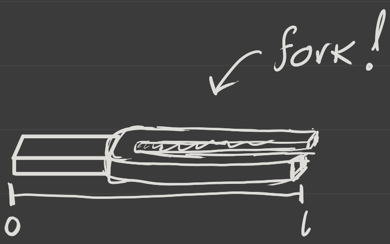
(1) A tuning fork may be regarded as a pair of vibrating flexible bars with a certain degree of stiffness. Each such bar is clamped at one end and is approximately modeled by the fourth-order PDE $u_{tt} + c^2 u_{xxxx} = 0$. It has initial conditions as for the wave equation.
Let's say that on the end x = 0 it is clamped (fixed), meaning that it satisfies $u(0,t) = u_x(0,t) = 0$.
On the other end x = l it is free, meaning that it satisfies $u_{xx}(l,t) = u_{xxx}(l,t) = 0$.
Thus there are a total of four boundary conditions, two at each end.
And also there are two intial conditions (for time).



$$u_{tt} + c^2 u_{xxxx} = 0$$

$$\boxed{\begin{matrix} u(0,t) = 0 & u_{xx}(l,t) = 0 \\ u_x(0,t) = 0 & u_{xxx}(l,t) = 0 \end{matrix}}$$

Boundary conditions (BCs)

$$\boxed{\begin{matrix} u(x,0) = u_0(x) \\ u_t(x,0) = V_0(x) \end{matrix}}$$

Initial conditions (ICs)

(1.1) Separation of variables method for

$$u_{tt} = -c^2 u_{xxxx}$$

$$\boxed{u(x,t) = T(t) \cdot X(x)}$$

$$\frac{d^2}{dt^2} u(x,t) = -c^2 \frac{d^4}{dx^4} u(x,t)$$

$$\frac{d^2}{dt^2}\left(T(t) \cdot X(x)\right) = -c^2 \frac{d^4}{dx^4}\left(T(t) \cdot X(x)\right)$$

$$T''(t) \cdot X(x) = -c^2 X^{(4)}(x)\, T(t)$$

$$\frac{T''(t)}{T(t)} = \frac{-c^2 X^{(4)}(x)}{X(x)} = \lambda \implies \frac{T''(t)}{T(t)} = \lambda \quad \& \quad \frac{-c^2 X^{(4)}(x)}{X(x)} = \lambda$$

Given that changing values of t does not affect $X^{(4)}/X$ and viseversa!

$$A e^{\omega t} + B e^{-\omega t}$$
$$\downarrow$$
$$\underline{A e^{\sqrt{\lambda} t} + B e^{-\sqrt{\lambda} t}}$$

Is it possible?

What sign does λ have?

Case 1: $\lambda > 0$

$$\frac{T''(t)}{T(t)} = \omega^2 \quad , \omega \neq 0 \qquad \boxed{T(t) = A e^{\omega t} + B e^{-\omega t}}$$

Case 2: $\lambda = 0$

$$\frac{T''(t)}{T(t)} = 0 \quad , \qquad T(t) = Bx + A$$

Not Physical!

Case 3: $\lambda < 0$

$$\frac{T''(t)}{T(t)} = -\omega^2 \quad , \omega \neq 0 \qquad \boxed{T(t) = A\cos(\omega t) + B\sin(\omega t)}$$

Therefore $\lambda < 0$

Now Solve for X

$$-c^2 \frac{X^{(4)}(x)}{X(x)} = -\underset{\overset{\downarrow}{\omega = 0}}{\omega^2} \Rightarrow X^{(4)}(x) = \frac{\omega^2}{c^2} X(x) \Rightarrow X^{(4)}(x) - \frac{\omega^2}{c^2} X(x) = 0$$

Assume $X(x) = e^{rx}$, we need to find r!

