Real Time Series analysis and modelling

Aid machine learning with dynamical insight

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All theorems are true, All models are wrong. All data are inaccurate. What are we to do?

The aim of this course is to teach you how to deal with real data, to increase your scepticism regarding reliable modelling in practice, and to expand the tool box you carry to include nonlinear techniques, both deterministic and stochastic with the aid of dynamical insight.

In short: to get you to think before you compute (and perhaps afterwards too.)

Lecture 4

Data analysis workshop

Space Time Separation Plot

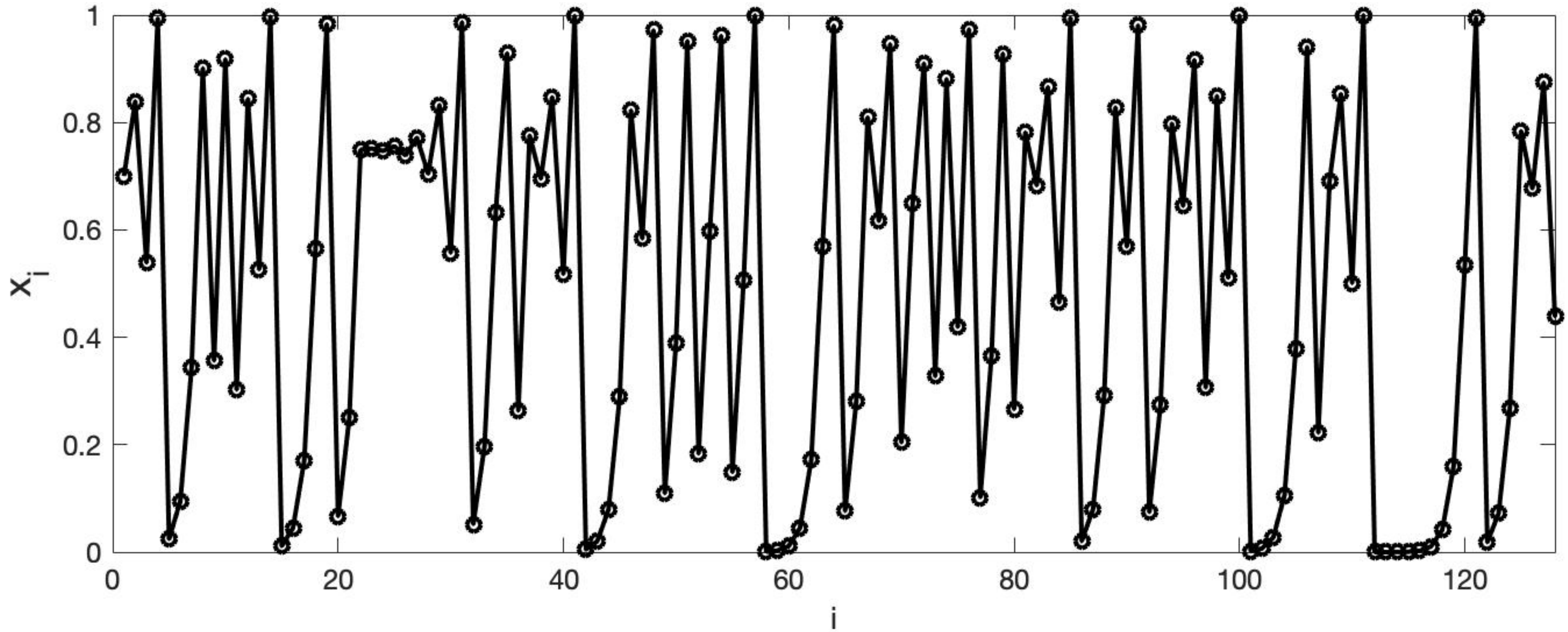
To further investigate the time dependence of a data set we can consider the space time separation plot. The space time separation plot considers explicitly the separation in time and space between pairs of points in a time series. This is done by plotting the contour maps of the fraction of points closer than a distance r at a given time separation as a function of Δt. Specifically we plot $P(|x(t+\Delta t) - x(t)| < r)$ for arbitrary t.

Each pair of points x_i and x_j, $i \neq j$ in a time series is separated by a distance ℓ_{ij} in observed state space and a distance t_{ij} in time. It is reasonable to ask how these two quantities: separation in time and separation in space, are related given various dynamical hypotheses.

