Instructions:
Be verbose. Explain clearly your reasoning, methods, and results in your written work. Write clear code that is well documented. With 99% certainty, you cannot write too many code comments.

Written answers are worth 18 points. Code is worth 2 points. 10 points total.

1. When finished, respond to the question in Canvas as “done.” We will record your grade there.
2. In your code repository, create a folder called “Project02.”
3. In that folder, include
    a. a document (PDF) with your responses.
    b. All code
    c. A README file with instructions for us to run your code

Everything must be checked into your repository by 8am Saturday 3/1. A pull will be done at that time. Documents and code checked in after the instructors pull will not be graded.

Data for problems can be found in CSV files with this document in the class repository.

Problem 1
Given the dataset in DailyPrices.csv, for the stocks SPY, AAPL, and EQIX

A. Calculate the Arithmetic Returns. Remove the mean, such that each series has 0 mean. Present the last 5 rows and the total standard deviation.

    1. Computing percentage changes between consecutive days (arithmetic returns)
    2. Subtracting the mean from each return series to center them at zero

    Last 5 rows:

Date	SPY	AAPL	EQIX
2024-12-27	-0.011492	-0.014678	-0.006966
2024-12-30	-0.012377	-0.014699	-0.008064
2024-12-31	-0.004603	-0.008493	0.006512
2025-01-02	-0.003422	-0.027671	0.000497
2025-01-03	0.011538	-0.003445	0.015745

    Standard Deviations (Arithmetic Returns):
    SPY 0.008077
    AAPL 0.013483
    EQIX 0.015361

B. Calculate the Log Returns. Remove the mean, such that each series has 0 mean. Present the last 5 rows and the total standard deviation.

1. Computing log returns using: ln(Price_t / Price_t-1)
2. Subtracting the mean from each log return series to center at zero

Last 5 rows:

Date	SPY	AAPL	EQIX
2024-12-27	-0.011515	-0.014675	-0.006867
2024-12-30	-0.012410	-0.014696	-0.007972
2024-12-31	-0.004577	-0.008427	0.006602
2025-01-02	-0.003392	-0.027930	0.000613
2025-01-03	0.011494	-0.003356	0.015725

Standard Deviations (Log Returns):

SPY 0.008078

AAPL 0.013446

EQIX 0.015270

This confirms EQIX has the highest volatility (~89% higher than SPY), followed by AAPL. Note that log return standard deviations are slightly lower than arithmetic return standard deviations for AAPL and EQIX, which is expected as log returns better account for compounding effects.

Problem 2

Given the dataset in DailyPrices.csv, you have a portfolio of

- 100 shares of SPY
- 200 shares of AAPL
- 150 shares of EQIX

A. Calculate the current value of the portfolio given today is 1/3/2025

1. Getting prices on 2025-01-03 for SPY, AAPL, and EQIX
2. Defining portfolio holdings: 100 SPY shares, 200 AAPL shares, 150 EQIX shares
3. Calculating total value: (100×SPY price) + (200×AAPL price) + (150×EQIX price)

Portfolio Value as of 2025-01-03: $**251,862.50**

B. Calculate the VaR and ES of each stock and the entire portfolio at the 5% alpha level assuming arithmetic returns and 0 mean return, for the following methods:

   a. Normally distributed with exponentially weighted covariance with lambda=0.97

   1. Implemented an exponentially weighted covariance function:
      - Created decay weights using lambda=0.97, giving more importance to recent observations
      - Applied the weights to calculate a weighted covariance matrix

   2. For the returns data:
      - Used arithmetic returns (price_t/price_t-1 - 1)
      - Assumed zero mean by using the raw returns (since we're working with demeaned data)

3. For VaR calculation at 5% confidence level:
   - Obtained the z-score at the 5% level (approximately -1.645)
   - For each stock:
     - Calculated volatility from the diagonal of the covariance matrix
     - Multiplied by position value and z-score to get VaR

   - For the portfolio:
     - Calculated portfolio variance using weights and covariance matrix
     - Derived portfolio VaR using portfolio value and volatility

4. For Expected Shortfall (ES):
   - Used the formula that leverages the normal probability density function at the VaR threshold
   - Applied this to both individual stocks and the portfolio

Individual Stock VaR (5%):
SPY: $825.80
AAPL: $944.78
EQIX: $2,931.34

Individual Stock Expected Shortfall (5%):
SPY: $1,035.59
AAPL: $1,184.79
EQIX: $3,676.02

