5.1. EIGENVECTORS AND EIGENVALUES.

EXAMPLE 1 Let $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. The images of $\mathbf{u}$ and $\mathbf{v}$ under multiplication by A are shown in Figure 1. In fact, $A\mathbf{v}$ is just $2\mathbf{v}$. So A only "stretches," or dilates, $\mathbf{v}$. ■

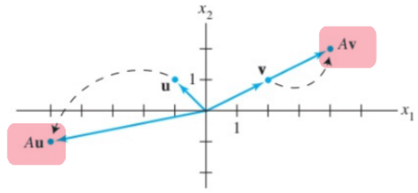

FIGURE 1 Effects of multiplication by A.

⇒ u와 v에 A를 곱했을 때 Av는 v와 동일한 line에 존재.

eigenvalue. eigenvector 정의.

DEFINITION An **eigenvector** of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an *eigenvector corresponding to* λ.[1]

Ax = λx를 만족하는 non zero vector x가 eigenvector,
Ax = λx에서 x가 nontrivial solution이 존재할 때 scalar λ가 eigenvalue,
이때 x는 λ에 상응하는 eigen vector 라고 함.

EXAMPLE 2 Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. Are $\mathbf{u}$ and $\mathbf{v}$ eigenvectors of A?

SOLUTION

$$A\mathbf{u} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4\begin{bmatrix} 6 \\ -5 \end{bmatrix} = -4\mathbf{u}$$

u는 A의 eigenvector.
-4는 A의 eigenvalue.

$$A\mathbf{v} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix} \neq \lambda\begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

Thus $\mathbf{u}$ is an eigenvector corresponding to an eigenvalue (-4), but $\mathbf{v}$ is not an eigenvector of A, because $A\mathbf{v}$ is not a multiple of $\mathbf{v}$. ■

EXAMPLE 3 Show that 7 is an eigenvalue of matrix A in Example 2, and find the corresponding eigenvectors.

$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix},$$

→ 7은 eigenvalue?

⇒ $Ax = 7x$를 만족해야함.

$Ax - 7x = 0.$

$(A-7I)x = 0$ 으로 정리할수있는데,

eigenvalue 정의에따라 x는 nonzero vector.

nontrivial solution이 존재하는지 파악해야됨.

(free variable이 존재하는가)

$$A - 7I = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}$$

$$\begin{bmatrix} -6 & 6 & 0 \\ 5 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ ⇒ $x_1 = x_2.$

⇒ $x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (general solution)

∴ nontrivial solution 존재 → 7은 A의 eigenvalue.

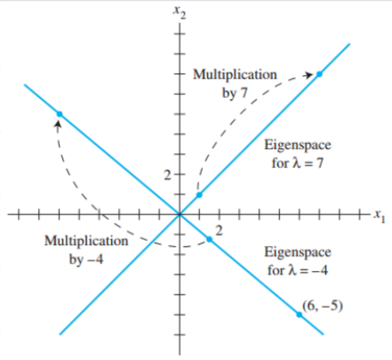

FIGURE 2 Eigenspaces for $\lambda = -4$ and $\lambda = 7$.

λ가 A의 eigenvalue라면 $(A-\lambda I)x = 0$은 nontrivial solution을 갖는데, x는 nonzero vector이므로 $A-\lambda I=0$의 solution을 찾으면되며, $A-\lambda I$ matrix의 null space가 A의 eigenspace가 됨.

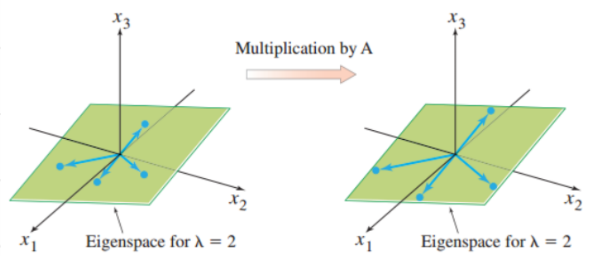

FIGURE 3 A acts as a dilation on the eigenspace.

Eigen space 위에 벡터들은 A로 linear transform 해도 방향은 변하지 않고 크기만 달라지므로 eigenspace도 유지됨.

THEOREM 1 The eigenvalues of a triangular matrix are the entries on its main diagonal.

triangular matrix의 eigenvalue는 대각원소.

- eigenvector는 nonzero vector 이어야하지만,

eigen value는 0일수도있음.

$Ax = \lambda x = 0x = 0.$

⇒ $Ax=0$ 식이 식가 0이아닌 nontrivial solution을 가지면 eigenvalue가 0일게 가능 · A is not Invertible.

THEOREM 2 If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent.

