$$u_{tt} - c^2 u_{xx} = 0$$

$$\hookrightarrow \left(\frac{d}{dt} - c\frac{d}{dx}\right)\left(\frac{d}{dt} + \frac{d}{dx}\right) u = 0$$

$$v = u_t + c u_x \Rightarrow v_t = u_{tt} + c u_{xt}$$

$$v_x = u_{xt} + c u_{xx}$$

$$-c v_x = -c u_{xt} - c^2 u_{xx}$$

$$v_t - c v_x = u_{tt} + \cancel{c u_{xt}} - \cancel{c u_{xt}} - c^2 u_{xx}$$

$$= u_{tt} - c^2 u_{xx} = 0$$

$$v_t - c v_x = 0 \text{ (a)} \quad \& \quad v = u_t + c u_x \text{ (b)}$$

$$\text{(a)} \Rightarrow v(x,t) = h(x+ct)$$

$$\text{(b)} \Rightarrow u_t + c u_x = h(x+ct)$$

$$\boxed{u(x,t) = f(x+ct) + g(x-ct)}$$

$$w = u - f(x-ct) \qquad w_t = u_t + c f'(x-ct)$$

$$w_x = u_x - f'(x-ct)$$

$$c w_x = c u_x - c f'(x-ct)$$

$$w_t + c w_x = u_t + \cancel{c f'(x-ct)} + c u_x - \cancel{c f'(x-ct)}$$

$$= u_t + c u_x$$

$$u_x = \frac{du}{dx} = \frac{d\tilde{x}}{dx}\frac{du}{d\tilde{x}} + \frac{d\tilde{t}}{dx}\frac{du}{d\tilde{t}} = u_{\tilde{x}} + u_{\tilde{t}}$$

$$u_t = \frac{du}{dt} = \frac{d\tilde{x}}{dt}\frac{du}{d\tilde{x}} + \frac{d\tilde{t}}{dt}\frac{du}{d\tilde{t}} = c\,u_{\tilde{x}} - c\,u_{\tilde{t}}$$

$\tilde{x} = x + ct$ & $\tilde{t} = x - ct$

$\tilde{x}_x = 1$ $\qquad$ $\tilde{t}_x = 1$

$\tilde{x}_t = c$ $\qquad$ $\tilde{t}_t = -c$

$$u_{xx} = (u_{\tilde{x}} + u_{\tilde{t}})_x = \tilde{x}_x (u_{\tilde{x}} + u_{\tilde{t}})_{\tilde{x}} + \tilde{t}_x (u_{\tilde{x}} + u_{\tilde{t}})_{\tilde{t}}$$

$$= u_{\tilde{x}\tilde{x}} + u_{\tilde{t}\tilde{x}} + u_{\tilde{x}\tilde{t}} + u_{\tilde{t}\tilde{t}} = u_{\tilde{x}\tilde{x}} + 2u_{\tilde{t}\tilde{x}} + u_{\tilde{t}\tilde{t}}$$

$$u_{tt} = c(u_{\tilde{x}} - u_{\tilde{t}})_t = c\,\tilde{x}_t (u_{\tilde{x}} - u_{\tilde{t}})_{\tilde{x}} + c\,\tilde{t}_t (u_{\tilde{x}} - u_{\tilde{t}})_{\tilde{t}}$$

$$= c^2 u_{\tilde{x}\tilde{x}} - c^2 u_{\tilde{t}\tilde{x}} - c^2 u_{\tilde{x}\tilde{t}} + c^2 u_{\tilde{t}\tilde{t}}$$

$$= c^2 (u_{\tilde{x}\tilde{x}} - 2u_{\tilde{x}\tilde{t}} + u_{\tilde{t}\tilde{t}})$$

$$u_{tt} - c^2 u_{xx} = c^2(\cancel{u_{\tilde{x}\tilde{x}}} - 2u_{\tilde{x}\tilde{t}} + \cancel{u_{\tilde{t}\tilde{t}}}) - c^2(\cancel{u_{\tilde{x}\tilde{x}}} \underset{\downarrow\,-}{+} 2u_{\tilde{x}\tilde{t}} + \cancel{u_{\tilde{t}\tilde{t}}})$$

