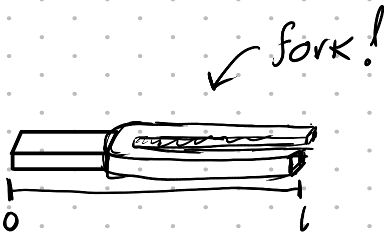
Exam 2



$$u_{tt} + c^2 u_{xxxx} = 0$$

$$\boxed{\begin{array}{ll} u(0,t) = 0 & u_{xx}(l,t) = 0 \\ u_x(0,t) = 0 & u_{xxx}(l,t) = 0 \end{array}}$$

Boundary conditions (BCs)

$$\boxed{\begin{array}{l} u(x,0) = u_0(x) \\ u_t(x,0) = V_0(x) \end{array}}$$

Initial conditions (ICs)

1.1 Separation of variables method for

$$u_{tt} = -c^2 u_{xxxx}$$

$$\frac{d^2}{dt^2} u(x,t) = -c^2 \frac{d^4}{dx^4} u(x,t)$$

We assume

$$u(x,t) = T(t) \cdot X(x)$$

$$\frac{d^2}{dt^2}\left(T(t) \cdot X(x)\right) = -c^2 \frac{d^4}{dx^4}\left(T(t) \cdot X(x)\right)$$

$$T''(t) \cdot X(x) = -c^2 X^{(4)}(x)\, T(t)$$

$$\frac{T''(t)}{-c^2 T(t)} = \frac{X^{(4)}(x)}{X(x)} = \lambda \quad \Rightarrow \quad -\frac{T''(t)}{c^2 T(t)} = \lambda \quad \& \quad \frac{X^{(4)}(x)}{X(x)} = \lambda$$

$$\Rightarrow \boxed{X^{(4)}(x) = \lambda X(x)}$$

(1.2) let $\lambda = 0$ then

$$-c^2 \frac{X^{(4)}(x)}{X(x)} = 0 \Rightarrow X^{(4)}(x) = 0$$

$$X^{(3)}(x) = c$$

$$X^{(2)}(x) = xc + d$$

$$X'(x) = x^2 c + xd + a$$

$$X(x) = x^3 c + x^2 d + xa + b$$

$X(0) = b \Rightarrow b = 0$

$X'(0) = a \Rightarrow a = 0$

$X''(l) = lc + d \Rightarrow lc + d = 0 \rightarrow d = -lc$

$X'''(l) = c \Rightarrow c = 0 \hookrightarrow d = 0$

$$\boxed{\text{Therefore for } \lambda = 0 \quad X(x) = 0}$$

(1.3) Let $\lambda = \beta^4$

Now let's assume that $X(x) = e^{\alpha x}$

Then $X^{(4)}(x) = \alpha^4 e^{\alpha x} = \lambda e^{\alpha x} = \beta^4 e^{\alpha x}$

$$\alpha^4 e^{\alpha x} = \beta^4 e^{\alpha x}$$

$$(\alpha^4 - \beta^4) e^{\alpha x} = 0$$

$$\alpha = \pm\beta \quad \& \quad \alpha = \pm i\beta$$

let $A = \frac{C+D}{2}$ & $B = \frac{C-D}{2}$

Now $X(x) = A e^{\beta x} + B e^{-\beta x} + C e^{\beta i x} + D e^{-\beta i x}$

$$= \frac{Ce^{\beta x}}{2} + \frac{Ce^{-\beta x}}{2} + \frac{De^{\beta x}}{2} - \frac{De^{\beta x}}{2} + C_3 \cos(\beta x) + C_4 \sin(\beta x)$$

$$= C_1 \cosh(\beta x) + C_2 \sinh(\beta x) + C_3 \cos(\beta x) + C_4 \sin(\beta x)$$

Now $X'(x) = C_1\beta \sinh(\beta x) + C_2\beta\cosh(\beta x) - C_3\beta\sin(\beta x) + C_4\beta\cos(\beta x)$

