Lecture 06 – Rotation

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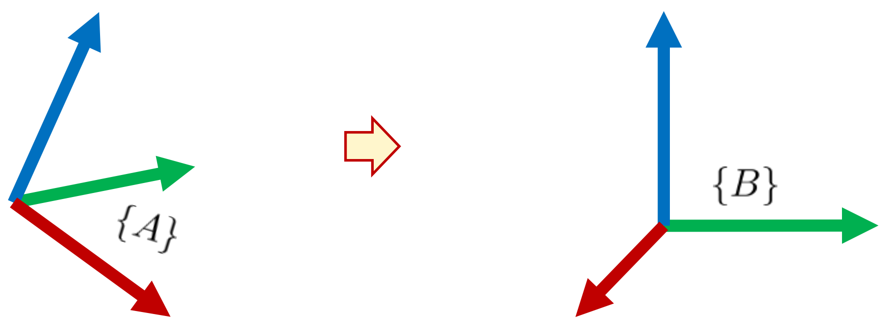
Outline

- About rotation
  - Lie group SO(3)
  - Lie algebra so(3)
  - Quaternion
  - Euler angles
  - Parameter perturbations (Update the variable with a small value)

Rotation matrix

- The rotation matrix consists of three directions of axes transformed into the target frame.

$$
\mathbf{R} = \begin{array}{ccc} \left[\begin{array}{ccc} R_1 & R_2 & R_3 \\ R_4 & R_5 & R_6 \\ R_7 & R_8 & R_9 \end{array}\right] & \begin{array}{c} {}^{a}x_{b} \\ {}^{a}y_{b} \\ {}^{a}z_{b} \end{array} \\ \begin{array}{ccc} {}^{b}x_{a} & {}^{b}y_{a} & {}^{b}z_{a} \end{array} & \end{array}
$$



$$1.\ \mathbf{R}\mathbf{R}^{\mathrm{T}} = \mathbf{I} \qquad 2.\ \det(\mathbf{R}) = 1$$

Rotation matrix

- Special Orthogonal group SO(n) is a set that satisfies

$$SO(n) = \{\mathbf{M} \in \mathbb{R}^{n \times n} | \mathbf{M}\mathbf{M}^{\mathrm{T}} = \mathbf{I}, \det(\mathbf{M}) = 1\}$$

- A 3x3 rotation matrix is in the SO(3) group:

$$\mathbf{R} \in SO(3)$$

What is a group ?

Group

- A **group** G is a **set** with **a binary operation** $\circ$ defined on the elements of G , if it satisfies :
  - Closure :

    $$g_1 \circ g_2 \in G$$

  - Identity :

    $$e \circ g = g \circ e = g$$

  - Inverse :

    $$g \circ g^{-1} = g^{-1} \circ g = e$$

  - Associativity (结合律）:

    $$g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3$$

Group

- Examples of a group
  - Real number + Multiplication
    - Identity element : 1
    - Inverse : $x \leftrightarrow 1/x$
    - Associativity : $(x \times y) \times z = x \times (y \times z)$
  - Real number + Addition
    - Identity element : 0
    - Inverse : $x \leftrightarrow -x$
    - Associativity : $(x + y) + z = x + (y + z)$
  - Matrices + Matrix multiplication
    - Identity element : $\mathbf{I}$
    - Inverse : $\mathbf{A} \leftrightarrow \mathbf{A}^{-1}$
    - Associativity : $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$

Lie group – SO(3)

- The **elements** of SO(3) are the **rotation matrices**. The **group operation** of SO(3) is the **matrix multiplication**.

$$SO(3) = \{\mathbf{R} | \mathbf{R}\mathbf{R}^{\mathrm{T}} = \mathbf{I} \det(\mathbf{R}) = 1\}$$

- We can verify the following axioms:
  - Closure : $\left.\begin{array}{l}(\mathbf{R}_1\mathbf{R}_2)(\mathbf{R}_1\mathbf{R}_2)^{\mathrm{T}} = \mathbf{I} \\ \det(\mathbf{R}_1\mathbf{R}_2) = \det(\mathbf{R}_1)\det(\mathbf{R}_2) = 1\end{array}\right\} \quad \mathbf{R}_1\mathbf{R}_2 \in SO(3)$
  - Identity : $\mathbf{I} \cdot \mathbf{R} = \mathbf{R} \cdot \mathbf{I} = \mathbf{R}$
  - Inverse : $\mathbf{R} \cdot \mathbf{R}^{-1} = \mathbf{R}^{-1} \cdot \mathbf{R} = \mathbf{I}$
  - Associativity: $(\mathbf{R}_1\mathbf{R}_2)\mathbf{R}_3 = \mathbf{R}_1(\mathbf{R}_2\mathbf{R}_3)$

Lie group - SO(3)

- Question : Think about if we change the group operation into the matrix addition, is this new algebra structure a SO(3) group? And why?

