Method

(BFGS) Broyden - Fletcher - Goldfarb Shanno

$$[B_{i+1}] = [B_i] + \left(1 + \frac{g_i^T [B_i] g_i}{d_i^T g_i}\right) \frac{d_i d_i^T}{d_i^T g_i} - \frac{d_i g_i^T [B_i]}{d_i^T g_i} - \frac{[B_i] g_i d_i^T}{d_i^T g_i}$$

$$d_i = X_{i+1} - X_i = \lambda_i^* S_i$$

$$g_i = \nabla f_{i+1} - \nabla f_i$$

Ex:-

$$f(x_1, x_2) = x_1 - x_2 + 2x_1^2 + 2x_1x_2 + x_2^2$$

$X_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ using BFGS method.

Sol:-

1st Iteration:- Let us Consider

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\nabla f_1 = \begin{Bmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \end{Bmatrix} = \begin{bmatrix} 1 + 4x_1 + 2x_2 \\ -1 + 2x_1 + 2x_2 \end{bmatrix}$$

$$\nabla f_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$S_1 = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{n_2} \begin{bmatrix} 1 \\ -1 \end{bmatrix}_{2\times 1}$$ $2\times 2 = 2\times 1$

2×1

$$S_1 = -\begin{bmatrix} 1+0 \\ 0-1 \end{bmatrix} = -\begin{bmatrix} 1 \\ -1 \end{bmatrix}_{2\times 1} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$f(X_1 - \cancel{f(X_2)} = f(X_1 + \lambda_1 S_1) = \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \lambda_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$$

$\cancel{X_2}$ $\Rightarrow$ $\begin{bmatrix} -\lambda_1 \\ \lambda_1 \end{bmatrix}$ $\rightarrow x_1$, $\rightarrow x_2$

$$f = \cancel{-\lambda_1 - \lambda_1 + 2\lambda_1^2 - 2\lambda_1^2} + \lambda_1^2$$

$$f = \lambda_1^2 - 2\lambda_1$$

$$\frac{\partial f}{\partial \lambda_1} = 0 \quad \Rightarrow 2\lambda_1 - 2 = 0$$

$$\boxed{\lambda_1 = 1}$$

$$X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

2nd Iteration:-

$$\nabla f_2 = \begin{bmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \end{bmatrix} = \begin{bmatrix} 1 + 4x_1 + 2x_2 \\ -1 + 2x_1 + 2x_2 \end{bmatrix}$$

$$\nabla f_2 = \begin{bmatrix} 1+4(-1)+2(1) \\ -1+2(-1)+2(1) \end{bmatrix} = \begin{bmatrix} 1-4+2 \\ -1-2+2 \end{bmatrix}$$

$$\nabla f_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

Iteration

$$d_1 = x_2 - x_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$d_1^T = [-1 \quad 1]$$

$$g_1 = \nabla f_2 - \nabla f_1 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$g_1 = \begin{bmatrix} -1-1 \\ -1+1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$$g_1^T = [-2 \quad 0]$$

$$B_2 = B_1 + \left(1 + \frac{g_1^T B_1 g_1}{d_1^T g_1}\right)\frac{d_1 d_1^T}{d_1^T g_1} - \frac{d_1 g_1^T B_1}{d_1^T g_1} - \frac{B_1 g_1 d_1^T}{d_1^T g_1}$$

$$B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \left(1 + [2 \quad 0]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$g_1^T B_1 g_1 = [-2 \quad 0]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$$= [-2 \quad 0]\begin{bmatrix} -2+0 \\ 0+0 \end{bmatrix} = [-2 \quad 0]_{1\times 2}\begin{bmatrix} -2 \\ 0 \end{bmatrix}_{2\times 1}$$

$$g_1^T B_1 g_1 = [+4+0] = [+4]_{1\times 1}$$

$$d_1^T g_1 = [-1 \quad 1]_{1\times 2}\begin{bmatrix} -2 \\ 0 \end{bmatrix}_{2\times 1} = [2+0] = [2]$$

$$d_1 d_1^T = \begin{bmatrix} -1 \\ 1 \end{bmatrix}_{2\times 1}[-1 \quad 1]_{1\times 2} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}_{2\times 2}$$

$$d_1^T g_1 = \begin{bmatrix} -1 & 1 \end{bmatrix}_{1\times 2} \begin{bmatrix} -2 \\ 0 \end{bmatrix}_{2\times 1} = [2+0] = [2]$$

$$d_1 g_1^T B_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}_{2\times 1} \begin{bmatrix} -2 & 0 \end{bmatrix}_{1\times 2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 0 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$d_1 g_1^T B_1 = \begin{bmatrix} 2+0 & 0+0 \\ -2+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -2 & 0 \end{bmatrix}$$

$$d_1^T g_1 = \begin{bmatrix} -1 & 1 \end{bmatrix}_{1\times 2} \begin{bmatrix} -2 \\ 0 \end{bmatrix}_{2\times 1} = [2+0] = [2]$$

$$B_1 g_1 d_1^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2} \begin{bmatrix} -2 \\ 0 \end{bmatrix}_{2\times 1} \begin{bmatrix} -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2+0 \\ 0+0 \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}_{2\times 1} \begin{bmatrix} -1 & 1 \end{bmatrix}_{1\times 2}$$

$$B_1 g_1 d_1^T = \begin{bmatrix} +2 & -2 \\ 0 & 0 \end{bmatrix}$$

$$d_1^T g_1 = \begin{bmatrix} -1 & 1 \end{bmatrix}_{1\times 2} \begin{bmatrix} -2 \\ 0 \end{bmatrix}_{2\times 1} = [2+0] = [2]$$

$$B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \left(1 + \frac{[4]}{[2]}\right) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \frac{\begin{bmatrix} 2 & 0 \\ -2 & 0 \end{bmatrix}}{2} - \frac{\begin{bmatrix} 2 & -2 \\ 0 & 0 \end{bmatrix}}{2}$$

$$B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 3/2 & -3/2 \\ -3/2 & 3/2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 5/2 \end{bmatrix}$$

$$S_2 = -[B_2]\nabla f_2$$

$$S_2 = -\begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 5/2 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$S_2 = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 5/2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$S_2 = \begin{bmatrix} 1/2 - 1/2 \\ -1/2 + 5/2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$