$X''(x) = C_1\beta^2\cosh(\beta x) + C_2\beta^2\sinh(\beta x) - C_3\beta^2\cos(\beta x) - C_4\beta^2\sin(\beta x)$

$X^{(3)}(x) = C_1\beta^3\sinh(\beta x) + C_2\beta^3\cosh(\beta x) + C_3\beta^3\sin(\beta x) - C_4\beta^3\cos(\beta x)$

With B.Cs

$X(0) = C_1 + C_3 = 0 \Rightarrow C_1 = -C_3$

$X'(0) = C_2 + C_4 = 0 \Rightarrow C_2 = -C_4$

$X''(l) = C_1\beta^2\cosh(\beta l) + C_2\beta^2\sinh(\beta l) - C_3\beta^2\cos(\beta l) - C_4\beta^2\sin(\beta l) = 0$

$= -C_3\beta^2\cosh(\beta l) - C_4\beta^2\sinh(\beta l) - C_3\beta^2\cos(\beta l) - C_4\beta^2\sin(\beta l)$

$= C_3\cosh(\beta l) + C_4\sinh(\beta l) + C_3\cos(\beta l) + C_4\sin(\beta l)$

$= C_3(\cosh(\beta l) + \cos(\beta l)) + C_4(\sinh(\beta l) + \sin(\beta l))$

$$-\frac{C_4}{C_3} = \frac{\cosh(\beta l) + \cos(\beta l)}{\sinh(\beta l) + \sin(\beta l)}$$

$X^{(3)}(l) = C_1\beta^3\sinh(\beta l) + C_2\beta^3\cosh(\beta l) + C_3\beta^3\sin(\beta l) - C_4\beta^3\cos(\beta l) = 0$

$= -C_3\beta^3\sinh(\beta l) - C_4\beta^3\cosh(\beta l) + C_3\beta^3\sin(\beta l) - C_4\beta^3\cos(\beta l)$

$= C_3\sinh(\beta l) + C_4\cosh(\beta l) - C_3\sin(\beta l) + C_4\cos(\beta l)$

$= C_3(\sinh(\beta l) - \sin(\beta l)) + C_4(\cosh(\beta l) + \cos(\beta l))$

$$-\frac{C_4}{C_3} = \frac{\sinh(\beta l) - \sin(\beta l)}{\cosh(\beta l) + \cos(\beta l)}$$

Therefore $$\frac{\cosh(\beta l) + \cos(\beta l)}{\sinh(\beta l) + \sin(\beta l)} = \frac{\sinh(\beta l) - \sin(\beta l)}{\cosh(\beta l) + \cos(\beta l)}$$