$$\mathbf{AB} \rightarrow \mathbf{A} + \mathbf{B}$$

- About '**special**' : $\det(\mathbf{R}) = 1$

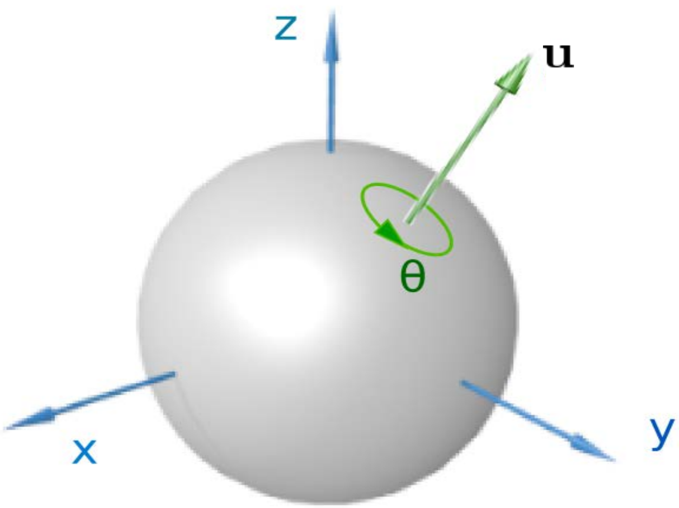
$$\rightarrow \quad \det(\mathbf{R}) = \boldsymbol{r}_1^{\mathrm{T}}(\boldsymbol{r}_2 \times \boldsymbol{r}_3) = 1$$

$$\rightarrow \quad \boldsymbol{r}_1 = \boldsymbol{r}_2 \times \boldsymbol{r}_3$$

$\rightarrow$ Right-handed coordinate frame

Lie algebra - so(3)

- Intuitively, a rotation can be described as a spinning action about an axis $\mathbf{u}, (\|\mathbf{u}\| = 1)$ with some amount of angle θ .



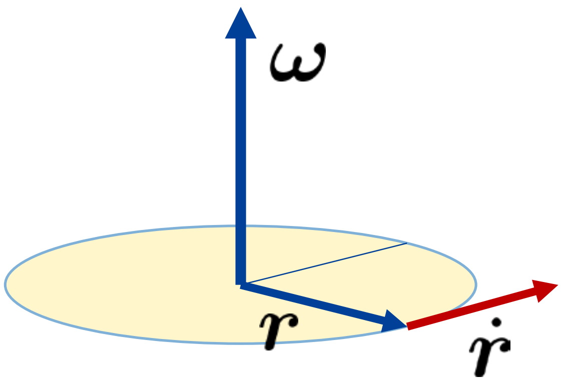
$$\begin{bmatrix} \theta \\ \boldsymbol{u} \end{bmatrix} \rightarrow \boxed{\theta \boldsymbol{u}} \in \mathbb{R}^{3\times 1} \rightarrow \boxed{\boldsymbol{\omega} \leftrightarrow \mathbf{R}}$$

?

Axis-angle representation

Exponential map

- How do we relate the rotation axis and angle to the rotation matrix?



$$\dot{\boldsymbol{r}} = \boldsymbol{\omega} \times \boldsymbol{r}$$

$$\frac{d\boldsymbol{r}}{dt} = \boldsymbol{\omega} \times \boldsymbol{r} = \boldsymbol{\omega}^{\wedge}\boldsymbol{r}$$

$$\boldsymbol{r}_t = \underbrace{\boxed{\exp((t\boldsymbol{\omega})^{\wedge})}}_{\mathbf{R}}\boldsymbol{r}_0$$

Here $\boldsymbol{x}^{\wedge} = [\boldsymbol{x}]_{\times}$ is a 3x3 skew symmetric matrix, $(\boldsymbol{x}^{\wedge})^{\mathrm{T}} = \boldsymbol{x}^{\wedge}$

Exponential map

- We can use an exponential map to map a skew symmetric matrix $\boldsymbol{x}^\wedge$ to a rotation matrix :

$$\mathbf{R} = \exp(\boldsymbol{x}^\wedge)$$

- Here $\exp : so(3) \rightarrow SO(3)$

$$\boldsymbol{x} \xrightarrow{(\cdot)^\wedge} [\boldsymbol{x}]_\times \xrightarrow{\exp(\cdot)} \mathbf{R}$$

- Note that $\boldsymbol{x}^\wedge \in so(3)$ - lie algebra for 3x3 skew symmetric matrices.

$$so(n) = \{\mathbf{S} \in \mathbb{R}^{n\times n} | \mathbf{S}^{\mathrm{T}} = -\mathbf{S}\}$$

Lie algebra – so(3)

- Lie algebra – A **vector space** with a binary operation named as **Lie bracket**, $[\cdot, \cdot] : G \times G \to G$
  - Bilinearity : given two scalars a, b

$$[a\boldsymbol{x} + b\boldsymbol{y}, \boldsymbol{z}] = a[\boldsymbol{x}, \boldsymbol{z}] + b[\boldsymbol{y}, \boldsymbol{z}]$$
$$[\boldsymbol{z}, a\boldsymbol{x} + b\boldsymbol{y}] = a[\boldsymbol{z}, \boldsymbol{x}] + b[\boldsymbol{z}, \boldsymbol{y}]$$

