Lecture 04- Geometry

EE382-Visual localization & Perception

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Last lecture

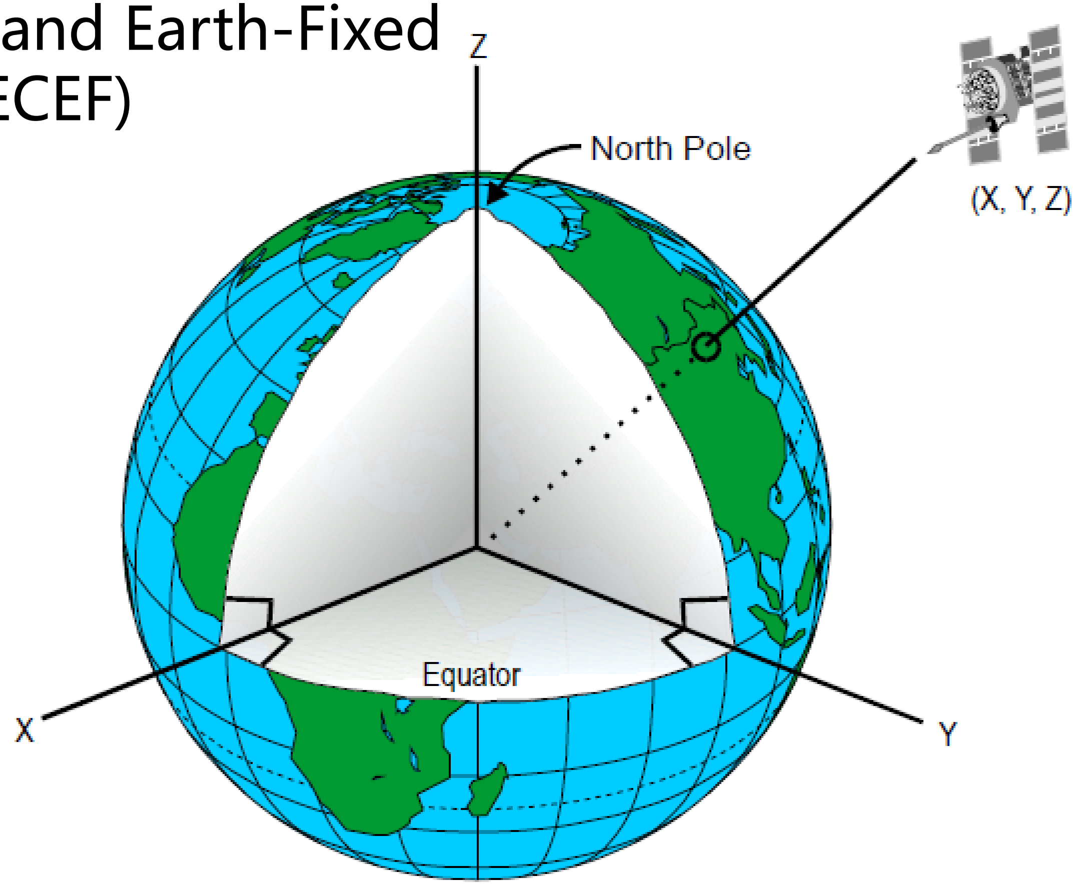
- CCD & CMOS
- Global shutter V.S rolling shutter
- Human color system & Bayer pattern
- Performance of Image Sensor
- Image as a 2D function (continuous) /2D matrix (discrete)
- Linear filtering
- Image derivatives
- Histogram equalization/specialization

Outline

- Reference Frame
  - Inertial Frame
  - Earth-fixed Frame (World Frame)
  - Body-fixed Frame (Body Frame)
- Rigid Transformation
  - Pose representation
    - Body-to-world transformation
    - World-to-body transformation
- Projective geometry
  - Homogenous coordinates
  - 2D projective geometry
  - 3D projective geometry

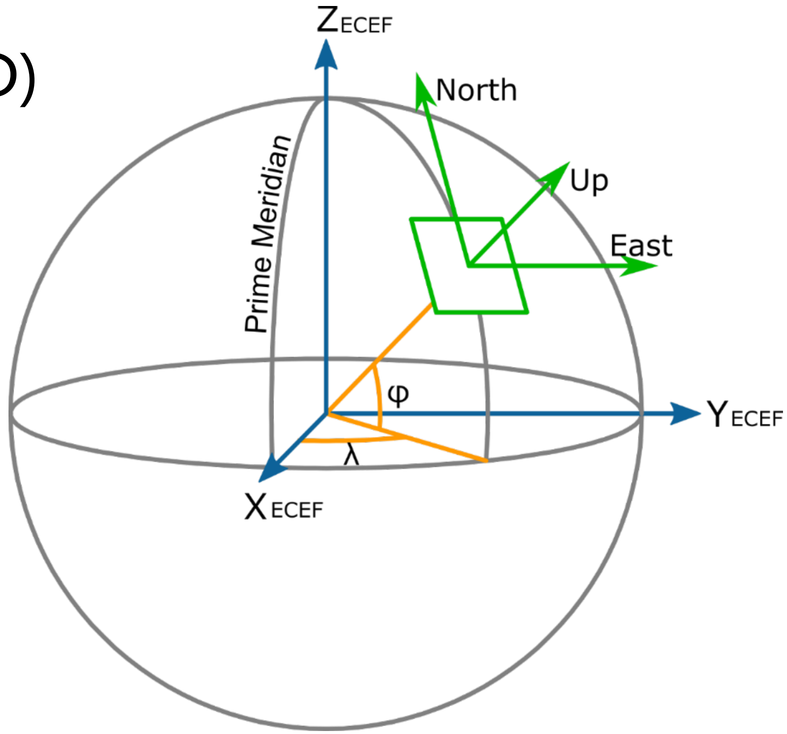
Reference Frame

- Inertial Frame
  - Earth-Centered Inertial frame (ECI)
  - Earth-Centered and Earth-Fixed Inertial Frame (ECEF)



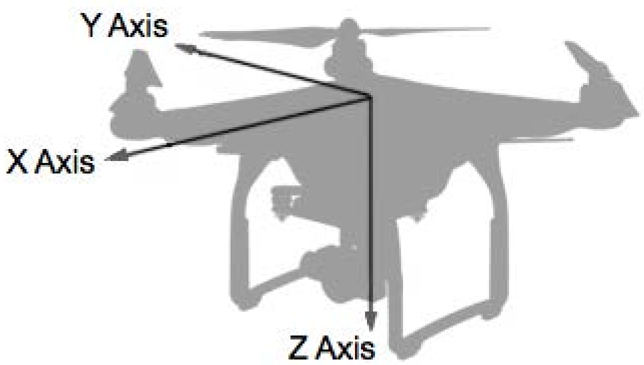
Reference Frame

- Earth-Fixed Frame (**Local Geodesic Frame** or **World Frame**)
  - East-North-Up (ENU)
  - North-East-Down (NED)