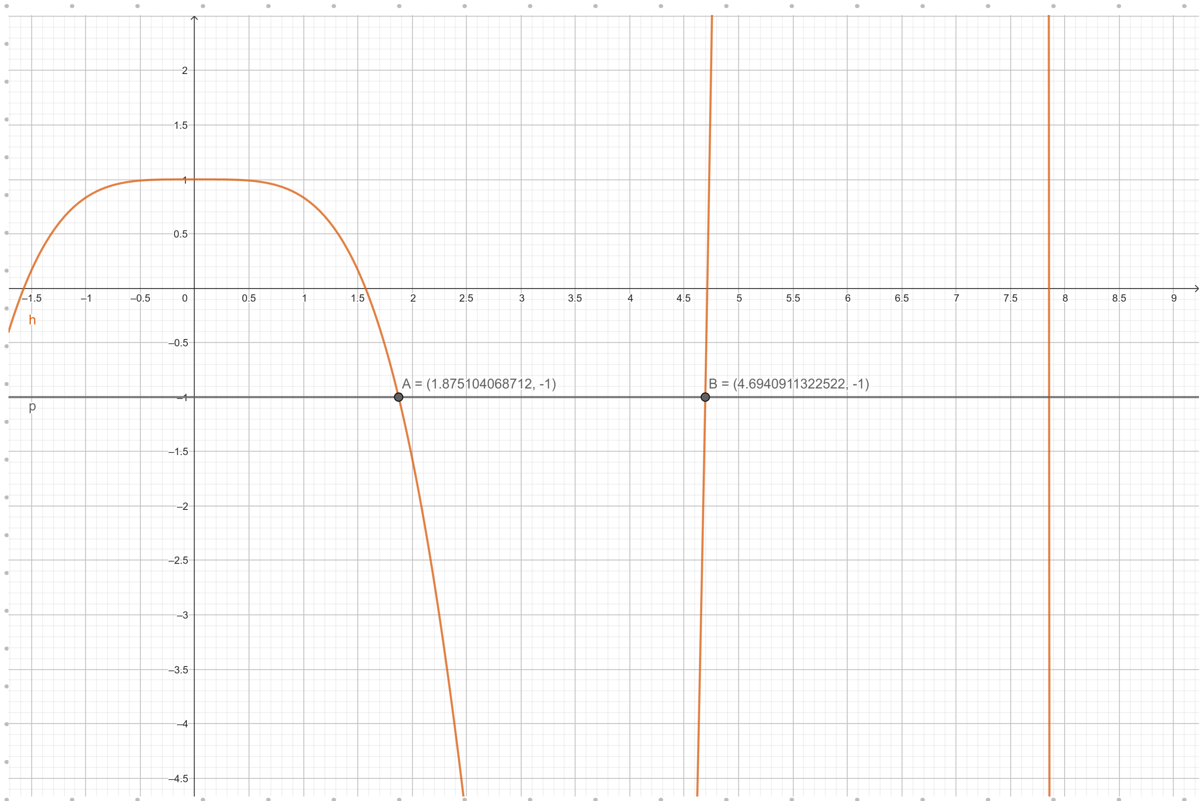
Then $\cosh^2(\beta l) + 2\cos(\beta l)\cosh(\beta l) + \cos^2(\beta l) = \sinh^2(\beta l) - \sin^2(\beta l)$

$\cosh^2(\beta l) - \sinh^2(\beta l) + \cos^2(\beta l) + \sin^2(\beta l) = -2\cos(\beta l)\cosh(\beta l)$

$2 = -2\cos(\beta l)\cosh(\beta l)$

$$\boxed{-1 = \cos(\beta l)\cosh(\beta l)}$$

1.4



$$l\beta_1 = A_x = 1.8751040 \quad \& \quad l\beta_2 = 4.69409 \quad \& \quad (l\beta_2 / l\beta_1)^2 = 6.26689$$

②

$u_t = i\, u_{xx}$ Assume $u(x,t) = T(t)\cdot X(x)$

Then $X(x)\,T'(t) = i\, T(t)\cdot X''(x)$

$$\frac{T'(t)}{T(t)} = \lambda \quad \& \quad i\,\frac{X''(x)}{X(x)} = \lambda$$

let $\lambda \neq 0$

$$\frac{T'(t)}{T(t)} = \lambda$$

$\frac{dT(t)}{\cancel{dt}} \cdot \cancel{dt}$?

Then $\displaystyle\int \frac{T'(t)}{T(t)}\,dt = \int \lambda\, dt$

$$\int \frac{1}{T(t)}\, dT(t) = \int \lambda\, dt$$

$$\ln(T(t)) = \lambda t + C_1$$

$$T(t) = A\, e^{\lambda t}$$

$\dfrac{X''(x)}{X(x)} = \dfrac{\lambda}{i}$ Then $X''(x) - \dfrac{\lambda}{i} X(x) = 0,$ Assume $X(x) = e^{\alpha x}$

$$\alpha^2 e^{\alpha x} - \frac{\lambda}{i} e^{\alpha x} = 0$$

$$\left(\alpha^2 - \frac{\lambda}{i}\right) e^{\alpha x} = 0$$

$$\Rightarrow \alpha^2 = \frac{\lambda}{i} \Rightarrow \alpha = \pm\sqrt{\frac{\lambda}{i}}$$