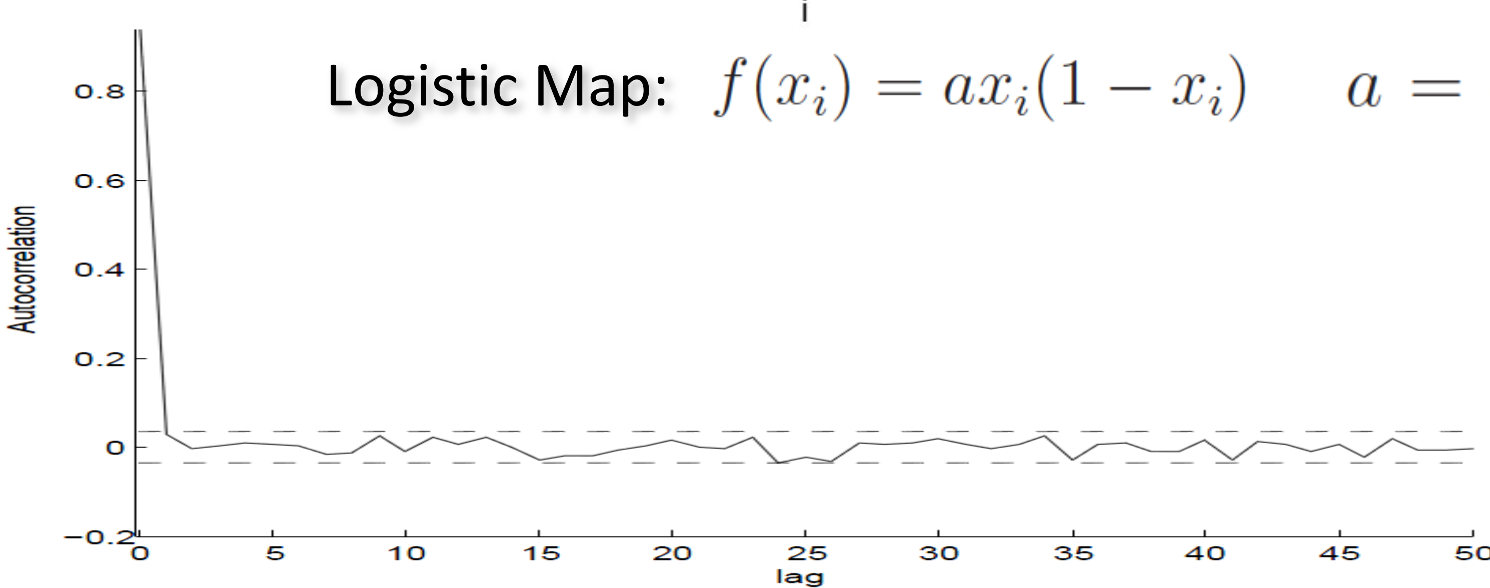
What does a periodic trajectory look like?
What does a recurrent non-periodic trajectory look like?
What does a non-stationary (drifting) trajectory look like?)

What is the point? We are starting to decide what model class to use!

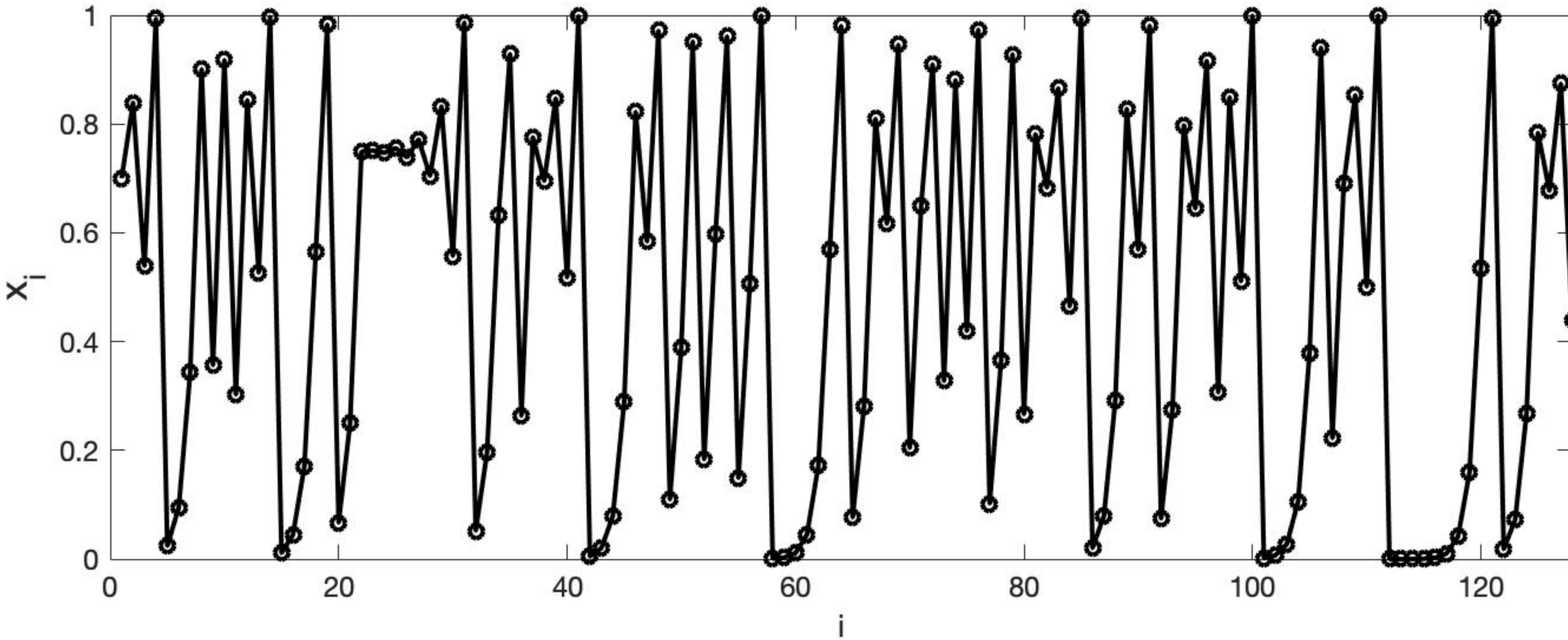
Apply Linear approach to Time Series of Nonlinear system

