Equation of Motion

Start with Newton's Second Law

$$\underline{F} = m\underline{a} = m\frac{d^2\underline{u}}{dt^2}$$

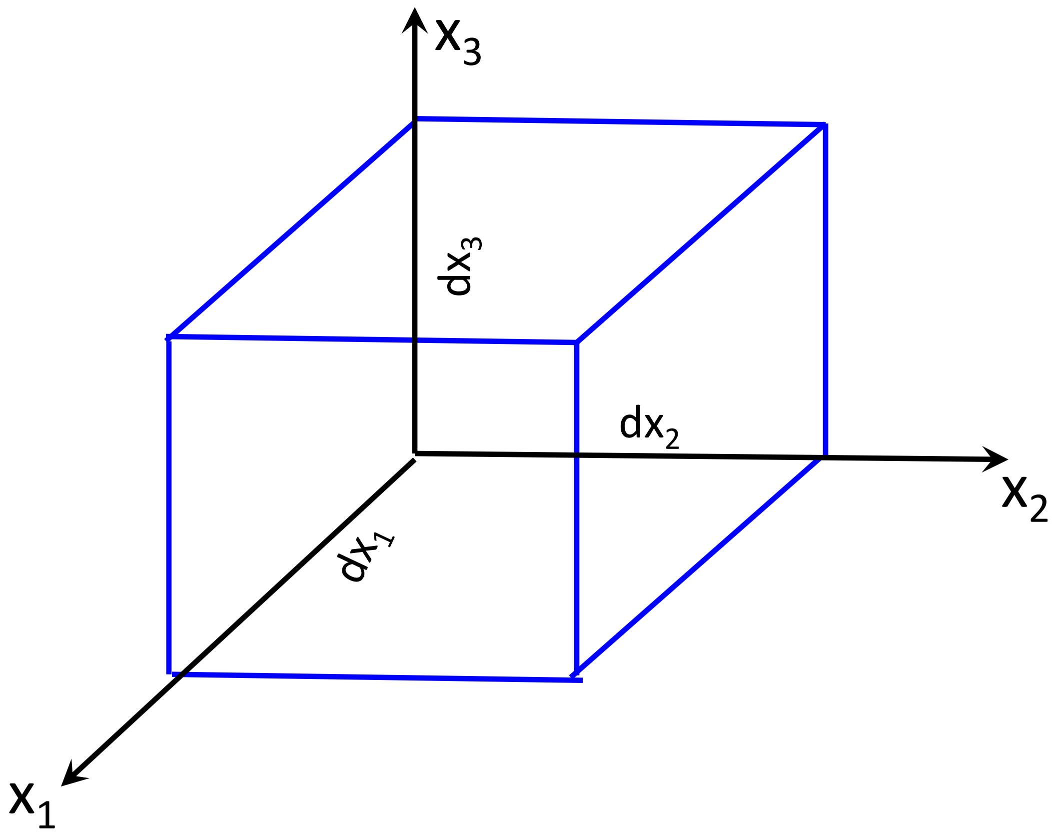
$$F_i = ma_i = m\frac{d^2u_i}{dt^2}$$

$$F_i: Froce; \;\; u_i: Displacement$$

Let's considering a block of material

Mass $m = \varrho dx_1 dx_2 dx_3$

$$F_i = \begin{cases} Body\ force: \quad f_i dx_1 dx_2 dx_3 \\ Surface\ force \end{cases}$$



Surface force

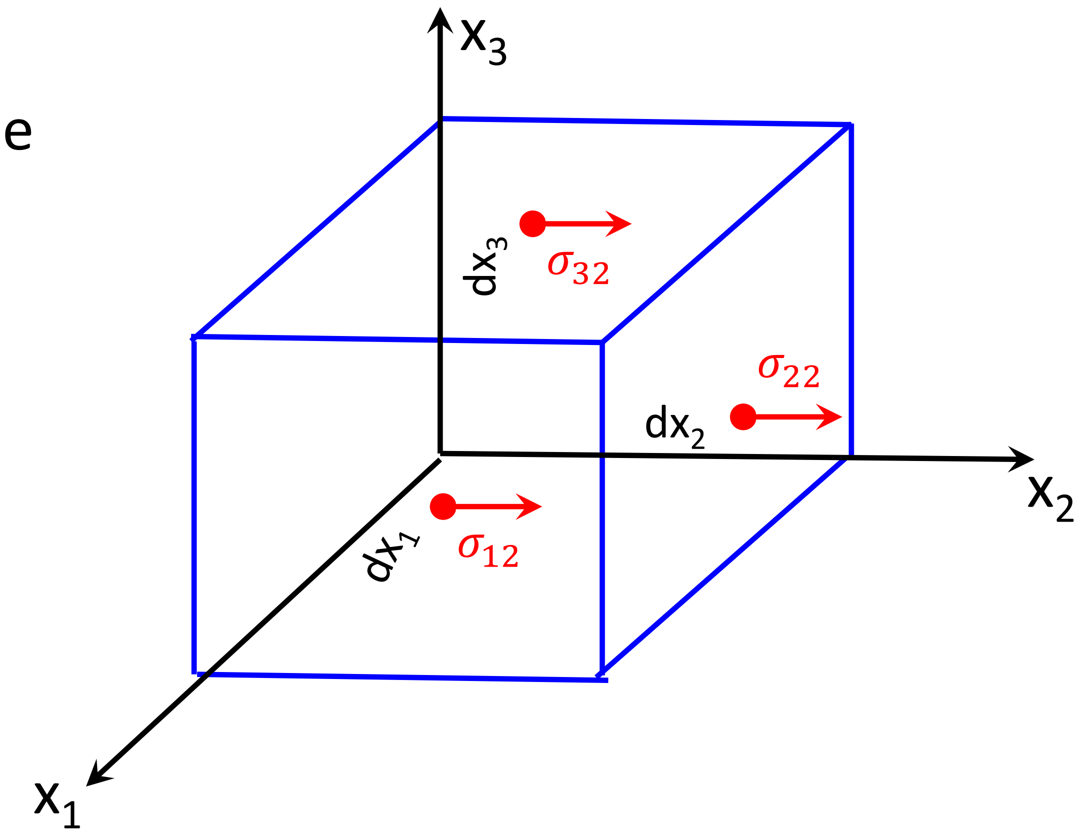
$Stress\ \sigma_{ij}$

i: direction of the normal to the plane
j: direction of the force

Acting on direction of x_2

σ_{22} — Normal stress

σ_{12} σ_{32} — Shear stress

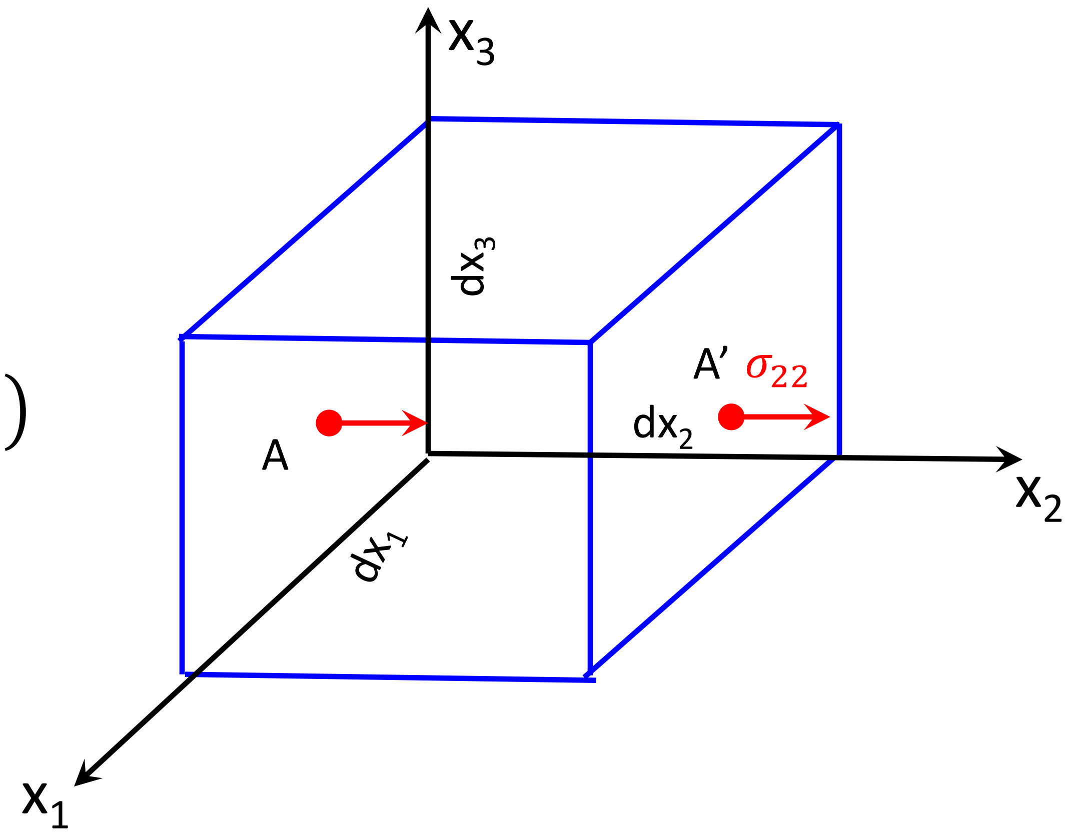


Surface force

$$Considering\ the\ \sigma_{22}$$

At A, Normal stress : $\sigma_{22}(\underline{x})$

At A', Normal stress : $\sigma_{22}(\underline{x} + dx_2 \cdot \hat{x}_2)$



Surface force

Considering the σ_{22}

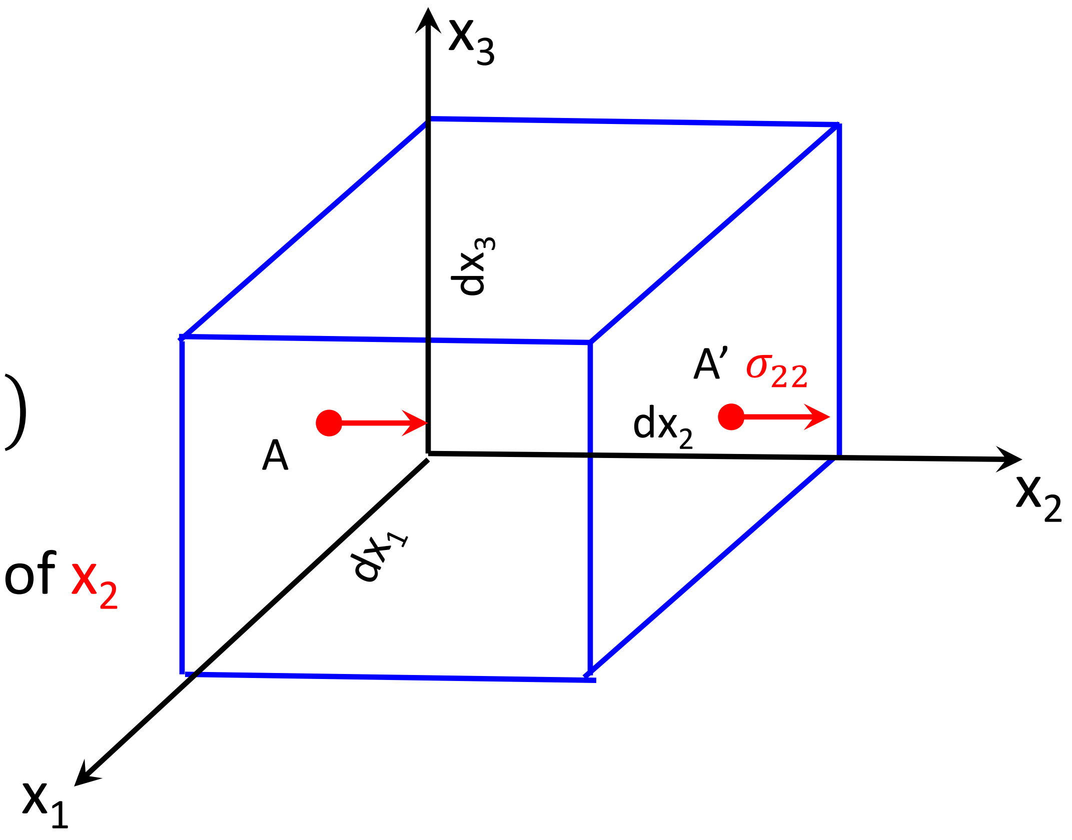
At A, Normal stress : $\sigma_{22}(\underline{x})$

At A', Normal stress : $\sigma_{22}(\underline{x} + dx_2 \cdot \hat{x}_2)$

Thus net surface force along the direction of x_2

$$\left[\sigma_{22}(\underline{x} + dx_2 \cdot \hat{x}_2) - \sigma_{22}(\underline{x})\right] dx_1 dx_3$$

Surface area



Surface force

Considering the σ_{22}

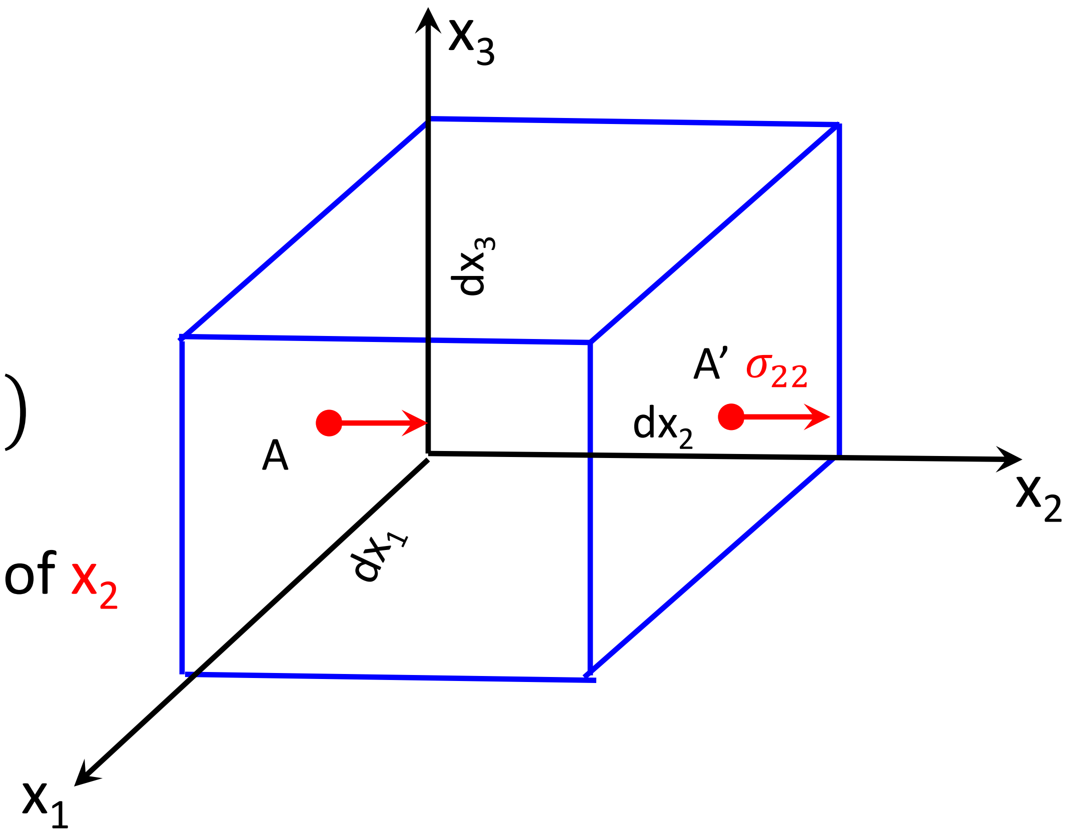
At A, Normal stress : $\sigma_{22}(\underline{x})$