$$= -4c^2 u_{\tilde{x}\tilde{t}} = 0$$

$$u_{\tilde{x}\tilde{t}} = 0$$

$$\Rightarrow u(\tilde{x}, \tilde{t}) = f(\tilde{x}) + g(\tilde{t})$$

$$u(x,t) = f(x+ct) + g(x-ct)$$

Von-Neumann Stability Analysis

$$u_{j+1,m} = \mu u_{j,m+1} + (1-2\mu) u_{j,m} + \mu u_{j,m-1} \qquad \mu = \frac{K \Delta t}{\Delta x^2}$$

$$\boxed{\varepsilon_{j,m} = N_{j,m} - u_{j,m}} \leftarrow \text{Error}$$

$$\Rightarrow \boxed{\varepsilon_{j+1,m} = \mu \varepsilon_{j,m+1} + (1-2\mu) \varepsilon_{j,m} + \mu \varepsilon_{j,m-1}}$$

$$\varepsilon(x,t) = \sum_{m=-M}^{M} E_m(t) e^{ik_m x} \qquad k_m = \frac{\pi m}{L}$$

in the interval L

$$m = -M, \dots, -1, 0, 1, \dots, M$$

$$M = \frac{L}{\Delta x}$$

$E_m(t) \leftarrow$ Amplitude of the error

$$\varepsilon(x,t) = \int_{-\frac{\pi}{\Delta x}}^{\frac{\pi}{\Delta x}} E_k(t) e^{ikx} \, dk \longrightarrow \varepsilon_m(x,t) = E_m(t) e^{ik_m x}$$

$$\varepsilon_{j,m} = E_n(t) e^{ik_n x} \qquad \varepsilon_{j+1,m} = E_n(t+\Delta t) e^{ik_n x}$$

$$\varepsilon_{j,m+1} = E_n(t) e^{ik_n(x+\Delta x)} \qquad \varepsilon_{j,m-1} = E_n(t) e^{ik_n(x-\Delta x)}$$

$$\Rightarrow \frac{E_n(t+\Delta t)}{E_n(t)} = 1 + \mu\left(e^{ik_n \Delta x} + e^{-ik_n \Delta x} - 2\right)$$

$$\theta = k_n \Delta x \in [-\pi, \pi]$$

$$\sin(\theta/2) = \frac{e^{i\theta/2} - e^{-i\theta/2}}{2i} \quad \& \quad \sin^2(\theta/2) = -\frac{e^{i\theta} + e^{-i\theta} - 2}{4}$$

$$\frac{E_n(t+\Delta t)}{E_n(t)} = 1 - 4\mu \sin^2(\theta/2) = G$$

Stability is $|G| \leq 1$

$$\Rightarrow \left|1 - \overbrace{4\mu \sin^2(\theta/2)}\right| \leq 1$$

Always positive!

$$\Rightarrow 4\mu \sin^2(\theta/2) \leq 2$$

$$\Downarrow$$

$$\mu \leq \frac{1}{2} \equiv \frac{K \Delta t}{\Delta x^2} \leq \frac{1}{2}$$

Task

2

$$u_t = k\, u_{xx}, \quad \text{for} \quad 0 < x < \pi \;\;\&\;\; 0 < t < T$$

$$u(x,0) = \phi(x) = \begin{cases} x & \text{if } x \in (0, \pi/2) \\ \pi - x & \text{if } x \in [\pi/2, \pi) \end{cases}$$

$$u(0,t) = u(L,t) = 0$$

$$u_{j+1,m} = \mu u_{j,m+1} + (1-2\mu) u_{j,m} + \mu u_{j,m-1}, \quad \mu = \frac{k \Delta t}{\Delta x^2}$$

With $\boxed{\Delta x = \frac{\pi}{20} \;\;\&\;\; T = \frac{3\pi^2}{80} \;\;\&\;\; k = 1 \;\;\&\;\; \Delta t = \frac{\pi^2}{1600}}$

Given that $\mu < \frac{1}{2} \;\Rightarrow\; \frac{\Delta t}{\pi^2} \cdot 20^2 < \frac{1}{2}$ $\quad \boxed{\Delta t < \frac{\pi^2}{800}}$

$$u(x,t) = \sum_{k=1}^{\infty} b_k \sin(kx) e^{-k^2 t}, \quad b_k = \begin{cases} \frac{4(-1)^{\frac{k+1}{2}}}{\pi k^2} & \text{if } k \text{ odd} \\ 0 & \text{if } k \text{ even} \end{cases}$$

③ $u(x,t) = \frac{1}{\sqrt{4\pi k}} e^{\frac{-x^2}{4kt}}$ ← This is the kernel

what!?