$$r^4 e^{rx} - \frac{\omega^2}{c^2} e^{rx} = \left(r^4 - \frac{\omega^2}{c^2}\right) e^{rx} = 0 \Rightarrow r^4 - \frac{\omega^2}{c^2} = 0$$

$$r^4 = \frac{\omega^2}{c^2} \Rightarrow r^2 = \pm \frac{\omega}{c}$$

$$r = \pm \sqrt{\pm \frac{\omega}{c}}$$

Therefore, if $\beta = \frac{\omega}{c}$, $r_1 = \sqrt{\beta}$, $r_2 = -\sqrt{\beta}$, $r_3 = i\sqrt{\beta}$, $r_4 = -i\sqrt{\beta}$

Then

$$\boxed{X(x) = C_1 e^{\sqrt{\beta} x} + C_2 e^{-\sqrt{\beta} x} + C_3 e^{i\sqrt{\beta} x} + C_4 e^{-i\sqrt{\beta} x}}$$

$$\boxed{u(x,t) = T(t) \cdot X(x) = \left[A\cos(\omega t) + B\sin(\omega t)\right] \cdot \left[C_1 e^{\sqrt{\beta} x} + C_2 e^{-\sqrt{\beta} x} + C_3 e^{i\sqrt{\beta} x} + C_4 e^{-i\sqrt{\beta} x}\right]}$$

(1.2) let $\lambda = 0$ then

$$-c^2 \frac{X^{(4)}(x)}{X(x)} = 0 \Rightarrow X^{(4)}(x) = 0$$

$$X^{(3)}(x) = c$$

$$X^{(2)}(x) = xc + d$$

$$X'(x) = x^2 c + xd + a$$

$$X(x) = x^3 C + x^2 d + xa + b$$

$$X(0) = b \Rightarrow b = 0$$

$$X'(0) = a \Rightarrow a = 0$$

$$X''(l) = lc + d \Rightarrow lc + d = 0 \rightarrow d = -lc$$

$$X'''(l) = c \Rightarrow c = 0$$

$$\hookrightarrow d = 0$$

Therefore for $\lambda = 0$ $X(x) = 0$

(1.3) Let $\lambda = \beta^4$

Now let's assume that $X(x) = e^{\alpha x}$

Then $X^{(4)}(x) = \alpha^4 e^{\alpha x} = \lambda e^{\alpha x} = \beta^4 e^{\alpha x}$ which means

$\alpha = \pm 1$ & $\alpha = \pm i$

let $A = \frac{C+D}{2}$ & $B = \frac{C-D}{2}$

Now $X(x) = A e^{\beta x} + B e^{-\beta x} + C e^{\beta i x} + D e^{-\beta i x}$

$$= \frac{Ce^{\beta x}}{2} + \frac{Ce^{-\beta x}}{2} + \frac{De^{\beta x}}{2} - \frac{De^{\beta x}}{2} + C_3 \cos(\beta x) + C_4 \sin(\beta x)$$

$$= C_1 \cosh(\beta x) + C_2 \sinh(\beta x) + C_3 \cos(\beta x) + C_4 \sin(\beta x)$$

Now $X'(x) = C_1 \beta \sinh(\beta x) + C_2 \beta \cosh(\beta x) - C_3 \beta \sin(\beta x) + C_4 \beta \cos(\beta x)$

$X''(x) = C_1 \beta^2 \cosh(\beta x) + C_2 \beta^2 \sinh(\beta x) - C_3 \beta^2 \cos(\beta x) - C_4 \beta^2 \sin(\beta x)$

