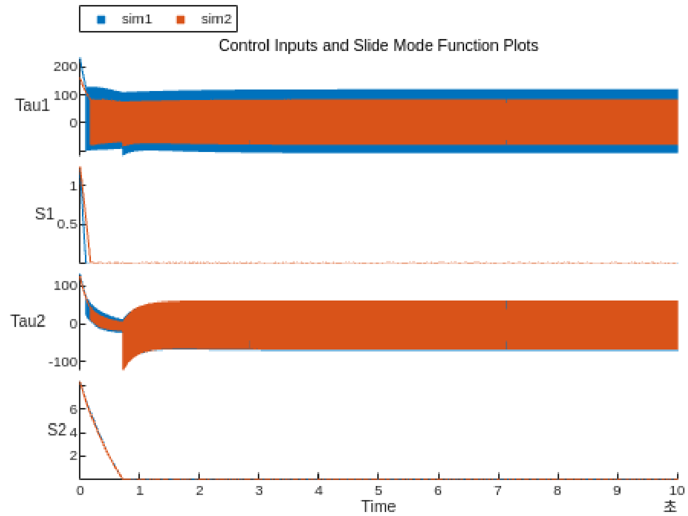


Figure 4: SMC with sign function

이제 control input chattering을 줄이기 위한 방법 중 하나인 sign function 대신에 saturation function을 사용해 보자. 아래 그림에서 sim3과 sim4가 saturation function을 쓴 결과이다. sim3은 마찬가지로 external disturbance 에서 robust하다는 것을 보이고, sim4는 model uncertainty에 robust하다는 것을 보인다.

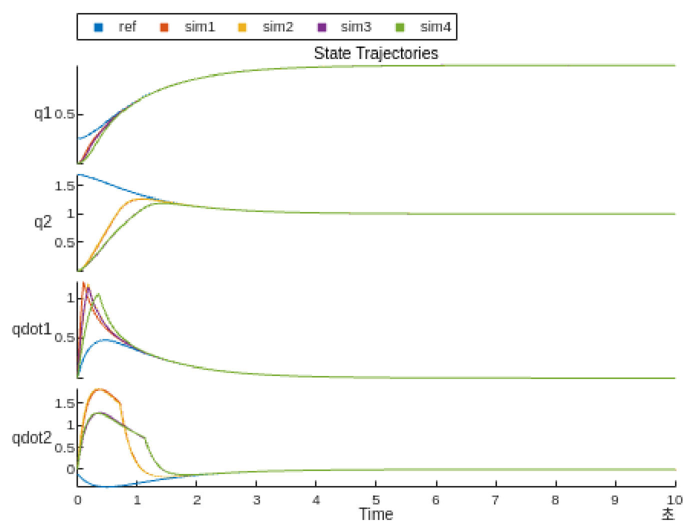


Figure 5: SMC with saturation function

또한 control input chattering을 보면 sign function을 썼을 때보다 확연하게 줄어든 것을 볼 수 있다.

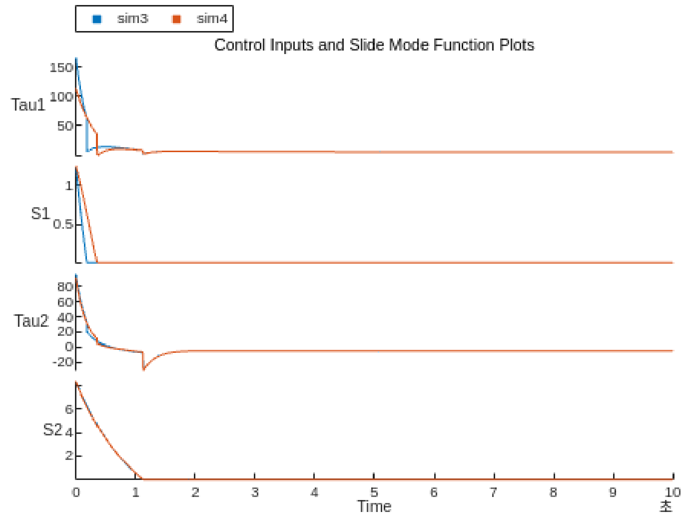
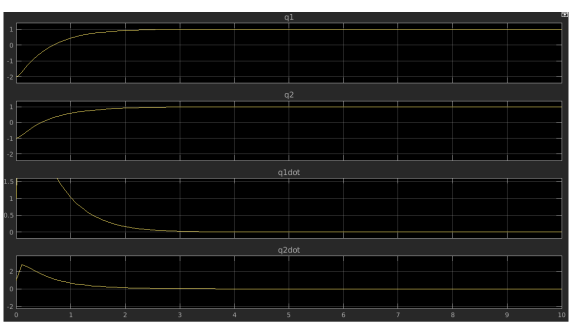