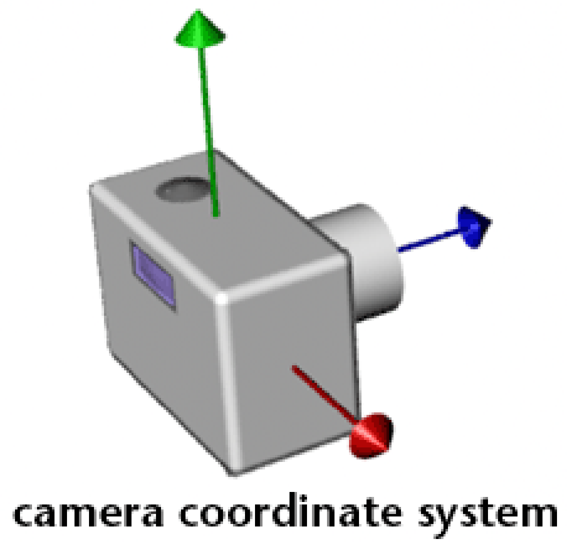


Reference Frame

- Body-fixed frame
  - Coordinate system attached to the rigid body



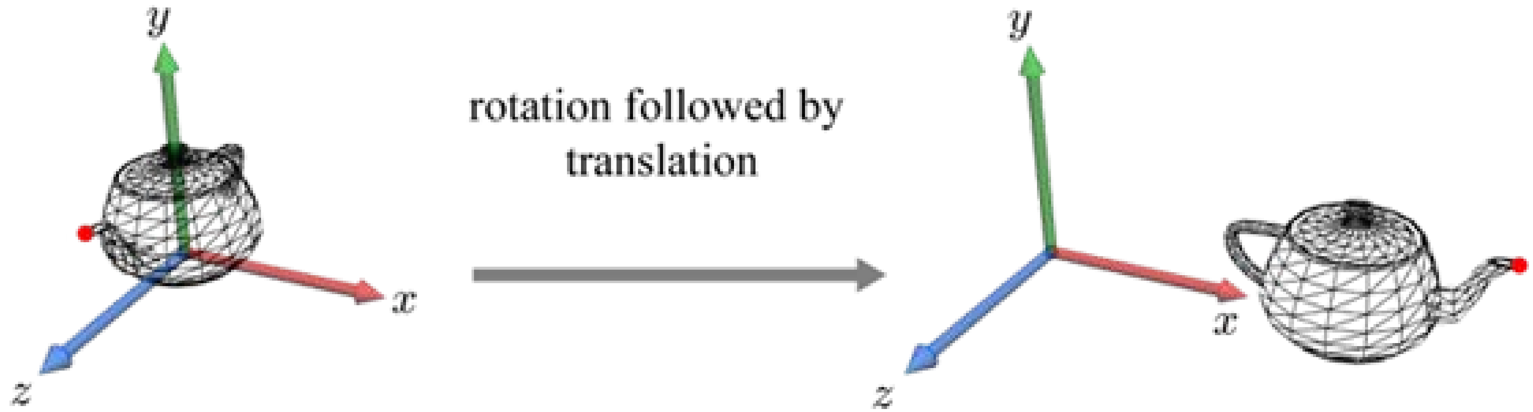
Body-fixed frame on a UAV



Body-fixed frame on a camera

Rigid transformation

- A rigid transformation is a transformation of a Euclidean space that preserves the **Euclidean distance** between every pair of point



- Shape will be preserved, conformal mapping

Rigid transformation

- A 3D rigid Transformation can be represented by a rotation plus a translation.
  - Give a point $\boldsymbol{x} \in \mathbb{R}^3$, its transformed point $\boldsymbol{x}' \in \mathbb{R}^3$ is given by

$$\boldsymbol{x}' = \mathbf{R}\boldsymbol{x} + \boldsymbol{t}$$

  - Here $\mathbf{R} \in \mathbb{R}^{3\times 3}$ is a rotation matrix, which satisfies：

$$\det(\mathbf{R}) = 1, \quad \mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I}_{3\times 3}$$

  - $\boldsymbol{t} \in \mathbb{R}^3$ is a translation vector

Rigid transformation

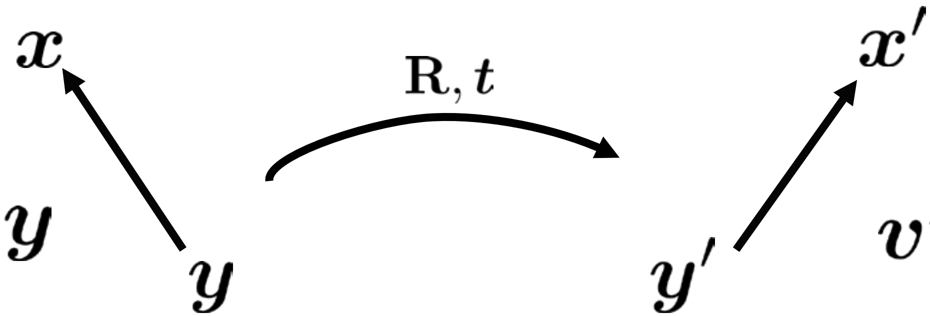
- We represent the rotation matrix and translation vector by

$$\mathbf{R} = \begin{bmatrix} R_1 & R_2 & R_3 \\ R_4 & R_5 & R_6 \\ R_7 & R_8 & R_9 \end{bmatrix} \qquad \mathbf{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}$$

Rigid transformation

- A direction can be transformed without $\boldsymbol{t}$

$$\boldsymbol{v} = \boldsymbol{x} - \boldsymbol{y}$$



$$\boldsymbol{v}' = \boldsymbol{x}' - \boldsymbol{y}'$$

$$\boldsymbol{x}' = \mathbf{R}\boldsymbol{x} + \boldsymbol{t}$$
$$\boldsymbol{y}' = \mathbf{R}\boldsymbol{y} + \boldsymbol{t}$$

$$\boldsymbol{x}' - \boldsymbol{y}' = \mathbf{R}(\boldsymbol{x} - \boldsymbol{y})$$

$$\boldsymbol{v}' = \mathbf{R}\boldsymbol{v}$$

Rigid transformation

- The reverse transformation can be computed from

$$\boldsymbol{x}' = \mathbf{R}\boldsymbol{x} + \boldsymbol{t}$$

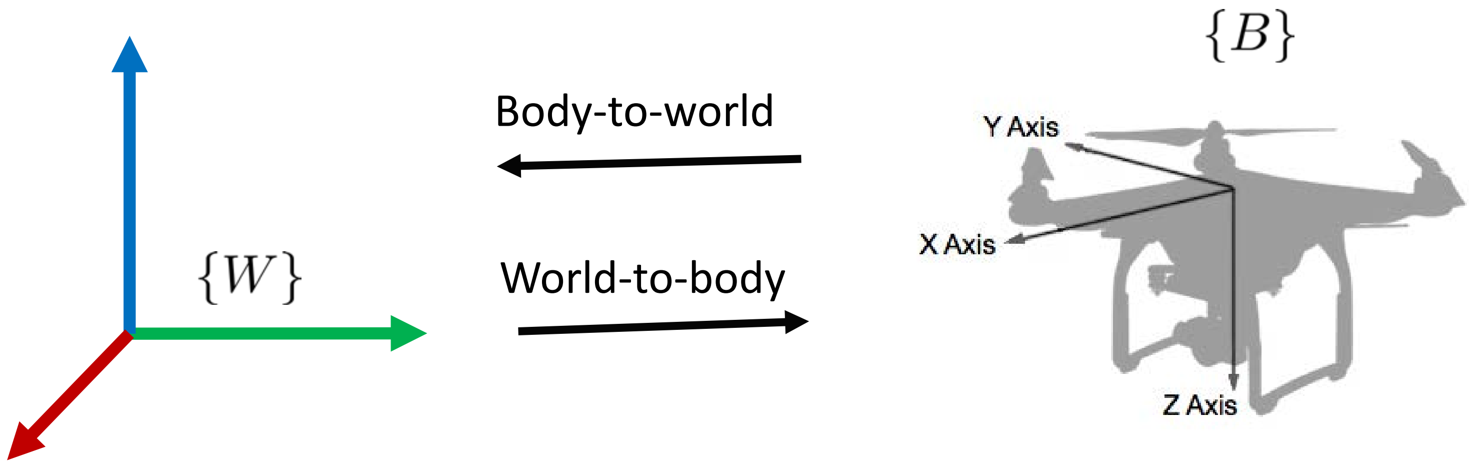
$$\boldsymbol{x} = \mathbf{R}^{\mathrm{T}}(\boldsymbol{x}' - \boldsymbol{t})$$

- Another form of rigid transformation

$$\boldsymbol{x} = \underbrace{\boxed{\mathbf{C}}}_{\text{Rotation matrix}}(\boldsymbol{x}' - \underbrace{\boxed{\boldsymbol{x}_0}}_{\text{Reference point}})$$