At A', Normal stress : $\sigma_{22}(\underline{x} + dx_2 \cdot \hat{x}_2)$

Thus net surface force along the direction of x_2

$$\left[\sigma_{22}(\underline{x} + dx_2 \cdot \hat{x}_2) - \sigma_{22}(\underline{x})\right] dx_1 dx_3$$

$$= \frac{\partial \sigma_{22}(\underline{x})}{\partial x_2} dx_1 dx_2 dx_3$$



Surface force

$$Considering\ the\ \sigma_{22}$$

$$\left[\sigma_{22}\left(\underline{x} + dx_2 \cdot \hat{x}_2\right) - \sigma_{22}\left(\underline{x}\right)\right] dx_1 dx_3$$

$$= \frac{\partial \sigma_{22}\left(\underline{x}\right)}{\partial x_2} dx_1 dx_2 dx_3$$

$$For\ \sigma_{32}$$

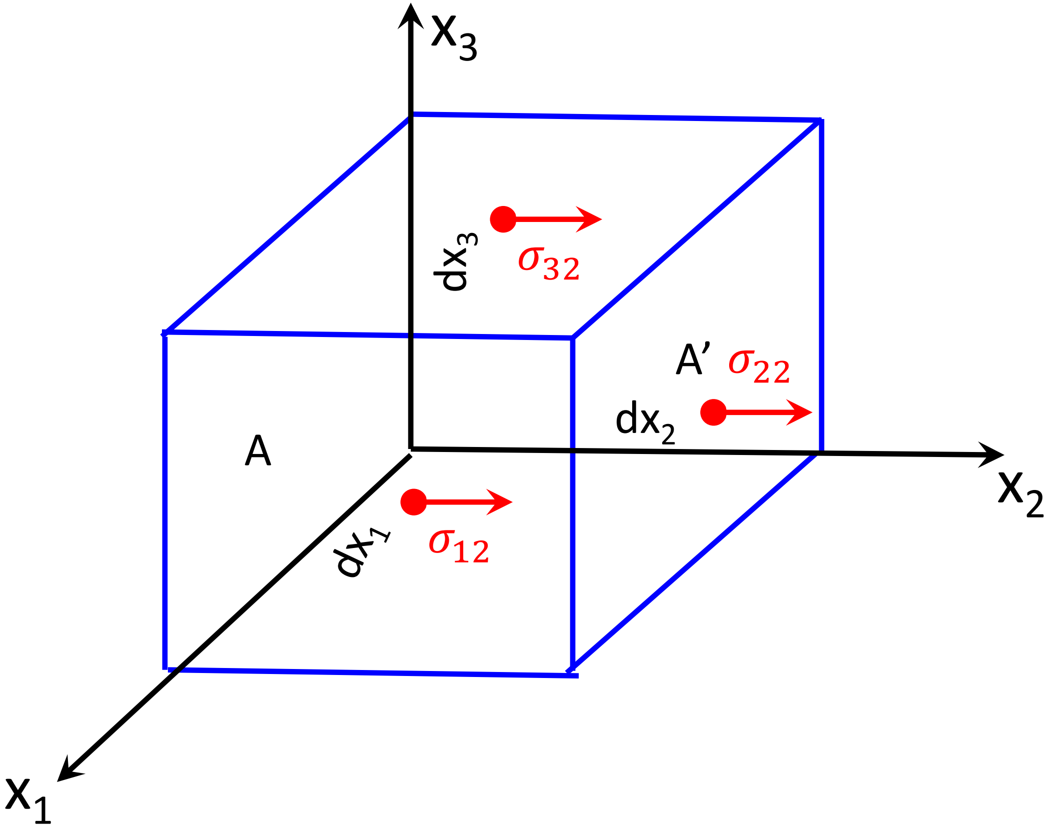
$$\left[\sigma_{32}\left(\underline{x} + dx_3 \cdot \hat{x}_3\right) - \sigma_{32}\left(\underline{x}\right)\right] dx_1 dx_2$$

$$= \frac{\partial \sigma_{32}\left(\underline{x}\right)}{\partial x_3} dx_1 dx_2 dx_3$$

$$For\ \sigma_{12}$$

$$\left[\sigma_{12}\left(\underline{x} + dx_1 \cdot \hat{x}_1\right) - \sigma_{12}\left(\underline{x}\right)\right] dx_2 dx_3$$

$$= \frac{\partial \sigma_{12}\left(\underline{x}\right)}{\partial x_1} dx_1 dx_2 dx_3$$



Surface force

Acting on direction of x_2 ,

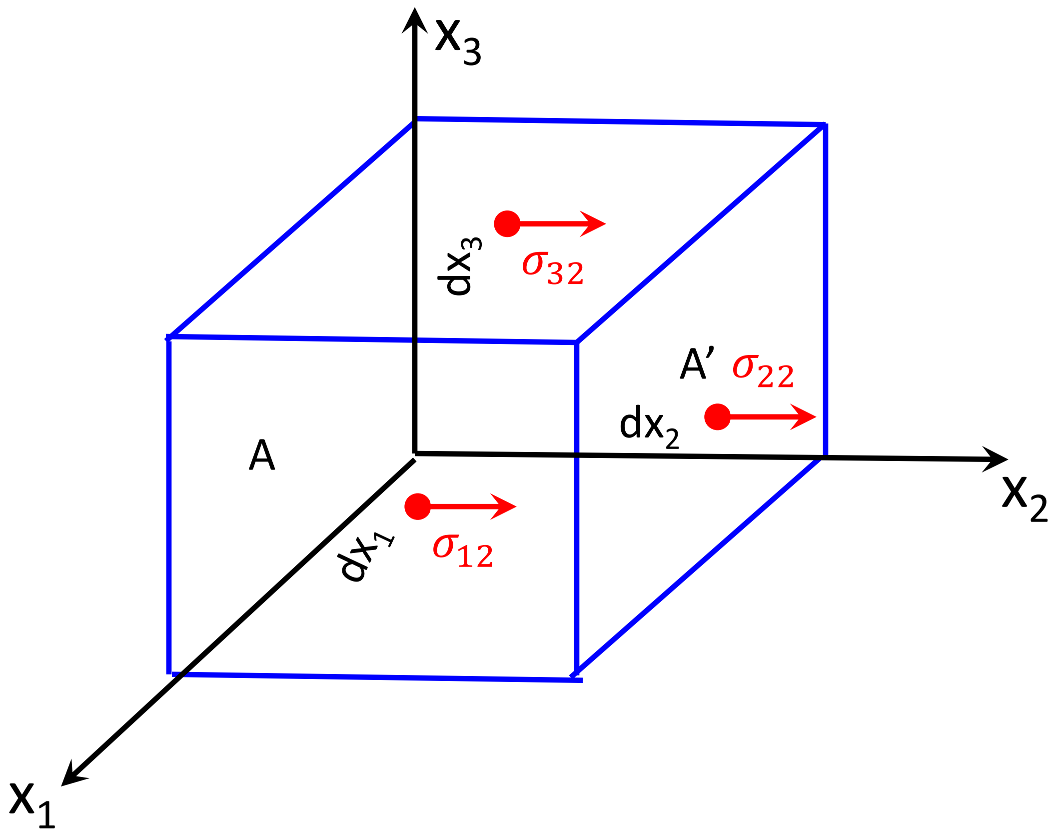
$$\frac{\partial \sigma_{22}(\underline{x})}{\partial x_2} dx_1 dx_2 dx_3$$

$$+ \frac{\partial \sigma_{32}(\underline{x})}{\partial x_3} dx_1 dx_2 dx_3$$

$$+ \frac{\partial \sigma_{12}(\underline{x})}{\partial x_1} dx_1 dx_2 dx_3$$

The total force

$$F_2 = \left(\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3}\right) dx_1 dx_2 dx_3 + f_2 \, dx_1 dx_2 dx_3$$



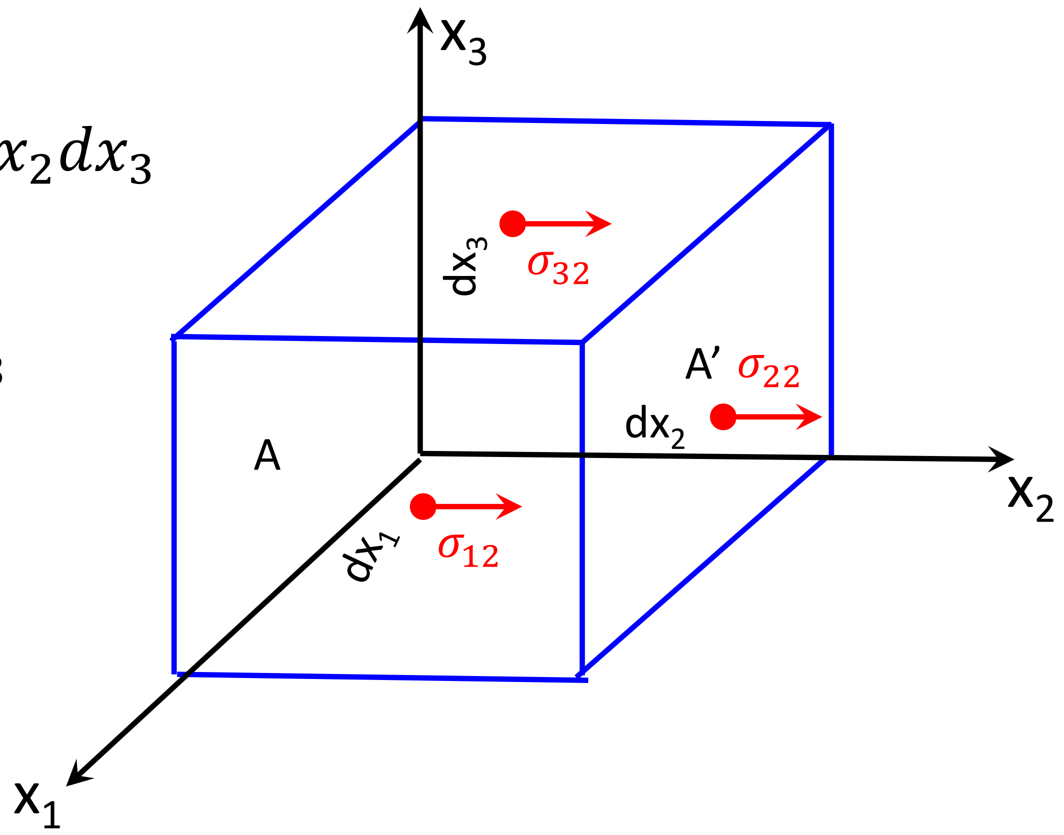
Equation of motion

$$F_i = ma_i = m\frac{d^2u_i}{dt^2}$$

$$mass: \quad m = \varrho dx_1 dx_2 dx_3$$

$$F_2 = \left(\frac{\partial\sigma_{12}}{\partial x_1} + \frac{\partial\sigma_{22}}{\partial x_2} + \frac{\partial\sigma_{32}}{\partial x_3}\right) dx_1 dx_2 dx_3 + f_2\, dx_1 dx_2 dx_3$$

$$= \varrho dx_1 dx_2 dx_3 \frac{d^2u_2}{dt^2}$$



Equation of motion

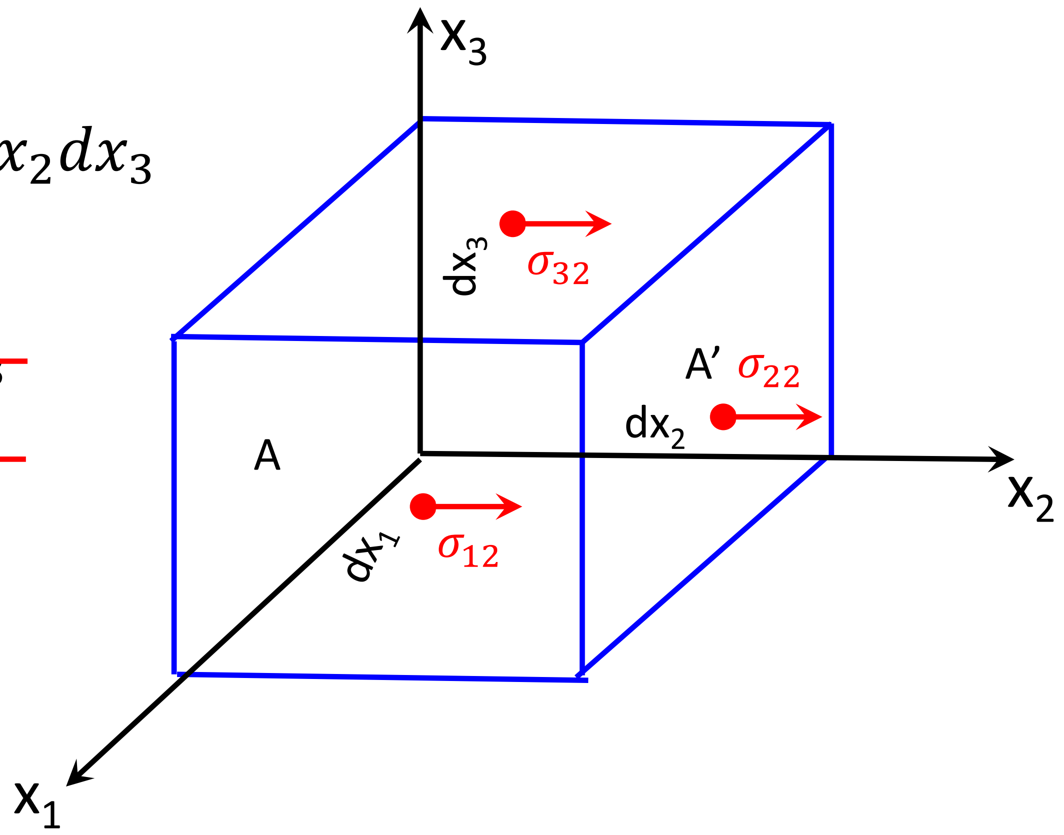
$$F_i = ma_i = m\frac{d^2u_i}{dt^2}$$

$$mass: \quad m = \varrho dx_1 dx_2 dx_3$$

$$F_2 = \left(\frac{\partial\sigma_{12}}{\partial x_1} + \frac{\partial\sigma_{22}}{\partial x_2} + \frac{\partial\sigma_{32}}{\partial x_3}\right) \cancel{dx_1 dx_2 dx_3}$$
$$+ f_2 \cancel{dx_1 dx_2 dx_3}$$

$$= \varrho \cancel{dx_1 dx_2 dx_3} \frac{d^2u_2}{dt^2}$$

$$\left(\frac{\partial\sigma_{12}}{\partial x_1} + \frac{\partial\sigma_{22}}{\partial x_2} + \frac{\partial\sigma_{32}}{\partial x_3}\right) + f_2 = \varrho\frac{d^2u_2}{dt^2}$$