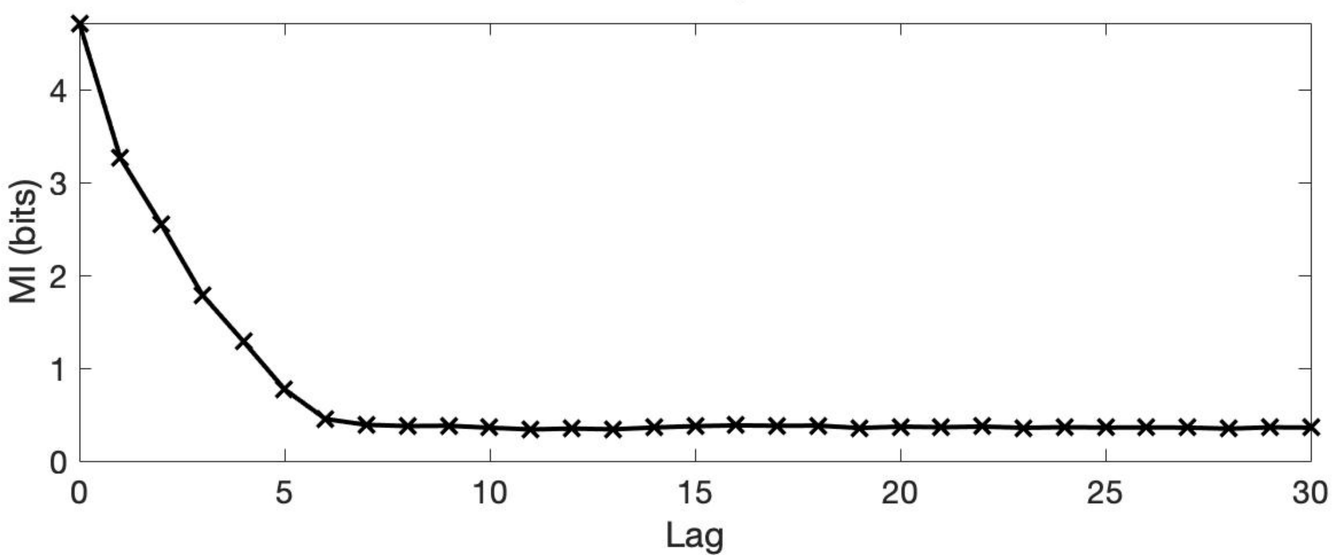


Logistic Map: $f(x_i) = ax_i(1 - x_i) \qquad a = 4$

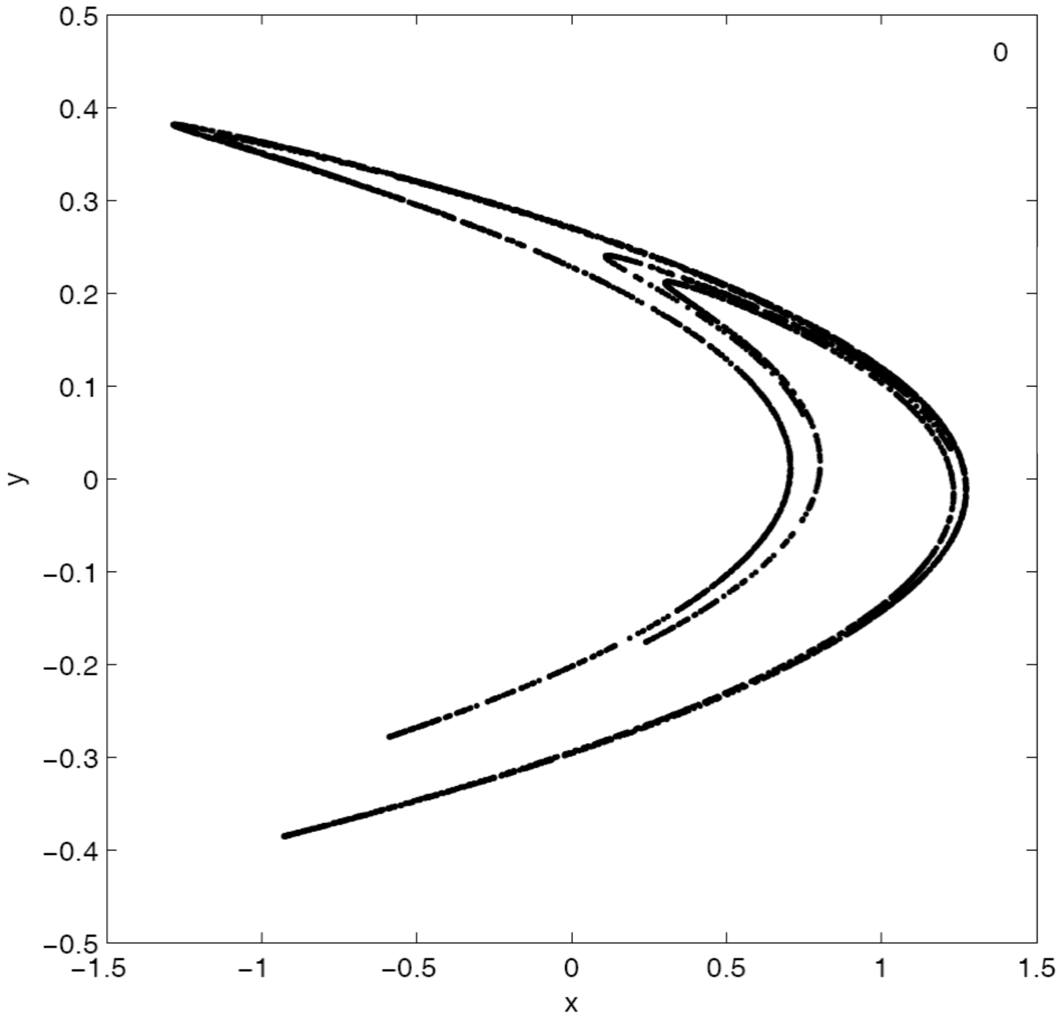


Mutual Information measure linear and nonlinear relationship





Delay reconstruction

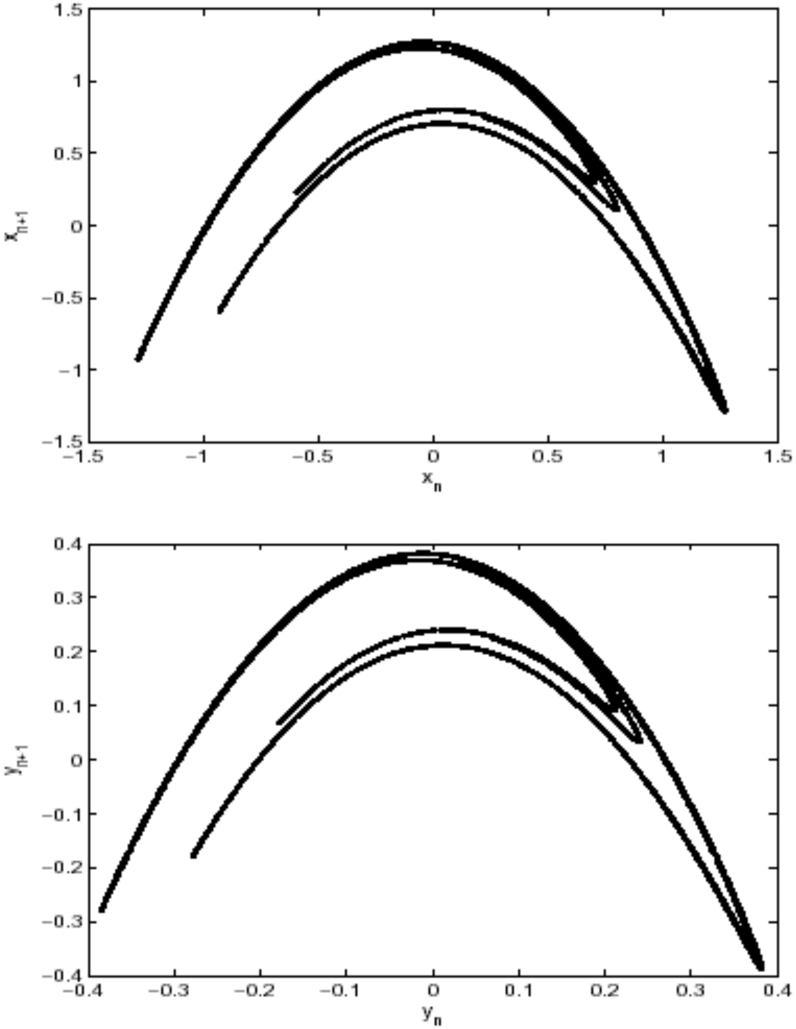