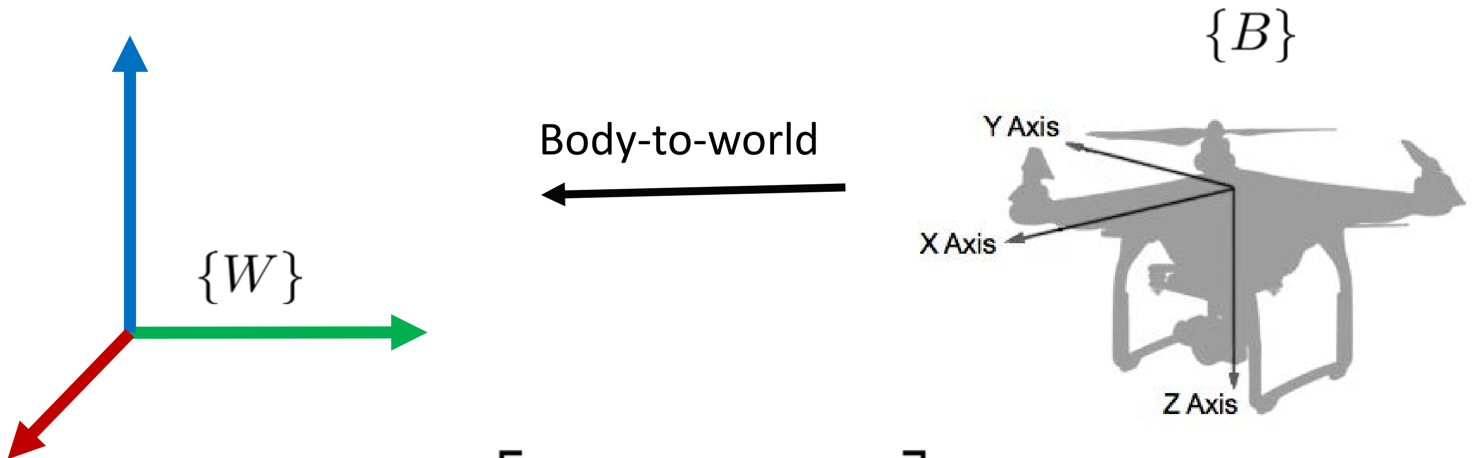
Pose representation

- The pose of a rigid body can be represented by the transformation from the world frame to the body-fixed frame or reverse



Pose representation

- Body-to-world transformation

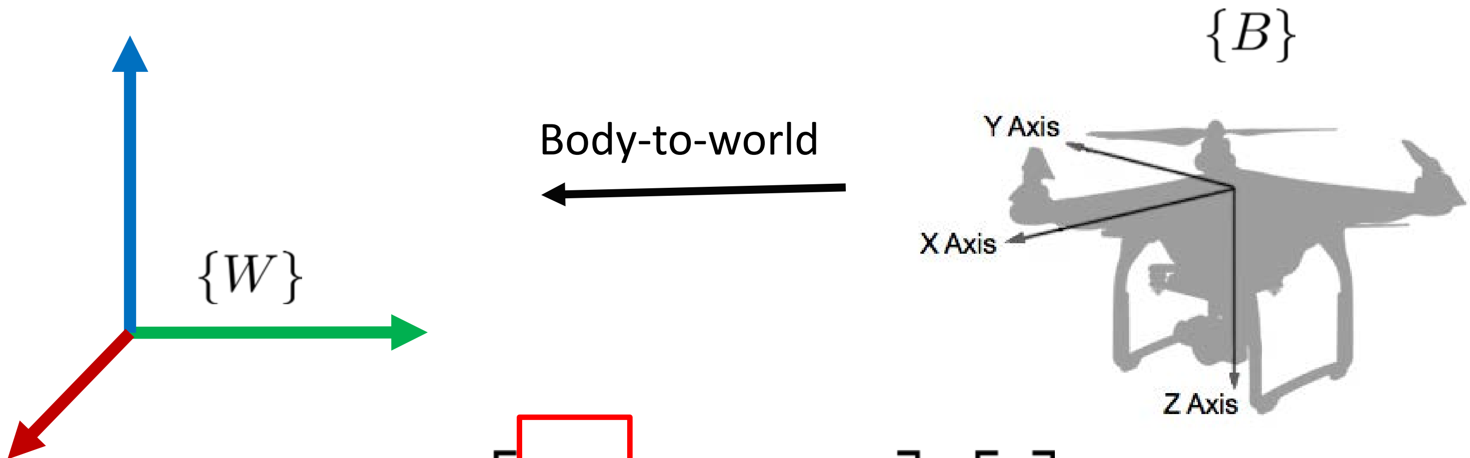


$$\boldsymbol{x}_w = \begin{bmatrix} R_1 & R_2 & R_3 \\ R_4 & R_5 & R_6 \\ R_7 & R_8 & R_9 \end{bmatrix} \boldsymbol{x} + \boldsymbol{t}$$

- We consider **three orthogonal directions (x,y,z)** and **the origin of the body frame** after transformation

Pose representation

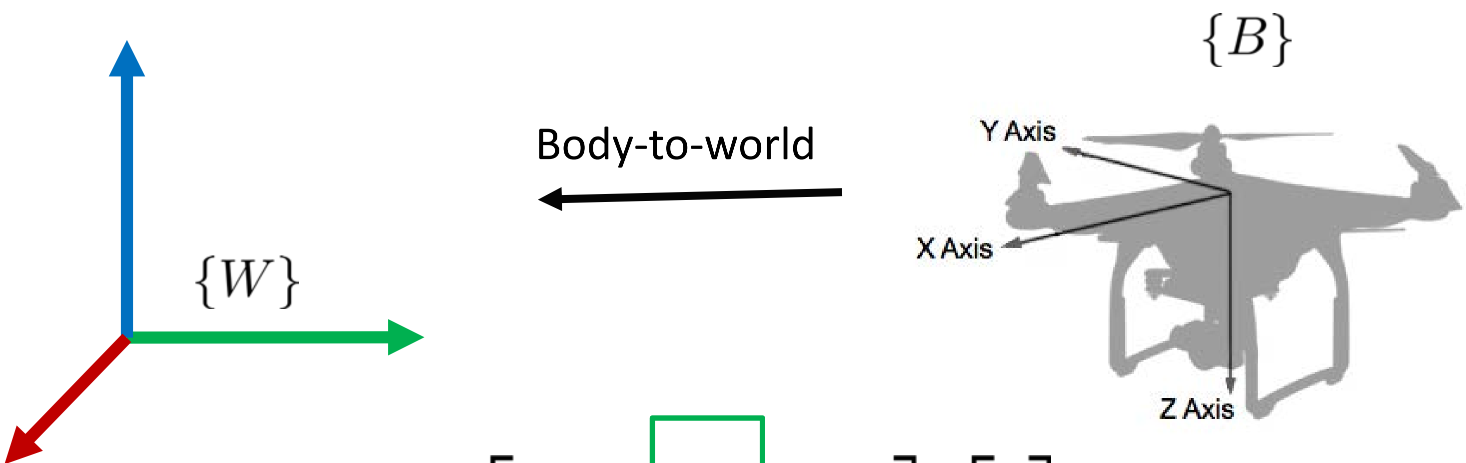
- X direction after transformation



$$\boldsymbol{v}_w^x = \begin{bmatrix} R_1 & R_2 & R_3 \\ R_4 & R_5 & R_6 \\ R_7 & R_8 & R_9 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Pose representation

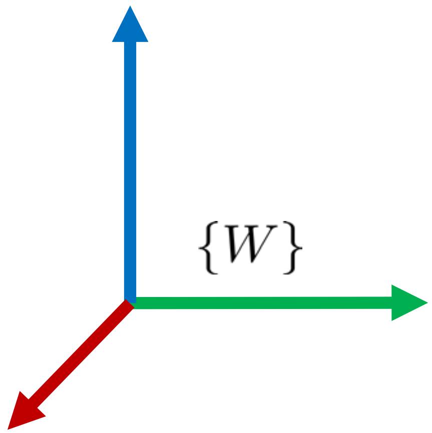
- Y direction after transformat



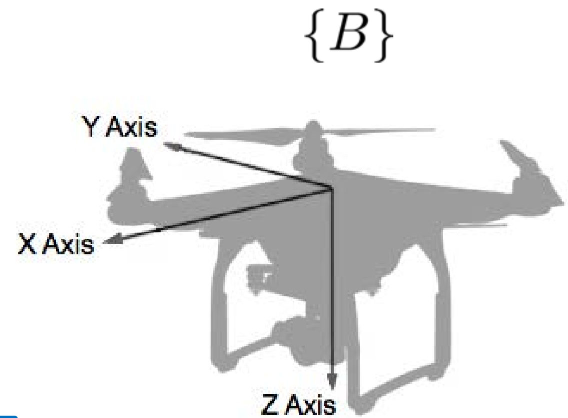
$$\boldsymbol{v}_w^y = \begin{bmatrix} R_1 & R_2 & R_3 \\ R_4 & R_5 & R_6 \\ R_7 & R_8 & R_9 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Pose representation

- Z direction after transformat



Body-to-world



$$\boldsymbol{v}_w^z = \begin{bmatrix} R_1 & R_2 & R_3 \\ R_4 & R_5 & R_6 \\ R_7 & R_8 & R_9 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Pose representation