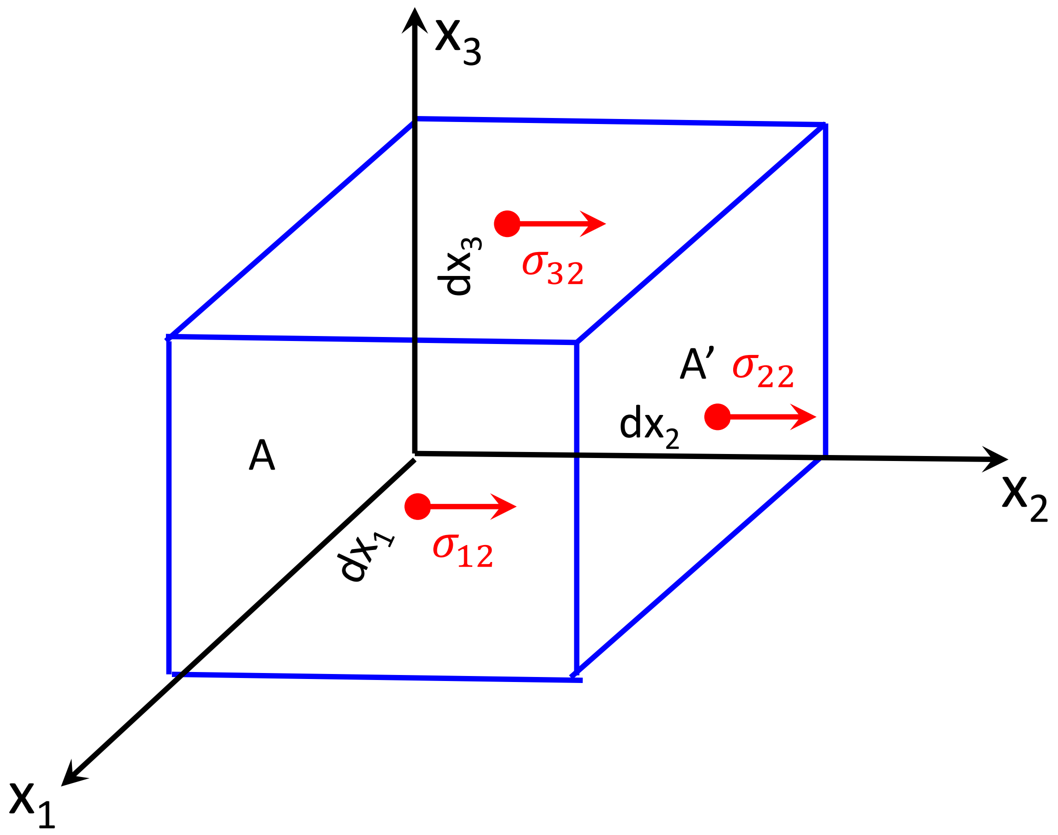


Equation of motion

$$\left(\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_1}\right) + f_1 = \varrho \frac{d^2 u_1}{dt^2}$$

$$\left(\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3}\right) + f_2 = \varrho \frac{d^2 u_2}{dt^2}$$

$$\left(\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3}\right) + f_3 = \varrho \frac{d^2 u_3}{dt^2}$$



In summation convention

$$\frac{\partial \sigma_{ji}}{\partial x_j} + f_i = \varrho \frac{d^2 u_i}{dt^2}$$

Note the stress tensor is symmetry: $\sigma_{ji} = \sigma_{ij}$

Equation of motion

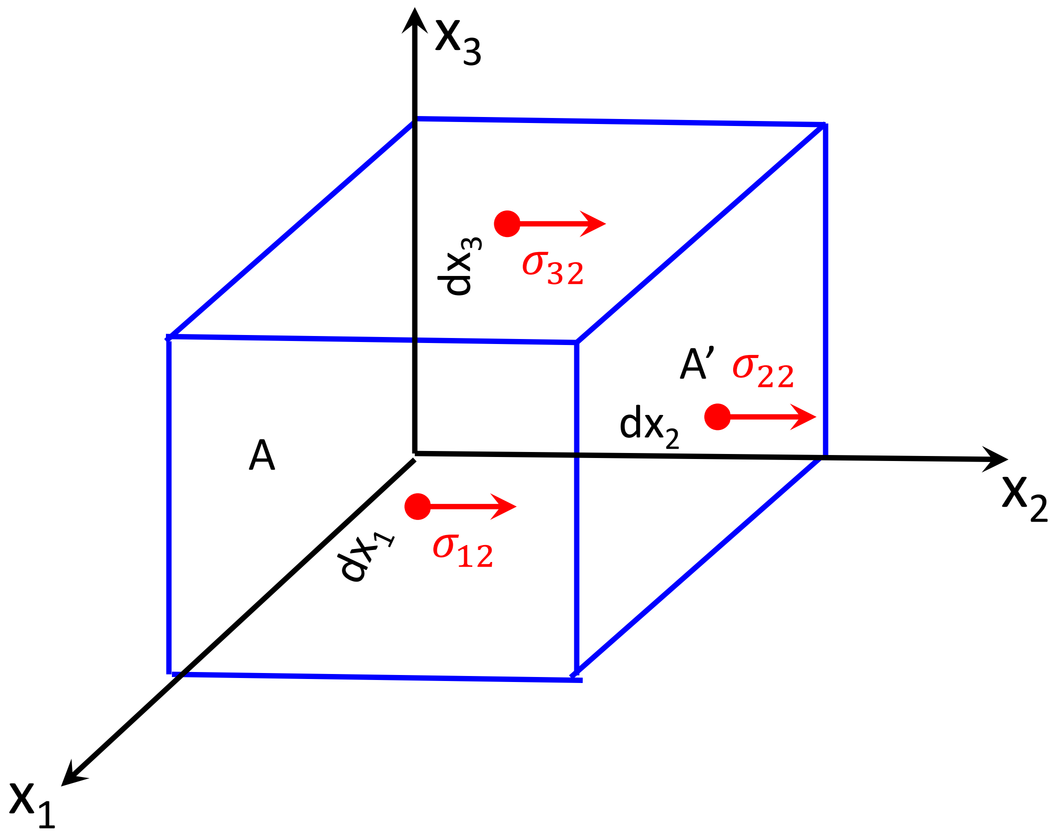
$$\frac{\partial \sigma_{ij}}{\partial x_j} + f_i = \varrho \frac{d^2 u_i}{dt^2}$$

In most cases: $f_i = 0$

$$\frac{\partial \sigma_{ij}}{\partial x_j} = \varrho \frac{d^2 u_i}{dt^2}$$

Homogeneous Equation of motion:

$$\boxed{\sigma_{ij,j} = \varrho \ddot{u}_i}$$



To solve this equation, we need another relation between σ_{ij} and u_i

Constitutive Equation

σ_{ij} <---------> u_i

Force <---------> Displacement

String as an example

Hooke's Law

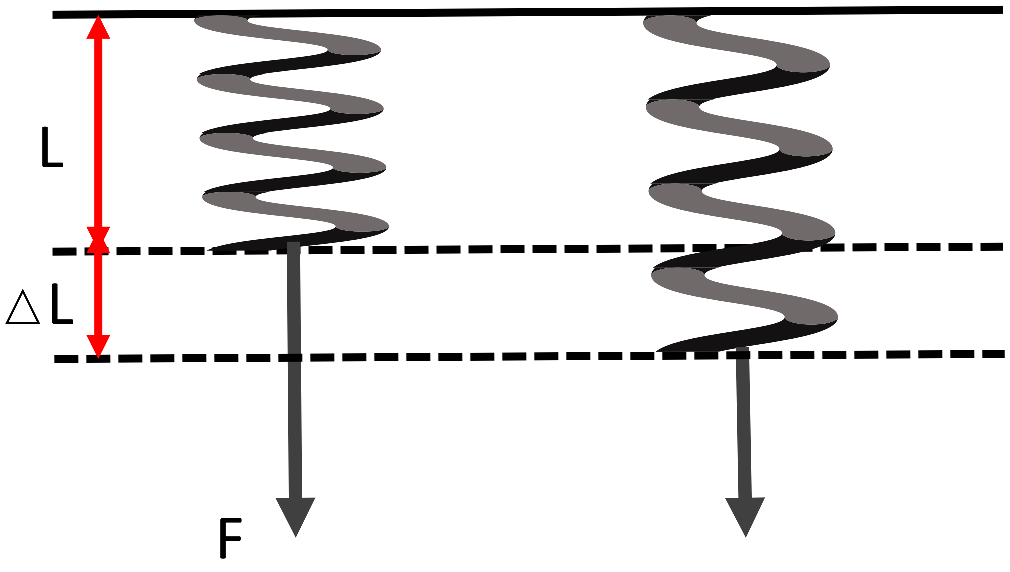
$$F = E \cdot \frac{\Delta L}{L}$$

Displacment

Force

Elastic constant

Strain



$$strain\ \varepsilon_{kl} = \frac{1}{2}\left(u_{k,l} + u_{l,k}\right)$$

Linear Hooker's Law

$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$

C_{ijkl}: Elastic Moduli

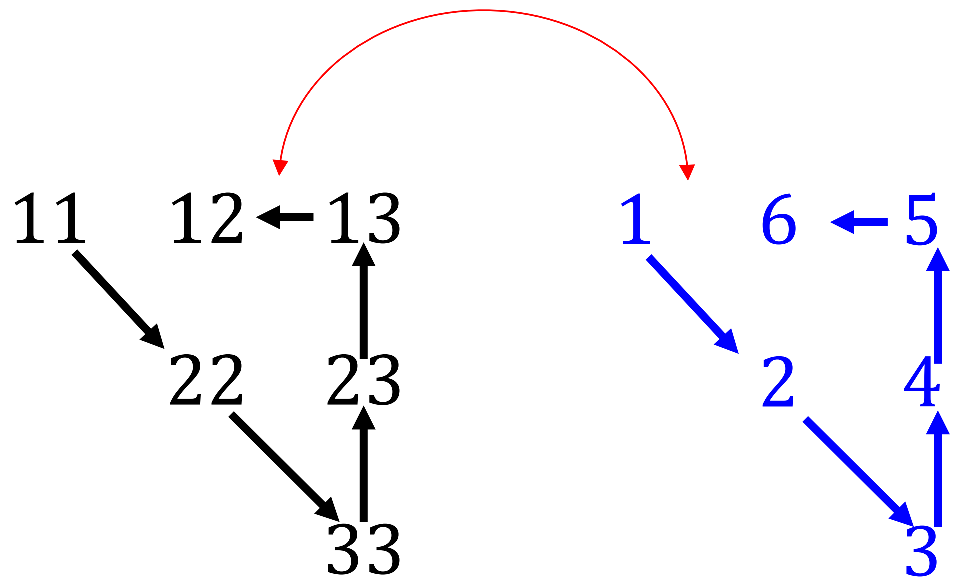
$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} = C_{ijlk}\varepsilon_{lk}$$
$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} = \sigma_{ji} = C_{jikl}\varepsilon_{kl}$$
$$C_{ijkl} = C_{klij}$$

Reduce to 21 parameters (general anisotropy)

C_{ijkl}: Elastic Moduli

Typically we write as:

$$\begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{pmatrix}$$



Isotropic case

$C_{2311} \rightarrow$?

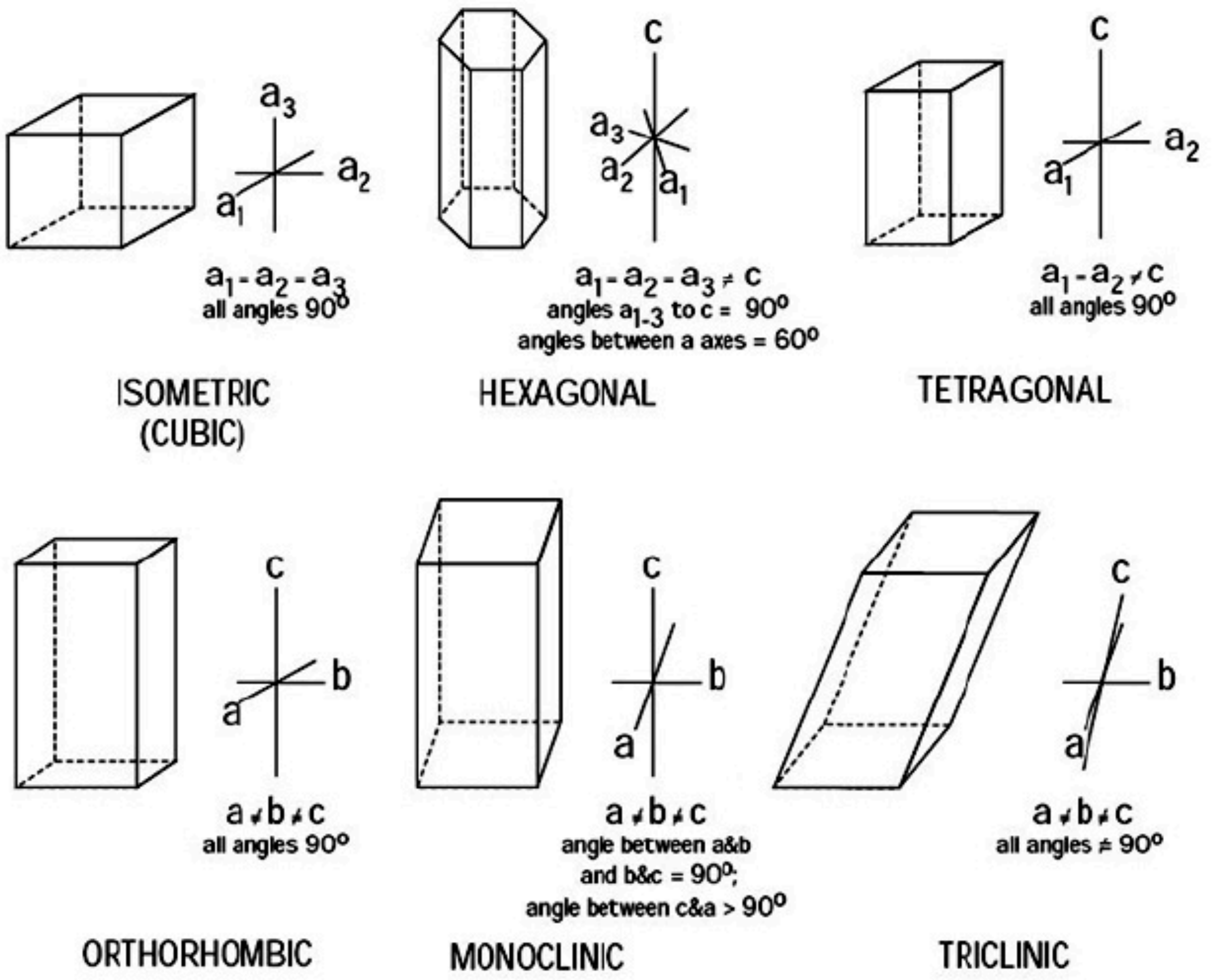
$$\begin{pmatrix} C_{11} & C_{12} & C_{12} & & & \\ C_{12} & C_{11} & C_{12} & & & \\ C_{12} & C_{12} & C_{11} & & & \\ & & & C_{55} & & \\ & & & & C_{55} & \\ & & & & & C_{55} \end{pmatrix}$$