Recall the Henon Map

$$\begin{aligned} x_{n+1} &= y_n + 1 - a x_n^2 \\ y_{n+1} &= b x_n \end{aligned}$$

Delay reconstruction

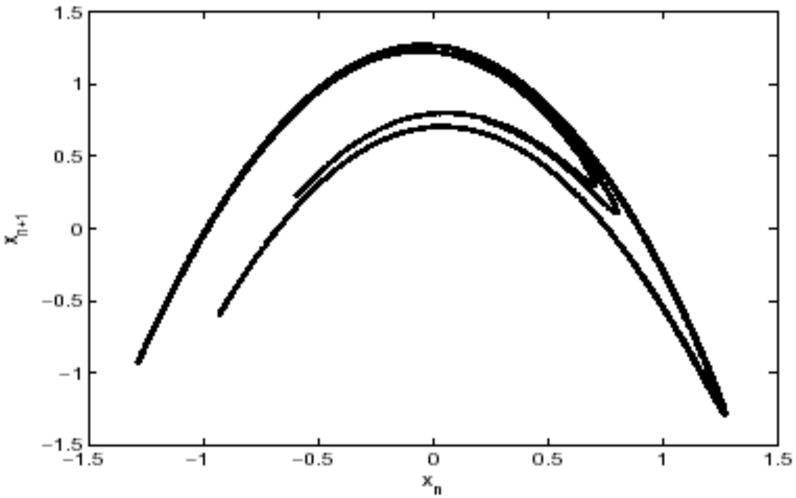
What if we only observe one of the state variable, x or y?