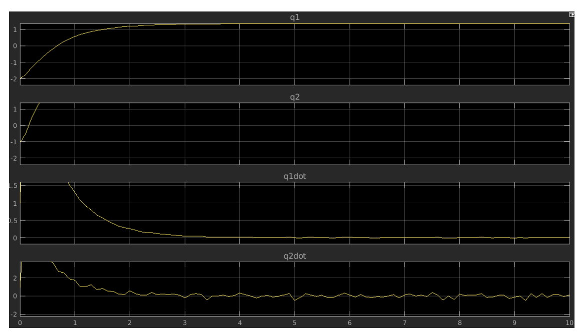
Figure 6: SMC with saturation function

2 Nonlinear model predictive controller

1번 문제에서의 robot manipulator dynamics를 이용해서 nonlinear model predictivve controller를 design해 보자. matlab example을 이용해서 disturbance가 존재할 때($\tau_D \neq 0$)와 존재하지 않을 때($\tau_d = 0$)를 비교하자. target state는 [1, 1, 0, 0]인 constant 값이고, 초기값은 [-2, -1, 1, 1]이다. 이때 disturbance는 평균이 400, 분산이 100인 gaussian distribution으로 주었다.



(a) States without disturbance



(b) States with disturbance

Figure 7: Simulation Results

위의 결과를 보면 disturbance가 없을 때는 target state로 수렴을 하는 것을 확인할 수 있었다. 하지만 disturbance를 주었을 때는 q2의 값을 제외하고 잘 수렴하는 것을 확인할 수 있다. 하지만 q2의 값은 target인 1이 아니라 4 근처로 수렴하였다.

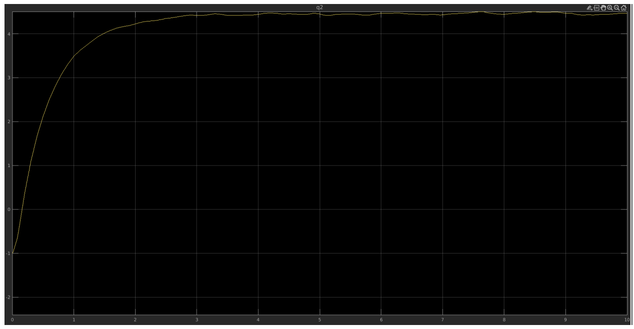


Figure 8: q2 state with disturbance

3 Backstepping control

(a)

(a) Let $a=1$

then $\begin{cases} \dot{x} = x - x^3 + \xi \\ \dot{\xi} = u \end{cases}$

$\Rightarrow$ let $x = x_1$, $\xi = x_2$ $\quad \begin{cases} \dot{x}_1 = x_1 - x_1^3 + x_2 \\ \dot{x}_2 = u \end{cases}$

$\Rightarrow x_2 = \alpha(x_1) = -x_1$

$\Rightarrow$ Let $x_1 = z_1$

$V = \frac{1}{2} z_1^2 \longrightarrow \dot{V} = z_1 \dot{z}_1 = z_1(-z_1^3) = -z_1^4 < 0$ 이기 때문에 stabilize.