  - Alternating :

$$[\boldsymbol{x}, \boldsymbol{x}] = 0$$

  - Jacobi Identity

$$[\boldsymbol{x}, [\boldsymbol{y}, \boldsymbol{z}]] + [\boldsymbol{z}, [\boldsymbol{x}, \boldsymbol{y}]] + [\boldsymbol{y}, [\boldsymbol{z}, \boldsymbol{x}]] = 0$$

  - Anti-commutativity

$$[\boldsymbol{x}, \boldsymbol{y}] = -[\boldsymbol{y}, \boldsymbol{x}]$$

Lie algebra – so(3)

- Example 1. $G=\mathbb{R}^3$, with the bracket defined as

$$[\boldsymbol{x},\boldsymbol{y}]=\boldsymbol{x}\times\boldsymbol{y}$$

- Example 2. 3x3 skew-symmetric matrix : $G=\{\Phi=\boldsymbol{x}^\wedge\}$

$$(\boldsymbol{x}^\wedge)^{\mathrm{T}}=\boldsymbol{x}^\wedge$$

$$\boldsymbol{x}^\wedge=[\boldsymbol{x}]_\times=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}^\wedge=\begin{bmatrix}0 & -x_3 & x_2\\x_3 & 0 & -x_1\\-x_2 & x_1 & 0\end{bmatrix}$$

with the bracket operation defined as

$$[\Phi_1,\Phi_2]=\Phi_1\Phi_2-\Phi_2\Phi_1 \quad (\Phi_1=\phi_1^\wedge,\Phi_2=\phi_2^\wedge)$$

Rodrigues' formula

- Given an axis-angle vector $\boldsymbol{\omega}$, we have its rotation matrix being

$$\mathbf{R} = \exp(\boldsymbol{\omega}^\wedge) = \exp(\theta[\boldsymbol{u}]_\times)$$

- By Taylor expansion, we have

$$\exp(\theta[\boldsymbol{u}]_\times) = \mathbf{I} + \theta[\boldsymbol{u}]_\times + \frac{\theta^2[\boldsymbol{u}]_\times^2}{2!} + \frac{\theta^3[\boldsymbol{u}]_\times^3}{3!} + \cdots$$

$$\boxed{\begin{aligned} [\boldsymbol{u}]_\times^2 &= \boldsymbol{u}\boldsymbol{u}^\mathrm{T} - \mathbf{I} \\ [\boldsymbol{u}]_\times^3 &= -[\boldsymbol{u}]_\times \end{aligned}}$$

$$[\mathbf{u}]_\times^4 = -[\mathbf{u}]_\times^2 \qquad [\mathbf{u}]_\times^5 = [\mathbf{u}]_\times \qquad [\mathbf{u}]_\times^6 = [\mathbf{u}]_\times^2 \qquad [\mathbf{u}]_\times^7 = -[\mathbf{u}]_\times \quad \cdots$$

Rodrigues' formula

- We therefore obtain

$$\exp(\theta[\boldsymbol{u}]_\times) = \mathbf{I} + \underbrace{\boxed{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)}}_{\sin(\theta)}[\boldsymbol{u}]_\times + \underbrace{\boxed{\left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} + \cdots\right)}}_{1-\cos(\theta)}[\boldsymbol{u}]_\times^2$$

- Finally we have the Rodrigues' formula

$$\mathbf{R} = \mathbf{I} + \sin(\theta)[\boldsymbol{u}]_\times + (1-\cos(\theta))[\boldsymbol{u}]_\times^2$$

- or

$$\mathbf{R} = \cos(\theta)\mathbf{I} + \sin(\theta)[\boldsymbol{u}]_\times + (1-\cos(\theta))\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}} \quad ([\boldsymbol{u}]_\times^2 = \boldsymbol{u}\boldsymbol{u}^{\mathrm{T}} - \mathbf{I})$$

Logarithmic maps

- The logarithmic map is the inverse of the exponential map

$$\log : SO(3) \to so(3) \qquad \log(\mathbf{R}) \to [\boldsymbol{u}\theta]_\times$$

$$\phi = \arccos\left(\frac{\operatorname{trace}(\mathbf{R}) - 1}{2}\right)$$

$$\mathbf{u} = \frac{(\mathbf{R} - \mathbf{R}^\top)^\vee}{2\sin\phi},$$

- Here, $(\cdot)^\vee$ is the inverse of $(\cdot)^\wedge$, namely $([\boldsymbol{x}]_\times)^\vee = \boldsymbol{x}$

$$\boldsymbol{x} \xleftarrow{(\cdot)^\vee} [\boldsymbol{x}]_\times \xleftarrow{\log(\cdot)} \mathbf{R}$$

Conversion

parameter

$$\text{vectorspace} : \phi \in \mathbb{R}^3$$
$$\text{field} : \mathbb{R}$$
$$\text{Cross product} : \phi_1 \times \phi_2$$

