Exam 3

Rules

- This is a written examination.
- Due date: May 25th.
- You can work in groups of up to 4 and you can consult any reference that you need.
- **Any form of plagiarism will be reported.**

Laplace equation in the cylinder

Consider the Laplace equation in an infinitely long pipe cylinder of radius a, which can be modeled by a cylinder. For $u(r, \theta, z)$, in cylindrical coordinates

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0, \qquad \text{for } (r, \theta, z) \in (0, a) \times (0, 2\pi) \times (-\infty, \infty). \tag{1}$$

Question 1: Give a simple justification on (1) being the corresponding Laplace equation in cylindrical coordinates. You can use the fact that the corresponding equations in Cartesian (3 dimensional) coordinates and Polar (2 dimensial) coordinates are respectively

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}, \qquad \text{Cartesian } (x, y, z),$$

and

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}, \qquad \text{Polar } (r, \theta).$$

Question 2: Using a separation of variables assumption on the solution, $u(r, \theta.z) = R(r)Q(\theta)Z(z)$, justify thoroughly that the following system of equations is obtained from (1)

$$\frac{d^2 Z}{dz^2} - k^2 Z = 0,$$
$$\frac{d^2 Q}{d\theta^2} + m^2 Q = 0,$$
$$\frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(k^2 - \frac{m^2}{r^2}\right) R = 0.$$

Question 3: Show thoroughly that $J_m(kr)$ forms a piece of the radial solution, given the condition that u is not a bounded function inside of the pipe and

$$u(r, \theta, 0) = 0.$$

With the functions $J_m(x)$ defined as

$$J_m(x) = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!(m+i)!} \left(\frac{x}{2}\right)^{2i+m}, \tag{2}$$