3.3. CRAMER'S RULE, VOLUME, AND LINEAR TRANSFORMATION

$$A_i(\mathbf{b}) = [\mathbf{a}_1 \quad \cdots \quad \underset{\substack{\uparrow \\ \text{col } i}}{\mathbf{b}} \quad \cdots \quad \mathbf{a}_n]$$

→ 행렬 A의 i번째 열을 b로 교체한 행렬

[Theorem 7]

Cramer's Rule

Let A be an invertible $n \times n$ matrix. For any $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has entries given by

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \qquad i = 1, 2, \ldots, n \tag{1}$$

A가 Invertible한 nxn matrix 일때, Ax=b의 해를 위식으로도 구할수있다.

(증명)

$A = [a_1 \cdots a_n]$, $I_i(x) = \begin{bmatrix} 1 & \cdots & x_1 & \cdots \\ \vdots & \ddots & \vdots & \vdots \\ \vdots & & x_i & \cdots \\ \cdots & & \vdots & \ddots \\ \vdots & \cdots & x_n & \cdots 1 \end{bmatrix}$ 일때,

$$\begin{aligned} A \cdot I_i(\mathbf{x}) &= A[\mathbf{e}_1 \quad \cdots \quad \mathbf{x} \quad \cdots \quad \mathbf{e}_n] = [A\mathbf{e}_1 \quad \cdots \quad A\mathbf{x} \quad \cdots \quad A\mathbf{e}_n] \\ &= [\mathbf{a}_1 \quad \cdots \quad \mathbf{b} \quad \cdots \quad \mathbf{a}_n] = A_i(\mathbf{b}) \end{aligned}$$

$\Rightarrow \det(A_i(b)) = (\det A)(\det I_i(x))$

$\Rightarrow \det(A_i(b)) = (\det A)\, x_i$

det $I_i(x)$는 Cofactor expansion하면 x_i.

$\therefore x_i = \dfrac{\det(A_i(b))}{\det A}$.

$$A\mathbf{x} = \mathbf{e}_j$$

⇒ x가 A^{-1}의 j번째 Column 일때 이식은 만족.

$AA^{-1} = I$ 이니까..

e_j는 Identity matrix의 j번째 col

x는 A^{-1}의 j번째 Col → x_i는 A^{-1}의 i행 j열 원소.

↓ Cramer's rule.

$$\{(i, j)\text{-entry of } A^{-1}\} = x_i = \frac{\det A_i(\mathbf{e}_j)}{\det A}$$

$\Rightarrow$ $\det A_i(\mathbf{e}_j) = (-1)^{i+j} \det A_{ji} = C_{ji}$

e_j가 Identity matrix라서
$A_i(e_j)$에서 j번째 행 i번째열
원제외하고 모두 '0'이라서..

$\Rightarrow$ $x_{i} = \dfrac{C_{ji}}{\det A}$ $\Rightarrow$ $A^{-1} = \dfrac{1}{\det A} C_{ij}$

$= \dfrac{1}{\det A} \underline{\text{Adj } A}$

A의 adjugate.

[Theorem 8]

An Inverse Formula

Let A be an invertible $n \times n$ matrix. Then

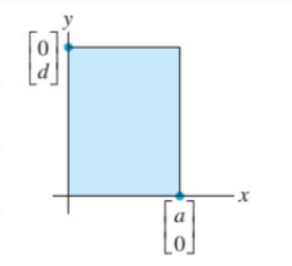
$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A$$

(비효율적)

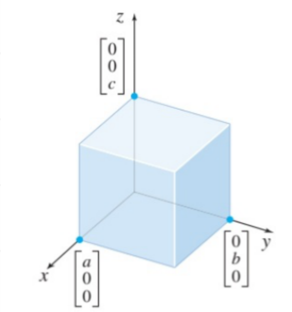
[Theorem 9]

If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $|\det A|$. If A is a 3×3 matrix, the volume of the parallelepiped determined by the columns of A is $|\det A|$.

: 2x2 matrix의 determinant 는 평행사변형의 면적.
3x3 matrix의 determinant는 평행육면체의 부피.