$\cdot^\wedge$ $\cdot^\vee$

Lie algebra

$$\begin{aligned} \text{vectorspace:} \quad & \mathfrak{so}(3) = \{\boldsymbol{\Phi} = \boldsymbol{\phi}^\wedge \in \mathbb{R}^{3\times 3} \mid \boldsymbol{\phi} \in \mathbb{R}^3, \}, \\ \text{field:} \quad & \mathbb{R}, \\ \text{Lie bracket:} \quad & [\boldsymbol{\Phi}_1, \boldsymbol{\Phi}_2] = \boldsymbol{\Phi}_1\boldsymbol{\Phi}_2 - \boldsymbol{\Phi}_2\boldsymbol{\Phi}_1, \end{aligned}$$

$\exp(\cdot)$ $\log(\cdot)$

$\mathrm{Exp}(\cdot)$ $\mathrm{Log}(\cdot)$

Lie group

$$SO(3) = \{\mathbf{C} \in \mathbb{R}^{3\times 3} \mid \mathbf{C}\mathbf{C}^T = \mathbf{1}, \det \mathbf{C} = 1\}$$

Summary

- Group
- Lie group – SO(3) : $SO(3) = \{\mathbf{R} \in \mathbb{R}^{3\times 3} \mid \mathbf{R}\mathbf{R}^{\mathrm{T}} = \mathbf{I}, \det(\mathbf{R}) = 1\}$
- Lie algebra – so(3) : $so(3) = \{\Phi \in \mathbb{R}^{3\times 3} \mid \Phi^{\mathrm{T}} = -\Phi\}$
- Exponential map : $\exp : so(3) \to SO(3)$
- Rodrigues formula : $\mathbf{R} = \cos(\theta)\mathbf{I} + \sin(\theta)[\boldsymbol{u}]_{\times} + (1-\cos(\theta))\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$
- Logarithmic map : $\log : SO(3) \to so(3)$

$$\boldsymbol{x} \xrightarrow{(\cdot)^{\wedge}} [\boldsymbol{x}]_{\times} \xrightarrow{\exp(\cdot)} \mathbf{R}$$

$$\boldsymbol{x} \xleftarrow{(\cdot)^{\vee}} [\boldsymbol{x}]_{\times} \xleftarrow{\log(\cdot)} \mathbf{R}$$

$$\mathbb{R}^3 \qquad\qquad so(3) \qquad\qquad SO(3)$$

Quaternion

- What is a quaternion?
  - A quaternion is a vector with four components:
  - A composite of a scalar and three different imaginary parts
- Mathematically, $\mathbf{q} = q_w + \underbrace{q_x\mathbf{i} + q_y\mathbf{j} + q_z\mathbf{k}}_{\text{Imaginary part}}$

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

$$\mathbf{ij} = \mathbf{k}, \mathbf{jk} = \mathbf{i}, \mathbf{ki} = \mathbf{j}$$

Quaternion

- Quaternion - an extension of a complex number
  - **What we learn from a complex number also applies to a quaternion**

- General quaternions : $\mathbf{q} = q_w + \mathbf{q}_v = \begin{bmatrix} q_w \\ \mathbf{q}_v \end{bmatrix}$

- Real quaternions : $q_w = \begin{bmatrix} q_w \\ \mathbf{0}_v \end{bmatrix}$

- Pure quaternions (3D vectors): $\mathbf{q}_v = \begin{bmatrix} 0 \\ \mathbf{q}_v \end{bmatrix}$

Quaternion

- The sum of two quaternion

$$\mathbf{p}+\mathbf{q}=\begin{bmatrix}p_w\\p_x\\p_y\\p_z\end{bmatrix}+\begin{bmatrix}q_w\\q_x\\q_y\\q_z\end{bmatrix}=\begin{bmatrix}p_w+q_w\\p_x+q_x\\p_y+q_y\\p_z+q_z\end{bmatrix}$$

- It is commutative and associative

$$\mathbf{p}+\mathbf{q}=\mathbf{q}+\mathbf{p}$$
$$\mathbf{p}+(\mathbf{q}+\mathbf{r})=(\mathbf{p}+\mathbf{q})+\mathbf{r}$$

Quaternion

- Identity : $\mathbf{q}_1 = 1 = \begin{bmatrix} 1 \\ \mathbf{0}_v \end{bmatrix}$

- Conjugate : $\mathbf{q}* = \begin{bmatrix} q_w \\ -\mathbf{q}_v \end{bmatrix}$ $\quad (\mathbf{p} \otimes \mathbf{q})^* = \mathbf{q}^* \otimes \mathbf{p}^*$

- Norm : $\|\mathbf{q}\| \triangleq \sqrt{\mathbf{q} \otimes \mathbf{q}^*} = \sqrt{\mathbf{q}^* \otimes \mathbf{q}} = \sqrt{q_w^2 + q_x^2 + q_y^2 + q_z^2} \in \mathbb{R}$

- Inverse : $\mathbf{q}^{-1} = \mathbf{q}^* / \|\mathbf{q}\|^2 \qquad \mathbf{q} \otimes \mathbf{q}^{-1} = \mathbf{q}^{-1} \otimes \mathbf{q} = \mathbf{q}_1$

- Dot product : $\mathbf{p}^T\mathbf{q} = p_w q_w + p_x q_x + p_y q_y + p_z q_z$

  $= \mathbf{p} \otimes \mathbf{q}^* \text{ or } = \mathbf{p}^* \otimes \mathbf{q}$

