Probability Distributions: In-Depth Explanation

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1 Introduction to Probability Distributions

A probability distribution is a mathematical function that describes the likelihood of obtaining the possible values that a random variable can assume. Distributions are broadly categorized into two types:

- **Discrete Distributions**: These deal with random variables that take on countable outcomes.
- **Continuous Distributions**: These are associated with random variables that take on an infinite number of possible values within a given range.

Each distribution has its own formula to compute probabilities, expected values, and variances. We'll discuss these in more detail in the following sections.

2 Discrete Distributions

2.1 Uniform Distribution

The **Discrete Uniform Distribution** assumes that all possible outcomes of a random variable are equally likely. This is often used in scenarios such as rolling a fair die or drawing a card from a shuffled deck.

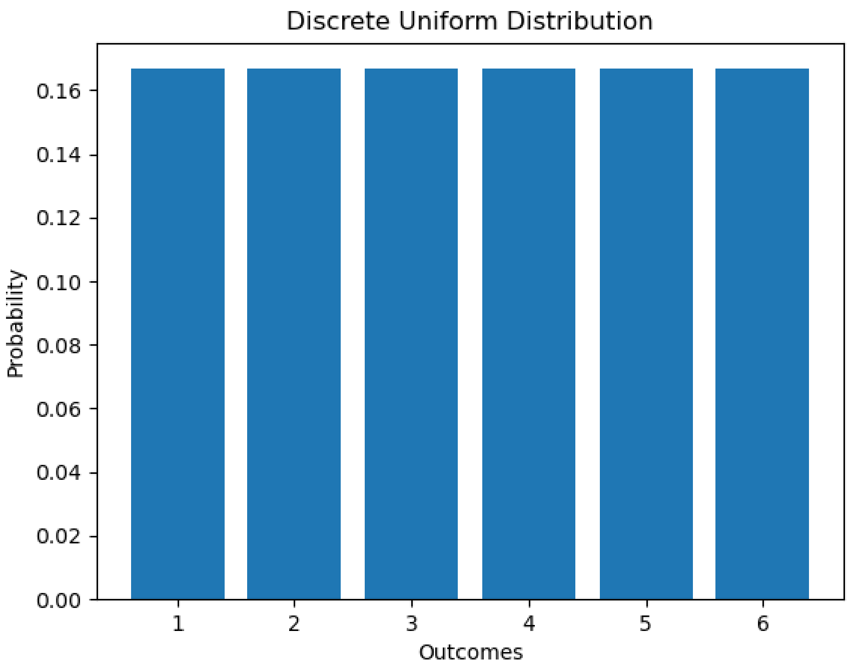
$$Y \sim U(a, b)$$

The mean and variance are given by:

$$E[Y] = \frac{a+b}{2}, \quad \text{Var}(Y) = \frac{(b-a+1)}{^2 - 1}12$$

Key properties:

- All outcomes are equally likely.
- The expected value gives the average of all possible outcomes.



2.2 Bernoulli Distribution

The **Bernoulli Distribution** is a discrete probability distribution for a random variable that takes on only two outcomes: success (with probability p) and failure (with probability $1 - p$). It models the outcome of a single trial.