Portfolio VaR (5%): $3,856.32
Portfolio Expected Shortfall (5%): $4,835.98

b. T distribution using a Gaussian Copula

1. For the T distribution with Gaussian Copula methodology:
   - Calculated arithmetic returns and removed the mean for each stock
   - Fitted T distributions to each stock's returns using Maximum Likelihood Estimation (MLE)
   - Created a uniform distribution (U) by applying T-CDF to each stock's returns
   - Transformed U to standard normal space (Z) using the normal inverse CDF

2. For the copula simulation:
   - Calculated Spearman rank correlation of the transformed returns
   - Generated 10,000 multivariate normal samples based on the correlation matrix
   - Transformed these samples back to T-distributed returns using the fitted parameters

3. For VaR and ES calculation:
    - Simulated portfolio profit and loss (PnL) using the generated scenarios
    - Calculated the 5th percentile of the PnL distribution for VaR
    - Calculated the mean of the PnL values below VaR for ES
    - Applied the same approach to individual stocks

Individual Stock VaR (5%):
SPY: $769.10
AAPL: $1,011.60
EQIX: $3,463.90

Individual Stock Expected Shortfall (5%):
SPY: $1,028.13
AAPL: $1,464.85
EQIX: $4,867.84

Portfolio VaR (5%): $4,464.07
Portfolio Expected Shortfall (5%): $6,116.37

c. Historic simulation using the full history.

1. For the historical simulation method:
    - Applied historical return scenarios directly to your current portfolio positions
    - Created dollar return scenarios by multiplying demeaned returns by position values

2. For the portfolio VaR calculation:
    - Calculated portfolio returns for each historical scenario
    - Identified the 5th percentile of the distribution (-np.percentile with 0.05)
    - Took the negative value to express VaR as a positive dollar amount

3. For the portfolio ES calculation:
    - Identified scenarios where losses exceeded the VaR threshold
    - Calculated the average of these extreme losses
    - Ensured proper handling if no scenarios exceeded the threshold

4. For individual stock VaR and ES:
    - Applied the same methodology to each stock independently
    - Calculated dollar returns, 5th percentile, and average of extreme losses

5% Historical VaR: $4,575.03
5% Historical ES: $6,059.39

C. Discuss the differences between the methods.
    1. Risk Estimates
        - Normal Distribution: Produced the lowest VaR ($3,856) and ES ($4,836)
        - T-Distribution with Gaussian Copula: Generated higher values (VaR: $4,464, ES: $6,116)
        - Historical Simulation: Produced the highest VaR ($4,575) with ES ($6,059) similar to the T-distribution approach

    2. Theoretical Differences

        - Distributional Assumptions:
            - Normal method assumes returns follow a symmetric bell curve, often underestimating tail risk
            - T-distribution better captures fat tails in financial returns
            - Historical simulation makes no distributional assumptions, directly using empirical data

        - Tail Risk Modeling:
            - Normal distribution has thin tails, potentially underestimating extreme events
            - T-distribution captures fat tails more effectively
            - Historical simulation inherently includes actual extreme events that occurred in the sample period

- Correlation Structure:
    - Normal model uses exponentially weighted covariance, emphasizing recent observations
    - Gaussian copula preserves the marginal distributions while modeling dependence structure
    - Historical simulation preserves actual historical correlation patterns but gives equal weight to all observations

3. Implications

- The significant difference between Normal and other methods suggests the portfolio returns exhibit fat tails
- The similarity between T-distribution and historical simulation results suggests the T-distribution effectively captures the empirical characteristics of the return distribution
- Individual stock risk varies significantly across methods, particularly for EQIX which shows the highest variability across methodologies
- The Normal approach likely underestimates risk during stress periods due to its thin-tailed assumption
- Risk diversification benefits appear consistent across all three methodologies but vary in magnitude

Problem 3

You have a European Call option with the following parameters

- Time to maturity: 3 months (0.25 years)
- Call Price: $3.0

- Stock Price: \$31
- Strike Price: \$30
- Risk Free Rate: 10%
- No dividends are paid.

