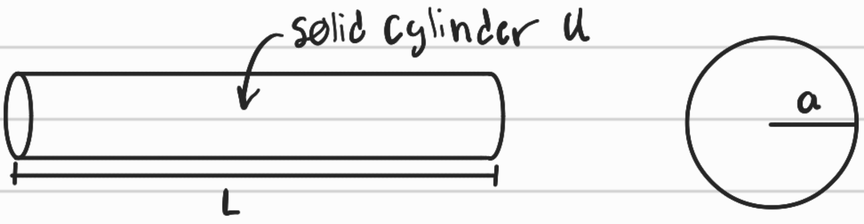
Laplace Equation in a Cylinder

$$\boxed{\nabla^2_{cyl} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}}$$

Laplacian Operator in cylindrical Coordinates



For $u(r,\theta,z)$, with $r \in (0,a)$ & $\theta \in (0,2\pi)$ & $z \in (0,L)$
the Laplace Equation is as follows

$$\nabla^2_{cyl} u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

① Notice that the Laplace equation in polar coordinates is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

Which turns out to be a Laplace equation for a disc of radius r, now if we want the Laplace equation for a cylinder, we just take the height of the cylinder as z from the cartesian form so we get the following equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

② By the method of Separation of Variables let's assume that

$$u(r,\theta,z) = R(r)\,Q(\theta)\,Z(z)$$

Then, the Laplacian Equation becomes

$$\frac{d}{dr^2} RQZ + \frac{1}{r}\frac{d}{dr} RQZ + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} RQZ + \frac{\partial^2}{\partial z^2} RQZ$$

$$= QZ\frac{dR}{dr^2} + \frac{QZ}{r}\frac{dR}{dr} + \frac{RZ}{r^2}\frac{\partial^2 Q}{\partial \theta^2} + RQ\frac{\partial^2 Z}{\partial z^2}$$

Dividing by $R(r)Q(\theta)Z(z)$ we get

$$\frac{1}{R}\frac{dR}{dr^2} + \frac{1}{Rr}\frac{dR}{dr} + \frac{1}{r^2 Q}\frac{\partial^2 Q}{\partial \theta^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = 0$$

Now we have that

$$\frac{1}{R}\frac{dR}{dr^2} + \frac{1}{Rr}\frac{dR}{dr} + \frac{1}{r^2 Q}\frac{\partial^2 Q}{\partial\theta^2} = -\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}$$

So, this equation can be solved if we have a constant c_1 such that

$$-\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = c_1$$

$$\frac{\partial^2 Z}{\partial z^2} = -c_1 Z \quad \Rightarrow \quad \frac{\partial^2 Z}{\partial z^2} + c_1 Z = 0$$

Now, the other equation needs to be further simplify, so

$$\frac{1}{R}\frac{dR}{dr^2} + \frac{1}{Rr}\frac{dR}{dr} + \frac{1}{r^2 Q}\frac{\partial^2 Q}{\partial\theta^2} = c_1$$

$$\frac{r^2}{R}\frac{dR}{dr^2} + \frac{r}{R}\frac{dR}{dr} + \frac{1}{Q}\frac{\partial^2 Q}{\partial\theta^2} = r^2 c_1$$

$$\frac{r^2}{R}\frac{dR}{dr^2} + \frac{r}{R}\frac{dR}{dr} - r^2 c_1 = -\frac{1}{Q}\frac{\partial^2 Q}{\partial\theta^2}$$

again, those need to be equal to a constant c_2, then

$$-\frac{1}{Q}\frac{\partial^2 Q}{\partial\theta^2} = c_2$$

$$\frac{\partial^2 Q}{\partial\theta^2} = -c_2 Q \quad \Rightarrow \quad \frac{\partial^2 Q}{\partial\theta^2} + c_2 Q = 0$$

and

$$\frac{r^2}{R}\frac{dR}{dr^2} + \frac{r}{R}\frac{dR}{dr} - r^2 c_1 = c_2$$

$$\frac{dR}{dr^2} + \frac{1}{r}\frac{dR}{dr} - Rc_1 = R\frac{c_2}{r^2} \quad \Rightarrow \quad \frac{dR}{dr^2} + \frac{1}{r}\frac{dR}{dr} - Rc_1 - R\frac{c_2}{r^2} = 0$$

$$\frac{dR}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \left(c_1 + \frac{c_2}{r^2}\right)R = 0$$

Now if $\overset{(c_1<0)}{c_1 = -k^2}$ & $\overset{(c_2>0)}{c_2 = m^2}$, our main equations become

$$\frac{\partial^2 Z}{\partial z^2} - k^2 Z = 0$$

$$\frac{\partial^2 Q}{\partial\theta^2} + m^2 Q = 0$$

$$\frac{dR}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(k^2 - \frac{m^2}{r^2}\right)R = 0$$

③ If we want to get a Solution for the radial Equation, we need to write it like a Bessel's differential Equation

$$\text{Let } x = Kr \Rightarrow r = \frac{x}{K}$$

$$dx = K dr \Rightarrow dr = \frac{1}{K} dx \Rightarrow dr^2 = \frac{1}{K^2} dx^2$$

Now Substituting in the radial equation we get

$$\frac{\frac{dR}{dx^2}}{K^2} + \frac{\frac{1}{x}\frac{dR}{dx}}{K\ K} + \left(\frac{K^2 - \frac{m^2}{x^2}}{K^2} \right) R = K^2 \frac{dR}{dx^2} + \frac{K^2}{x} \frac{dR}{dx} + K^2 \left(1 - \frac{m^2}{x^2} \right) R$$

$$= \frac{dR}{dx^2} + \frac{1}{x}\frac{dR}{dx} + \left(1 - \frac{m^2}{x^2} \right) R = x^2 \frac{dR}{dx^2} + x \frac{dR}{dx} + (x^2 - m^2) R = 0 \quad (1)$$