$\Rightarrow$ Let $z_2 = x_2 - \alpha(x_1)$

$\Rightarrow \dot{z}_1 = z_1 - z_1^3 + z_2 + \alpha(x_1) = -z_1^3 + z_2$

$\dot{z}_2 = u - \dot{\alpha}(x_1) = v$

$\Rightarrow V_2 = \frac{1}{2} z_1^2 + \frac{1}{2} z_2^2$

$\dot{V}_2 = z_1 \dot{z}_1 + z_2 \dot{z}_2$

$= -z_1^4 + z_2(u - \dot{\alpha}(x_1))$

여기서 $u = \dot{\alpha}(x_1) - z_2$로 하면 $\dot{V}_2 = -z_1^4 - z_2^2 < 0$ 으로 stabilize.

즉, $u = -\dot{x}_1 - (x_2 + x_1) = -\dot{x}_1 - x_2 - x_1$

Figure 9: (a)

(b)

$$\text{b)}\begin{cases}\dot{x}_1 = a x_1 - x_1^3 + x_2 \\ \dot{x}_2 = u\end{cases}$$

$\Rightarrow x_2 = \alpha(x_1) = -a x_1$

$\Rightarrow$ let $x_1 = z_1$ then $V_1 = \frac{1}{2} z_1^2 = z_1 \dot{z}_1 = z_1(-z_1^3) = -z_1^4 < 0$

하지만 $\alpha(x_1)$은 unknown value.

$z_2 = x_2 - \hat{\alpha}_1(x_1) = x_2 - (-\hat{a} x_1) = x_2 + \hat{a} x_1$

$V_2 = \frac{1}{2} z_1^2 + \frac{1}{2} z_2^2 \Rightarrow \dot{V}_2 = z_1 \dot{z}_1 + z_2 \dot{z}_2 = -z_1^4 + z_2(u + \dot{\hat{a}} z_1 + \hat{a} \dot{z}_1)$

$V_3 = \frac{1}{2} z_1^2 + \frac{1}{2} z_2^2 + \frac{1}{2} \tilde{a}^2$ where $\tilde{a} = \hat{a} - a$

$\dot{V}_3 = z_1 \dot{z}_1 + z_2 \dot{z}_2 + \tilde{a} \dot{\tilde{a}}$

$= -z_1^4 + z_2(u + \dot{\hat{a}} z_1 + \hat{a} \dot{z}_1) + \tilde{a} \dot{\tilde{a}}$

$\Rightarrow u = -z_2 - \dot{\hat{a}} z_1 - \hat{a} \dot{z}_1$

$\dot{\hat{a}} = -\gamma x_1 e$

Figure 10: (b)

(c)

x_d를 sin함수로 주었다. 이렇게 했을 때 adaptive backstepping control을 이용해 추종을 하는지 simulation했다.

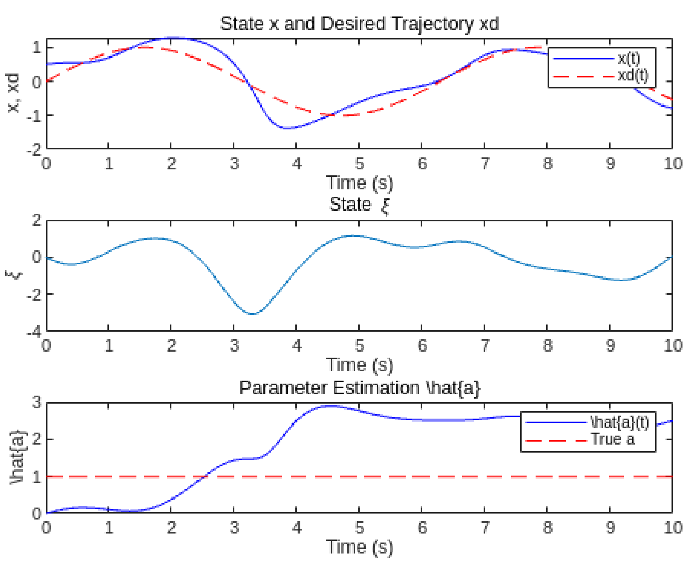


Figure 11: (c)