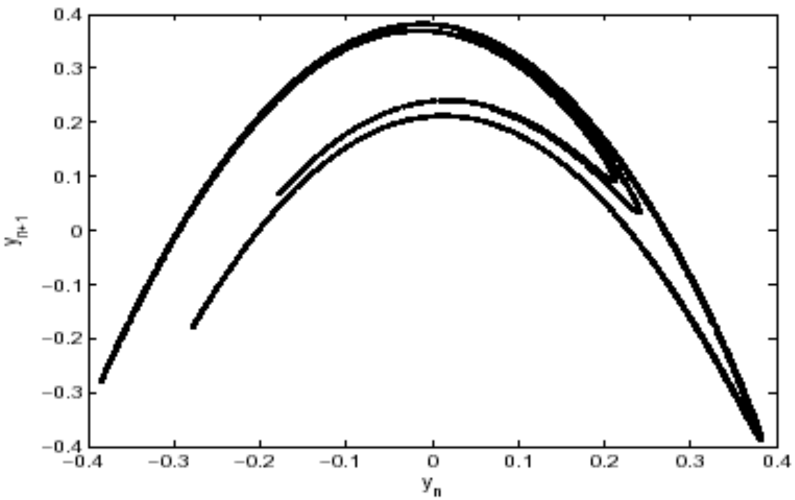


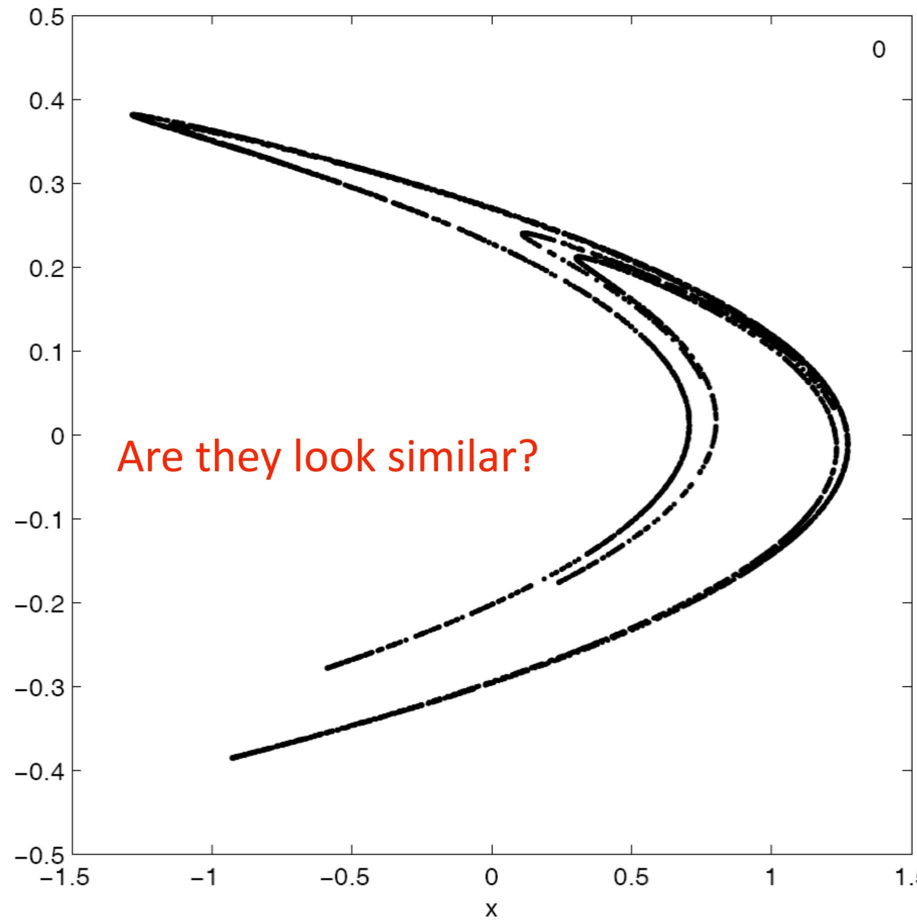
Are they look similar?

Delay reconstruction

What if we only observe one of the state variable, x or y?







Delay reconstruction

In reality, we never have access to all the state space variables. Often only a few (or even one) components are measured.

For scalar time series, for example, the observation space $\mathbb{O}$ is one dimensional. Often this space is not sufficient to express the range of dynamics of the system unambiguously.

Rather than model in observation space $\mathbb{O}$ it is therefore usual to reconstruct the dynamics of the system in a further space: the modelling space $\mathbb{M}$.

If we measure only a scalar value, how can we construct a higher dimensional model state space $\mathbb{M}$ in order to mimic the true state space $\mathbb{S}$?

Delay reconstruction

Model states **y** can be generated from the observations using a reconstruction function H of all observations:

$$\mathrm{y}_i = H(s_1, s_2, \ldots, s_N, i)$$

The function need not use all values of s_j. It is more usual to use an interval, or segment, of observations to generate y_i, where the size of the interval reflects the assumed complexity of the underlying system.

One of the most popular reconstruction functions is so called Delay Reconstruction.

Delay reconstruction

A delay coordinate function H simply builds an m dimensional vector, $\mathbf{y}(t) \in \mathbf{R}^m$, from m measurements separated by a delay time τ_d. In symbols

$$\begin{aligned}
\mathbf{y}(t) &= H(\mathbf{x}(t)) \\
&= (h(\mathbf{x}(t)), h(\mathbf{x}(t-\tau_{d_1})), \ldots, h(\mathbf{x}(t-\tau_{d_{m-1}}))) \\
&= (s(t), s(t-\tau_{d_1}), \ldots, s(t-\tau_{d_{m-1}})).
\end{aligned}$$

where $\tau_{d_1} < \tau_{d_2} < \ldots < \tau_{d_{m-1}}$. The delay times τ_{d_k} are restricted to being integer multiples of the sampling time τ_s. It is usual to select 'uniform' delay times by setting $\tau_{d_k} = k\tau_d$ so that each delay time is an integer multiple of some delay time τ_d.

How can we be sure the measurement function contains the information we need to reconstruct the dynamics? Enter the Takens' Theorem.....

Takens' Theorem

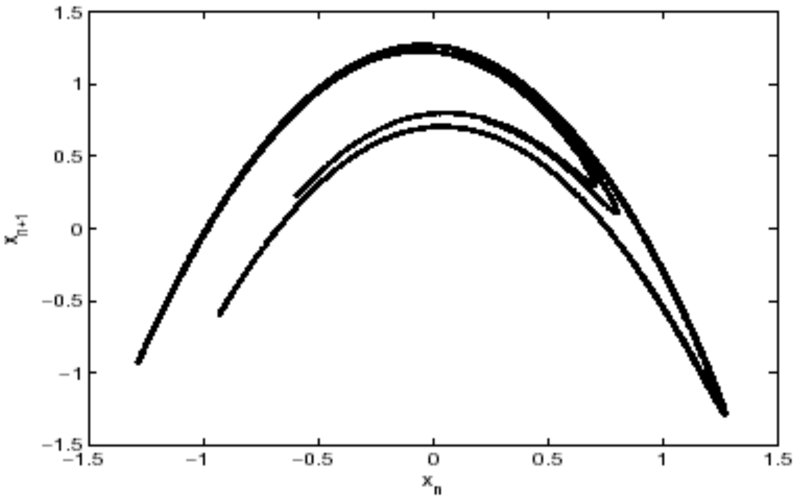
Takens' Theorem[13] tells us that, given a continuous time dynamical system with a compact invariant smooth manifold A, such that A