Notice that we arrive at the Bessel's equation of order m, these equation's solutions are defined by the Bessel's functions $J_m(x)$ & $Y_m(x)$, we're going to find $J_m(x)$.

$$\text{let } R = \sum_{n=0}^{\infty} a_n x^{n+m}, \text{ then}$$

$$R' = \sum_{n=1}^{\infty} (n+m) a_n x^{n+m-1} \quad \& \quad R'' = \sum_{n=2}^{\infty} (n+m)(n+m-1) a_n x^{n+m-2}$$

now substituting in (1) we get

$$\sum_{n=0}^{\infty} (n+m)(n+m-1) a_n x^{n+m} + \sum_{n=0}^{\infty} (n+m) a_n x^{n+m} + (x^2 - m^2) \sum_{n=0}^{\infty} a_n x^{n+m}$$

$$= \sum_{n=2}^{\infty} (n+m)(n+m-1) a_n x^{n+m} + \sum_{n=1}^{\infty} (n+m) a_n x^{n+m} + \sum_{n=0}^{\infty} a_n x^{n+m+2} - m^2 \sum_{n=0}^{\infty} a_n x^{n+m}$$

$$= \sum_{n=0}^{\infty} (n+m)(n+m-1) a_n x^{n+m} + \sum_{n=0}^{\infty} (n+m) a_n x^{n+m} + \sum_{n=2}^{\infty} a_{n-2} x^{n+m} - m^2 \sum_{n=0}^{\infty} a_n x^{n+m}$$

$$= \sum_{n=0}^{\infty} (n+m)(n+m-1) a_n x^{n+m} + (n+m) a_n x^{n+m} - m^2 a_n x^{n+m} + \sum_{n=2}^{\infty} a_{n-2} x^{n+m}$$

$$= \sum_{n=0}^{\infty} \left((n+m)(n+m-1) + (n+m) - m^2 \right) a_n x^{n+m} + \sum_{n=2}^{\infty} a_{n-2} x^{n+m}$$

$$= \sum_{n=0}^{\infty} \left(\overbrace{(m+n)^2 - m^2}^{(m+n-m)(m+n+m)} \right) a_n x^{n+m} + \sum_{n=2}^{\infty} a_{n-2} x^{n+m}$$

$$= \sum_{n=0}^{\infty} n(2m+n)\, a_n x^{n+m} + \sum_{n=2}^{\infty} a_{n-2} x^{n+m}$$

$$= 0 + (2m+1)a_1 x^{m+1} + \sum_{n=2}^{\infty} n(2m+n)\, a_n x^{n+m} + \sum_{n=2}^{\infty} a_{n-2} x^{n+m}$$

$$= (2m+1)a_1 x^{m+1} + \sum_{n=2}^{\infty} n(2m+n)\, a_n x^{n+m} + a_{n-2} x^{n+m}$$

$$= (2m+1)a_1 x^{m+1} + \sum_{n=2}^{\infty} \left(a_{n-2} + n(2m+n)\, a_n\right) x^{n+m} = 0$$

Now, if $m \neq 1/2$ then

$$(2m+1)a_1 = 0$$

$$\left(a_{n-2} + n(2m+n)\, a_n\right) x^{n+m} = 0$$

For $n \geq 2$, $n \in \mathbb{Z}$:

$$a_1 = 0$$

$$a_{n-2} + n(2m+n)\, a_n = 0 \Rightarrow a_n = -\frac{a_{n-2}}{n(2m+n)}$$

Now if $n = 2i+1$, $i > 0$, $i \in \mathbb{Z}$, then (n odd)

$$a_{2i+1} = -\frac{a_{2i-1}}{(2i+1)(2(m+i)+1)} = \frac{(-1)^{i+1} a_1^{\,0}}{[(2i+1)(2(m+i)+1) \cdots (6m+9)]} = 0$$

Now if $n = 2i$, $i > 0$, $i \in \mathbb{Z}$, then (n even)

$$a_{2i} = -\frac{a_{2i-2}}{2i(2m+2i)} = -\frac{a_{2i-2}}{4i(m+i)} = \frac{(-1)^i a_0}{[4i(m+i)(4(i-1)(m+i-1)) \cdots 4(m+1)]}$$

By replacing those values in $R = \sum_{n=0}^{\infty} a_n x^{n+m}$ we get

$$R = \cancel{\sum_{i=0}^{\infty} a_{2i+1} x^{2i+m+1}}^{\,0} + \sum_{i=0}^{\infty} a_{2i} x^{2i+m}$$

$$= \sum_{i=0}^{\infty} \frac{(-1)^i a_0}{[4i(m+i)(4(i-1)(m+i-1)) \cdots 4(m+1)]} x^{2i+m}$$

$$= \sum_{i=0}^{\infty} \frac{(-1)^i a_0\, m!}{[4i(m+i)(4(i-1)(m+i-1)) \cdots 4(m+1)]} \frac{x^{2i+m}}{m!}$$

Then $R = a_0 \sum_{i=0}^{\infty} \frac{(-1)^i m!}{2^{2i}\, i!\,(m+i)!} x^{2i+m}$, but with $J_m(x) = \sum \frac{(-1)^i x^{2i+m}}{2^{2i+m}\, i!\,(m+i)!}$

we have $R = a_0\, 2^m\, m!\, J_m(x)$.

Given that $x = kr$, then $R = a_0\, 2^m\, m!\, J_m(kr)$