$X^{(3)}(x) = C_1 \beta^3 \sinh(\beta x) + C_2 \beta^3 \cosh(\beta x) + C_3 \beta^3 \sin(\beta x) - C_4 \beta^3 \cos(\beta x)$

with B.Cs

$X(0) = C_1 + C_3 = 0 \Rightarrow C_1 = -C_3$

$X'(0) = C_2 + C_4 = 0 \Rightarrow C_2 = -C_4$

$$X''(l) = C_1 \beta^2 \cosh(\beta l) + C_2 \beta^2 \sinh(\beta l) - C_3 \beta^2 \cos(\beta l) - C_4 \beta^2 \sin(\beta l) = 0$$

$$= -C_3 \beta^2 \cosh(\beta l) - C_4 \beta^2 \sinh(\beta l) - C_3 \beta^2 \cos(\beta l) - C_4 \beta^2 \sin(\beta l)$$

$$= C_3 \cosh(\beta l) + C_4 \sinh(\beta l) + C_3 \cos(\beta l) + C_4 \sin(\beta l)$$

$$= C_3(\cosh(\beta l) + \cos(\beta l)) + C_4(\sinh(\beta l) + \sin(\beta l))$$

$$-\frac{C_4}{C_3} = \frac{\cosh(\beta l) + \cos(\beta l)}{\sinh(\beta l) + \sin(\beta l)}$$

$$X^{(3)}(l) = C_1 \beta^3 \sinh(\beta l) + C_2 \beta^3 \cosh(\beta l) + C_3 \beta^3 \sin(\beta l) - C_4 \beta^3 \cos(\beta l) = 0$$

$$= -C_3 \beta^3 \sinh(\beta l) - C_4 \beta^3 \cosh(\beta l) + C_3 \beta^3 \sin(\beta l) - C_4 \beta^3 \cos(\beta l)$$

$$= C_3 \sinh(\beta l) + C_4 \cosh(\beta l) - C_3 \sin(\beta l) + C_4 \cos(\beta l)$$

$$= C_3(\sinh(\beta l) - \sin(\beta l)) + C_4(\cosh(\beta l) + \cos(\beta l))$$

$$-\frac{C_4}{C_3} = \frac{\sinh(\beta l) - \sin(\beta l)}{\cosh(\beta l) + \cos(\beta l)}$$

Therefore $$\frac{\cosh(\beta l) + \cos(\beta l)}{\sinh(\beta l) + \sin(\beta l)} = \frac{\sinh(\beta l) - \sin(\beta l)}{\cosh(\beta l) + \cos(\beta l)}$$

Then

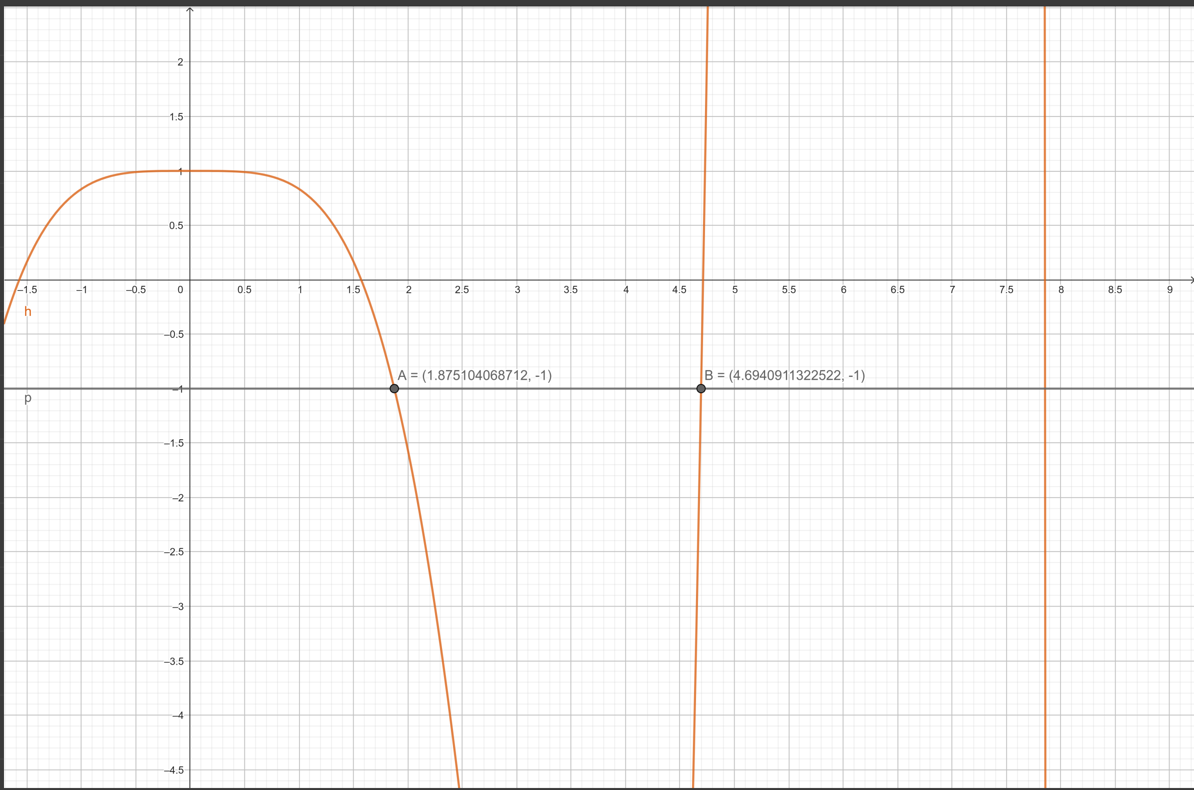
$$\cosh^2(\beta l) + 2\cos(\beta l)\cosh(\beta l) + \cos^2(\beta l) = \sinh^2(\beta l) - \sin^2(\beta l)$$