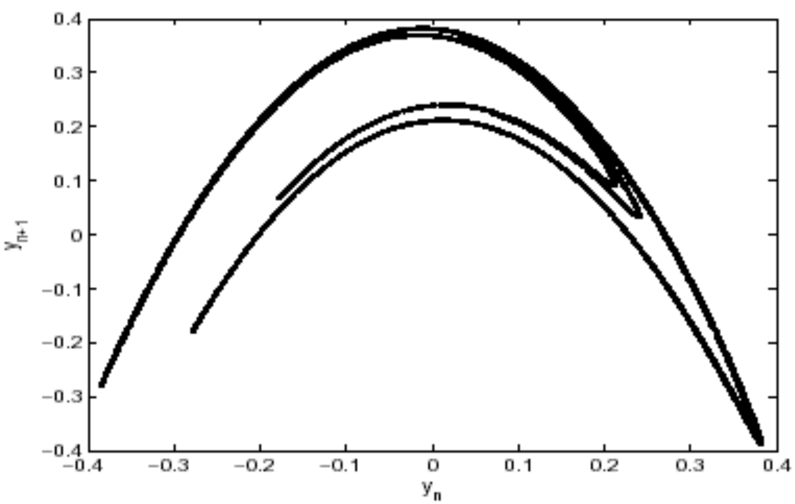
- is of dimension d_A,
- contains only a finite number of equilibria,
- contains no periodic orbits of period τ_d or $2\tau_d$
- contains only a finite number of periodic orbits of period $p\tau_d$, $3 \leq p \leq m$

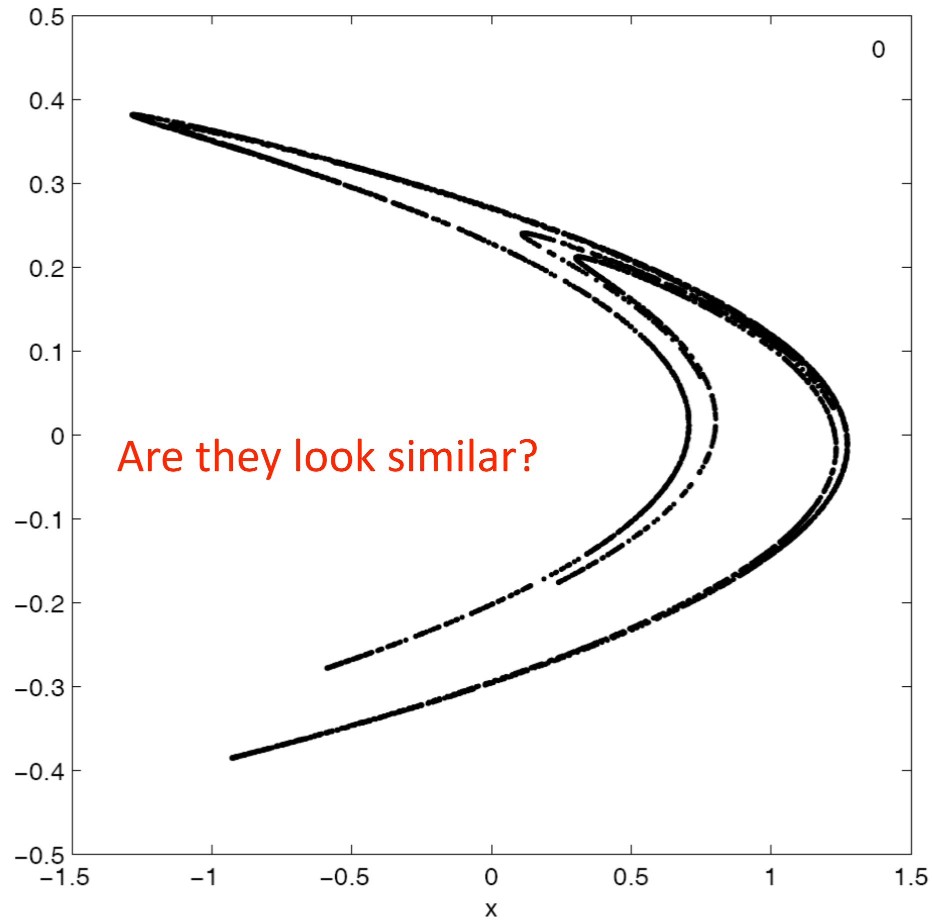
and if the Jacobians of the return maps of those periodic orbits have distinct eigenvalues, then *with probability one*, a C^1 measurement function h will yield a delay coordinate function H which is a differentiable embedding from A to $H(A)$ for $m > 2d_A$.

Delay reconstruction

What if we only observe one of the state variable, x or y?