$C_{12} = C_{11}$ - 2C_{55}, 2 independent parameters

Anisotropic materials

Type of symmetry	Number of independent elastic coefficients	Typical mineral
isotropic solid	2	volcanic glas
cubic	3	garnet
hexagonal	5	ice
trigonal I	7	ilmenite
trigonal II	6	quartz
tetragonal	6	stishovite
orthorhombic	9	olivine
monoclinic	13	hornblende
triclinic	21	plagioclase

Symmetry planes



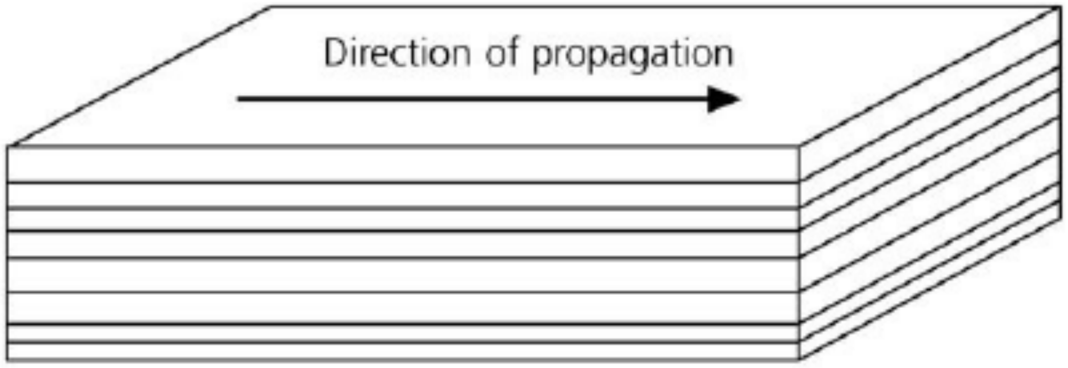
Hexagonal symmetry (5 independent elastic coefficients)

A vertical symmetry axis: Transversely anisotropy

VTI

$$\begin{pmatrix} C_{11} & C_{12} & C_{13} & & & \\ C_{12} & C_{11} & C_{13} & & & \\ C_{13} & C_{13} & C_{33} & & & \\ & & & C_{44} & & \\ & & & & C_{44} & \\ & & & & & C_{66} \end{pmatrix}$$

$$C_{12} = C_{11} - 2C_{66}$$



Shale sediments

Hexagonal symmetry (5 independent elastic coefficients)

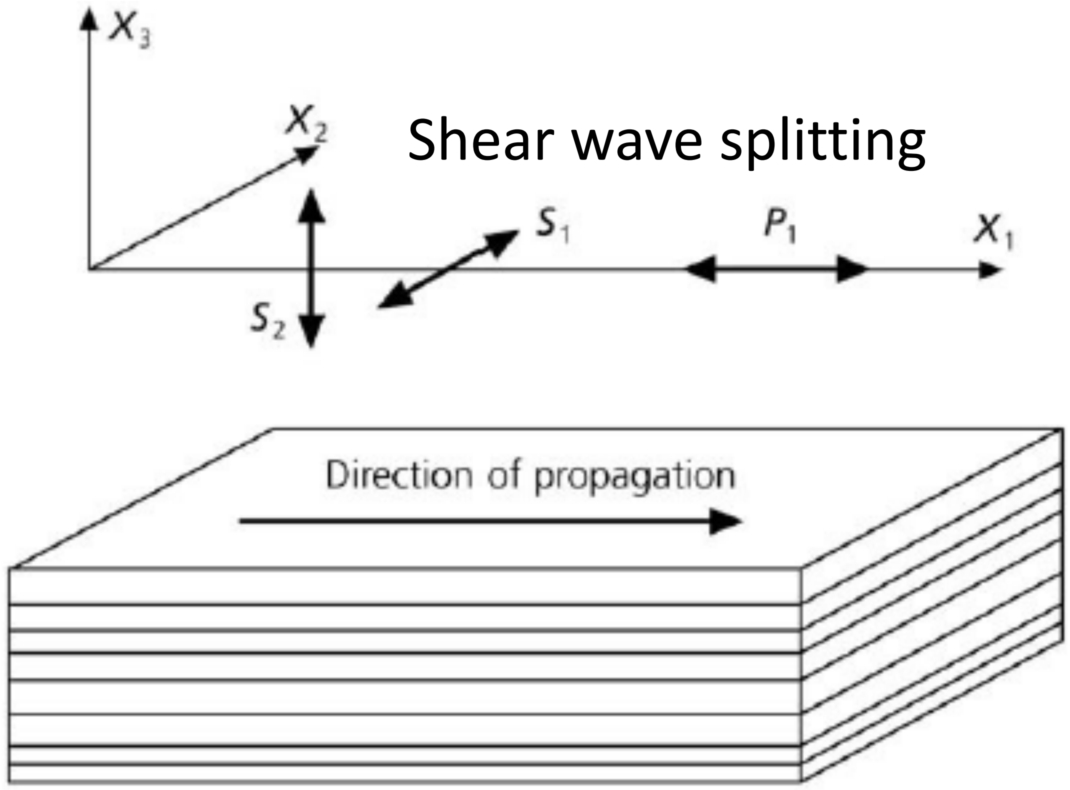
A vertical symmetry axis: Transversely anisotropy

VTI

$$
\begin{pmatrix}
C_{11} & C_{12} & C_{13} & & & \\
C_{12} & C_{11} & C_{13} & & & \\
C_{13} & C_{13} & C_{33} & & & \\
 & & & C_{44} & & \\
 & & & & C_{44} & \\
 & & & & & C_{66}
\end{pmatrix}
$$

$$C_{12} = C_{11} - 2C_{66}$$

Shear wave splitting



Hexagonal symmetry (5 independent elastic coefficients)

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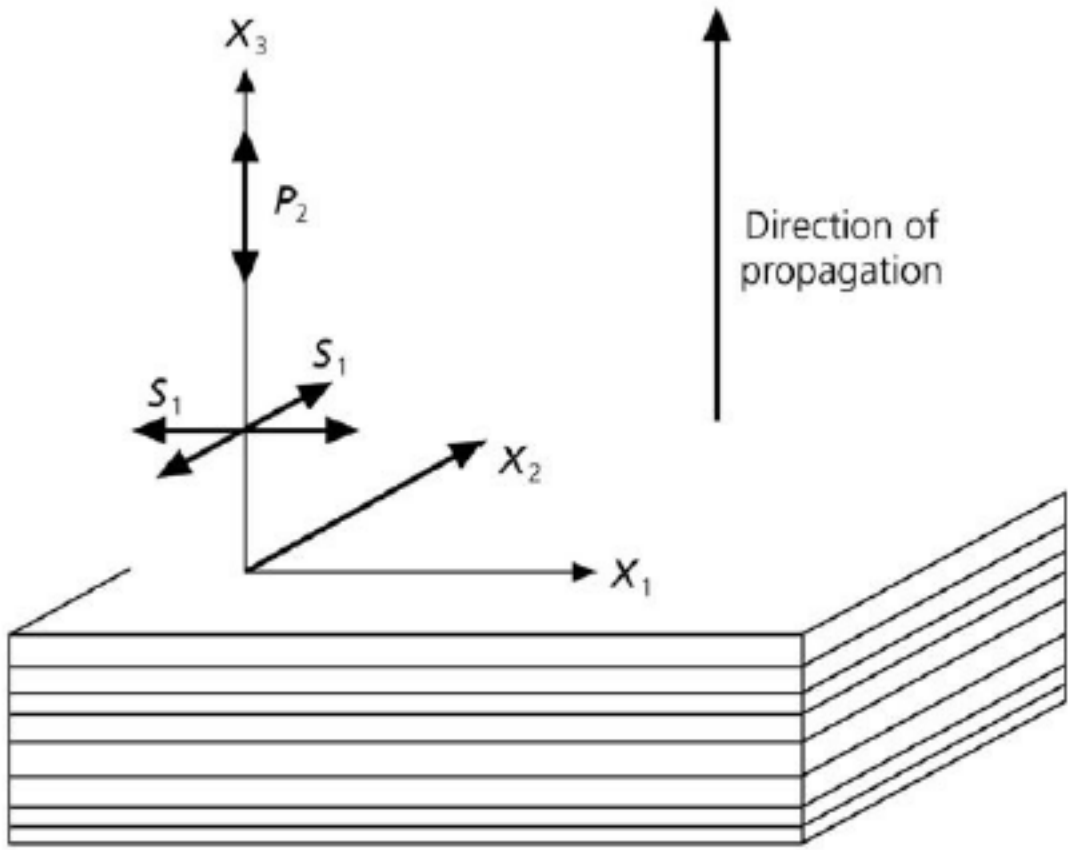
$$\begin{pmatrix} C_{11} & C_{12} & C_{13} & & & \\ C_{12} & C_{11} & C_{13} & & & \\ C_{13} & C_{13} & C_{33} & & & \\ & & & C_{44} & & \\ & & & & C_{44} & \\ & & & & & C_{66} \end{pmatrix}$$

$$C_{12} = C_{11} - 2C_{66}$$

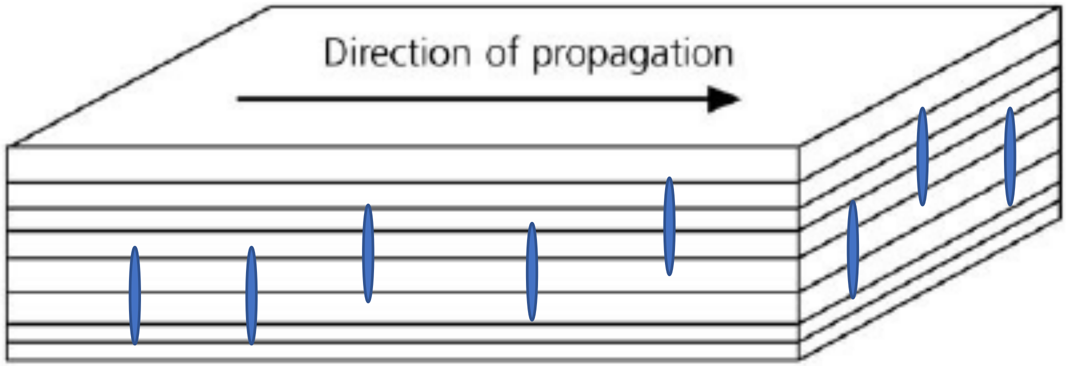


Hexagonal symmetry (5 independent elastic coefficients)

Orthohomic symmetry (9 parameters)

$$\begin{pmatrix} C_{11} & C_{12} & C_{13} & & & \\ C_{12} & C_{22} & C_{23} & & & \\ C_{13} & C_{23} & C_{33} & & & \\ & & & C_{44} & & \\ & & & & C_{55} & \\ & & & & & C_{66} \end{pmatrix}$$

With cracks



Isotropic case

$$\begin{pmatrix} C_{11} & C_{12} & C_{12} & & & \\ C_{12} & C_{11} & C_{12} & & & \\ C_{12} & C_{12} & C_{11} & & & \\ & & & C_{55} & & \\ & & & & C_{55} & \\ & & & & & C_{55} \end{pmatrix}$$

$$C_{55} = \mu$$
$$C_{11} = \lambda + 2\mu$$
$$C_{12} = C_{11} - 2C_{55} = \lambda$$

λ, μ: Lame's constant
μ: Shear modulus

$K = \lambda + \frac{2}{3}\mu$ Bulk Modulus

$\gamma = \frac{\lambda}{2(\lambda+\mu)}$ Possion ratio

Isotropic case

$$\begin{pmatrix} C_{11} & C_{12} & C_{12} & & & \\ C_{12} & C_{11} & C_{12} & & & \\ C_{12} & C_{12} & C_{11} & & & \\ & & & C_{55} & & \\ & & & & C_{55} & \\ & & & & & C_{55} \end{pmatrix}$$

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$\gamma = \frac{\lambda}{2(\lambda+\mu)}$ Possion ratio

liquid	$\mu = 0$	$\gamma = 0.5$
Typical solid		$\gamma < 0.5$
Possion’s body	$\lambda = \mu$	$\gamma = 0.25$

Isotropic case

$$\begin{pmatrix} C_{11} & C_{12} & C_{12} & & & \\ C_{12} & C_{11} & C_{12} & & & \\ C_{12} & C_{12} & C_{11} & & & \\ & & & C_{55} & & \\ & & & & C_{55} & \\ & & & & & C_{55} \end{pmatrix}$$

$$C_{55} = \mu$$
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$$C_{12} = C_{11} - 2C_{55} = \lambda$$

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liquid	$\mu = 0$	$\gamma = 0.5$
Typical solid		$\gamma < 0.5$
Possion's body	$\lambda = \mu$	$\gamma = 0.25$

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

Constitutive Eq. $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$

$$
\begin{aligned}
\theta &= \varepsilon_{kk} \\
&= \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \\
&= \underline{\nabla} \cdot \underline{u}
\end{aligned}
$$

$$
\begin{aligned}
&= [\lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})]\varepsilon_{kl} \\
&= \lambda\delta_{ij}\,\varepsilon_{kk} + \mu(\varepsilon_{ij} + \varepsilon_{ij}) \\
&= \lambda\delta_{ij}\,\theta + 2\mu\varepsilon_{ij}
\end{aligned}
$$

Constitutive Eq.: $\sigma_{ij} = \lambda\delta_{ij}\,\theta + 2\mu\varepsilon_{ij}$

Equation of Motion: $\varrho\ddot{u}_i = \sigma_{ij,j}$

Constitutive Eq.: $\sigma_{ij} = \lambda\delta_{ij}\,\theta + 2\mu\varepsilon_{ij}$

Equation of Motion: $\varrho\ddot{u}_i = \sigma_{ij,j}$

Use u_1 as an example

Constitutive Eq.: $\sigma_{ij} = \lambda\delta_{ij}\,\theta + 2\mu\varepsilon_{ij}$