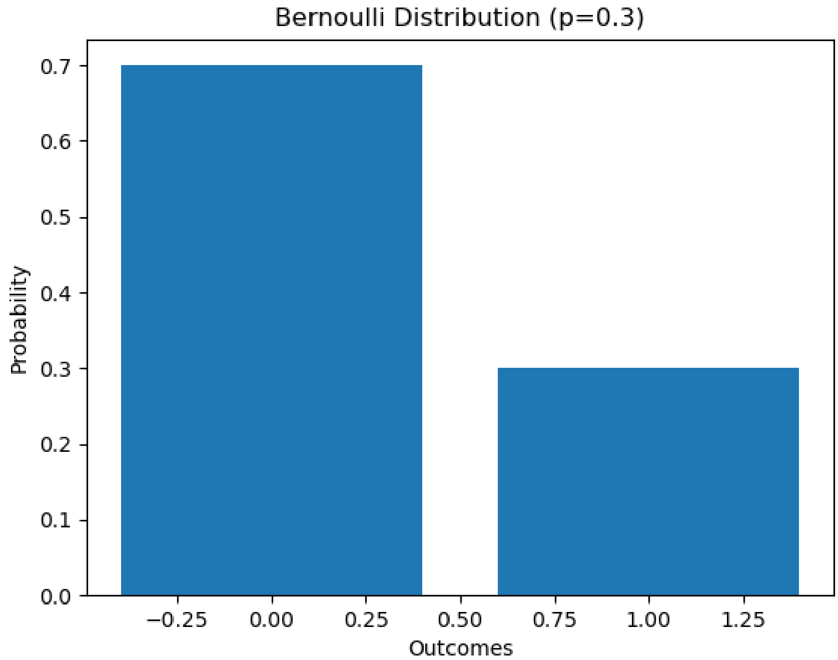
$$Y \sim \mathrm{Bern}(p)$$

The expected value and variance are:

$$E[Y] = p, \quad \mathrm{Var}(Y) = p(1 - p)$$

Key properties:

- This is used to model binary outcomes (e.g., flipping a coin).
- The mean p is the probability of success.



2.3 Binomial Distribution

The **Binomial Distribution** extends the Bernoulli distribution to multiple independent trials (e.g., flipping a coin multiple times). It models the number of successes in n trials.

$$Y \sim B(n, p)$$

The probability mass function (PMF) is:

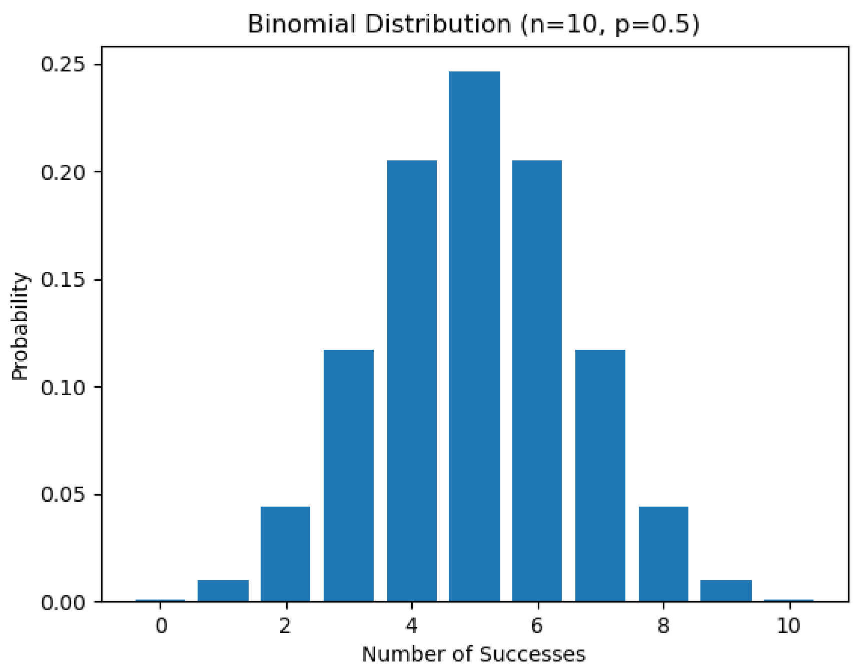
$$P(Y = y) = \binom{n}{y} p^y (1 - p)^{n-y}$$

The expected value and variance are:

$$E[Y] = np, \quad \mathrm{Var}(Y) = np(1 - p)$$

Key properties:

- Used in scenarios where we have a fixed number of independent trials.
- The number of successes follows a binomial pattern.