- If we use the body-to-world transformation to represent the pose
  - Column vectors in $\mathbf{R}$ are the directions of body axes in the world frame

$$\mathbf{R} = \begin{array}{c} \begin{bmatrix} R_1 & R_2 & R_3 \\ R_4 & R_5 & R_6 \\ R_7 & R_8 & R_9 \end{bmatrix} \\ \begin{matrix} \text{X} & \quad \text{Y} & \quad \text{Z} \end{matrix} \end{array}$$

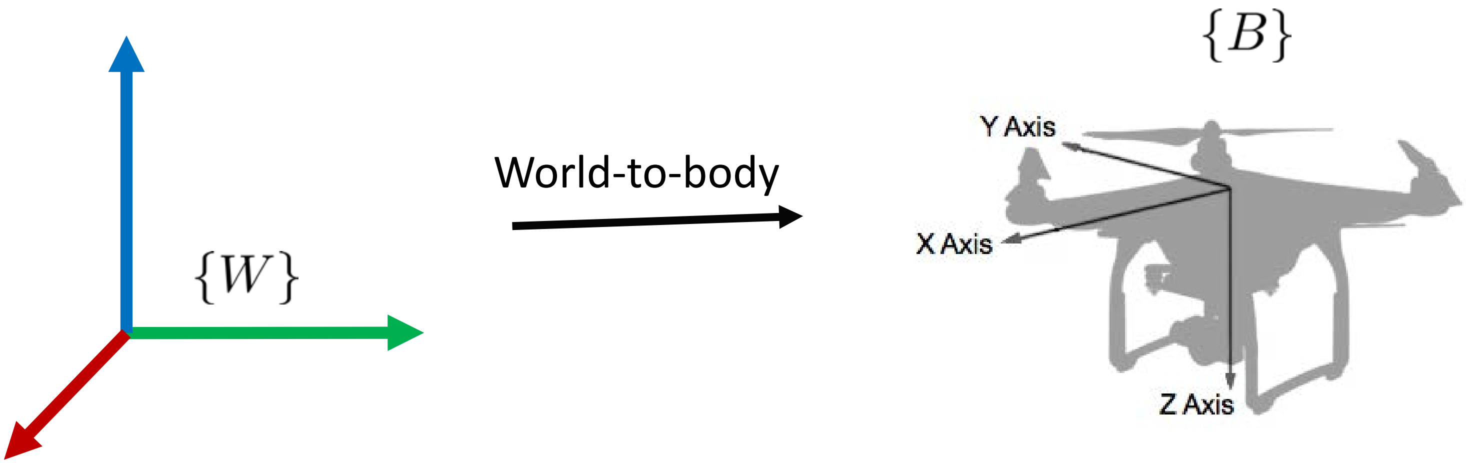
Pose representation

- The origin of the body frame is given by

$$\boldsymbol{x}_w^o = \begin{bmatrix} R_1 & R_2 & R_3 \\ R_4 & R_5 & R_6 \\ R_7 & R_8 & R_9 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \boldsymbol{t}$$
$$= \boldsymbol{t}$$

Pose representation

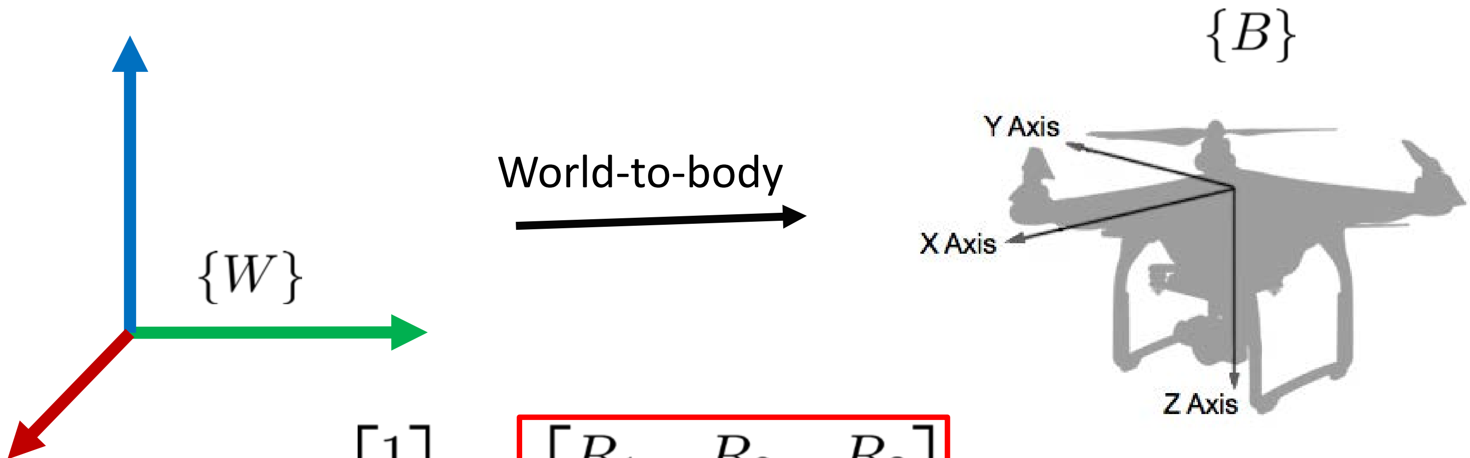
- World-to-body transformation



$$\boldsymbol{x} = \begin{bmatrix} R_1 & R_2 & R_3 \\ R_4 & R_5 & R_6 \\ R_7 & R_8 & R_9 \end{bmatrix} \boldsymbol{x}_w + \boldsymbol{t}$$

Pose representation

- World-to-body transformation

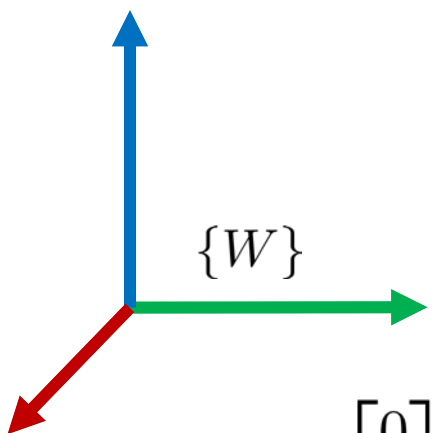


$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} R_1 & R_2 & R_3 \\ R_4 & R_5 & R_6 \\ R_7 & R_8 & R_9 \end{bmatrix} \boldsymbol{v}_w^x$$

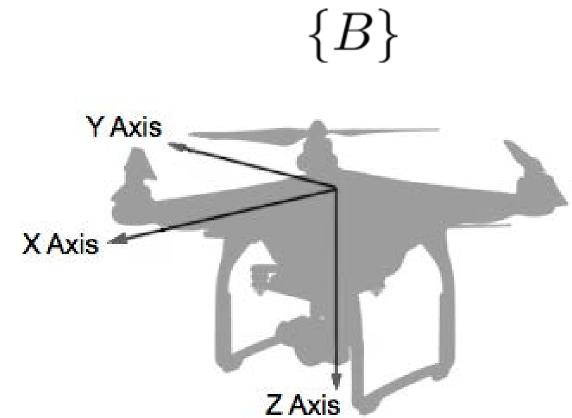
$$\boldsymbol{v}_w^x = \begin{bmatrix} R_1 & R_4 & R_7 \\ R_2 & R_5 & R_8 \\ R_3 & R_6 & R_9 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Pose representation

- World-to-body transformation



World-to-body

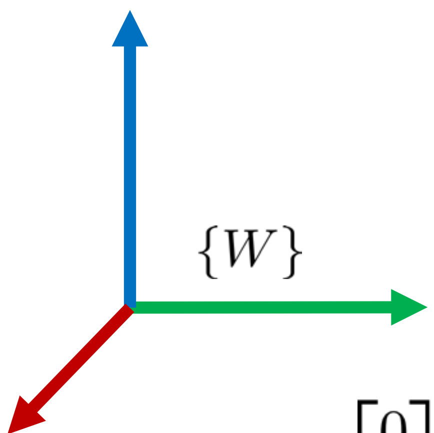


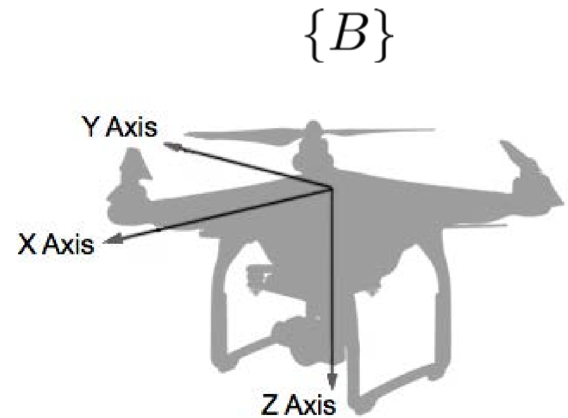
$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} R_1 & R_2 & R_3 \\ R_4 & R_5 & R_6 \\ R_7 & R_8 & R_9 \end{bmatrix} \boldsymbol{v}_w^y$$

$$\boldsymbol{v}_w^y = \begin{bmatrix} R_1 & R_4 & R_7 \\ R_2 & R_5 & R_8 \\ R_3 & R_6 & R_9 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Pose representation