Equation of Motion: $\varrho\ddot{u}_i = \sigma_{ij,j}$

Use u_1 as an example

$$\varrho\frac{\partial^2 u_1}{\partial t^2} = \frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{12}}{\partial x_2} + \frac{\partial\sigma_{13}}{\partial x_3} = \frac{\partial(\lambda\,\theta + 2\mu\varepsilon_{11})}{\partial x_1} + \frac{\partial(2\mu\varepsilon_{12})}{\partial x_2} + \frac{\partial(2\mu\varepsilon_{13})}{\partial x_3}$$

$$= \lambda\frac{\partial\theta}{\partial x_1} + 2\mu\left(\frac{\partial\varepsilon_{11}}{\partial x_1} + \frac{\partial\varepsilon_{12}}{\partial x_1} + \frac{\partial\varepsilon_{13}}{\partial x_1}\right)$$

$$= \lambda\frac{\partial\theta}{\partial x_1} + \mu\left[2\frac{\partial\left(\frac{\partial u_1}{\partial x_1}\right)}{\partial x_1} + \frac{\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right)}{\partial x_2} + \frac{\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right)}{\partial x_3}\right]$$

$$= \lambda\frac{\partial\theta}{\partial x_1} + \mu\frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right) + \mu\left(\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2}\right)$$

$$= (\lambda + \mu)\frac{\partial\theta}{\partial x_1} + \mu\underline{\nabla}^2\,u_1$$

$$\varrho \frac{\partial^2 u_1}{\partial t^2} = (\lambda + \mu) \frac{\partial \theta}{\partial x_1} + \mu \underline{\nabla}^2 u_1$$

$$\varrho \frac{\partial^2 u_2}{\partial t^2} = (\lambda + \mu) \frac{\partial \theta}{\partial x_2} + \mu \underline{\nabla}^2 u_2$$

$$\varrho \frac{\partial^2 u_3}{\partial t^2} = (\lambda + \mu) \frac{\partial \theta}{\partial x_3} + \mu \underline{\nabla}^2 u_3$$

$$\theta = \underline{\nabla} \cdot \underline{u}$$

$$\varrho \underline{\ddot{u}} = (\lambda + \mu)\, \underline{\nabla}\left(\underline{\nabla} \cdot \underline{u}\right) + \mu \underline{\nabla}^2\, \underline{u}$$

Equation of Motion for Isotropic case

Equation of Motion for Isotropic case

$$\varrho \underline{\ddot{u}} = (\lambda + \mu)\, \underline{\nabla}\left(\underline{\nabla} \cdot \underline{u}\right) + \mu \underline{\nabla}^2\, \underline{u}$$

$$\underline{\nabla}^2\, \underline{u} = \underline{\nabla}\left(\underline{\nabla} \cdot \underline{u}\right) - \underline{\nabla} \times \underline{\nabla} \times \underline{u}$$

$$= (\lambda + \mu)\, \underline{\nabla}(\underline{\nabla} \cdot \underline{u}) + \mu\left(\underline{\nabla}\left(\underline{\nabla} \cdot \underline{u}\right) - \underline{\nabla} \times \underline{\nabla} \times \underline{u}\right)$$

$$= (\lambda + 2\mu)\, \underline{\nabla}\left(\underline{\nabla} \cdot \underline{u}\right) - \mu \underline{\nabla} \times \underline{\nabla} \times \underline{u}$$

Numerical solution: FD, SEM, …

Helmholtz's Theorem

Any vector

$$\underline{u}(\underline{x},t) = \underline{\nabla}\phi(\underline{x},t) + \underline{\nabla}\times\underline{\psi}(\underline{x},t)$$

ϕ: scalar potentials; $\underline{\nabla}\phi$: curl free, no rotation

$$\underline{\nabla}\times(\underline{\nabla}\phi) = 0$$

$\underline{\psi}$: zero divergence $\underline{\nabla}\cdot\underline{\psi} = 0$

No volume change

divergenceless vector $\underline{\nabla}\cdot(\underline{\nabla}\times\underline{\psi}) = 0$

Equation of Motion for Isotropic case

$$\underline{\nabla}\times(\underline{\nabla}\phi) = 0$$

$$\underline{\nabla}\cdot\underline{\psi} = 0$$

$$\underline{\nabla}\cdot(\underline{\nabla}\times\underline{\psi}) = 0$$

$$\varrho\underline{\ddot{u}} = (\lambda + 2\mu)\,\underline{\nabla}(\underline{\nabla}\cdot\underline{u}) - \mu\underline{\nabla}\times\underline{\nabla}\times\underline{u}$$

$$\underline{u}(\underline{x}, t) = \underline{\nabla}\phi(\underline{x}, t) + \underline{\nabla}\times\underline{\psi}(\underline{x}, t)$$

Equation of Motion for Isotropic case

$$\underline{\nabla} \times (\underline{\nabla}\phi) = 0$$

$$\underline{\nabla} \cdot \underline{\psi} = 0$$

$$\underline{\nabla} \cdot (\underline{\nabla} \times \underline{\psi}) = 0$$

$$\varrho \underline{\ddot{u}} = (\lambda + 2\mu)\, \underline{\nabla}(\underline{\nabla} \cdot \underline{u}) - \mu \underline{\nabla} \times \underline{\nabla} \times \underline{u}$$

Left: $\varrho \underline{\ddot{u}} = \varrho(\, \underline{\nabla}\ddot{\phi} + \underline{\nabla} \times \underline{\ddot{\psi}})$

Right: $(\lambda + 2\mu)\, \underline{\nabla}(\underline{\nabla} \cdot \underline{u}) - \mu \underline{\nabla} \times \underline{\nabla} \times \underline{u}$

$$= (\lambda + 2\mu)\, \underline{\nabla}\left[\underline{\nabla} \cdot (\underline{\nabla}\phi + \underline{\nabla} \times \underline{\psi})\right] - \mu \underline{\nabla} \times \underline{\nabla} \times (\underline{\nabla}\phi + \underline{\nabla} \times \underline{\psi})$$

$$= (\lambda + 2\mu)\, \underline{\nabla}(\underline{\nabla}^2\, \phi + \underline{\nabla} \cdot \underline{\nabla} \times \underline{\psi})$$

$$-\mu \underline{\nabla} \times \underline{\nabla} \times \underline{\nabla}\phi - \mu \underline{\nabla} \times \underline{\nabla} \times \underline{\nabla} \times \underline{\psi}$$

Equation of Motion for Isotropic case

$$\varrho(\underline{\nabla}\ddot{\phi} + \underline{\nabla}\times\underline{\ddot{\psi}}) = (\lambda + 2\mu)\,\underline{\nabla}(\underline{\nabla}^2\,\phi) + \mu\underline{\nabla}\times\underline{\nabla}^2\underline{\psi}$$

$$\underline{\nabla}\left[\varrho\ddot{\phi} - (\lambda + 2\mu)\underline{\nabla}^2\,\phi\right] + \underline{\nabla}\times\left(\varrho\underline{\ddot{\psi}} - \mu\underline{\nabla}^2\underline{\psi}\right) = 0$$

We have

$$\begin{cases} \dfrac{\lambda + 2\mu}{\varrho}\underline{\nabla}^2\,\phi - \ddot{\phi} = 0 \\ \\ \dfrac{\mu}{\varrho}\underline{\nabla}^2\underline{\psi} - \underline{\ddot{\psi}} = 0 \end{cases}$$

Equation of Motion for Isotropic case

$$\begin{cases} \dfrac{\lambda + 2\mu}{\varrho} \underline{\nabla}^2 \phi - \ddot{\phi} = 0 \\ \\ \dfrac{\mu}{\varrho} \underline{\nabla}^2 \underline{\psi} - \underline{\ddot{\psi}} = 0 \end{cases}$$

$$\alpha = \sqrt{\frac{\lambda + 2\mu}{\varrho}} \qquad P\ velocity$$

$$\beta = \sqrt{\frac{\mu}{\varrho}} \qquad S\ velocity$$

$\alpha > \beta$

For possion's solid

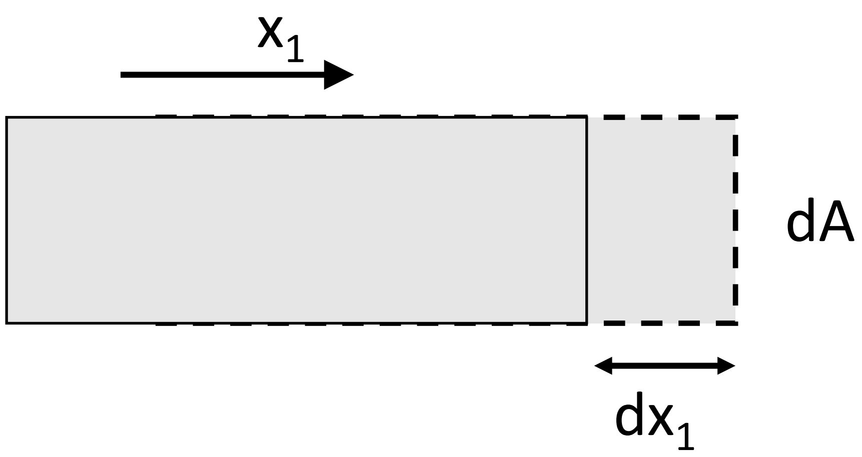
$\alpha =? \beta$

Equation of Motion for Isotropic case

$$\begin{cases} \underline{\nabla}^2 \phi - \dfrac{1}{\alpha^2}\ddot{\phi} = 0 \\ \underline{\nabla}^2 \underline{\psi} - \dfrac{1}{\beta^2}\underline{\ddot{\psi}} = 0 \end{cases}$$

Scalar Potential	$\underline{\nabla}^2 \phi - \frac{1}{\alpha^2}\ddot{\phi} = 0$	$P\ wave$
Vector Potential	$\underline{\nabla}^2 \underline{\psi} - \frac{1}{\beta^2}\underline{\ddot{\psi}} = 0$	$S\ wave$

Check the 1D case



E: Young's Modulus

$$\rho \ddot{u}_1 = \frac{\partial \sigma_{11}}{\partial x_1}$$

$$\sigma_{11} = E\varepsilon_{11} = E\frac{\partial u_1}{\partial x_1}$$

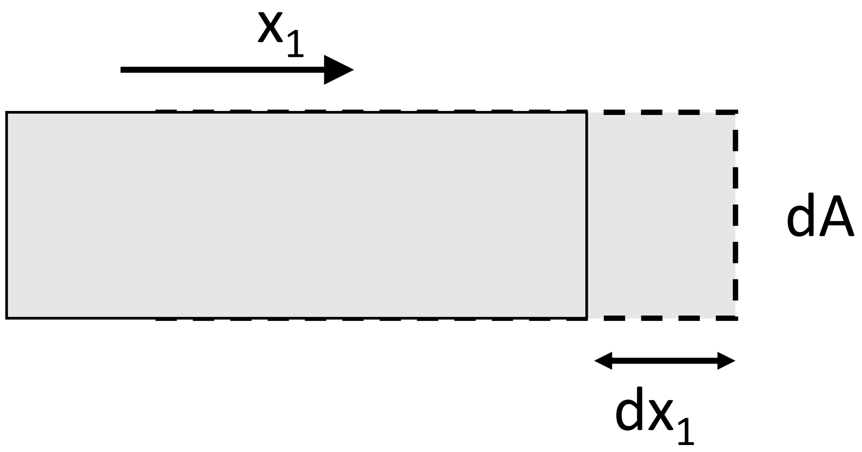
$$\rho \ddot{u}_1 = E\frac{\partial^2 u_1}{\partial x_1{}^2}$$

$$c = \sqrt{\frac{E}{\rho}}$$

Sound velocity

$$\frac{\partial^2 u_1}{\partial x_1{}^2} - \frac{1}{c^2}\frac{\partial^2 u_1}{\partial t^2} = 0$$

Check the 1D case



E: Young's Modulus

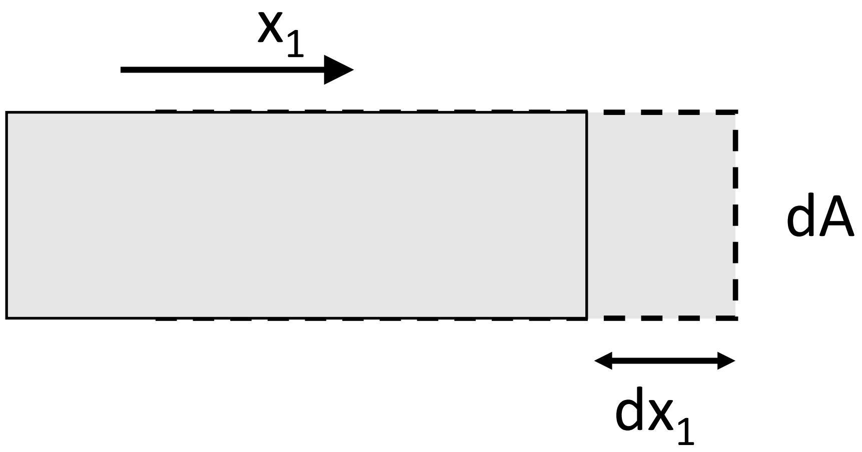
$$\frac{\partial^2 u_1}{\partial x_1{}^2} - \frac{1}{c^2}\frac{\partial^2 u_1}{\partial t^2} = 0$$

In general form

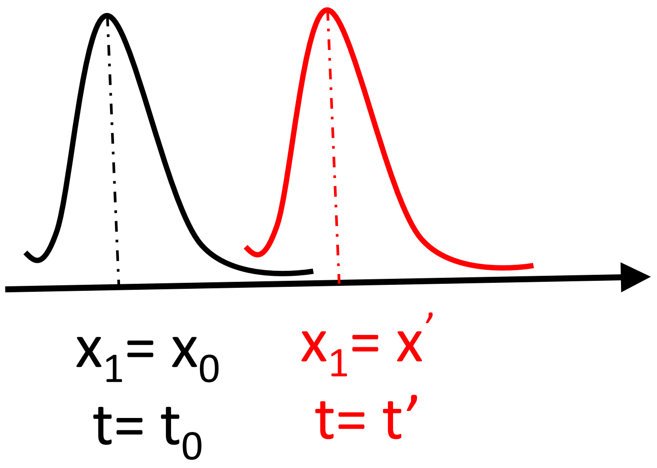
$$u_1\left(x_1, t\right) = f_1\left(x_1 - ct\right) + f_2\left(x_1 + ct\right)$$

f_1, f_2: Any solutions satisfy the initial condition

Check the 1D case



E: Young's Modulus



$$\frac{\partial^2 u_1}{\partial x_1^{\,2}} - \frac{1}{c^2}\frac{\partial^2 u_1}{\partial t^2} = 0$$

In general form

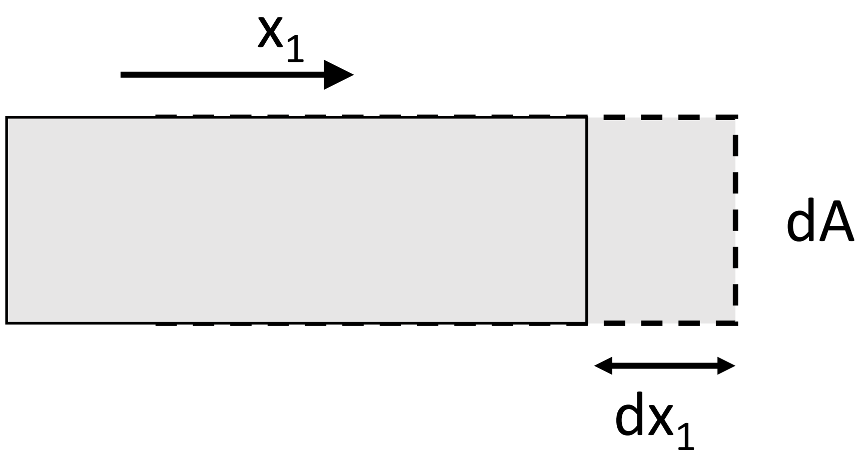
$$u_1(x_1, t) = f_1(x_1 - ct) + f_2(x_1 + ct)$$