서로다른(중복되지않는) eigenvalue 들에 해당하는 eigenvector set은 linearly independent 하다.

5.1. EXERCISES.

1. Is $\lambda = 2$ an eigenvalue of $\begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}$? Why or why not?

sol). $Ax = \lambda x = 2x.$

$Ax = 2x.$

$(A - 2I)x = 0.$

$$\begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}.$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 3 & 6 & 0 \end{bmatrix}$$ row1 × (−3) + row2.

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ x_2 is free variable.

∴ $\lambda = 2$는 eigenvalue.

5.2. THE CHARACTERISTIC EQUATION.

A scalar λ is an eigenvalue of an $n \times n$ matrix A if and only if λ satisfies the characteristic equation

$$\det(A - \lambda I) = 0$$

λ가 characteristic equation $\det(A-\lambda I)=0$을 만족하면 λ는 행렬 A의 eigenvalue 이다.

- Similarity ⇒ 어떤 invertible matrix P가 있을때
  $B = PAP^{-1}$ 이 성립하면 A가 B에 similar 하다고함.
  equivalently, $A = PBP^{-1}$.

  A와 B가 similar할때,
  A를 PAP^{-1}로 변환하는것을 similarity transform 이라함.

THEOREM 4 If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).

A와 B가 similar 할때, 같은 eigen value를 가진다.

WARNINGS:

1. The matrices

$$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

are not similar even though they have the same eigenvalues.

2. Similarity is not the same as row equivalence. (If A is row equivalent to B, then $B = EA$ for some invertible matrix E.) Row operations on a matrix usually change its eigenvalues.

A와 B가 similar 하면, eigenvalue가 같지만.
eigen value가 같다고해서 similar 한것은 아니다.

5.2. EXERCISES

In Exercises 21 and 22, A and B are $n \times n$ matrices. Mark each statement True or False. Justify each answer.

21. a. The determinant of A is the product of the diagonal entries in A.

b. An elementary row operation on A does not change the determinant.

c. $(\det A)(\det B) = \det AB$

d. If $\lambda + 5$ is a factor of the characteristic polynomial of A, then 5 is an eigenvalue of A.

Sol).

a. det A는 A의 대각원소들의 곱인가?

$\Rightarrow$ False.

triangular matrix의 경우에만 대각원소들의 곱으로 det 구함.

b. row operation은 determinant 값을 변경시키지 않는다?

$\Rightarrow$ False.

row operation
- replacement $\rightarrow$ same det
- interchange, scaling $\rightarrow$ chang det.

c. $(\det A)(\det B) = \det AB$?

$\Rightarrow$ True.

(Theorem 3).

d. A의 characteristic polynomial이 $(\lambda+5)$로 구성될때, A의 eigenvalue는 5 ?

$\Rightarrow$ False.

$(\lambda-(-5))$ 형태이므로

eigenvalue $= -5$,,

EXAMPLE 3 Find the characteristic equation of

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

SOLUTION Form $A - \lambda I$, and use Theorem 3(d):

$$\det(A - \lambda I) = \det \begin{bmatrix} 5-\lambda & -2 & 6 & -1 \\ 0 & 3-\lambda & -8 & 0 \\ 0 & 0 & 5-\lambda & 4 \\ 0 & 0 & 0 & 1-\lambda \end{bmatrix}$$
$$= (5-\lambda)(3-\lambda)(5-\lambda)(1-\lambda)$$

The characteristic equation is

$$(5-\lambda)^2(3-\lambda)(1-\lambda) = 0$$

or

$$(\lambda-5)^2(\lambda-3)(\lambda-1) = 0$$

Expanding the product, we can also write

$$\lambda^4 - 14\lambda^3 + 68\lambda^2 - 130\lambda + 75 = 0$$

In Examples 1 and 3, $\det(A - \lambda I)$ is a polynomial in λ. It can be shown that if A is an $n \times n$ matrix, then $\det(A - \lambda I)$ is a polynomial of degree n called the **characteristic polynomial** of A.

5.3. DIAGONALIZATION.

[EX.1].

$$\boxed{D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}}$$

diagonal matrix.

$$D^2 = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 5^2 & 0 \\ 0 & 3^2 \end{bmatrix}$$

$$D^3 = DD^2 = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 5^2 & 0 \\ 0 & 3^2 \end{bmatrix} = \begin{bmatrix} 5^3 & 0 \\ 0 & 3^3 \end{bmatrix}$$

$$\boxed{D^k = \begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix} \quad \text{for } k \geq 1}$$

diagonal matrix의 제곱
= 대각원소들의 제곱.

만약, $A = PDP^{-1}$ 에서 D가 diagonal matrix 라면

(A가 diagonal matrix와 similar 하다면) A^k 도 쉽게 구할수있다.