- World-to-body transformation





$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} R_1 & R_2 & R_3 \\ R_4 & R_5 & R_6 \\ R_7 & R_8 & R_9 \end{bmatrix} \boldsymbol{v}_w^z$$

$$\boldsymbol{v}_w^z = \begin{bmatrix} R_1 & R_4 & R_7 \\ R_2 & R_5 & R_8 \\ R_3 & R_6 & R_9 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

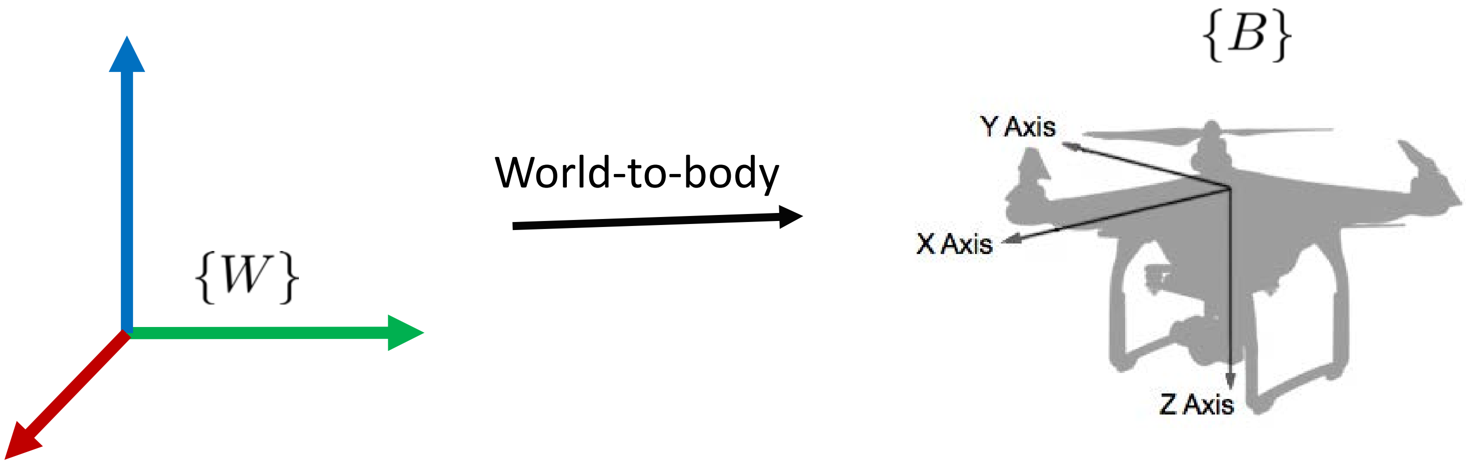
Pose representation

- If we use the world-to-body transformation to represent the pose
  - Row vectors in $\mathbf{R}$ are the directions of body axes in the world frame

$$\mathbf{R} = \begin{array}{cc} \begin{bmatrix} R_1 & R_2 & R_3 \\ R_4 & R_5 & R_6 \\ R_7 & R_8 & R_9 \end{bmatrix} & \begin{array}{c} \mathrm{X} \\ \mathrm{Y} \\ \mathrm{Z} \end{array} \end{array}$$

Pose representation

- The origin of the body frame



$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} R_1 & R_2 & R_3 \\ R_4 & R_5 & R_6 \\ R_7 & R_8 & R_9 \end{bmatrix} \boldsymbol{x}_w^0 + \boldsymbol{t}$$

$$\Rightarrow \quad \boldsymbol{x}_w^0 = -\mathbf{R}^{\mathrm{T}} \boldsymbol{t}$$

Summary

- Rigid transformation

$$\boldsymbol{x}' = \mathbf{R}\boldsymbol{x} + \boldsymbol{t} \quad \text{or} \quad \boldsymbol{x} = \mathbf{R}(\boldsymbol{x} - \boldsymbol{x}_0)$$

- Pose representation

Pose representation	Axis directions (in world frame)	Origin (in world frame)
Body-to-world	Column vectors of $\mathbf{R}$	$\boldsymbol{t}$
World-to-body	Row vectors of $\mathbf{R}$	$-\mathbf{R}^{\mathrm{T}}\boldsymbol{t}$

Body-to-world

$$\mathbf{R} = \begin{array}{c} \begin{bmatrix} \boxed{\begin{matrix} R_1 \\ R_4 \\ R_7 \end{matrix}} & \boxed{\begin{matrix} R_2 \\ R_5 \\ R_8 \end{matrix}} & \boxed{\begin{matrix} R_3 \\ R_6 \\ R_9 \end{matrix}} \end{bmatrix} \\ \begin{matrix} \mathrm{X} & \quad \mathrm{Y} & \quad \mathrm{Z} \end{matrix} \end{array}$$

World-to-body

$$\mathbf{R} = \begin{array}{cc} \begin{bmatrix} \boxed{R_1 \quad R_2 \quad R_3} \\ \boxed{R_4 \quad R_5 \quad R_6} \\ \boxed{R_7 \quad R_8 \quad R_9} \end{bmatrix} & \begin{matrix} \mathrm{X} \\ \mathrm{Y} \\ \mathrm{Z} \end{matrix} \end{array}$$

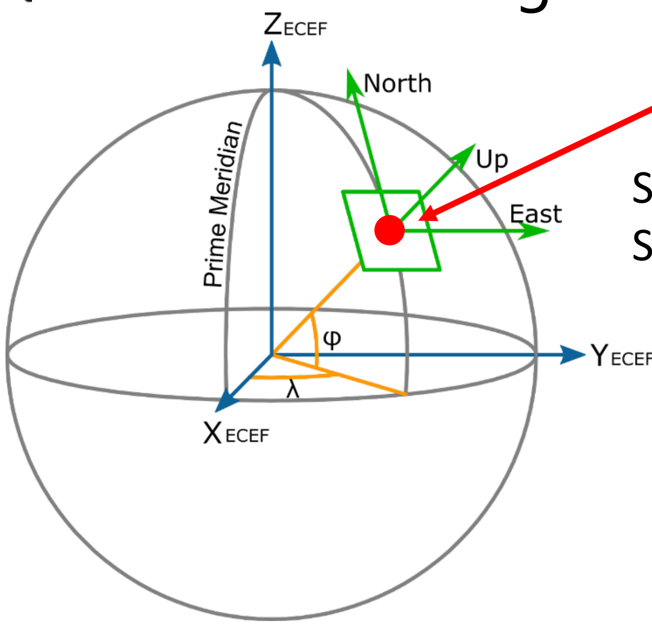
Example of transformation

- ECEF Frame <-> NED Frame

$$\boldsymbol{x}_{NED} = \boxed{\mathbf{R}}(\boldsymbol{x}_{ECEF} - \boldsymbol{x}_{REF})$$

$$\boldsymbol{R} = \begin{bmatrix} -\sin(\phi)\cos(\lambda) & -\sin(\lambda) & -\cos(\phi)\cos(\lambda) \\ -\sin(\phi)\sin(\lambda) & \cos(\lambda) & -\cos(\phi)\sin(\lambda) \\ \cos(\phi) & 0 & -\sin(\phi) \end{bmatrix}$$

  - ϕ, λ latitude and longitude of $\boldsymbol{x}_{REF}$



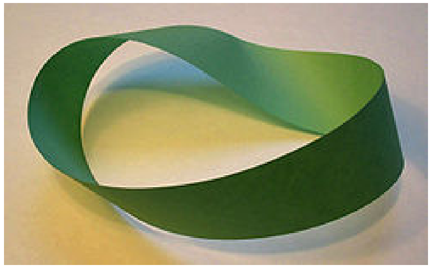
Step1. Select a reference point
Step2. Compute the rotation matrix
- 3 directions in the ECEF frame