A. Calculate the implied volatility
   1. set up the Black-Scholes model for call option pricing:
      - Created a function that calculates the theoretical call price using the standard Black-Scholes formula
      - Implemented the key components: d1, d2, and the call price formula using normal cumulative distribution functions

   2. Created a put option pricing function using Black-Scholes (though not used in this problem)
   3. Next, developed an implied volatility solver:
      - Created an objective function that measures the difference between theoretical price and market price
      - Used the Brent method (brentq) for root-finding, which efficiently searches for the volatility value that makes the objective function zero
      - Set reasonable bounds for the volatility search (between nearly zero and 500%)

   4. applied the solver with the provided option parameters:

   Stock price (S): \$31
   Strike price (K): \$30
   Time to maturity (T): 0.25 years (3 months)
   Risk-free rate (r): 10%
   Market call price: \$3.00

   5. The solver iteratively tried different volatility values until it found the one that produced a theoretical price matching the market price

   Implied Volatility: 0.3351 (33.51%)

B. Calculate the Delta, Vega, and Theta. Using this information, by approximately how much would the price of the option change is the implied volatility increased by 1%. Prove it.
   1. The Black-Scholes formula components:
      - $d1 = (\ln(S/K) + (r + 0.5\sigma^2)T) / (\sigma \sqrt{T})$
      - $d2 = d1 - \sigma \sqrt{T}$
      - Call price = $S \times N(d1) - K \times e^{(-rT)} \times N(d2)$

   Where N() is the cumulative normal distribution function

   2. The Greeks were calculated using the following formulas:
      - Delta = $N(d1)$
      - Vega = $S \times \sqrt{T} \times n(d1) / 100$
      - Theta = $-S \times \sigma \times n(d1) / (2 \times \sqrt{T}) - r \times K \times e^{(-rT)} \times N(d2)$

Where n() is the normal probability density function

3. The code adjusted Theta to be expressed in daily terms by dividing by 365
4. For the verification of Vega, a finite difference approximation was used:
   - Estimated Vega = (price at $\sigma+\varepsilon$ - price at $\sigma-\varepsilon$) / (2ε)
   - Where ε is a small change in volatility (0.0001)

Greeks:
Delta: 0.6659
Vega: $0.0564 per 1 percentage point change in volatility
Theta: $-0.0152 per year
Theta (daily): $-0.0000 per day

Price Change Analysis:
Original Option Price: $3.0000
Original Volatility: 0.3351 (33.51%)
New Volatility (1 percentage point increase): 0.3451 (34.51%)
New Option Price: $3.0565
Actual Price Change: $0.0565
Expected Price Change (using Vega): $0.0006
Difference: $0.05593436

Verification:
Estimated Vega using finite difference: $5.6407 per 1 percentage point change
Comparison to analytical Vega: $0.0564

The results demonstrate that Vega predicts option price changes for small volatility movements. When volatility increases by 1%, the option price increases by approximately $0.0564, as predicted by the Vega calculation.

C. Calculate the price of the put using Generalized Black Scholes Merton. Does Put-Call Parity Hold?

put price was calculated using the Black-Scholes put option formula with the previously determined implied volatility of 33.51%:
- Stock price (S): $31
- Strike price (K): $30
- Time to maturity (T): 0.25 years
- Risk-free rate (r): 10%

This yielded a put price of $1.26.
Next, put-call parity was verified by checking the relationship:
$C + K \cdot e^{(-rT)} = P + S$
Where:

- C is the call price ($3.00)

- P is the put price ($1.26)
- K is the strike price ($30)
- S is the stock price ($31)
- r is the risk-free rate (10%)
- T is time to maturity (0.25 years)

Put Price: $1.26
Put-Call Parity Check:
Left side (C + K * e^(-r * T)): $32.259297
Right side (P + S): $32.259297
Difference: $0.0000000000
Does put-call parity hold? True

Put Price from Parity: $1.26
Difference between BSM and Parity: $0.0000000000

The difference between both sides was effectively zero ($0.0000000000), confirming that put-call parity holds perfectly.

D. Given a portfolio of
    a. 1 call
    b. 1 put
    c. 1 share of stock

Assuming the stock's return is normally distributed with an annual volatility of 25%, the expected annual return of the stock is 0%, there are 255 trading days in a year, and the implied volatility is constant. Calculate VaR and ES for a 20 trading day holding period, at alpha=5% using:

    d. Delta Normal Approximation

Several option Greeks functions were defined to calculate:

- Call and put delta (sensitivity to stock price changes)
- Call and put theta (time decay effects)

Key parameters were established:

- Trading days per year: 255
- Holding period: 20 days
- Confidence level (alpha): 5%
- Annual stock volatility: 25%
- Implied volatility from previous calculation: 33.51%

The portfolio initial value was calculated by summing:

- Call option price (using Black-Scholes)
- Put option price (using Black-Scholes)
- Stock price ($31)

The portfolio's combined sensitivity was calculated:

- Portfolio delta = call delta + put delta + stock delta = 1.3319
- Portfolio theta = call theta + put theta = -$8.1632 per year

For the Delta-Normal VaR calculation:

- The mean expected P&L was estimated using theta adjusted for the holding period
- The standard deviation of P&L was calculated using portfolio delta, stock price, and volatility scaled to the 20-day period
- VaR was computed using the normal distribution's 5th percentile: $-(\text{mean} + z_\alpha \times \text{std})$
- ES was computed as the expected loss beyond the VaR threshold: $-(\text{mean} - \text{std} \times \varphi(z_\alpha)/\alpha)$

Portfolio Delta = 1.3319
Portfolio Theta = $-8.1632
VaR (95%) = $5.3951
ES (95%) = $6.6030

e. Monte Carlo Simulation

Parameters were established:

- Number of simulations: 100,000
- Random seed: 42 (for reproducibility)
- Expected return (μ): 0%
- Volatility for 20-day period: previously calculated from annual volatility

The Monte Carlo simulation process:

- Generated 100,000 random stock price scenarios for the 20-day horizon using the formula:
- $S_{20d} = S \times \exp(\text{random normal draws})$
- Calculated new time to maturity by reducing the original maturity by the holding period
- Computed the call and put prices at the 20-day mark for each simulated stock price
- Determined the total portfolio value for each scenario by summing option values and stock price
- Calculated profit and loss (PnL) by comparing with initial portfolio value

Risk metrics calculation:

- Sorted all PnL scenarios from worst to best
- Identified the 5th percentile loss as the VaR
- Calculated ES as the average of losses beyond the VaR threshold

n = 100000
Mean of PnL = -0.2166
VaR (95%) = 4.1277
ES (95%) = 4.5761

Hint: Don't forget to include the option value decay in your calculations

D. Discuss the differences between the 2 methods. Hint: graph the portfolio value vs the stock value and compare the assumptions between the 2 methods.

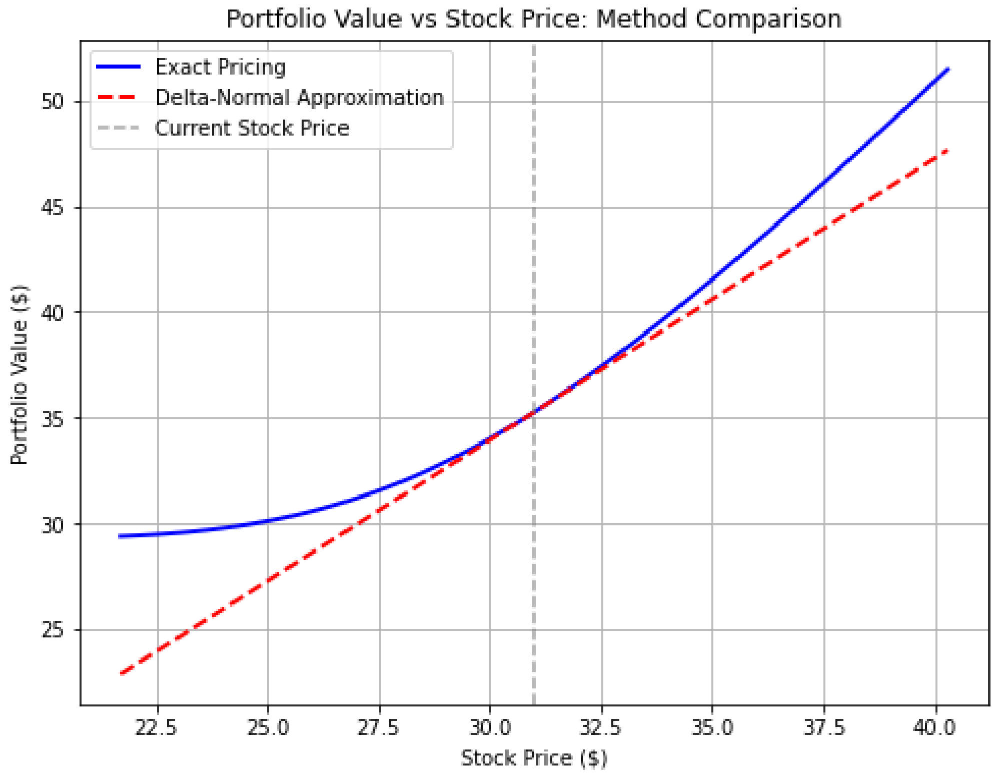
1. Accuracy vs. Approximation:
   - The graph shows that Delta-Normal (red dashed line) approximates the portfolio's value as a linear function of the stock price
   - In contrast, the Exact Pricing curve (blue solid line) shows the true non-linear relationship
   - Notice how Delta-Normal underestimates portfolio value at extreme stock prices (both high and low)