$t = t_0$: $\quad f_1(x_1 - ct_0)$

$t = t' > t_0$: $\quad f_1(x_1 - ct')$

Wave move to the right

Check the 1D case



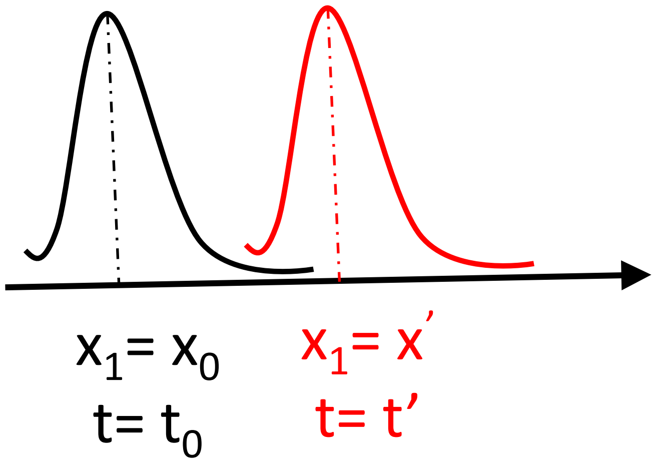
E: Young's Modulus

$$\boxed{\frac{\partial^2 u_1}{\partial x_1^{\,2}} - \frac{1}{c^2}\frac{\partial^2 u_1}{\partial t^2} = 0}$$

In general form

$$u_1(x_1,t) = f_1(x_1 - ct) + f_2(x_1 + ct)$$

Right $\rightarrow$ $\leftarrow$ Left



Phase: $(x_1 \pm ct)$

Wavefront: describing a surface of constant phase

$$\boxed{\frac{\partial^2 u_1}{\partial {x_1}^2} - \frac{1}{c^2}\frac{\partial^2 u_1}{\partial t^2} = 0}$$

Separation of Variables

$$u_1(x_1, t) = X(x_1)T(t)$$

$$\frac{d^2 X}{dx_1^2}T(t) - \frac{1}{c^2}X(x_1)\frac{d^2 T}{dt^2} = 0$$

$$c^2\frac{1}{X}\frac{d^2 X}{dx_1^2} - \frac{1}{T}\frac{d^2 T}{dt^2} = 0$$

Assuming

$$c^2\frac{1}{X}\frac{d^2 X}{dx_1^2} = -\omega^2 = \frac{1}{T}\frac{d^2 T}{dt^2}$$

Assuming $$c^2 \frac{1}{X}\frac{d^2X}{dx_1^2} = -\omega^2 = \frac{1}{T}\frac{d^2T}{dt^2}$$

$$\frac{d^2X}{dx_1^2} + \frac{\omega^2}{c^2}\cdot X = 0 \longrightarrow X(x_1) = A_1 e^{i\left(\frac{\omega}{c}\right)x_1} + A_2 e^{-i\left(\frac{\omega}{c}\right)x_1}$$

$$\frac{d^2T}{dt^2} + \omega^2 T = 0 \longrightarrow T(t) = B_1 e^{i\omega t} + B_2 e^{-i\omega t}$$

$$u_1(x_1,t) = X(x_1)T(t)$$

$$= C_1 e^{i\omega\left(t+\frac{x_1}{c}\right)} + C_2 e^{i\omega\left(t-\frac{x_1}{c}\right)} + C_3 e^{-i\omega\left(t+\frac{x_1}{c}\right)} + C_4 e^{-i\omega\left(t-\frac{x_1}{c}\right)}$$

C_1-C_4 are determined by initial and boundary conditions.

$$\boxed{\frac{\partial^2 u_1}{\partial x_1^{\,2}} - \frac{1}{c^2}\frac{\partial^2 u_1}{\partial t^2} = 0}$$

Check $u_1(x_1, t) = Ae^{i\omega\left(t \pm \frac{x_1}{c}\right)}$

ω: angular frequency

$T = \frac{2\pi}{\omega}$: period

$\lambda = c \cdot T = c \cdot \frac{2\pi}{\omega}$: wavelength

$k = \frac{\omega}{c} = \frac{2\pi}{\lambda}$: wavenumber

T

Exploration
100 Hz

Body wave
1 sec

Surface wave
100 sec

Earth free oscillation
1000 sec

Assume c = 5 km/s

50 m

5000 km

λ

3D case

$$\underline{u}(\underline{x},t) = \underline{\nabla}\phi(\underline{x},t) + \underline{\nabla}\times\underline{\psi}(\underline{x},t)$$

Equation of Motion for Isotropic case

$$\begin{cases} \underline{\nabla}^2\,\phi - \dfrac{1}{\alpha^2}\ddot{\phi} = 0 & \quad P\ wave \\ \underline{\nabla}^2\underline{\psi} - \dfrac{1}{\beta^2}\underline{\ddot{\psi}} = 0 & \quad S\ wave \end{cases}$$

$$\underline{\nabla}^2 \phi - \frac{1}{\alpha^2}\ddot{\phi} = 0$$

$$\frac{\partial^2 \phi}{\partial t^2} = \alpha^2 \left(\frac{\partial^2 \phi}{\partial {x_1}^2} + \frac{\partial^2 \phi}{\partial {x_2}^2} + \frac{\partial^2 \phi}{\partial {x_3}^2} \right)$$

Separation of Variables $\phi(x_1, x_2, x_3, t) = X(x_1)Y(x_2)Z(x_3)T(t)$

$$XYZ\frac{d^2 T}{dt^2} = \alpha^2 T \left(YZ\frac{\partial^2 X}{\partial {x_1}^2} + XZ\frac{\partial^2 Y}{\partial {x_2}^2} + XY\frac{\partial^2 Z}{\partial {x_3}^2} \right)$$

$$\frac{1}{T}\frac{d^2 T}{dt^2} = \alpha^2 \left(\frac{1}{X}\frac{\partial^2 X}{\partial {x_1}^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial {x_2}^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial {x_3}^2} \right)$$

$$\frac{1}{T}\frac{d^2T}{dt^2}=\alpha^2\left(\underbrace{\frac{1}{X}\frac{\partial^2 X}{\partial {x_1}^2}}_{-k_1^2}+\underbrace{\frac{1}{Y}\frac{\partial^2 Y}{\partial {x_2}^2}}_{-k_2^2}+\underbrace{\frac{1}{Z}\frac{\partial^2 Z}{\partial {x_3}^2}}_{-k_3^2}\right)$$

Define $\omega^2=\alpha^2\left(k_1^2+k_2^2+k_3^2\right)$

$$\begin{cases}\dfrac{d^2T}{dt^2}+\omega^2T=0\\[2ex]\dfrac{d^2X}{d{x_1}^2}+k_1^2X=0\\[2ex]\dfrac{d^2Y}{d{x_2}^2}+k_2^2Y=0\\[2ex]\dfrac{d^2Z}{d{x_3}^2}+k_3^2Z=0\end{cases}$$

$$\phi\left(\underline{x},t\right)=Ae^{\pm i\left(\omega t\pm k_1x_1\pm k_2x_2\pm k_3x_3\right)}$$

$$\phi(\underline{x},t) = Ae^{\pm i(\omega t \pm k_1 x_1 \pm k_2 x_2 \pm k_3 x_3)}$$

Sum of a set of waves propagating in any directions.

$$\omega^2 = \alpha^2 \left(k_1^2 + k_2^2 + k_3^2\right)$$

Wavenumber $\underline{k_\alpha} = |k_\alpha|\hat{k} = \left(\frac{\omega}{\alpha}\right)\hat{k} = k_1\widehat{x_1} + k_2\widehat{x_2} + k_3\widehat{x_3}$

Let's only consider x_1-x_3 plane

At wavefront $\omega t - k_1 x_1 - k_3 x_3 = 0$

At t = 0;

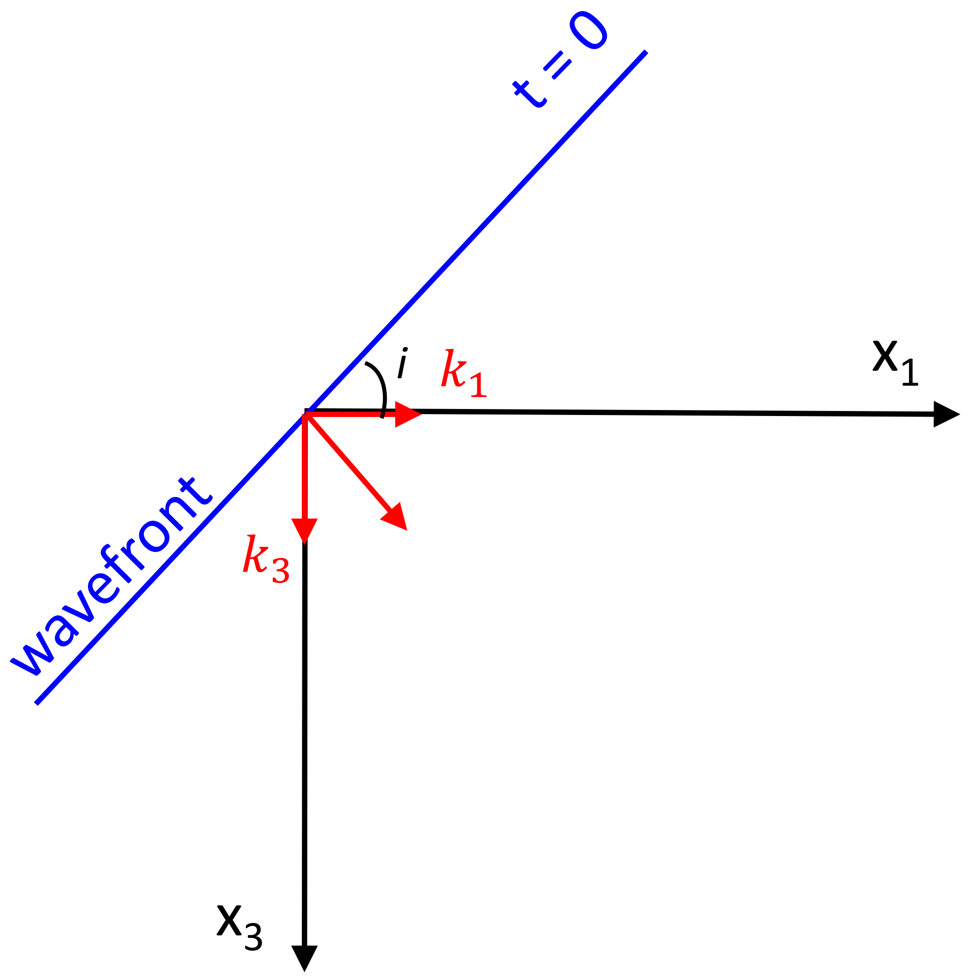
$$k_1 x_1 + k_3 x_3 = 0$$

$$x_3 = -\frac{k_1}{k_3} x_1$$

$$k_1^2 + k_3^2 = \frac{\omega^2}{\alpha^2} \qquad \tan i = \frac{k_1}{k_3}$$

$$k_1 = \frac{\omega}{\alpha} \sin i = \omega \frac{\sin i}{\alpha} = \omega p$$

$p = \frac{\sin i}{\alpha}$ ray parameter / horizontal slowness



Let's only consider x_1-x_3 plane

At wavefront $\omega t - k_1 x_1 - k_3 x_3 = 0$

At t = 0;

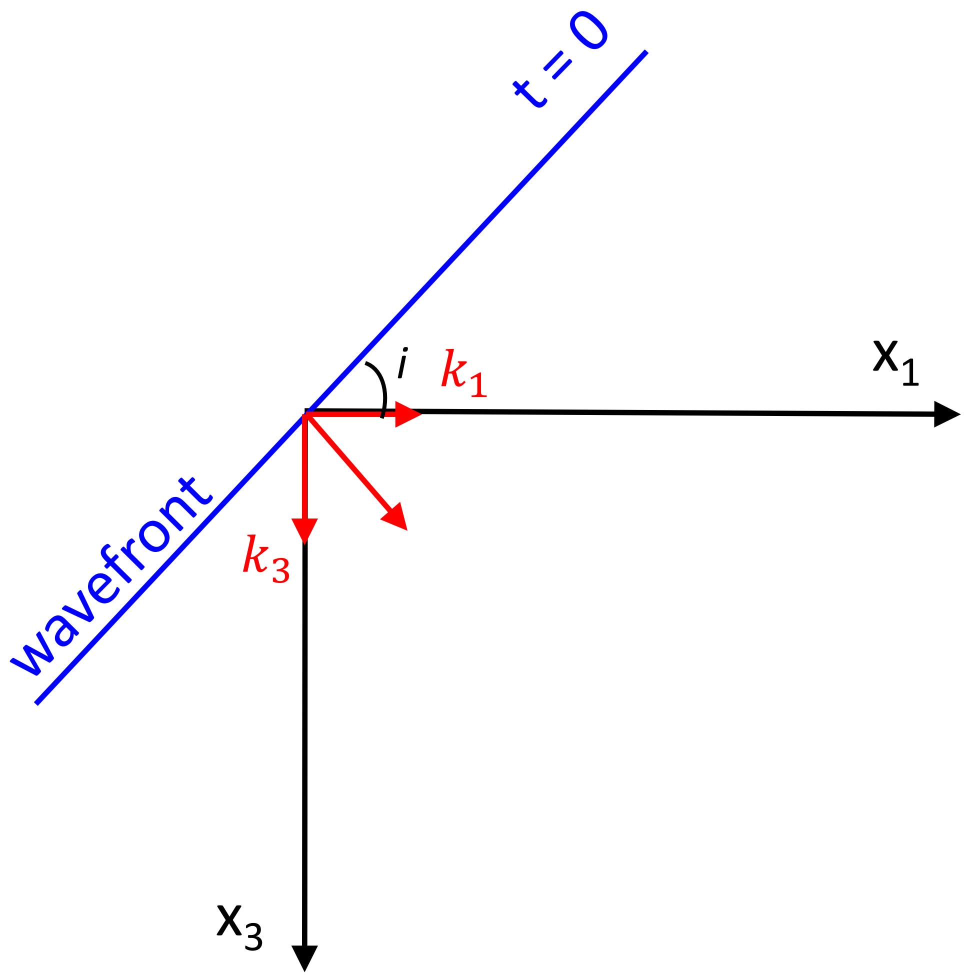
$$k_1 x_1 + k_3 x_3 = 0$$

$$x_3 = -\frac{k_1}{k_3} x_1$$

$$k_1^2 + k_3^2 = \frac{\omega^2}{\alpha^2} \qquad \tan i = \frac{k_1}{k_3}$$

$$k_3 = \frac{\omega}{\alpha}\cos i = \omega \frac{\cos i}{\alpha} = \omega \eta_\alpha$$

$$\eta_\alpha = \frac{\cos i}{\alpha} = \frac{\sqrt{1-\alpha^2 p^2}}{\alpha} = \sqrt{\frac{1}{\alpha^2} - p^2}$$ vertical slowness