What is the probability that at $t=1$ the particle lies in (a,b)?

$$\mathcal{N}(0,\Delta t) = \frac{1}{\sqrt{2\pi\Delta t}} e^{-\frac{x^2}{2\Delta t}}$$

$$\boxed{\int_a^b \frac{1}{\sqrt{2\pi\Delta t}} e^{-\frac{x^2}{2\Delta t}} dx}$$

1?

$$P(X \leq x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\Delta t}} e^{-\frac{t^2}{2\Delta t}} dt$$

$\Downarrow$

$$P(a < X \leq b) = F_X(b) - F_X(a)$$

④ What is the value of the diffusion Coefficient (the Constant k) Corresponding c to this Case?

What is c? which case? The heat kernel interpretation?

The case of task 2?

Monte Carlo method for the heat Equation

Generate a random Sample of Size $N = 10^3$ from a Uniform distribution from $-\frac{1}{2} \le x \le \frac{1}{2}$

(bin Counting ...)

5 $P(a \le x \le b)$ at $T = 1$ $\leftarrow$ Using Python Stuff

6 Exact expression for the probability distribution of the particles at $T = 1$, and compare with 5!

$u(\xi, \mu)$ & $\xi(x,t)$ & $\mu(x,t)$

$$\frac{du}{dx} = \frac{du}{d\xi} \cdot \frac{d\xi}{dx} + \frac{du}{d\mu} \cdot \frac{d\mu}{dx}$$ ← Chain Rule

$u_{tt} - c^2 u_{xx} = 0$

$u_{tt} = c^2 u_{xx}$ ← wave equation (Simplest second-order equation)

$C = \frac{T}{\rho}$ ← string tension, ← string density

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u = 0$$

We start from $u(x,t)$, then

compute $u_t + c u_x = v$, then

compute $v_t - c v_x$ and should be the zero function

$$u(x,t) = f(x - ct) + g(x + ct)$$

General Solution

$$\boxed{u_{tt} = c^2 u_{xx}, \quad -\infty < x < \infty, \quad u(x,0) = \phi(x) \quad \& \quad u_t(x,0) = \psi(x)}$$

Initial value Problem

$$u(x,t) = \frac{1}{2}\left[\phi(x - ct) + \phi(x + ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds$$

solution!

Exersise

① $u_{tt} = c^2 u_{xx}$, $\underbrace{u(x,0)}_{}= e^x$, $u_t(x,0) = \sin(x)$

$\phi(x) = e^x$, $\psi(x) = \sin(x)$

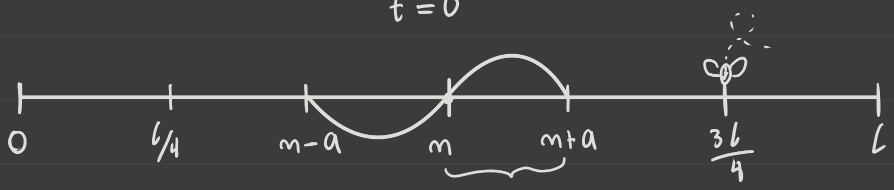
$$u(x,t) = \frac{1}{2}\left[e^{x+ct} + e^{x-ct} \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} \sin(s)\, ds$$

$$= \frac{1}{2}\left[e^x e^{ct} + e^x e^{-ct} \right] + \frac{1}{2c}\left[-\cos(s) \Big|_{x-ct}^{x+ct} \right]$$

$$= \frac{e^x}{2}\left[e^{ct} + e^{-ct} \right] + \frac{1}{2c}\left[\cos(x-ct) - \cos(x+ct) \right]$$

② $a < \frac{l}{4} \Rightarrow 2a < \frac{l}{2}$

$t = 0$



$\boxed{c^2 = \frac{T}{\rho}}$ ← Speed of the wave

$\downarrow$

$x = vt$

Mid Point! $\longrightarrow \frac{l}{2} = m$

$$\Rightarrow x = \frac{3l}{4} - \left(\underbrace{\frac{l}{2}}_{\frac{2l}{4}} + a\right) = \frac{l}{4} - a$$

$$c = \sqrt{\frac{T}{\rho}} = v$$

$$\boxed{\Rightarrow t = \frac{\frac{l}{4} - a}{\sqrt{\frac{T}{\rho}}}}$$