Projective geometry

- **Key concept - Homogenous coordinates**
  - Represent an n-dimensional vector by a $n+1$ dimensional coordinate

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \\ w \end{pmatrix} = \lambda \mathbf{x} \sim \begin{pmatrix} x_1/w \\ x_2/w \\ \dots \\ x_n/w \end{pmatrix}$$

Homogenous coordinate **Cartesian coordinate**



  - Can represent **infinite** points or lines

August Ferdinand Möbius
1790-1868

Projective geometry

- Using homogenous coordinates, we can write a rigid transformation in a compact form :

$$\boldsymbol{x}' = \mathbf{R}\boldsymbol{x} + \boldsymbol{t}$$

$$\begin{bmatrix} \boldsymbol{x}' \\ 1 \end{bmatrix} \sim \begin{bmatrix} \mathbf{R} & \boldsymbol{t} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ 1 \end{bmatrix}$$

$$\boxed{\boldsymbol{X}' \sim \mathbf{T}\boldsymbol{X}}$$

$$\boldsymbol{X} \sim \mathbf{T}^{-1}\boldsymbol{X}'$$

2D projective geometry

- Point representation:
  - A point $(x,y)^T$ can be represented by a homogenous coordinate: $(x,y,1)^T$

Homogeneous coordinates: $(x,y,1)^T \quad \sim \quad (\lambda x, \lambda y, \lambda)^T$

$$\begin{bmatrix} x_1/x_3 \\ x_2/x_3 \end{bmatrix}$$

Inhomogeneous coordinates: $(x,y)^T$

2D projective geometry

- Line representation:
  - A line is represented by a line equation:

$$ax + by + c = 0$$

Hence a line can be naturally represented by a homogeneous coordinate:

$$\boxed{\mathbf{l} = (a, b, c)^T}$$

It has the same scale equivalence relationship:

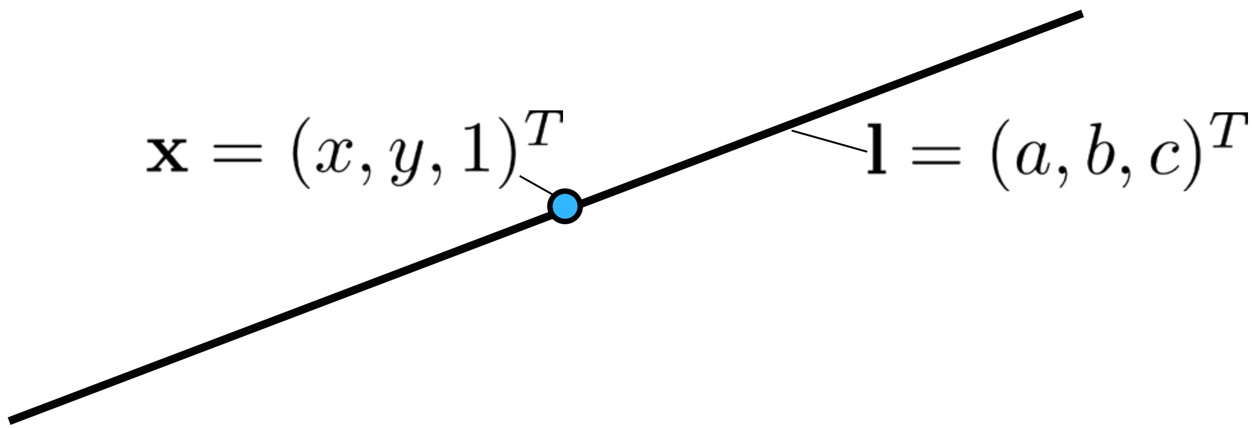
$$\mathbf{l} = (a, b, c)^T \sim \lambda \mathbf{l} = (\lambda a, \lambda b, \lambda c)^T$$

2D projective geometry

- A point lie on a line is simply described as:

$$\mathbf{x}^T\mathbf{l} = 0 \text{ , or } \mathbf{l}^T\mathbf{x} = 0$$

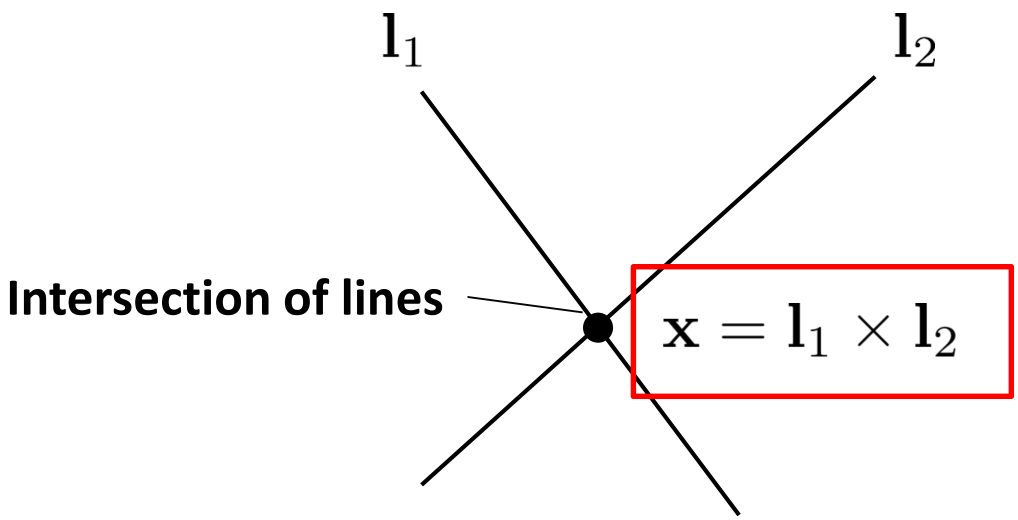
where $\mathbf{x} = (x, y, 1)^T$ and $\mathbf{l} = (a, b, c)^T$.



A point or a line has only two degree of freedom (DoF, 自由度).

2D projective geometry

- Intersection of lines：
  - The intersection of two lines is computed by the cross production of the two homogenous coordinates of the two lines:



Here, ‘$\times$‘ represents cross production

2D projective geometry

- Example of intersection of two lines: Let the two lines be

$$\mathbf{l}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{l}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

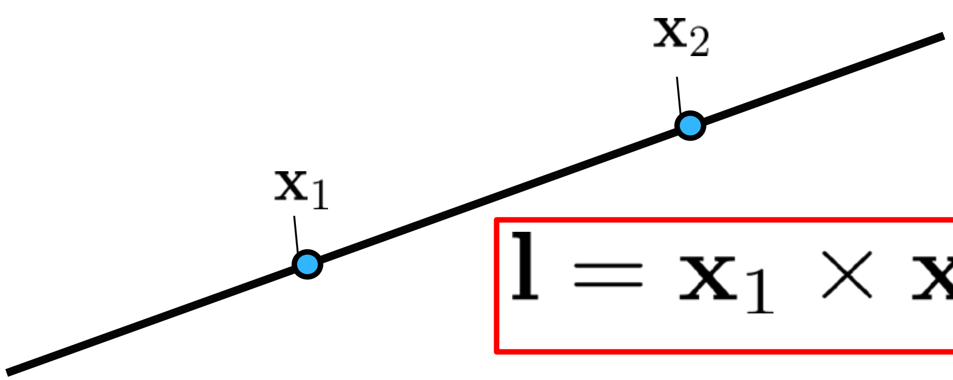
- The intersection point is computed as :

$$\mathbf{x} = \mathbf{l}_1 \times \mathbf{l}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{vmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- The inhomogeneous coordinate is $(1, 1)^T$

