4.2. NULL SPACES, COLUMN SPACES, AND LINEAR TRANSFORMATION.

Null spaces.

The **null space** of an $m \times n$ matrix A, written as Nul A, is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. In set notation,

$$\text{Nul } A = \{\mathbf{x} : \mathbf{x} \text{ is in } \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}$$

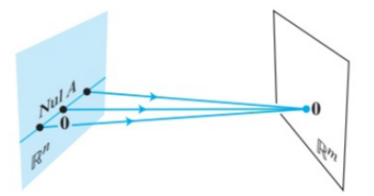


FIGURE 1

- null spaces는 R^n 의 subspace $\xrightarrow{\text{따라서}}$ ① 0을 포함 ② 덧셈에 닫힘 ③ 곱셈에 닫힘 — subspace 조건.

m×n matrix 있을때
n개의 미지수와
m개의 homogeneous linear equation 의 모든 해의 집합.

$$\underset{m \times n}{A} * \underset{n \times 1}{X} = \text{zero matrix.}$$

(Example 3) Find a $\overset{=\text{basis.}}{\text{spanning set}}$ for the null space of the matrix.

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{aligned} x_1 - 2x_2 \quad - x_4 + 3x_5 &= 0 \\ x_3 + 2x_4 - 2x_5 &= 0 \\ 0 &= 0 \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \underset{\uparrow\ \mathbf{u}}{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}} + x_4 \underset{\uparrow\ \mathbf{v}}{\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}} + x_5 \underset{\uparrow\ \mathbf{w}}{\begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}}$$

Free variable이 3개라 벡터도 3개 (u, v, w) 나옴.
Null space는 3개의 벡터의 linear combination으로 이루어져있음.
{u, v, w} is spanning set for Null A.

Column spaces.

The **column space** of an $m \times n$ matrix A, written as Col A, is the set of all linear combinations of the columns of A. If $A = [\mathbf{a}_1 \ \cdots \ \mathbf{a}_n]$, then

$$\text{Col } A = \text{Span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$$

The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

$$\boxed{\text{Col } A = \{\mathbf{b} : \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \text{ in } \mathbb{R}^n\}}$$

⇒ Colum space는 matrix A의 열들의 선형결합 집합.
A의 열들을 span하는 공간이 Column space이며,
R^m의 Subspace.

- A가 $m \times n$ matrix 일때
  R^m 공간의 모든 b에 대해서 $Ax=b$의 식을 만족하면,
  A의 Column space 와 m차원공간 (R^m)이 같다.

- A가 $n \times n$ matrix 일때, 경우에따라
  어떤벡터가 Null / Column space에 동시에속할수도 있다.
  Null A는 R^n에, Col A는 R^m의 Subspace 이지만,
  (때때로) 무조건 적으로 딱! 분리되어있는건 아니다.

Contrast Between Nul A and Col A for an $m \times n$ Matrix A

Nul A	Col A
1. Nul A is a subspace of $\mathbb{R}^n$.	**1.** Col A is a subspace of $\mathbb{R}^m$.
2. Nul A is implicitly defined; that is, you are given only a condition ($A\mathbf{x} = \mathbf{0}$) that vectors in Nul A must satisfy.	**2.** Col A is explicitly defined; that is, you are told how to build vectors in Col A.
x가 알아야지 계산을 해봐야 알수있음 (암시적)	(명시적) A의 열벡터들이므로 바로 알수있음.
3. It takes time to find vectors in Nul A. Row operations on $[\,A \quad \mathbf{0}\,]$ are required.	**3.** It is easy to find vectors in Col A. The columns of A are displayed; others are formed from them.
4. There is no obvious relation between Nul A and the entries in A.	**4.** There is an obvious relation between Col A and the entries in A, since each column of A is in Col A.
5. A typical vector $\mathbf{v}$ in Nul A has the property that $A\mathbf{v} = \mathbf{0}$.	**5.** A typical vector $\mathbf{v}$ in Col A has the property that the equation $A\mathbf{x} = \mathbf{v}$ is consistent.
Null space 에 속하는 어떤 벡터 v는 Av=0 특징을 가짐.	Column space 에 속하는 어떤벡터 v는 Ax=v를 만족하는 해를 가짐.
6. Given a specific vector $\mathbf{v}$, it is easy to tell if $\mathbf{v}$ is in Nul A. Just compute $A\mathbf{v}$.	**6.** Given a specific vector $\mathbf{v}$, it may take time to tell if $\mathbf{v}$ is in Col A. Row operations on $[\,A \quad \mathbf{v}\,]$ are required.
v가 Null A에 속하는지 보려면 Av=0인지 Av를 계산해보면됨 (간단)	
7. Nul $A = \{\mathbf{0}\}$ if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.	**7.** Col $A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^m$.
8. Nul $A = \{\mathbf{0}\}$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.	**8.** Col $A = \mathbb{R}^m$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ *onto* $\mathbb{R}^m$.
Nul A가 0벡터뿐이면 trivial solution 이고 one-to-one.	Col A가 R^m 과 같으면, R^m 공간의 모든 b가 solution 이자 linear transform 에서 onto.