2.4 Poisson Distribution

The **Poisson Distribution** is used to model the number of occurrences of an event over a fixed interval of time or space, when these events occur with a known constant mean rate.

$$Y \sim \text{Poisson}(\lambda)$$

The PMF is:

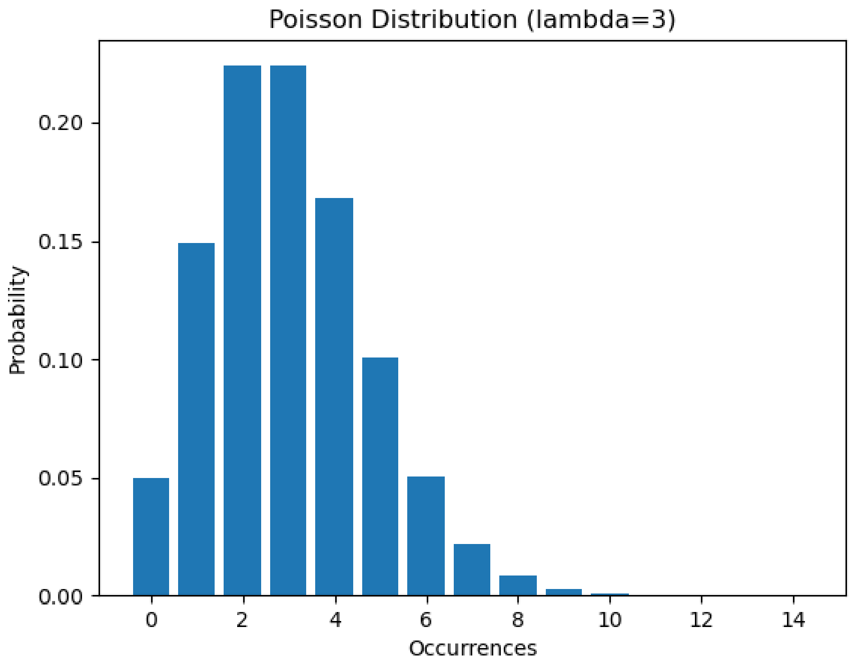
$$P(Y = y) = \frac{\lambda^y e^{-\lambda}}{y!}$$

The expected value and variance are:

$$E[Y] = \lambda, \quad \text{Var}(Y) = \lambda$$

Key properties:

- Useful for modeling rare events, such as the number of emails received in an hour.



3 Continuous Distributions

3.1 Normal Distribution

The **Normal Distribution**, also known as the Gaussian distribution, is the most commonly encountered continuous probability distribution. Its bell-shaped curve is symmetric about the mean.

$$Y \sim N(\mu, \sigma^2)$$

The PDF is given by:

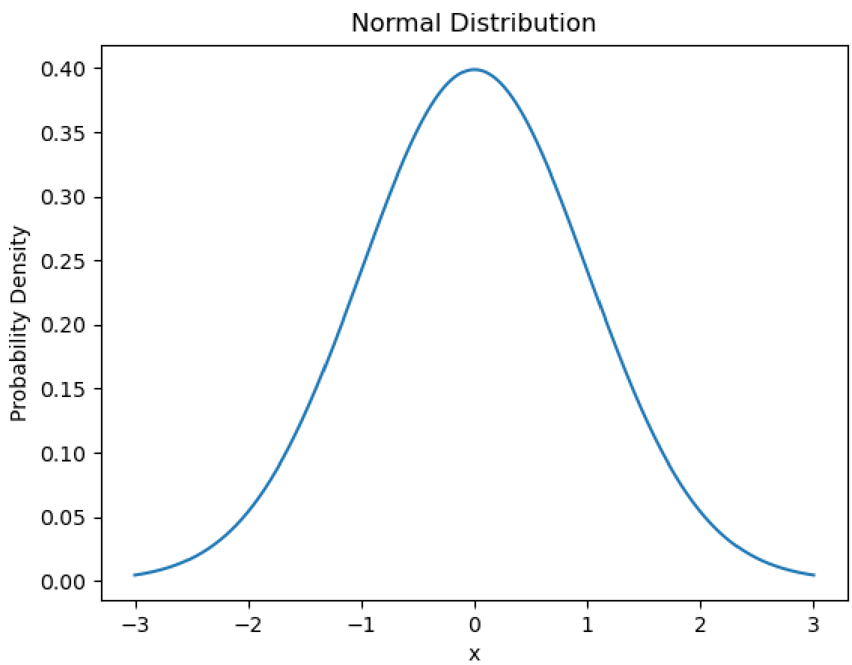
$$f(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$$

The expected value and variance are:

$$E[Y] = \mu, \quad \mathrm{Var}(Y) = \sigma^2$$

Key properties:

- The Normal distribution is used in many natural phenomena (e.g., height of individuals).
- 68% of values fall within one standard deviation from the mean, 95% within two, and 99.7% within three.