Quaternion

- Product of two quaternions $\mathbf{p} = p_w + p_x\mathbf{i} + p_y\mathbf{j} + p_z\mathbf{k}$

$$\mathbf{q} = q_w + q_x\mathbf{i} + q_y\mathbf{j} + q_z\mathbf{k}$$

$$\mathbf{p} \otimes \mathbf{q} = (p_w + p_x\mathbf{i} + p_y\mathbf{j} + p_z\mathbf{k})(q_w + q_x\mathbf{i} + q_y\mathbf{j} + q_z\mathbf{k})$$

$$\Rightarrow \quad \mathbf{p} \otimes \mathbf{q} = \begin{bmatrix} p_w q_w - p_x q_x - p_y q_y - p_z q_z \\ p_w q_x + p_x q_w + p_y q_z - p_z q_y \\ p_w q_y - p_x q_z + p_y q_w + p_z q_x \\ p_w q_z + p_x q_y - p_y q_x + p_z q_w \end{bmatrix}$$

  - Not commutative: $\mathbf{p} \otimes \mathbf{q} \neq \mathbf{q} \otimes \mathbf{p}$
  - Associative
  - Distributive over sum
  - Bi-linear

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Quaternion

- Left/right quaternion-product

$$[\mathbf{q}]_L = \begin{bmatrix} q_w & -q_x & -q_y & -q_z \\ q_x & q_w & -q_z & q_y \\ q_y & q_z & q_w & -q_x \\ q_z & -q_y & q_x & q_w \end{bmatrix}, \qquad [\mathbf{q}]_R = \begin{bmatrix} q_w & -q_x & -q_y & -q_z \\ q_x & q_w & q_z & -q_y \\ q_y & -q_z & q_w & q_x \\ q_z & q_y & -q_x & q_w \end{bmatrix}$$

- Or briefly $[\mathbf{q}]_L = q_w \mathbf{I} + \begin{bmatrix} 0 & -\mathbf{q}_v^\top \\ \mathbf{q}_v & [\mathbf{q}_v]_\times \end{bmatrix}, \qquad [\mathbf{q}]_R = q_w \mathbf{I} + \begin{bmatrix} 0 & -\mathbf{q}_v^\top \\ \mathbf{q}_v & -[\mathbf{q}_v]_\times \end{bmatrix}$

where $\mathbf{q}_v = [q_x, q_y, q_z]^T$

$$\boxed{\mathbf{p} \otimes \mathbf{q} = [\mathbf{p}]_L \, \mathbf{q} = [\mathbf{q}]_R \, \mathbf{p}}$$

Quaternion

- Quaternion multiplication matrix is orthogonal :

$$[\mathbf{q}]_L[\mathbf{q}]_L^T = \mathbf{I}_{4\times 4} \qquad [\mathbf{p}]_R[\mathbf{p}]_R^T = \mathbf{I}_{4\times 4}$$

- Dot product is preserved ：

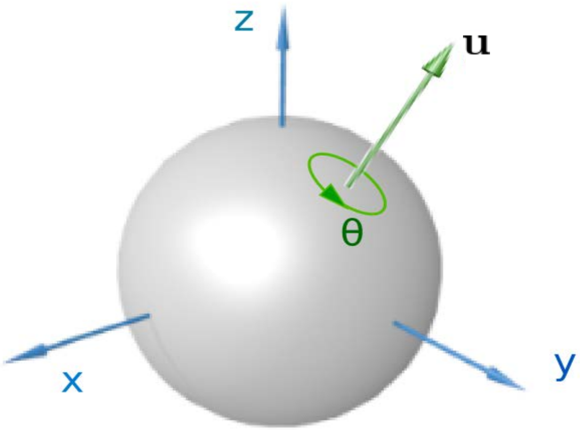
$$\mathbf{r}^{\mathrm{T}}\mathbf{t} = ([\mathbf{q}]_L\mathbf{r})^{\mathrm{T}}([\mathbf{q}]_L\mathbf{r}) = (\mathbf{q}\otimes\mathbf{r})^{\mathrm{T}}(\mathbf{q}\otimes\mathbf{t})$$

$$\mathbf{r}^{\mathrm{T}}\mathbf{t} = ([\mathbf{q}]_R\mathbf{r})^{\mathrm{T}}([\mathbf{q}]_R\mathbf{r}) = (\mathbf{r}\otimes\mathbf{q})^{\mathrm{T}}(\mathbf{t}\otimes\mathbf{q})$$