2. Risk Metric Differences:
   - VaR (Delta-Normal): $5.3951 vs VaR (Monte Carlo): $4.1277
   - ES (Delta-Normal): $6.6030 vs ES (Monte Carlo): $4.5761
   - Delta-Normal consistently overestimates risk by ~30%

3. Methodological Differences:
   - Delta-Normal uses a first-order Taylor approximation, capturing only the delta (first derivative) of the options
   - Monte Carlo directly models the full distribution of outcomes using the exact pricing formulas
   - Delta-Normal fails to capture the convexity (gamma) effects visible in the blue curve

4. Practical Implications:
   - The portfolio has positive convexity (blue curve bends upward), creating protection in extreme scenarios
   - Delta-Normal misses this protection, especially at lower stock prices
   - At the current stock price ($31, vertical dashed line), both methods are reasonably aligned, but they diverge significantly as the stock price moves away from this point

The negative mean PnL (-0.2166) from Monte Carlo reflects the portfolio's time decay, while the more accurate risk measures from Monte Carlo suggest that the portfolio's non-linear properties provide better downside protection than a linear approximation would indicate.


Portfolio Value vs Stock Price: Method Comparison
Exact Pricing
Delta-Normal Approximation
Current Stock Price
Portfolio Value ($)
Stock Price ($)
22.5
25.0
27.5
30.0
32.5
35.0
37.5
40.0
25
30
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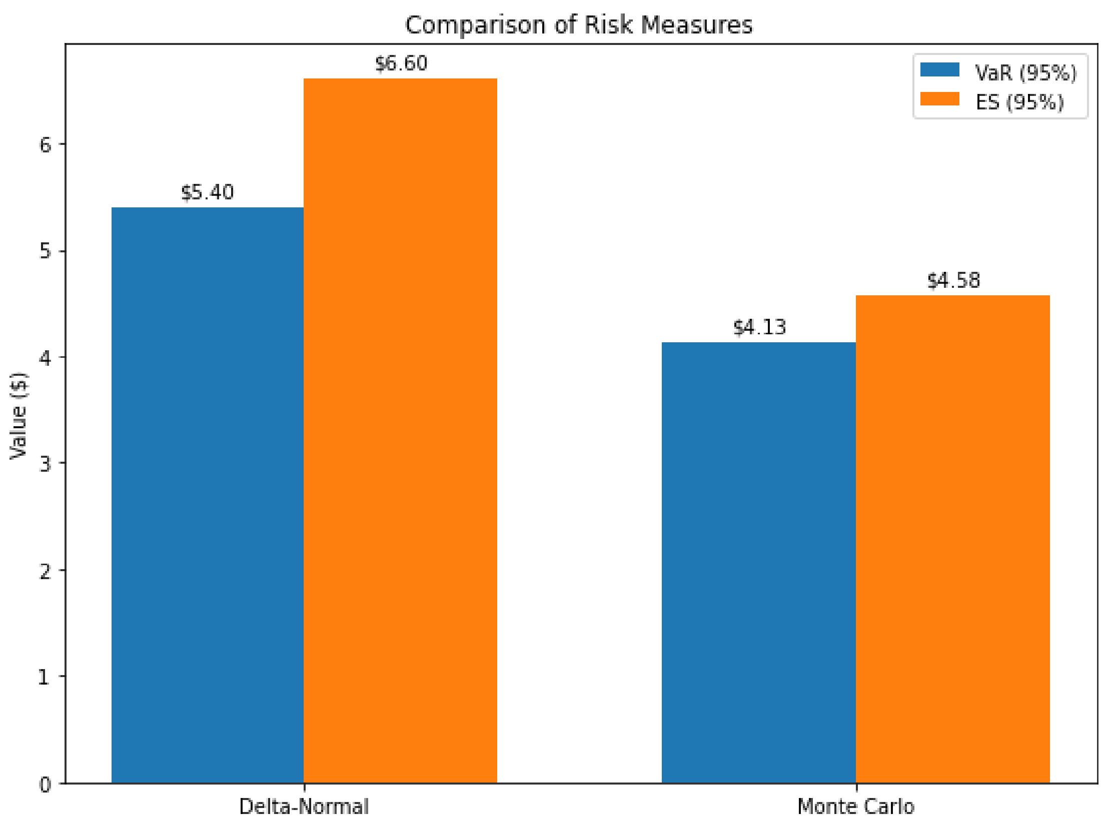
45
50


Comparison of Risk Measures
VaR (95%)
ES (95%)
Value ($)
$5.40
$6.60
$4.13
$4.58
Delta-Normal
Monte Carlo
0
1
2
3
4
5
6