3.2 Student's T-Distribution

The **Student's T-Distribution** is used when estimating the mean of a normally distributed population when the sample size is small and the population standard deviation is unknown.

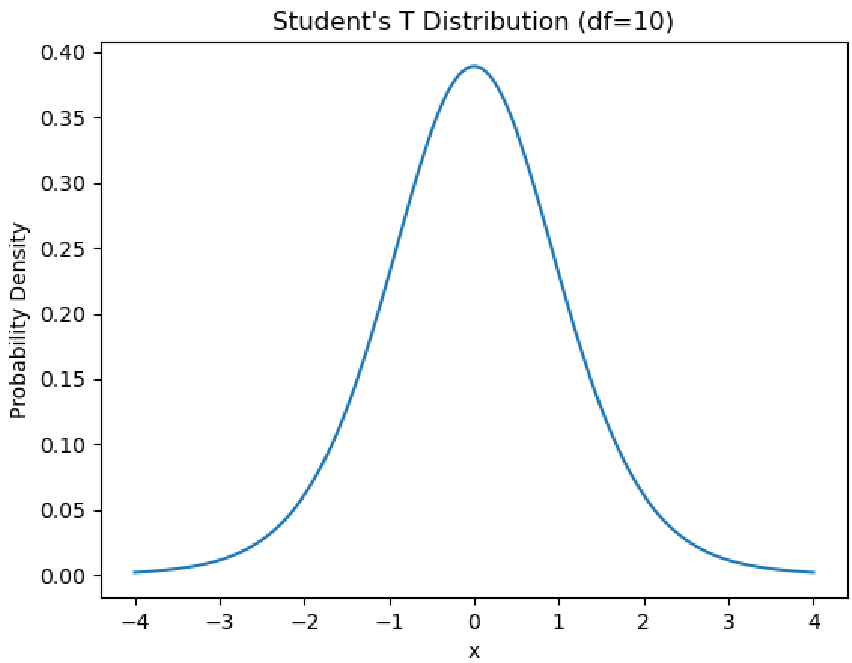
$$Y \sim t(k)$$

The expected value and variance are:

$$E[Y] = 0, \quad \mathrm{Var}(Y) = \frac{k}{k-2}, \quad k > 2$$

Key properties:

- It is similar to the normal distribution but with heavier tails.
- Useful for confidence intervals and hypothesis testing in small samples.



3.3 Chi-Squared Distribution

The **Chi-Squared Distribution** is primarily used in hypothesis testing, especially in tests of goodness of fit and tests for independence in contingency tables.