Unit quaternion

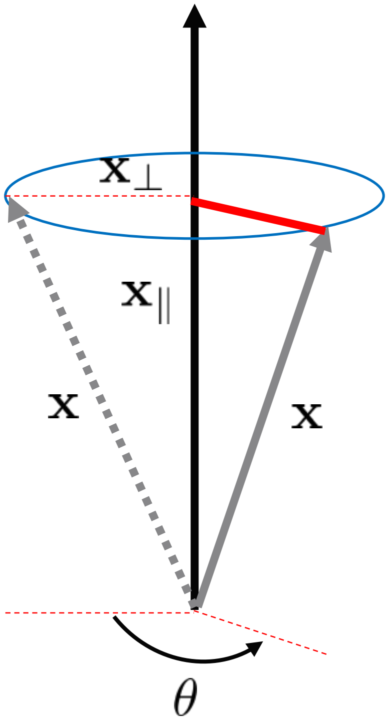
- For unit quaternion, we have $\|\mathbf{q}\| = 1$ and $\mathbf{q}^{-1} = \mathbf{q}^*$
- The unit quaternions also forms a group with respect to quaternion multiplication.
- For a rotation $\mathbf{R} = \exp([\boldsymbol{u}\theta]^\wedge)$, its corresponding unit quaternion is given by

$$\mathbf{q} = \begin{bmatrix} \cos(\theta/2) \\ \mathbf{u}\sin(\theta/2) \end{bmatrix}$$



Vector rotation

- Transformation of a vector by an axis-angle $\boldsymbol{u}, \theta$



$$\mathbf{x}_{||} = \mathbf{u}\,\mathbf{u}^\top \mathbf{x}$$

$$\mathbf{x}_\perp = \mathbf{x} - \mathbf{u}\,\mathbf{u}^\top \mathbf{x}$$

⇨ $$\mathbf{x}'_\perp = \mathbf{x}_\perp \cos(\theta) + (\mathbf{u} \times \mathbf{x}) \sin(\theta)$$

⇨ $$\mathbf{x}' = \mathbf{x}'_\perp + \mathbf{x}_{||}$$

⇨ $$\mathbf{x}' = \mathbf{x}_\perp \cos(\theta) + (\mathbf{u} \times \mathbf{x}) \sin(\theta) + \mathbf{x}_{||}$$

⇨ $$\mathbf{x}' = (\cos(\theta)\mathbf{I} + \sin(\theta)[\boldsymbol{u}]_\times + (1 - \cos(\theta))\boldsymbol{u}\boldsymbol{u}^\mathrm{T})\mathbf{x}$$

Unit quaternion

- Transformation of a vector

$$\mathbf{x}' = \mathbf{q} \otimes \mathbf{x} \otimes \mathbf{q}^*$$

$$\mathbf{q} = \begin{bmatrix} \cos(\theta/2) \\ \mathbf{u}\sin(\theta/2) \end{bmatrix}$$

$$\mathbf{x}' = \mathbf{x}_\perp \cos(\theta) + (\mathbf{u} \times \mathbf{x})\sin(\theta) + \mathbf{x}_{||}$$

Unit quaternion

- Convert a unit quaternion to a rotation matrix

$$\mathbf{x}' = \mathbf{q} \otimes \mathbf{x} \otimes \mathbf{q}^*$$

$$\Downarrow$$

$$\mathbf{x}' = \mathbf{R}\mathbf{x}$$

$$\boxed{\mathbf{R}\{\mathbf{q}\} = (q_w^2 - \mathbf{q}_v^\top \mathbf{q}_v)\,\mathbf{I} + 2\,\mathbf{q}_v\mathbf{q}_v^\top + 2\,q_w\,[\mathbf{q}_v]_\times}$$

Summary

- Quaternion : $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$
- Quaternion multiplication :

$$\mathbf{p} \otimes \mathbf{q} = (p_w + p_x\mathbf{i} + p_y\mathbf{j} + p_z\mathbf{k})(q_w + q_x\mathbf{i} + q_y\mathbf{j} + q_z\mathbf{k})$$

$$\mathbf{p} \otimes \mathbf{q} = [\mathbf{p}]_L \, \mathbf{q} = [\mathbf{q}]_R \, \mathbf{p}$$

- Unit quaternion : $\mathbf{q} = \begin{bmatrix} \cos(\theta/2) \\ \mathbf{u}\sin(\theta/2) \end{bmatrix}$
- Vector rotation : $\mathbf{x}' = \mathbf{x}'_{\perp} + \mathbf{x}_{||}$
- Vector rotation by a unit quaternion : $\mathbf{x}' = \mathbf{q} \otimes \mathbf{x} \otimes \mathbf{q}^*$

$$\mathbf{x}' = \mathbf{x}_{\perp} \cos(\theta) + (\mathbf{u} \times \mathbf{x}) \sin(\theta) + \mathbf{x}_{||}$$

- Convert a quaternion to a rotation matrix

$$\mathbf{R}\{\mathbf{q}\} = (q_w^2 - \mathbf{q}_v^\top \mathbf{q}_v)\,\mathbf{I} + 2\,\mathbf{q}_v\mathbf{q}_v^\top + 2\,q_w\,[\mathbf{q}_v]_\times$$