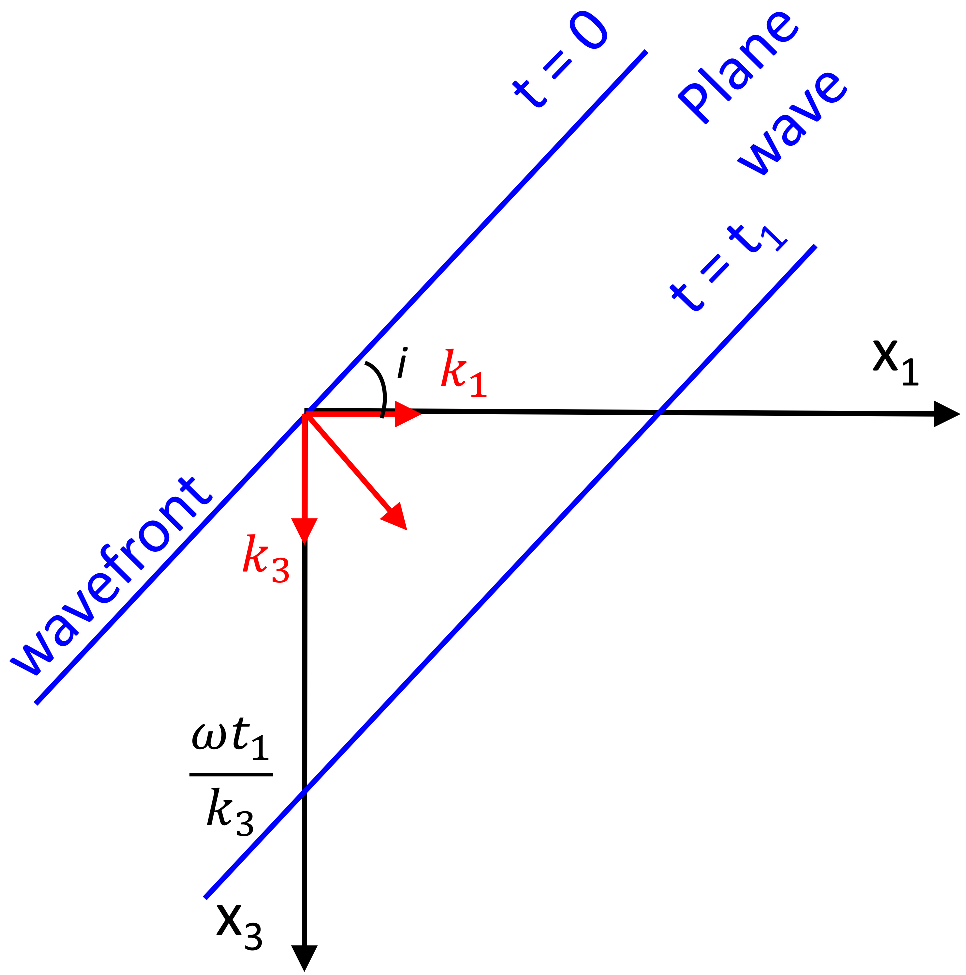
Let's only consider x_1-x_3 plane

At wavefront $\omega t - k_1 x_1 - k_3 x_3 = 0$

At time $t_1 > 0$;

$$\omega t_1 - k_1 x_1 - k_3 x_3 = 0$$

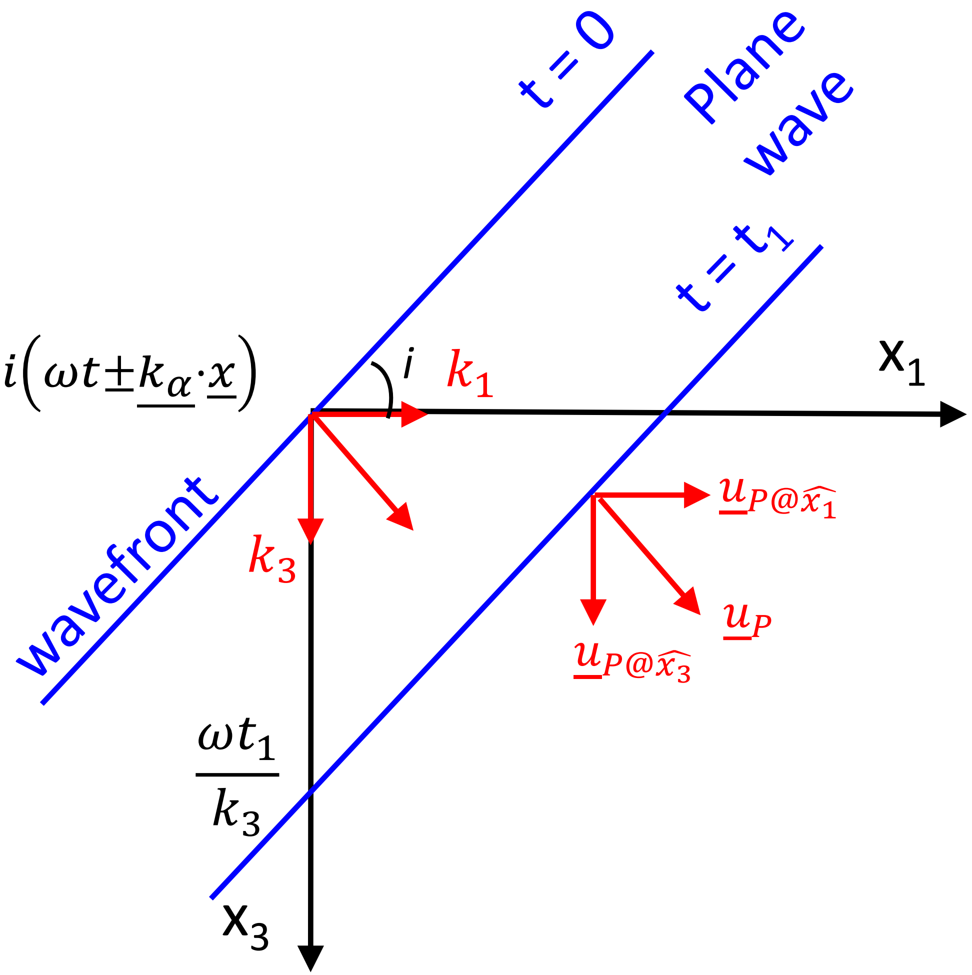
$$x_3 = -\frac{k_1}{k_3} x_1 + \frac{\omega t_1}{k_3}$$



$$\phi = Ae^{\pm i(\omega t \pm \underline{k}_\alpha \cdot \underline{x})}$$

Displacement

$$\underline{u_P} = \underline{\nabla}\phi = \left(\frac{\partial}{\partial x_1}\widehat{x_1} + \frac{\partial}{\partial x_2}\widehat{x_2} + \frac{\partial}{\partial x_3}\widehat{x_3}\right) Ae^{\pm i(\omega t \pm \underline{k_\alpha} \cdot \underline{x})}$$



Here,

$$\phi = e^{i(\omega t - k_1 x_1 - k_3 x_3)}$$

$$\underline{u_P} = -ik_1 Ae^{i(\omega t - k_1 x_1 - k_3 x_3)}\widehat{x_1}$$

$$-ik_3 Ae^{i(\omega t - k_1 x_1 - k_3 x_3)}\widehat{x_3}$$

$$\frac{\underline{u}_{P@\widehat{x_3}}}{\underline{u}_{P@\widehat{x_1}}} = \frac{k_3}{k_1} = \frac{\eta_\alpha}{p}$$

For S wave $$\underline{\nabla}^2\underline{\psi} - \frac{1}{\beta^2}\underline{\ddot{\psi}} = 0$$

$$\frac{\partial^2\underline{\psi}}{\partial t^2} = \beta^2\left(\frac{\partial^2\underline{\psi}}{\partial x_1^2} + \frac{\partial^2\underline{\psi}}{\partial x_2^2} + \frac{\partial^2\underline{\psi}}{\partial x_3^2}\right)$$

$$\underline{\psi}(\underline{x}, t) = \underline{B}e^{i(\omega t - \underline{\mathrm{k}}_\beta \cdot \underline{x})} \qquad \underline{\mathrm{k}}_\beta = \left|\frac{\omega}{\beta}\right|\hat{\mathrm{k}}_\beta$$

$$\underline{u}_S = \underline{\nabla}\times\underline{\psi}$$

$$= \left(\frac{\partial\psi_3}{\partial x_2} - \frac{\partial\psi_2}{\partial x_3}\right)\widehat{x_1} + \left(\frac{\partial\psi_1}{\partial x_3} - \frac{\partial\psi_3}{\partial x_1}\right)\widehat{x_2} + \left(\frac{\partial\psi_2}{\partial x_1} - \frac{\partial\psi_1}{\partial x_2}\right)\widehat{x_3}.$$

For S wave $$\underline{\nabla}^2\underline{\psi} - \frac{1}{\beta^2}\underline{\ddot{\psi}} = 0$$

$$\frac{\partial^2\underline{\underline{\psi}}}{\partial t^2} = \beta^2\left(\frac{\partial^2\underline{\underline{\psi}}}{\partial x_1^2} + \frac{\partial^2\underline{\underline{\psi}}}{\partial x_2^2} + \frac{\partial^2\underline{\underline{\psi}}}{\partial x_3^2}\right)$$

$$\underline{\psi}(\underline{x}, t) = \underline{B}e^{i(\omega t - \underline{\mathrm{k}}_\beta \cdot \underline{x})} \qquad \underline{\mathrm{k}}_\beta = \left|\frac{\omega}{\beta}\right|\hat{\mathrm{k}}_\beta$$

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For S wave $\underline{\nabla}^2 \underline{\psi} - \frac{1}{\beta^2} \underline{\ddot{\psi}} = 0$

$$\frac{\partial^2 \underline{\underline{\psi}}}{\partial t^2} = \beta^2 \left(\frac{\partial^2 \underline{\underline{\psi}}}{\partial x_1^2} + \frac{\partial^2 \underline{\underline{\psi}}}{\partial x_2^2} + \frac{\partial^2 \underline{\underline{\psi}}}{\partial x_3^2} \right)$$

$$\underline{\psi}(\underline{x}, t) = \underline{B} e^{i(\omega t - \underline{\mathrm{k}}_\beta \cdot \underline{x})} \qquad \underline{\mathrm{k}}_\beta = \left| \frac{\omega}{\beta} \right| \hat{\mathrm{k}}_\beta$$

$$\underline{u}_S = \underline{\nabla} \times \underline{\psi}$$

$$= \left(\frac{\partial \psi_3}{\partial x_2} - \frac{\partial \psi_2}{\partial x_3} \right) \widehat{x_1} + \left(\frac{\partial \psi_1}{\partial x_3} - \frac{\partial \psi_3}{\partial x_1} \right) \widehat{x_2} + \left(\frac{\partial \psi_2}{\partial x_1} - \frac{\partial \psi_1}{\partial x_2} \right) \widehat{x_3}.$$

only in x_1-x_3 plane

$\underline{\psi}(x_1, x_3)$

$$= -\frac{\partial \psi_2}{\partial x_3} \widehat{x_1} + \frac{\partial \psi_2}{\partial x_1} \widehat{x_3} + \left(\frac{\partial \psi_1}{\partial x_3} - \frac{\partial \psi_3}{\partial x_1} \right) \widehat{x_2}$$

For S wave

$$\underline{\nabla}^2 \underline{\psi} - \frac{1}{\beta^2} \underline{\ddot{\psi}} = 0$$

$$\frac{\partial^2 \underline{\underline{\psi}}}{\partial t^2} = \beta^2 \left(\frac{\partial^2 \underline{\underline{\psi}}}{\partial x_1^2} + \frac{\partial^2 \underline{\underline{\psi}}}{\partial x_2^2} + \frac{\partial^2 \underline{\underline{\psi}}}{\partial x_3^2} \right)$$

$$\underline{\psi}(\underline{x}, t) = \underline{B} e^{i(\omega t - \underline{\mathrm{k}}_\beta \cdot \underline{x})} \qquad \underline{\mathrm{k}}_\beta = \left| \frac{\omega}{\beta} \right| \hat{\mathrm{k}}_\beta$$

$$\underline{u}_S = \underline{\nabla} \times \underline{\psi}$$

$$= \left(\frac{\partial \psi_3}{\partial x_2} - \frac{\partial \psi_2}{\partial x_3} \right) \widehat{x_1} + \left(\frac{\partial \psi_1}{\partial x_3} - \frac{\partial \psi_3}{\partial x_1} \right) \widehat{x_2} + \left(\frac{\partial \psi_2}{\partial x_1} - \frac{\partial \psi_1}{\partial x_2} \right) \widehat{x_3}.$$

only in x_1-x_3 plane

$\underline{\psi}(x_1, x_3)$

$$= \underbrace{-\frac{\partial \psi_2}{\partial x_3} \widehat{x_1} + \frac{\partial \psi_2}{\partial x_1} \widehat{x_3}}_{\text{SV}} + \underbrace{\left(\frac{\partial \psi_1}{\partial x_3} - \frac{\partial \psi_3}{\partial x_1} \right) \widehat{x_2}}_{\text{SH}}$$

$$\underline{u}_P = \frac{\partial\phi}{\partial x_1}\widehat{x_1} + \frac{\partial\phi}{\partial x_3}\widehat{x_3}$$

$$\underline{u}_S = -\frac{\partial\psi_2}{\partial x_3}\widehat{x_1} + \frac{\partial\psi_2}{\partial x_1}\widehat{x_3} + \left(\frac{\partial\psi_1}{\partial x_3} - \frac{\partial\psi_3}{\partial x_1}\right)\widehat{x_2}$$

$$\underline{u} = \underline{u}_P + \underline{u}_S$$

$$= \left(\frac{\partial\phi}{\partial x_1} - \frac{\partial\psi_2}{\partial x_3}\right)\widehat{x_1} + \left(\frac{\partial\phi}{\partial x_3} + \frac{\partial\psi_2}{\partial x_1}\right)\widehat{x_3}$$

P-SV system

$$+ \left(\frac{\partial\psi_1}{\partial x_3} - \frac{\partial\psi_3}{\partial x_1}\right)\widehat{x_2}$$