$$\cosh^2(\beta l) - \sinh^2(\beta l) + \cos^2(\beta l) + \sin^2(\beta l) = -2\cos(\beta l)\cosh(\beta l)$$

$$2 = -2\cos(\beta l)\cosh(\beta l)$$

$$-1 = \cos(\beta l)\cosh(\beta l)$$

1.4



$l\beta_1 = A_x = 1.8751040$ & $l\beta_2 = 4.69409$

$c = \frac{x(B)}{x(A)}$

$= 2.5033763248547$

$d = c^2$

$= 6.2668930238428$

② $u_t = i\,u_{xx}$, Assume $u(x,t) = T(t)\cdot X(x)$

Then $X(x)T'(t) = i\,T(t)\cdot X''(x)$

$$\frac{T'(t)}{T(t)} = \lambda \quad \& \quad i\,\frac{X''(x)}{X(x)} = \lambda$$

let $\lambda \neq 0$

$$\frac{T'(t)}{T(t)} = \lambda$$

$\dfrac{dT(t)}{\cancel{dt}} \cdot \cancel{dt}$?

Then $$\int \frac{T'(t)}{T(t)}\,dt = \int \lambda\,dt$$

$$\int \frac{1}{T(t)}\,dT(t) = \int \lambda\,dt$$

$$\ln(T(t)) = \lambda t + C_1$$

$$T(t) = A\,e^{\lambda t}$$

$$\frac{X''(x)}{X(x)} = \frac{\lambda}{i}$$

Then $X''(x) - \frac{\lambda}{i}X(x) = 0$, Assume $X(x) = e^{\alpha x}$

$$\alpha^2 e^{\alpha x} - \frac{\lambda}{i}e^{\alpha x} = 0$$

$$\left(\alpha^2 - \frac{\lambda}{i}\right)e^{\alpha x} = 0$$

$$\Rightarrow \alpha^2 = \frac{\lambda}{i} \Rightarrow \alpha = \pm\sqrt{\frac{\lambda}{i}}$$

$$X(x) = B\,e^{\sqrt{\lambda/i}\,x} \quad \text{or} \quad X(x) = C\,e^{-\sqrt{\lambda/i}\,x}$$

$$X(x) = B\,e^{\sqrt{\lambda/i}\,x} + C\,e^{-\sqrt{\lambda/i}\,x}$$

Now $u(x,t) = A\,e^{\lambda t}\left(B\,e^{\sqrt{\lambda/i}\,x} + C\,e^{-\sqrt{\lambda/i}\,x}\right)$ / $e^{ax} = \sum_{k=0}^{\infty}\frac{(ax)^k}{k!}$

