Combinatorics: Understanding Permutations, Variations, and Combinations

by Juan Suazo Verger

1 Permutations (No Repetition)

1.1 Definition

Permutations count the number of ways to arrange all elements of a set where **order matters** and **no element repeats**.

1.2 Formula

$$P(n) = n!$$

Where $n!$ (n factorial) is the product of all positive integers up to n.

1.3 Example

Imagine you have 3 different books and you want to arrange them on a shelf. The number of possible arrangements is:

$$P(3) = 3! = 3 \times 2 \times 1 = 6$$

Arrangements: (Book1, Book2, Book3), (Book1, Book3, Book2), etc.

1.4 When to Apply

- Use permutations when you need to arrange all items in a specific order. - Example scenarios: Arranging people in a line, shuffling cards.

2 Permutations (With Repetition)

2.1 Definition

Permutations with repetition allow **elements to be repeated** in the arrangement. The order still matters.

2.2 Formula

$$P(n, r) = n^r$$

Where n is the number of distinct items, and r is the number of positions.

2.3 Example

Imagine you have 3 different letters (A, B, C) and you want to create a 2-letter code, where letters can repeat. The number of possible 2-letter codes is:

$$P(3, 2) = 3^2 = 9$$

Possible codes: AA, AB, AC, BA, BB, BC, etc.

2.4 When to Apply

- When order matters and elements can repeat. - Example scenarios: Creating codes or passwords where characters can repeat.

3 Variations (No Repetition)

3.1 Definition

Variations (or arrangements) deal with selecting and ordering a subset of elements, where **order matters**, but **elements are not repeated**.

3.2 Formula

$$V(n,r) = \frac{n!}{(n-r)!}$$

Where n is the total number of elements, and r is the number of elements to arrange.

3.3 Example

Imagine you have 5 people and you want to select 3 of them to form a committee with specific roles. The number of ways to assign the roles is:

$$V(5,3) = \frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{120}{2} = 60$$

Possible combinations: (Person1, Person2, Person3), (Person2, Person3, Person1), etc.

3.4 When to Apply

- Use when you are selecting a subset and arranging it in a specific order. - Example scenarios: Choosing teams for different roles (e.g., leader, assistant, etc.).

4 Variations (With Repetition)

4.1 Definition

This allows selecting and arranging a subset where **order matters**, and **elements can repeat**.

4.2 Formula

$$V(n, r) = n^r$$

4.3 Example

Imagine you have 4 different colors and want to paint 3 rooms, where each room can be painted the same or a different color. The number of ways to paint the rooms is:

$$V(4, 3) = 4^3 = 64$$

Possible combinations: (Red, Red, Blue), (Green, Yellow, Green), etc.

4.4 When to Apply

- Use when selecting a subset with order and allowing repetition. - Example scenarios: Choosing ice cream flavors for scoops where the same flavor can be used multiple times.

5 Combinations (No Repetition)

5.1 Definition

Combinations are used when **order doesn't matter** and **elements cannot repeat**.

5.2 Formula

$$C(n,r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Where n is the total number of elements, and r is the number of elements to select.

5.3 Example

Imagine you have 5 different fruits and want to pick 3 for a fruit salad, where the order in which you pick them doesn't matter. The number of ways to choose the fruits is:

$$C(5,3) = \frac{5!}{3!(5-3)!} = \frac{120}{6 \times 2} = 10$$

Possible combinations: (Apple, Banana, Cherry), (Apple, Banana, Grape), etc.

5.4 When to Apply

- Use combinations when selecting a subset without regard to order. - Example scenarios: Choosing lottery numbers, selecting a team from a group where roles don't matter.

6 Combinations (With Repetition)

6.1 Definition

Combinations with repetition allow **elements to be repeated**, but **order does not matter**.

6.2 Formula

$$C(n+r-1, r) = \frac{(n+r-1)!}{r!(n-1)!}$$

6.3 Example

Imagine you have 3 types of cookies (chocolate, vanilla, strawberry) and want to select 4 cookies, allowing repeats. The number of ways to choose the cookies is:

$$C(3+4-1, 4) = C(6, 4) = \frac{6!}{4!2!} = \frac{720}{24 \times 2} = 15$$

Possible combinations: (2 chocolate, 1 vanilla, 1 strawberry), etc.

6.4 When to Apply

- Use when you need to select a subset allowing repetition but without regard to order. - Example scenarios: Dividing items into groups where duplicates are allowed (e.g., scooping flavors for ice cream).

7 Symmetry in Combinations

The **binomial coefficient** $\binom{n}{k}$ has an important property called **symmetry**. This is expressed as:

$$\binom{n}{k} = \binom{n}{n-k}$$

This means that the number of ways to choose k elements from a set of n elements is the same as the number of ways to choose $n-k$ elements.

7.1 Example

If $n = 5$ and $k = 2$, then:

$$\binom{5}{2} = \binom{5}{3} = 10$$

8 Alternative Notations for Combinations

Combinations are often represented in different notations. For example:

$$\binom{n}{k} = C_k^n = C(n,k)$$

All of these notations represent the same concept: choosing k elements from n.
- $\binom{n}{k}$ is the standard binomial coefficient notation.
- C_k^n is an alternative notation.
- $C(n,k)$ is another alternative used in some contexts.

9 How to Recognize and Apply Each Formula

- **Order Matters?**
  - Yes: Use **permutations** or **variations**.
  - No: Use **combinations**.
- **Repetition Allowed?**
  - Yes: Use **permutations** or **combinations with repetition**.
  - No: Use **permutations** or **combinations without repetition**.