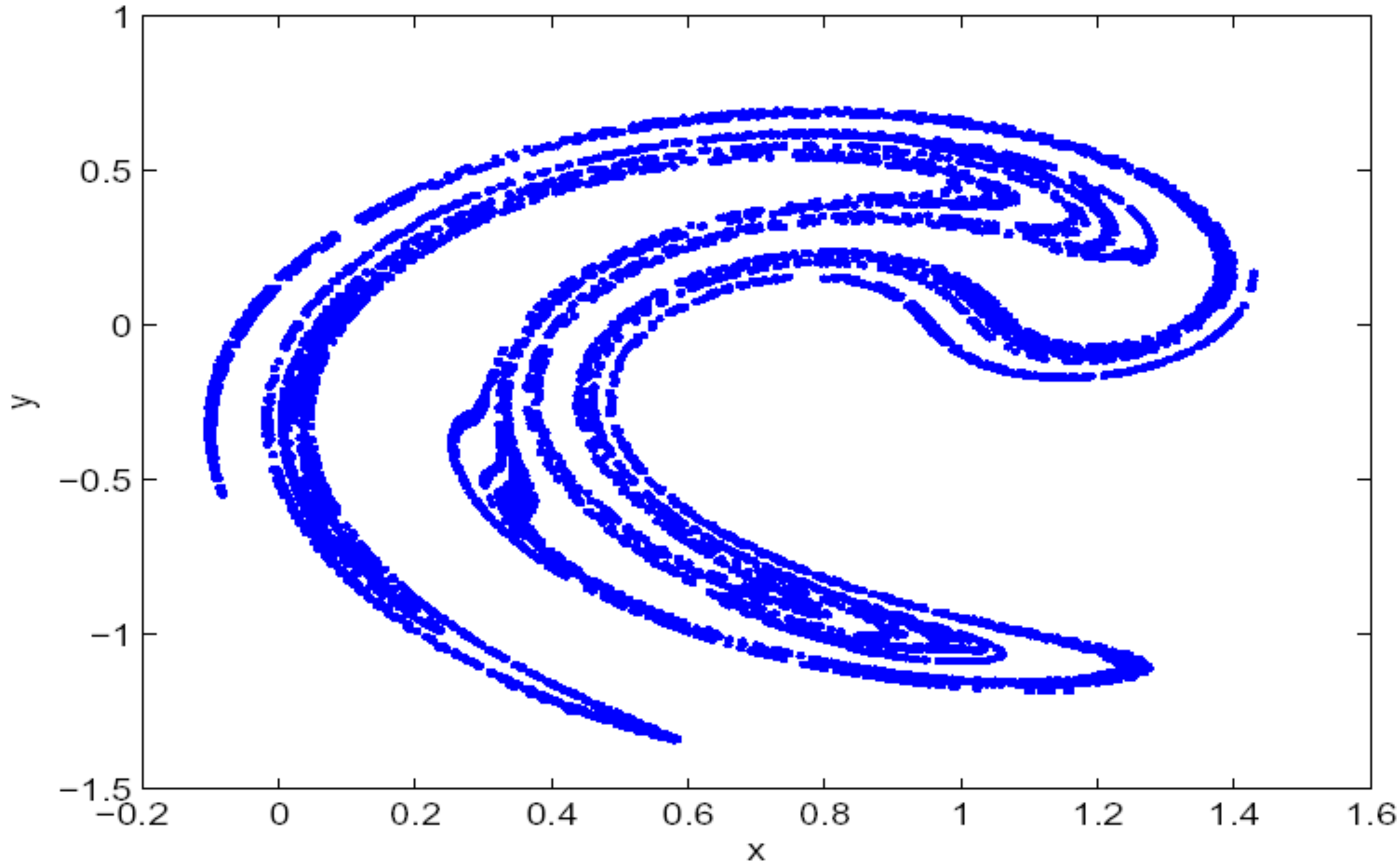


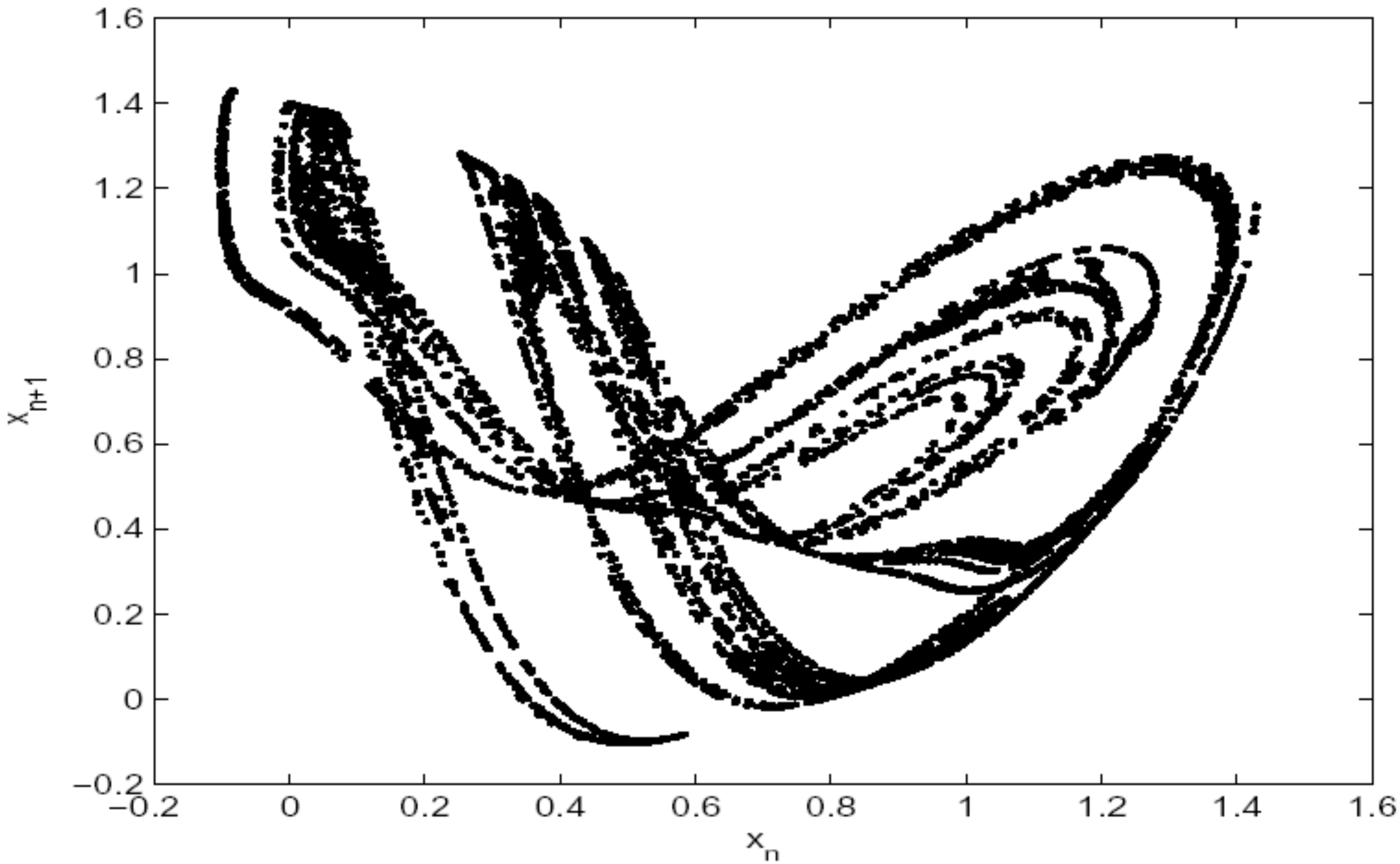


Are they look similar?

Delay reconstruction

Ikeda Map





1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.2
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
x_n
x_{n+1}

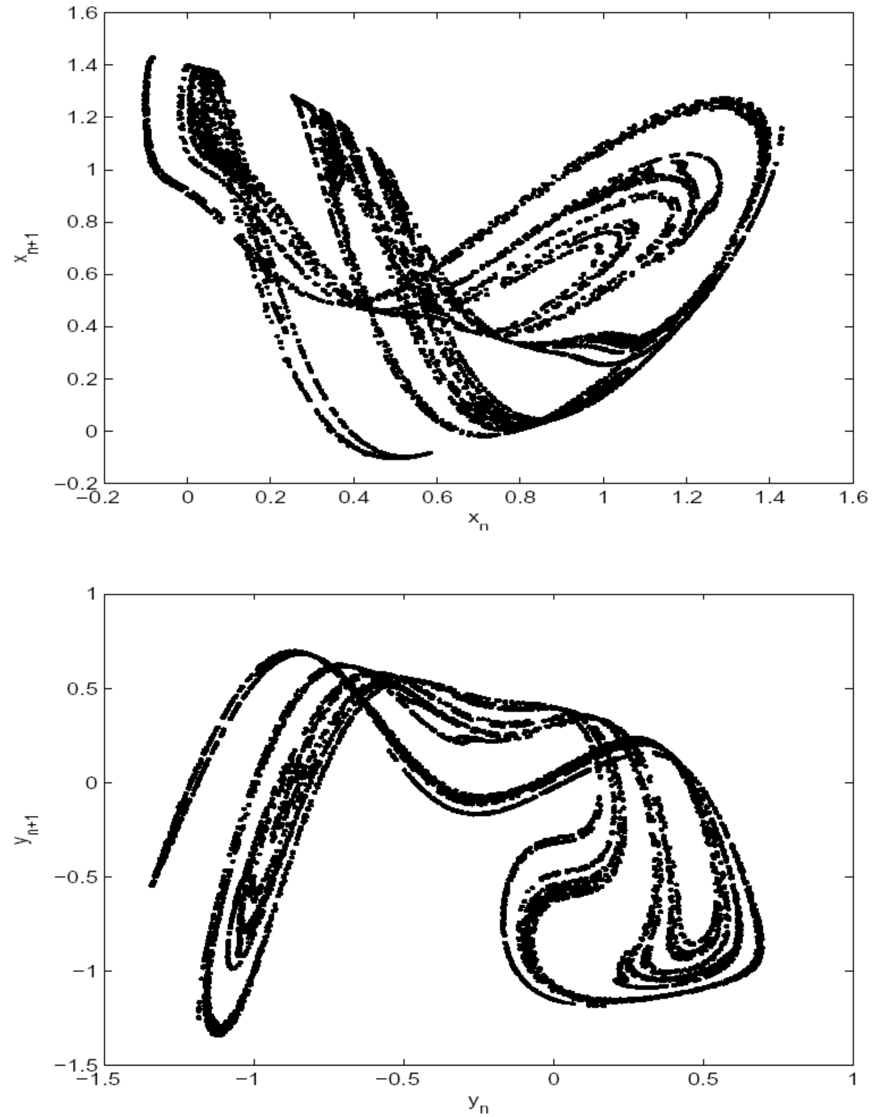


Figure 2: Delay plot of Ikeda map for both x and y

Q: given a one dimensional time series, how can we tell how many delays (how large the model dimension) should be considered?

One simple diagnostic approach is to use Principal Component Analysis....

Principal Component Analysis

1. Principal components are linear combinations of original variables

2. Principal components are orthogonal (uncorrelated)

3. Principal components are “rank” ordered according to the proportion of the total variance they explained, the first few PCs may dominate and therefore leads to dimension reduction.