$\Rightarrow$ $\left| \det \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \right| = |ad| = \left\{ \begin{matrix} \text{area of} \\ \text{rectangle} \end{matrix} \right\}$



$\Rightarrow$ $\left| \det \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \right| = |abc|.$

[Theorem 10]

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation determined by a 2×2 matrix A. If S is a parallelogram in $\mathbb{R}^2$, then

$$\{\text{area of } T(S)\} = |\det A| \cdot \{\text{area of } S\} \tag{5}$$

If T is determined by a 3×3 matrix A, and if S is a parallelepiped in $\mathbb{R}^3$, then

$$\{\text{volume of } T(S)\} = |\det A| \cdot \{\text{volume of } S\} \tag{6}$$

$\mathbb{R}^2$ 공간에 있는 S를 2x2 matrix A로 linear transform

⇒ area3 = area2 × area1

⇒ det(AB) = det A × det B.

· Theorem 10은 평행사변형 ($\mathbb{R}^2$) 혹은 평행육면체 ($\mathbb{R}^3$)가 아닌 일반적인 도형에도 적용할 수 있음.

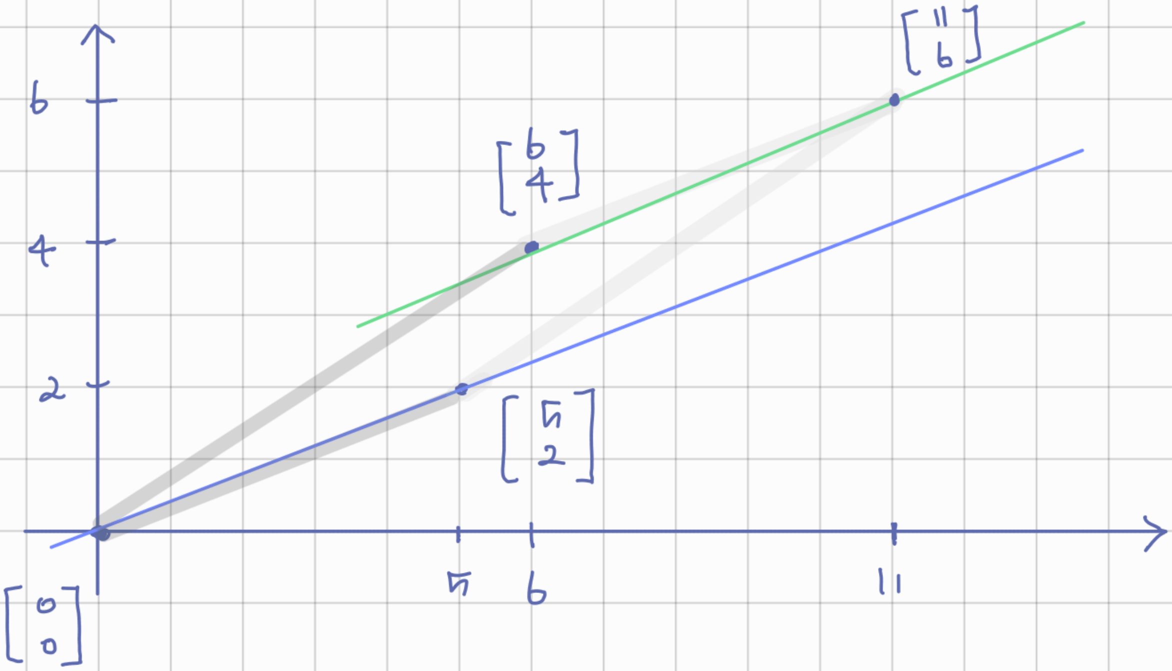
⇒ 일반적인 도형도 아주 작은 사각형들의 집합으로 근사할 수 있기 때문.

3.3. EXERCISES.

Q. In Exercises 19–22, find the area of the parallelogram whose vertices are listed.

19. (0, 0), (5, 2), (6, 4), (11, 6)

Sol).



$\Rightarrow$ matrix $A = \begin{bmatrix} 5 & 6 \\ 2 & 4 \end{bmatrix}$

$\Rightarrow \det A = (5 \times 4) - (6 \times 2) = 20 - 12 = 8.$