Rotation matrix vs Quaternion

	Rotation matrix, $\mathbf{R}$	Quaternion, $\mathbf{q}$
Parameters	$3 \times 3 = 9$	$1 + 3 = 4$
Degrees of freedom	3	3
Constraints	6	1
Constraints	$\mathbf{R}\mathbf{R}^\top = \mathbf{I}\,,\ \det(\mathbf{R}) = +1$	$\mathbf{q} \otimes \mathbf{q}^* = 1$
Identity	$\mathbf{I}$	1
Inverse	$\mathbf{R}^\top$	$\mathbf{q}^*$
Composition	$\mathbf{R}_1\,\mathbf{R}_2$	$\mathbf{q}_1 \otimes \mathbf{q}_2$
Rotation operator	$\mathbf{R} = \mathbf{I} + \sin\phi\,[\mathbf{u}]_\times + (1-\cos\phi)\,[\mathbf{u}]_\times^2$	$\mathbf{q} = \cos\phi/2 + \mathbf{u}\sin\phi/2$
Rotation action	$\mathbf{R}\,\mathbf{x}$	$\mathbf{q} \otimes \mathbf{x} \otimes \mathbf{q}^*$

Euler angle

- Use three angles to represent a rotation $(\psi, \theta, \phi) \rightarrow \mathbf{R}$
- Rotation about the x axis

$$\mathbf{R}(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

- Rotation about the y axis

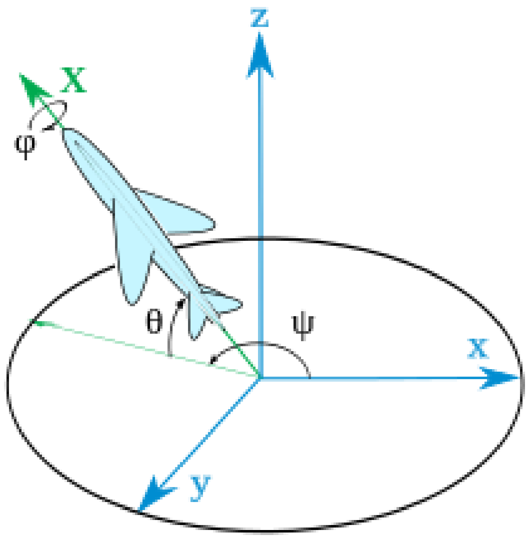
$$\mathbf{R}(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -sin\theta & 0 & \cos\theta \end{bmatrix}$$

- Rotation about the z axis

$$\mathbf{R}(\psi) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Euler angle

- Tait-Bryan convention （z-y-x) : Yaw-pitch-roll
- Describes the pose of an aerial vehicle in the ENU coordinate system



$$\mathbf{R}_{wb} = \mathbf{R}_z(\psi)\mathbf{R}_y(\theta)\mathbf{R}_x(\phi)$$

Euler angle

- A quaternion to a z-y-x Euler angle

$$\boldsymbol{q}_{wb} = \begin{bmatrix} \cos(\psi/2) \\ 0 \\ 0 \\ \sin(\psi/2) \end{bmatrix} \begin{bmatrix} \cos(\theta/2) \\ 0 \\ \sin(\theta/2) \\ 0 \end{bmatrix} \begin{bmatrix} \cos(\phi/2) \\ \sin(\phi/2) \\ 0 \\ 0 \end{bmatrix}$$

- A z-y-x Euler angle to a quaternion

$$\begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} = \begin{bmatrix} \mathrm{atan2}(2(q_w q_x + q_y q_z), 1 - 2(q_x^2 + q_y^2)) \\ \mathrm{asin}(2(q_w q_y - q_z q_x)) \\ \mathrm{atan2}(2(q_w q_z + q_x q_y), 1 - 2(q_y^2 + q_z^2)) \end{bmatrix}$$

About parameter perturbations

- As we mentioned in the previous lecture, in an iterative solver, we need to update the variable repeatedly :

$$\boldsymbol{x} \leftarrow \boldsymbol{x} + \Delta \boldsymbol{x}$$

- If the variable is not in a vector space, like a rotation, it cannot be updated by a simple vector addition :

$$\boldsymbol{x} \not\leftarrow \boldsymbol{x} + \Delta x$$

- Sometimes, the dimension of the parameter space of a rotation is not the same as the dimension of the parameters :

$$\boldsymbol{x} \in \mathbb{R}^m; \quad \Delta \boldsymbol{x} \in \mathbb{R}^n$$

$$m \neq n$$

Parameter perturbations

- We define the operator $\boxplus$ to represent "**adding**" a perturbation to our parameters.
- Suppose the parameter space is denoted by $\mathcal{X}$ and the perturbation space is $\mathbb{R}^n$. The operator is a mapping defined as :

$$\boxplus : \mathcal{X} \times \mathbb{R}^n \rightarrow \mathcal{X}$$

Parameter perturbations

- If the parameter space is a vector space, $\mathcal{X} = \mathbf{R}^n$,

$$\boldsymbol{x} \boxplus \Delta\boldsymbol{x} = \boldsymbol{x} + \Delta\boldsymbol{x}$$

- If the parameter is a Lie group (like rotation).

$$\boldsymbol{x} \boxplus \Delta\boldsymbol{x} = \boldsymbol{x} \otimes \exp(\Delta\boldsymbol{x}^\wedge) \quad \text{(Right multiplication)}$$

$$= \exp(\Delta\boldsymbol{x}^\wedge) \otimes \boldsymbol{x} \quad \text{(Left multiplication)}$$