2D projective geometry

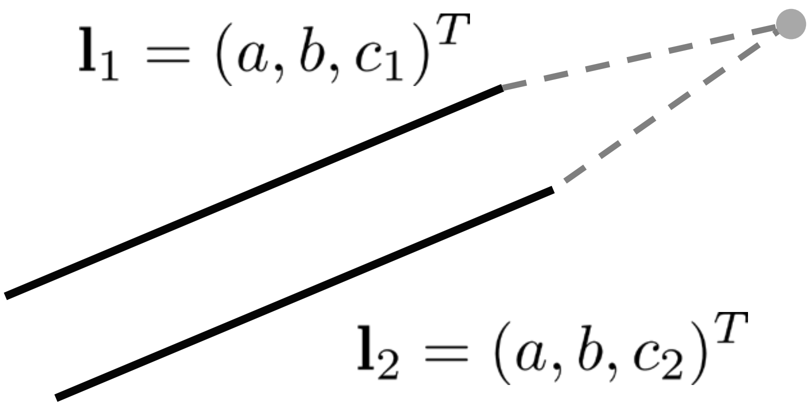
- Line across two points:
  - The line across the two points is obtained by the cross production of the two homogenous coordinates of the two points:



$$\boxed{\mathbf{l} = \mathbf{x}_1 \times \mathbf{x}_2}$$

2D Projective Geometry

- The intersection of two parallel lines can also be computed in the same way :



$$\mathbf{l}_1 = (a, b, c_1)^T$$

$$\mathbf{l}_2 = (a, b, c_2)^T$$

$$\mathbf{x} = \mathbf{l}_1 \times \mathbf{l}_2$$

$$\mathbf{x} = \begin{bmatrix} b(c_2 - c_1) \\ -a(c_2 - c_1) \\ 0 \end{bmatrix} = \begin{bmatrix} b \\ -a \\ 0 \end{bmatrix}$$

- Note that the third element of the intersection point is zero, which means this point is an infinite point.

2D Project Geometry

- A linear transformation in 2D space is named as a **homography** matrix
- Given a homogenous vector of a vector 2D point, its transformed homogenous vector is

$$\boldsymbol{x}' = \mathbf{H}\boldsymbol{x}$$

- Here, $\mathbf{H}$ is a 3×3 matrix, which is defined as

$$\mathbf{H} = \begin{bmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{bmatrix}$$

2D Project Geometry

- Homography matrix is scale-invariant

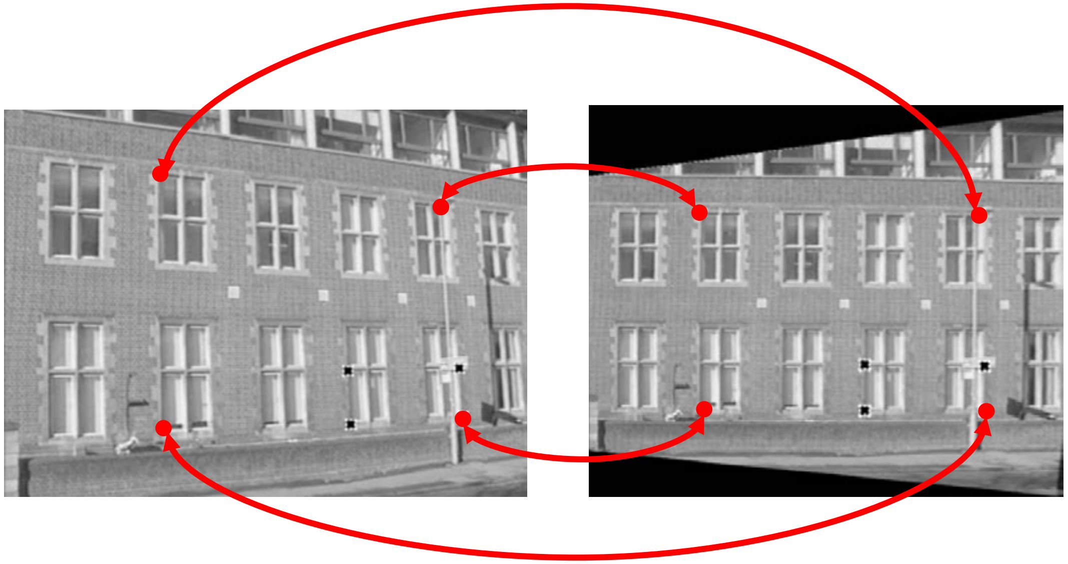
$$\mathbf{H} \sim \lambda \mathbf{H}$$

- Four points determine a sole homography transformation
  - Given four pairs of transformed 2D points
  - Let $h_9 = 1$

$$\begin{aligned} \boldsymbol{x}_1' &= \mathbf{H}\boldsymbol{x}_1 \\ \boldsymbol{x}_2' &= \mathbf{H}\boldsymbol{x}_2 \\ \boldsymbol{x}_3' &= \mathbf{H}\boldsymbol{x}_3 \\ \boldsymbol{x}_4' &= \mathbf{H}\boldsymbol{x}_4 \end{aligned} \quad \Rightarrow \quad \boxed{\mathbf{A}\begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \\ h_7 \\ h_8 \end{bmatrix} = \boldsymbol{b}} \quad \Rightarrow \quad \mathbf{H}$$

2D projective geometry

- An example of projective transformation
  - Use homography transformation to remove the perspective distortion of a planar object



$$\mathbf{H} = \begin{bmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{bmatrix}$$

2D projective geometry

- Given a transformation of points $\boxed{\mathbf{x}' = H\mathbf{x}}$,

the transformation of lines is given by $\boxed{\mathbf{l}' = \mathbf{H}^{-T}\mathbf{l}}$.

$$\mathbf{l}^T\mathbf{x} = 0$$

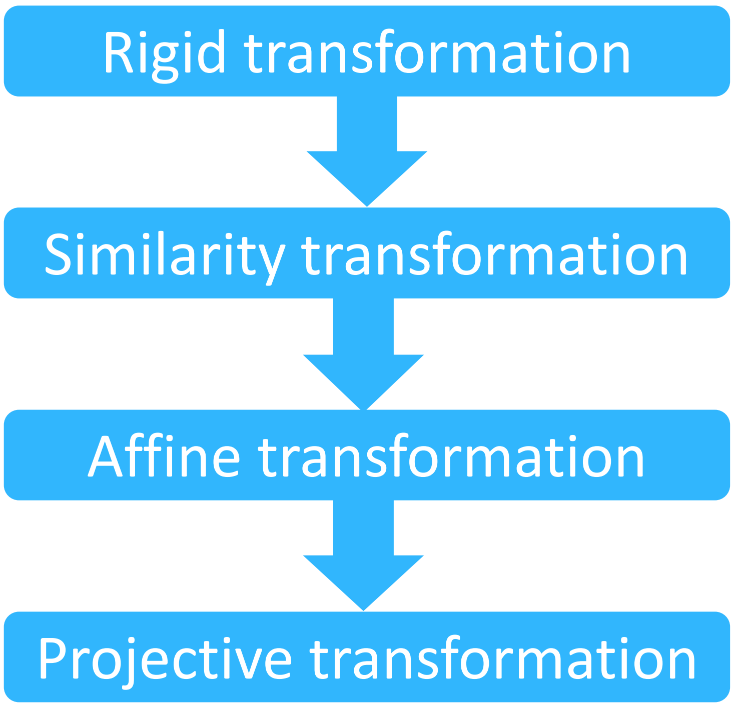
$$\Rightarrow \quad \mathbf{l}^T H^{-1}\mathbf{x}' = 0 \quad (\mathbf{x} = H^{-1}\mathbf{x}')$$

$$\Rightarrow \quad \boxed{\mathbf{l}^T H^{-1}}\mathbf{x}' = 0$$

$$\Rightarrow \quad \mathbf{l}' = \mathbf{H}^{-T}\mathbf{l}$$

2D projective geometry

- A hierarchy of transformations



$$\begin{bmatrix} \epsilon\cos\theta & -\sin\theta & t_x \\ \epsilon\sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$

2D projective geometry

- Isometrics : Rigid transformation [+ Reflection]

Rotation

Translation

$$\mathbf{H} = \begin{bmatrix} \cos\theta & -sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \mathbf{H} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

$(\theta, t_x, t_y)^T$

3DoF

Euclidean transformation

$$\mathbf{H} = \begin{bmatrix} -\cos\theta & -sin\theta & t_x \\ -\sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Reflection + Euclidean transformation

A isometric transformation has *3* degree of freedom, whose invariants include ***length***, ***angle*** and ***area***.

2D projective geometry

- Similarity transformation

$$\mathbf{H} = \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \mathbf{H} = \begin{bmatrix} s\mathbf{R} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

A isometric transformation has *4* degree of freedom, with one more degree of freedom on the scaling.

It preserves the 'shape', angle.