SH system

For SH wave: $\underline{u}_{SH} = \frac{\partial \psi_1}{\partial x_3} - \frac{\partial \psi_3}{\partial x_1}$

$$\begin{cases} \frac{\partial^2 \psi_1}{\partial t^2} = \beta^2 \underline{\nabla}^2 \psi_1 \\ \frac{\partial^2 \psi_3}{\partial t^2} = \beta^2 \underline{\nabla}^2 \psi_3 \end{cases}$$

$$\frac{\partial^2 u_{SH}}{\partial t^2} = \frac{\partial^2 \left(\frac{\partial \psi_1}{\partial x_3} - \frac{\partial \psi_3}{\partial x_1} \right)}{\partial t^2}$$

$$= \frac{\partial}{\partial x_3} \left(\frac{\partial^2 \psi_1}{\partial t^2} \right) - \frac{\partial}{\partial x_1} \left(\frac{\partial^2 \psi_3}{\partial t^2} \right) = \beta^2 \left(\frac{\partial}{\partial x_3} \underline{\nabla}^2 \psi_1 - \frac{\partial}{\partial x_1} \underline{\nabla}^2 \psi_3 \right)$$

$$\frac{\partial^2 u_{SH}}{\partial t^2} = \beta^2 \underline{\nabla}^2 u_{SH}$$

For SH wave: $\underline{u}_{SH} = \frac{\partial \psi_1}{\partial x_3} - \frac{\partial \psi_3}{\partial x_1}$

$$\begin{cases} \frac{\partial^2 \psi_1}{\partial t^2} = \beta^2 \underline{\nabla}^2 \psi_1 \\ \frac{\partial^2 \psi_3}{\partial t^2} = \beta^2 \underline{\nabla}^2 \psi_3 \end{cases}$$

$$\frac{\partial^2 u_{SH}}{\partial t^2} = \frac{\partial^2 \left(\frac{\partial \psi_1}{\partial x_3} - \frac{\partial \psi_3}{\partial x_1} \right)}{\partial t^2}$$

$$= \frac{\partial}{\partial x_3}\left(\frac{\partial^2 \psi_1}{\partial t^2}\right) - \frac{\partial}{\partial x_1}\left(\frac{\partial^2 \psi_3}{\partial t^2}\right) = \beta^2 \left(\frac{\partial}{\partial x_3} \underline{\nabla}^2 \psi_1 - \frac{\partial}{\partial x_1} \underline{\nabla}^2 \psi_3 \right)$$

SH wave equation

$$\frac{\partial^2 u_{SH}}{\partial t^2} = \beta^2 \underline{\nabla}^2 u_{SH}$$

Sound wave (no shear wave) equation

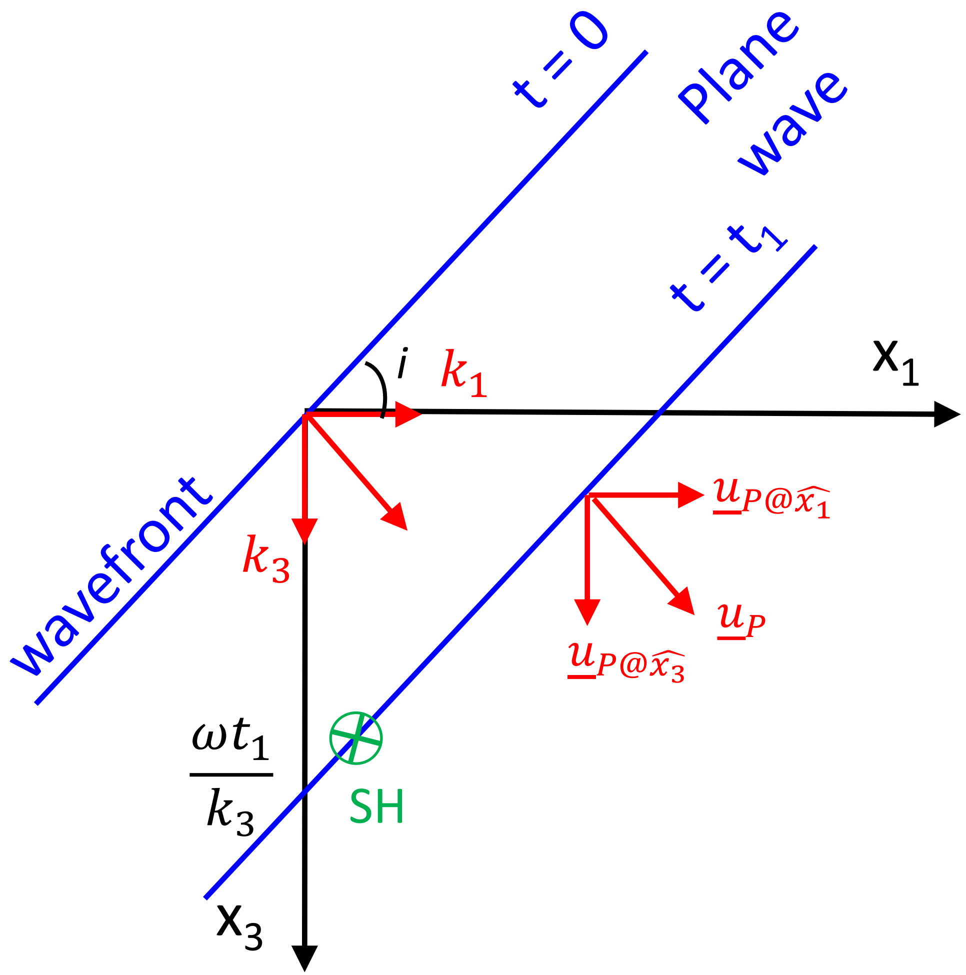
$$\underline{\nabla}^2 \phi - \frac{1}{\alpha^2} \ddot{\phi} = 0$$

SH wave equation

$$\frac{\partial^2 u_{SH}}{\partial t^2} = \beta^2 \underline{\nabla}^2 u_{SH}$$

$$U_{SH}(x_1, x_3, t) = A' e^{i(\omega t \pm k_{\beta 1} x_1 \pm k_{\beta 2} x_3)}$$

Move in x_2 direction
Follow the wavefront.



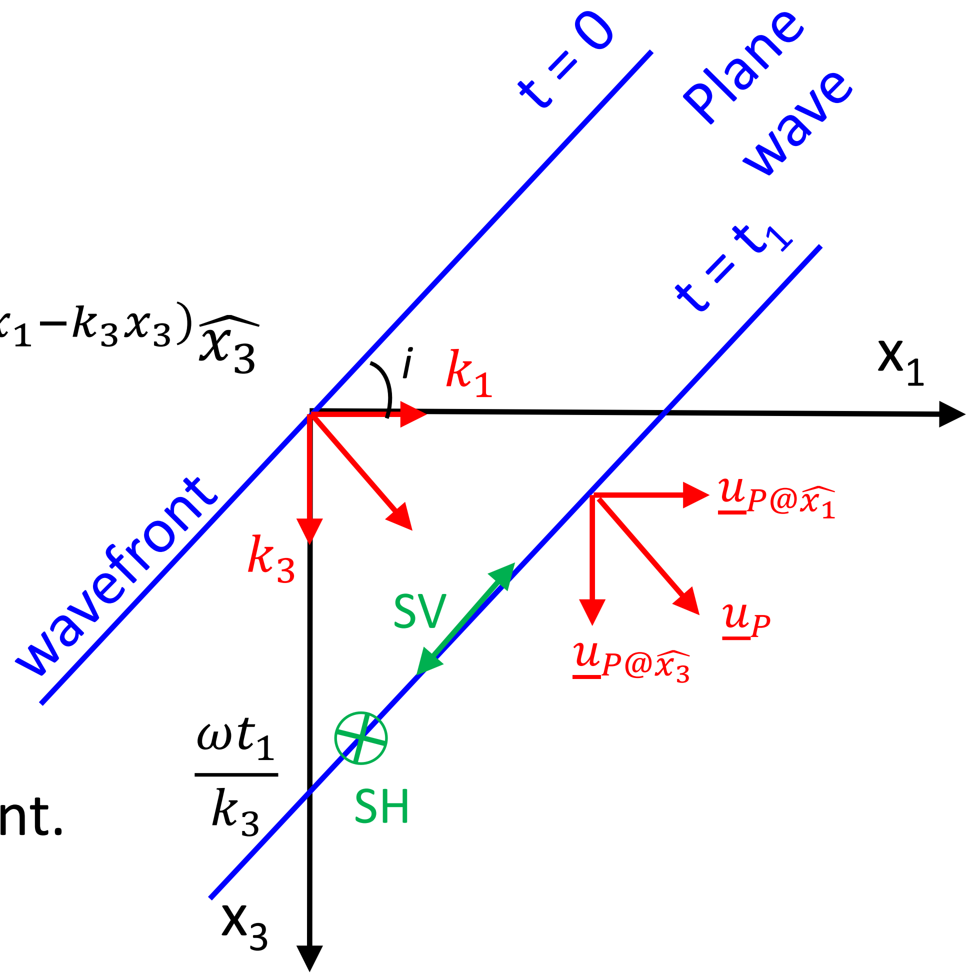
For SV

$$\psi_2 = B' e^{i(\omega t - k_1 x_1 - k_3 x_3)}$$

$$\underline{u}_{SV} = k_3 B' e^{i(\omega t - k_1 x_1 - k_3 x_3)} \widehat{x_1} - k_1 B' e^{i(\omega t - k_1 x_1 - k_3 x_3)} \widehat{x_3}$$

$$\frac{\underline{u}_{SV@\widehat{x_3}}}{\underline{u}_{SV@\widehat{x_1}}} = -\frac{k_1}{k_3}$$

Particle motion is parallel with the wavefront.

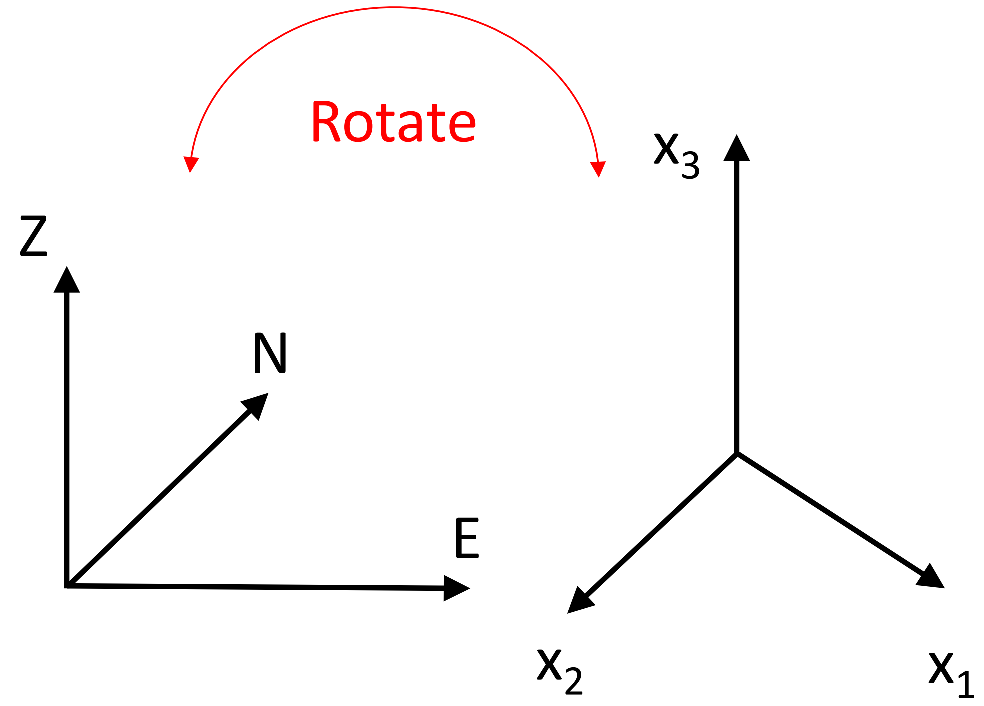


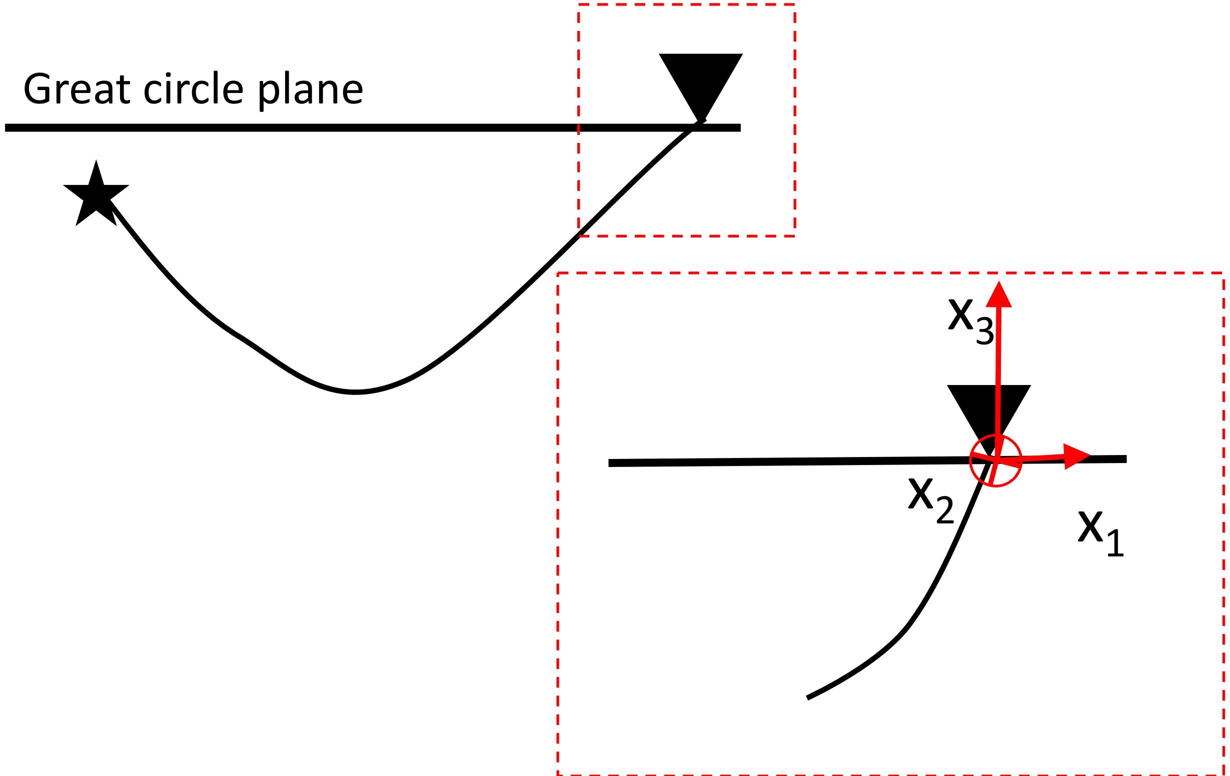
Why bother to rotate?

$$\underline{u} = \underline{u}_P + \underline{u}_S$$

$$= \left(\frac{\partial\phi}{\partial x_1} - \frac{\partial\psi_2}{\partial x_3}\right)\widehat{x_1} + \left(\frac{\partial\phi}{\partial x_3} + \frac{\partial\psi_2}{\partial x_1}\right)\widehat{x_3} \quad \text{P-SV system}$$

$$+ \left(\frac{\partial\psi_1}{\partial x_3} - \frac{\partial\psi_3}{\partial x_1}\right)\widehat{x_2} \quad \text{SH system}$$





Why bother to rotate?

$$\underline{u} = \underline{u}_P + \underline{u}_S$$

$$= \left(\frac{\partial\phi}{\partial x_1} - \frac{\partial\psi_2}{\partial x_3}\right)\widehat{x_1} + \left(\frac{\partial\phi}{\partial x_3} + \frac{\partial\psi_2}{\partial x_1}\right)\widehat{x_3}$$ P-SV system

$$+ \left(\frac{\partial\psi_1}{\partial x_3} - \frac{\partial\psi_3}{\partial x_1}\right)\widehat{x_2}$$ SH system