$X(x) = B e^{\sqrt{\lambda/i}\, x}$ or $X(x) = C e^{-\sqrt{\lambda/i}\, x}$

$$X(x) = B\, e^{\sqrt{\lambda/i}\, x} + C e^{-\sqrt{\lambda/i}\, x}$$

$$\boxed{e^{ax} = \sum_{k=0}^{\infty} \frac{(ax)^k}{k!}}$$

Serie for the e^{ax}

Now $u(x,t) = A e^{\lambda t}\left(B\, e^{\sqrt{\lambda/i}\, x} + C e^{-\sqrt{\lambda/i}\, x}\right)$

Then $$\boxed{u(x,t) = A \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!}\left(B \sum_{k=0}^{\infty} \frac{(\sqrt{\lambda/i}\, x)^k}{k!} + C \sum_{k=0}^{\infty} \frac{(-\sqrt{\lambda/i}\, x)^k}{k!}\right)}$$

(3) $g(x) = e^x \quad , x \in [-1, 1] \quad , L = 1$

$$C_n = \frac{1}{2L} \int_{-L}^{L} g(x)\, e^{(in\pi x)/L}\, dx \qquad g(x) = \sum_{n=-\infty}^{\infty} C_n\, e^{-(in\pi x)/L}$$

Then $C_n = \frac{1}{2} \int_{-1}^{1} e^{(in\pi x)+x}\, dx = \frac{1}{2} \int_{-1}^{1} e^{(in\pi+1)x}\, dx$

$$\left.\begin{aligned} a &= (in\pi+1)x \\ \frac{da}{dx} &= in\pi+1 \\ dx &= \frac{1}{in\pi+1}\, da \end{aligned}\right\} \longrightarrow \frac{1}{2} \int_{-1}^{1} \frac{e^a}{in\pi+1}\, da = \frac{1}{2} \frac{1}{in\pi+1} \left(e^{(in\pi+1)x} \right) \Big|_{-1}^{1} = \frac{1}{\pi n i+1} \frac{e^{(in\pi+1)} - e^{-(in\pi+1)}}{2}$$

$$= \frac{\sinh(in\pi+1)}{in\pi+1}$$

Now

$$g(x) = \sum_{n=-\infty}^{\infty} \frac{\sinh(in\pi+1)}{in\pi+1} \cdot e^{-in\pi x}$$

Let's use this result for solving

$$u_{tt} = c^2 u_{xx}$$

$$u(x,0) = e^x \quad \& \quad u_t(x,0) = 0 \,, \quad \forall\, x \in [-1,1]$$

We know that

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s)\, ds$$

where $\phi(x) = u(x,0)$ & $\psi(x) = u_x(x,0)$

$$u(x,t) = \frac{1}{2}\left[\sum_{n=-\infty}^{\infty} \frac{\sinh(in\pi+1)}{in\pi+1} \cdot e^{-in\pi(x+ct)} + \sum_{n=-\infty}^{\infty} \frac{\sinh(in\pi+1)}{in\pi+1} \cdot e^{-in\pi(x-ct)} \right]$$

$$= \sum_{n=-\infty}^{\infty} \frac{\sinh(in\pi+1)}{in\pi+1} \left(e^{-in\pi x} e^{-in\pi ct} + e^{-in\pi x} e^{in\pi ct} \right) \cdot \frac{1}{2}$$

$$= \sum_{n=-\infty}^{\infty} \frac{\sinh(in\pi+1)}{in\pi+1} \cdot \frac{\left(e^{-in\pi ct} + e^{in\pi ct} \right)}{2} \cdot e^{-in\pi x}$$

$$\boxed{= \sum_{n=-\infty}^{\infty} \frac{\sinh(in\pi+1) \cdot \cosh(in\pi ct)}{in\pi+1} \cdot e^{-in\pi x}}$$