2D projective geometry

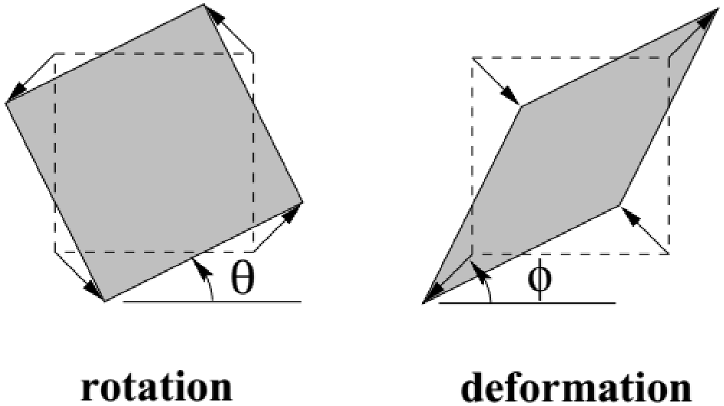
- An Affine transformation is a non-singular linear transformation which has the following form:

$$\mathbf{H} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \mathbf{H} = \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

6DoF

$$\mathbf{A} = R(\theta)R(-\phi)\mathbf{D}R(\phi)$$

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$



2D projective geometry

- Projective transformation

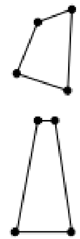
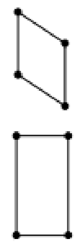
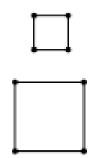
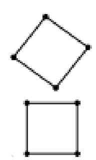
$$\mathbf{H} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ v_1 & v_2 & 1 \end{bmatrix} \Rightarrow \mathbf{H} = \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{v}^T & 1 \end{bmatrix}$$

8DoF

- Can be decomposed into a chain of essential transformations:

$$\mathbf{H} = \underbrace{v}_{\text{Scale}}\mathbf{H}_S\mathbf{H}_A\mathbf{H}_P = v \underbrace{\begin{bmatrix} s\mathbf{R} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}}_{\text{Similarity}} \underbrace{\begin{bmatrix} \mathbf{K} & 0 \\ \mathbf{0}^T & 1 \end{bmatrix}}_{\text{Pure Affine}} \underbrace{\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{v}^T & 1 \end{bmatrix}}_{\text{Pure projective}}$$

Summary

Group	Matrix	Distortion	Invariant properties
Projective 8 dof	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$		Concurrency, collinearity, **order of contact**: intersection (1 pt contact); tangency (2 pt contact); inflections (3 pt contact with line); tangent discontinuities and cusps. cross ratio (ratio of ratio of lengths).
Affine 6 dof	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Parallelism, ratio of areas, ratio of lengths on collinear or parallel lines (e.g. midpoints), linear combinations of vectors (e.g. centroids). The line at infinity, $\mathbf{l}_\infty$.
Similarity 4 dof	$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Ratio of lengths, angle. The circular points, **I**, **J** (see section 2.7.3).
Euclidean 3 dof	$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Length, area

* Page 44 Multiple-view Geometry for Computer Vision

Summary

- Homogenous/inhomogenous coordinates
- 2D points vs 2D lines
- Points / lines at infinity
- Projective transformation, Homography
- Hierarchy of transformations:
  – Isometrics (Euclidean) ->similarity->affine->projective

3D projective geometry

- The homogeneous a 3D point is represented by

$$\mathbf{x} = (x_1, x_2, x_3, x_4)^T$$

- Its inhomogeneous coordinates is

$$(x_1/x_4, x_2/x_4, x_3/x_4)^T$$

- The projective transformation in 3D space is a linear transformation on homogeneous 4-vectors, represented by a non-singular 4×4 matrix:

$$\mathbf{X}' = \mathbf{H}\mathbf{X}$$

3D projective geometry

- A plane in 3D space is written as

$$\underbrace{\pi_1 x + \pi_2 y + \pi_3 z}_{\text{Plane normal}} + \pi_4 = 0$$

$$\mathbf{n}^T \tilde{\mathbf{x}} + d = 0$$

- Let the point on the plane be $\mathbf{x} = (x, y, z, 1)^T$, we have :

$$\boxed{\pi^T \mathbf{x} = 0}$$

3D projective geometry

- Transformation of planes :

$$\pi' = \mathbf{H}^{-T}\pi$$

$$\Rightarrow \quad \pi^T \mathbf{H}^{-1}\mathbf{x}' = 0 \quad (\mathbf{x} = \mathbf{H}^{-1}\mathbf{x}')$$

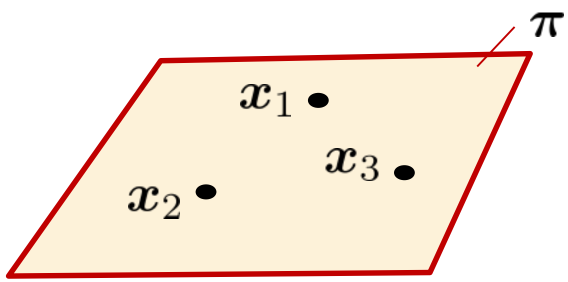
$$\Rightarrow \quad \boxed{\pi^T \mathbf{H}^{-1}}\mathbf{x}' = 0$$

$$\Rightarrow \quad \pi' = \mathbf{H}^{-T}\pi$$

3D Projective Geometry

- A plane from three points

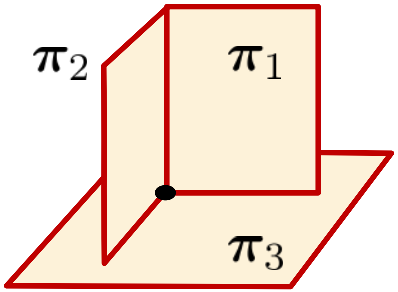
$$\begin{aligned}\boldsymbol{\pi}^{\mathrm{T}}\boldsymbol{x}_1 &= 0\\ \boldsymbol{\pi}^{\mathrm{T}}\boldsymbol{x}_2 &= 0\\ \boldsymbol{\pi}^{\mathrm{T}}\boldsymbol{x}_3 &= 0\end{aligned} \Rightarrow \underbrace{\begin{bmatrix}\boldsymbol{x}_1^{\mathrm{T}}\\ \boldsymbol{x}_2^{\mathrm{T}}\\ \boldsymbol{x}_3^{\mathrm{T}}\end{bmatrix}}_{\mathbb{R}^{3\times 4}} \underbrace{\boldsymbol{\pi}}_{\mathbb{R}^{4\times 1}} = \mathbf{0}$$



An under-determined linear system, solved by SVD decomposition

3D Projective Geometry

- Three planes define a point



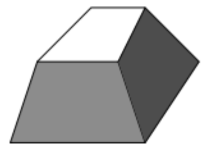
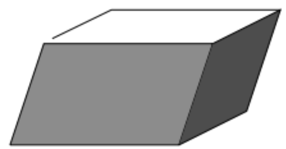
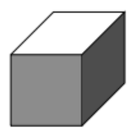
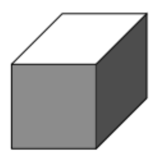
$$\boldsymbol{\pi}_1 \boldsymbol{x} = 0$$
$$\boldsymbol{\pi}_2 \boldsymbol{x} = 0$$
$$\boldsymbol{\pi}_3 \boldsymbol{x} = 0$$

$\Rightarrow$

$$\begin{bmatrix} \boldsymbol{\pi}_1^{\mathrm{T}} \\ \boldsymbol{\pi}_2^{\mathrm{T}} \\ \boldsymbol{\pi}_3^{\mathrm{T}} \end{bmatrix} \boldsymbol{x} = \mathbf{0}$$

3D projective geometry

- Hierarchy of 3D transformation

Group	Matrix	Distortion	Invariant properties
Projective 15 dof	$\begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{v}^\mathsf{T} & v \end{bmatrix}$		Intersection and tangency of surfaces in contact. Sign of Gaussian curvature.
Affine 12 dof	$\begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$		Parallelism of planes, volume ratios, centroids. The plane at infinity, $\boldsymbol{\pi}_\infty$, (see section 3.5).
Similarity 7 dof	$\begin{bmatrix} s\mathtt{R} & \mathbf{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$		The absolute conic, Ω_∞, (see section 3.6).
Euclidean 6 dof	$\begin{bmatrix} \mathtt{R} & \mathbf{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$		Volume.

Summary

- 3D points vs. 3D Planes
- Transformations of 3D points and 3D planes
- Three planes define a point / Three points define a plane
- Hierarchy of 3D transformation