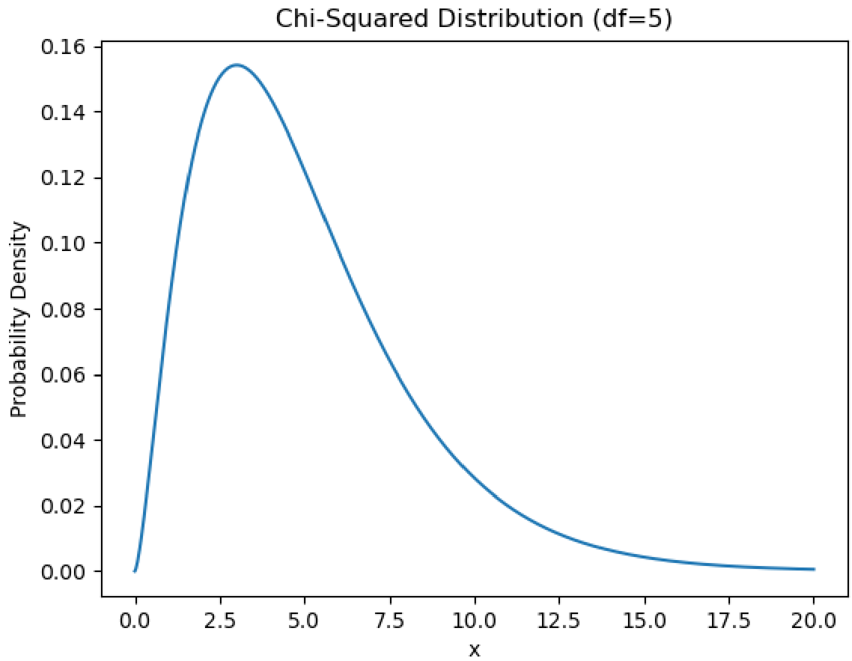
$$Y \sim \chi^2(k)$$

The expected value and variance are:

$$E[Y] = k, \quad \mathrm{Var}(Y) = 2k$$

Key properties:

- The distribution is skewed to the right, and its shape depends on the degrees of freedom k.



3.4 Exponential Distribution

The **Exponential Distribution** describes the time between events in a Poisson process, where events occur continuously and independently at a constant average rate.

$$Y \sim \text{Exp}(\lambda)$$

The PDF is given by:

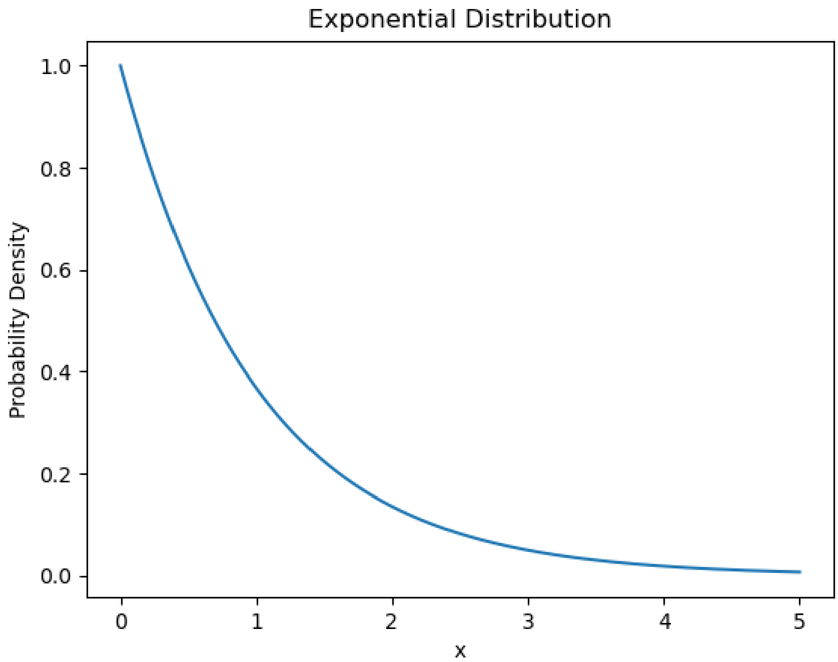
$$f(y) = \lambda e^{-\lambda y}, \quad y \geq 0$$

The expected value and variance are:

$$E[Y] = \frac{1}{\lambda}, \quad \text{Var}(Y) = \frac{1}{\lambda^2}$$

Key properties:

- Frequently used to model time-to-failure in reliability analysis.



3.5 Logistic Distribution

The **Logistic Distribution** is similar to the normal distribution but has heavier tails. It is commonly used in logistic regression models.

$$Y \sim \text{Logistic}(\mu, s)$$

The PDF is given by:

$$f(y) = \frac{e^{-(y-\mu)/s}}{s(1 + e^{-(y-\mu)/s})^2}$$

The CDF is:

$$F(y) = \frac{1}{1 + e^{-(y-\mu)/s}}$$

Key properties:

- It is used to model growth in social science, biology, and economics.