[EX.2]

$$A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$$

$$A = PDP^{-1}$$

$$P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$A^2 = (PDP^{-1})(PDP^{-1}) = PD\underbrace{(P^{-1}P)}_{I}DP^{-1} = PDDP^{-1}$$

$$= PD^2P^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5^2 & 0 \\ 0 & 3^2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$A^3 = (PDP^{-1})A^2 = (PD\underbrace{P^{-1})P}_{I}D^2P^{-1} = PDD^2P^{-1} = PD^3P^{-1}$$

$$A^k = PD^kP^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \cdot 5^k - 3^k & 5^k - 3^k \\ 2 \cdot 3^k - 2 \cdot 5^k & 2 \cdot 3^k - 5^k \end{bmatrix}$$

⇒ Square matrix A가 diagonal matrix 와 similar 하면
A는 Diagonalizable 이라함.

(Eigenvalue decomposition 이라고도 많이부름).

THEOREM 5 **The Diagonalization Theorem**

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

「A가 diagonalizable 하려면
A는 n개의 linearly Independent eigenvector를 갖고있어야함.」

D가 diagonal matrix이고 $A=PDP^{-1}$ 이면, P의 Column은
A의 n개의 linearly Independent eigenvector 로 이루어짐.
이경우, D의 diagonal entries는 P들 각각의 eigenvector에
상응하는 A의 eigen value.

- $A=PDP^{-1}$,

D is Diagonal matrix, diagonal entry = eigenvalue of A 일때

① A와 D는 similar ⇒ A의 eigenvalue = D의 eigenvalue.
A의 determinant = D의 determinant.

② D가 diagonal matrix ⇒ diagonal entry = eigen value.
diagonal entry 곱 = determinant.

A의 eigenvalue = D의 eigenvalue = D의 대각원소.
⇒ det(A) = det(D) = D의 대각원소들의 곱.
= D의 eigenvalue 들의 곱
= A의 eigenvalue 들의 곱.

• Diagonalization 하는 방법.

① eigenvalue를 찾는다.
② eigenvector를 찾는다.
③ Column이 eigenvector로 구성된 행렬 P를 만든다.
④ ①에서 구한 eigenvalue로 diagonal entry가 eigenvalue인 Diagonal matrix D를 만든다.

THEOREM 6 An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

= L·I eigenvector

n개의 eigenvalue가 distinct 하면 diagonalizable 하지만 diagonalizable 한 모든행렬이 n개의 다른 eigenvalue를 가지는것은 아니다.

$p \le n$

THEOREM 7 Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \ldots, \lambda_p$.

a. For $1 \le k \le p$, the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of the eigenvalue λ_k.

b. The matrix A is diagonalizable if and only if the sum of the dimensions of the eigenspaces equals n, and this happens if and only if (*i*) the characteristic polynomial factors completely into linear factors and (*ii*) the dimension of the eigenspace for each λ_k equals the multiplicity of λ_k.

c. If A is diagonalizable and $\mathcal{B}_k$ is a basis for the eigenspace corresponding to λ_k for each k, then the total collection of vectors in the sets $\mathcal{B}_1, \ldots, \mathcal{B}_p$ forms an eigenvector basis for $\mathbb{R}^n$.

↳ $\dim \boxed{\text{Nul}(A-\lambda_k I)} \le \lambda_k$의 중복횟수.

= eigenspace for λ_k.

A is diagonalizable

$\equiv \sum_k \dim \text{Nul}(A-\lambda_k I) = n$

$\equiv \dim \text{Nul}(A-\lambda_k I) = \lambda_k$ 중복횟수

$\equiv$ Eigenvector들이 $\mathbb{R}^n$의 basis 형성.

5.3. EXERCISES.

In Exercises 1 and 2, let $A = PDP^{-1}$ and compute A^4.

1. $P = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

sol). $A = PDP^{-1}$

$$P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$P^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\therefore P^{-1} = \underbrace{\frac{1}{(5\times3)-(7\times2)}}_{=1} \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$$

$A^k = PD^kP^{-1}$ 을 활용할수있으므로

$$A^4 = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2^4 & 0 \\ 0 & 1^4 \end{bmatrix} \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 16 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 5\times16 + 7\times0 & 5\times0 + 7\times1 \\ 2\times16 + 3\times0 & 2\times0 + 3\times1 \end{bmatrix} \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 80 & 7 \\ 32 & 3 \end{bmatrix} \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 80\times3 + 7\times(-2) & 80\times(-7) + 7\times5 \\ 32\times3 + 3\times(-2) & 32\times(-7) + 3\times5 \end{bmatrix} = \begin{bmatrix} 226 & -525 \\ 90 & -209 \end{bmatrix}$$