4 Nonlinear System

(a), (b), (c)

(a) $y = x_1 - x_2$

$\dot{y} = \dot{x}_1 - \dot{x}_2 = x_1 + \cancel{x_1 x_2} - \cancel{x_2^2 + u} - \cancel{x_1 x_2} + \cancel{x_2^2} - \cancel{u}$

$\ddot{y} = \dot{x}_1 = x_1 + x_1 x_2 - x_2^2 + u$

$\Rightarrow$ Relative degree $= 2$

(b) Let's define the internal variable

$$\begin{cases} z_1 = y = x_1 - x_2 \\ z_2 = \dot{y} = x_1 \end{cases}$$

$$\Rightarrow \begin{cases} \dot{z}_1 = z_2 = x_1 \\ \dot{z}_2 = \dot{x}_1 = x_1 + x_1 x_2 - x_2^2 + u \end{cases}$$

여기서 $\begin{cases} z_2 = x_1 \\ z_1 = x_1 - x_2 \end{cases} \longrightarrow x_2 = x_1 - z_1 = z_2 - z_1$

$$\Rightarrow \begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = z_2 + z_2(z_2 - z_1) - (z_2 - z_1)^2 + u \end{cases}$$

$\Rightarrow$ Internal variable $\eta = x_3$

$\dot{\eta} = x_1 + x_1 x_2 - x_2^2 - (x_3 - x_1)^3 + u$

$\quad = z_2 + z_2(z_2 - z_1) - (z_2 - z_1)^2 - (\eta - z_2)^3 + u$

$\Rightarrow$ Normal Form

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = z_2 + z_2(z_2 - z_1) - (z_2 - z_1)^2 + u \\ \dot{\eta} = z_2 + z_2(z_2 - z_1) - (z_2 - z_1)^2 - (\eta - z_2)^3 + u \end{cases}$$

(c) To find the zero dynamics, $z_1 = z_2 = 0$

$\Rightarrow x_1 = x_2, \; x_2 = 0 \longrightarrow x_1 = 0 \;\&\; x_2 = 0$

$\Rightarrow \dot{\eta} = -\eta^3$

$\Rightarrow$ Let Lyapunov function $V(\eta) = \frac{1}{2}\eta^2$; positive-definite

$\dot{V}(\eta) = \eta\dot{\eta} = -\eta^4 < 0$; negative-definite

$\Rightarrow$ globally asymtotically stable

Figure 12: (a), (b), (c)

(d)

(d) let control law

$$u = -k_1 z_1 - k_2 z_2 \quad \text{where } y = z_1,\ \dot{y} = z_2$$

Lyapunov function

$$V = \frac{1}{2} z_1^2 + \frac{1}{2} z_2^2 + \frac{1}{2} \eta^2$$

$$\begin{aligned}
\dot{V} &= z_1 \dot{z}_1 + z_2 \dot{z}_2 + \eta \dot{\eta} \\
&= z_1 z_2 + z_2^2 + \cancel{z_2^3} - z_2^2 z_1 - z_2(\cancel{z_2^2} - 2 z_1 z_2 + z_1^2) - k_1 z_1 z_2 - k_2 z_2^2 \\
&\quad - \eta^4 \\
&= z_1 z_2 + z_2^2 (1 - 3 z_1 - k_2) - z_1^2 z_2 - k_1 z_1 z_2 \\
&= (1 - k_1) z_1 z_2 + (1 - k_2) z_2^2 - 3 z_1 z_2^2 - z_1^2 z_2
\end{aligned}$$

$\Rightarrow k_1 < 1,\ k_2 < 1$

$\Rightarrow u = -k_1 z_1 - k_2 z_2$ where $k_1 < 1,\ k_2 < 1$

Figure 13: (d)

Submitted by Jusuk Lee on June 2, 2024.