Then
$$u(x,t) = A\sum_{k=0}^{\infty}\frac{(\lambda t)^k}{k!}\left(B\sum_{k=0}^{\infty}\frac{(\sqrt{\lambda/i}\,x)^k}{k!} + C\sum_{k=0}^{\infty}\frac{(-\sqrt{\lambda/i}\,x)^k}{k!}\right)$$

3

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \cos\left(\frac{n\pi x}{L}\right) \right]$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cdot \cos\left(\frac{n\pi x}{L}\right) dx \qquad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Let $f(x) = e^x$ & $L = 1$, then $\boxed{\int f\,dg = f \cdot g - \int g \cdot df}$ ← Integration by Parts!

$$a_n = \int_{-1}^{1} e^x \cos(n\pi x)\, dx = \frac{(1+e^2)\pi n \sin(\pi n) + (e^2-1)\cos(\pi n)}{e\pi^2 n^2 + e}$$

$$b_n = \int_{-1}^{1} e^x \sin(n\pi x)\, dx = \frac{(1+e^2)\sin(\pi n) - (e^2-1)\pi n \cos(\pi n)}{e\pi^2 n^2 + e}$$

$$a_0 = \frac{e^2 - 1}{e\pi^2 n^2 + e}$$

$$e^x = \frac{1}{2} \frac{e^2-1}{e\pi^2 n^2 + e} + \sum_{n=1}^{\infty} \left[\frac{(1+e^2)\pi n \sin(\pi n) + (e^2-1)\cos(\pi n)}{e\pi^2 n^2 + e} \cos(n\pi x) + \frac{(1+e^2)\sin(\pi n) - (e^2-1)\pi n \cos(\pi n)}{e\pi^2 n^2 + e} \sin(n\pi x) \right]$$

Now, we're going to use that result to solve

$$u_{tt} = c^2 u_{xx}, \text{ with } u(x,0) \overset{\phi(x)}{=} e^x \ \& \ u_t(x,0) \overset{\psi(x)}{=} 0 \quad \forall x \in [-1,1]$$

We know that

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s)\, ds$$

Now, because $\psi(s) = 0$

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right]$$

$$g(x) = \frac{1}{2} \cdot \frac{e^2-1}{e\pi^2 n^2 + e} + \sum_{n=1}^{\infty} \left[\frac{(1+e^2)\pi n \sin(\pi n) + (e^2-1)\cos(\pi n)}{e\pi^2 n^2 + e} \cos(n\pi x) + \frac{(1+e^2)\sin(\pi n) - (e^2-1)\pi n \cos(\pi n)}{e\pi^2 n^2 + e} \sin(n\pi x) \right]$$

$$\phi(x) = (e\pi^2 n^2 + e)^{-1} \left[\frac{e^2-1}{2} + \sum_{n=1}^{\infty} \left[\left[(1+e^2)\pi n \sin(\pi n) + (e^2-1)\cos(\pi n)\right] \cos(n\pi x) + \left[(1+e^2)\sin(\pi n) - (e^2-1)\pi n \cos(\pi n)\right] \sin(n\pi x) \right] \right]$$

$$U(x,t) = \frac{1}{2} \left[\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(n\pi(x+ct)) + b_n \sin(n\pi(x+ct)) \right] + \right.$$

$$\left. \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(n\pi(x-ct)) + b_n \sin(n\pi(x-ct)) \right] \right]$$

$$= \frac{1}{2} \left[a_0 + \sum_{n=1}^{\infty} a_n \left[\cos(n\pi(x+ct)) + \cos(n\pi(x-ct)) \right] + b_n \left[\sin(n\pi(x+ct)) + \sin(n\pi(x-ct)) \right] \right]$$