- Here $\boldsymbol{x} \in SO(3), \Delta\boldsymbol{x} \in \mathbb{R}^3$, we have

$$\boxplus : SO(3) \times \mathbb{R}^3 \rightarrow SO(3)$$

Rotation perturbations

- When the variable is represented by a rotation matrix :

$$\boldsymbol{x} \boxplus \Delta \boldsymbol{x} \leftrightarrow \mathbf{R} \cdot \delta \mathbf{R}$$

- Matrix exponential : $A \in \mathbb{R}^{n \times n}$

$$\exp(\mathbf{A}) = \mathbf{I} + \mathbf{A} + \frac{1}{2!}\mathbf{A}^2 + \frac{1}{3!}\mathbf{A}^3 + \dots$$

- The incremental rotation can be approximated from the perturbation vector by :

$$\begin{aligned} \delta \mathbf{R} &= \exp([\Delta \boldsymbol{x}]_\times) = \mathbf{I} + [\Delta \boldsymbol{x}]_\times + [\Delta \boldsymbol{x}]_\times [\Delta \boldsymbol{x}]_\times + \dots \\ &\approx \mathbf{I} + [\Delta \boldsymbol{x}]_\times \end{aligned}$$

$$\boxed{\boldsymbol{x} \boxplus \Delta \boldsymbol{x} \approx \mathbf{R}(\mathbf{I} + [\Delta \boldsymbol{x}]_\times)}$$

Rotation perturbations

- When the variable is represented by a unit quaternion:

$$\boldsymbol{x} \boxplus \Delta\boldsymbol{x} \leftrightarrow \boldsymbol{q} \otimes \delta\boldsymbol{q}$$

- Definition of a unit quaternion:

$$\boldsymbol{q} = \begin{bmatrix} \cos(\theta/2) \\ \boldsymbol{u}\sin(\theta/2) \end{bmatrix} \xrightarrow{\theta \to 0} \boldsymbol{q} \approx \begin{bmatrix} 1 \\ \boldsymbol{u}\theta/2 \end{bmatrix}$$

- Let the perturbation vector $\Delta\boldsymbol{x} \triangleq \boldsymbol{u}\theta$, we have

$$\boldsymbol{x} \boxplus \Delta\boldsymbol{x} \approx \boldsymbol{q}\{\boldsymbol{x}\} \otimes \begin{bmatrix} 1 \\ \Delta\boldsymbol{x}/2 \end{bmatrix}$$

Jacobian matrix with respect to non-vector parameters

- How do we compute the Jacobian matrix if the parameter is not a vector?
- The change of the function value after a small perturbation is described by

$$\frac{\partial}{\partial \Delta \boldsymbol{x}} \boldsymbol{h} = \frac{\boldsymbol{h}(\boldsymbol{x} \boxplus \Delta \boldsymbol{x}) - \boldsymbol{h}(\mathbf{x})}{\Delta \mathbf{x}} \Big|_{\Delta \boldsymbol{x} \to 0}$$

Numerical differentiation

- The Jacobian matrix can be evaluated by numerical differentiation if the analytic approach is too complicated.
- At each time, the perturbation is only enabled in a single dimension.

$$\Delta \boldsymbol{x}^{(j)} = \left[0 \cdots \delta \cdots 0\right]^T \in \mathbb{R}^n$$

δ is a small value: $\max(|10^{-4} x_i|, 10^{-6})$

$$\boxed{\mathbf{H}(:, j) \approx \frac{\boldsymbol{h}(\boldsymbol{x} \boxplus \Delta \boldsymbol{x}^{(j)}) - \boldsymbol{h}(\boldsymbol{x})}{\delta}}$$

Summary

- Non-vector parameters, like rotations, cannot be simply added
- Perturbation operator $\boxplus : \mathcal{X} \times \mathbb{R}^n \rightarrow \mathcal{X}$;
- Rotation perturbation :

$$\boldsymbol{x} \boxplus \Delta \boldsymbol{x} \leftrightarrow \mathbf{R} \cdot \delta \mathbf{R} \qquad \boldsymbol{x} \boxplus \Delta \boldsymbol{x} \approx \mathbf{R}(\mathbf{I} + [\Delta \boldsymbol{x}]_\times)$$

$$\boldsymbol{x} \boxplus \Delta \boldsymbol{x} \leftrightarrow \boldsymbol{q} \otimes \delta \boldsymbol{q} \qquad \boldsymbol{x} \boxplus \Delta \boldsymbol{x} \approx \boldsymbol{q}\{x\} \otimes \begin{bmatrix} 1 \\ \Delta \boldsymbol{x}/2 \end{bmatrix}$$

- Jacobian matrix with respect to non-vector parameters:

$$\frac{\partial}{\partial \Delta \boldsymbol{x}} \boldsymbol{h} = \frac{\boldsymbol{h}(\boldsymbol{x} \boxplus \Delta \boldsymbol{x}) - \boldsymbol{h}(\boldsymbol{x})}{\Delta \boldsymbol{x}} \Big|_{\Delta \boldsymbol{x} \rightarrow 0}$$

- Numerical differentiation