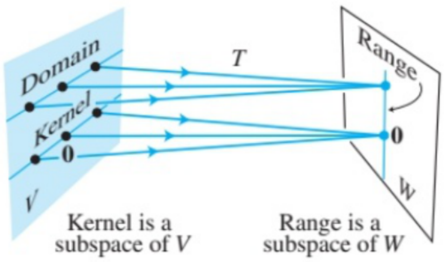


FIGURE 2 Subspaces associated with a linear transformation.

A **linear transformation** T from a vector space V into a vector space W is a rule that assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W, such that

(i) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in V, and

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u}$ in V and all scalars c.

↳ Kernel : T에 의하여 0으로 보내지는 V의 모든벡터집합.

= Null A

= V의 subspace.

Range : $T(v)=w$에서 W에 속하는 벡터 w.

= Col A

= W의 subspace.

4.2. EXERCISES.

In Exercises 3–6, find an explicit description of Nul A by listing vectors that span the null space.

4. $A = \begin{bmatrix} 1 & -6 & 4 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$

sol). Row operation $\Rightarrow \begin{bmatrix} 1 & -6 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$ row1 − row2×2.

$\begin{bmatrix} 1 & 6 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ row2 × $\frac{1}{2}$

$\therefore\ x_1 = 6x_2$

$x_3 = 0.$

$\Rightarrow x = \begin{bmatrix} 6x_2 \\ x_2 \\ 0 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 6 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \text{span} \begin{bmatrix} 6 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

4.3. LINEARLY INDEPENDENT SETS ; BASES.

$j>1$일때, 어떤 벡터 v_j 앞 index $(1, \dots, v_{j-1})$벡터들의 조합으로 표현할수있는 set.

Let H be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_p\}$ in V is a **basis** for H if

(i) $\mathcal{B}$ is a linearly independent set, and

(ii) the subspace spanned by $\mathcal{B}$ coincides with H; that is,

$$H = \mathrm{Span}\{\mathbf{b}_1, \dots, \mathbf{b}_p\}$$

H가 V의 subspace 이고, V에 속하는 Indexed set B가 (i), (ii) 조건을 만족하면 V의 subspace H의 basis 이다.

Spanning set에서 불필요한 벡터를 제거하면 basis를 구할수있다.

⇓

The Spanning Set Theorem

Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ be a set in V, and let $H = \mathrm{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$.

a. If one of the vectors in S—say, $\mathbf{v}_k$—is a linear combination of the remaining vectors in S, then the set formed from S by removing $\mathbf{v}_k$ still spans H.

b. If $H \neq \{\mathbf{0}\}$, some subset of S is a basis for H.

a. S 중 어떤 벡터 v_k S중 다른 벡터의 linear combination 이라면 v_k를 set에서 제외해도 S는 H를 span 한다.

b. H의 basis는 S의 subset 이다.

• 어떤 matrix A를 Row reduction해서 matrix B를 얻었을때,
matrix A와 B의 원소(숫자)는 서로 달라도,
$Ax=0$, $Bx=0$ 식에서 같은 Solution set 을가진다.

⇒ 같은 x에대하여 식을 만족하므로
A가 linear dependent 하면 B도 linear dependent 하고
반대 Case도 동일하다.

• The pivot columns of a matrix A form a basis for Col A.

⇒ non-pivot column들은 pivot column들의
linear combination 으로 표현할수있기때문.

• basis set이 다음과 같을때

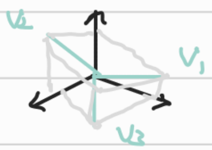
① $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}2\\3\\0\end{bmatrix}\right\}$
Linearly independent but does not span $\mathbb{R}^3$

② $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}2\\3\\0\end{bmatrix},\begin{bmatrix}4\\5\\6\end{bmatrix}\right\}$
A basis for $\mathbb{R}^3$

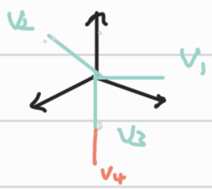
③ $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}2\\3\\0\end{bmatrix},\begin{bmatrix}4\\5\\6\end{bmatrix},\begin{bmatrix}7\\8\\9\end{bmatrix}\right\}$
Spans $\mathbb{R}^3$ but is linearly dependent

① basis 개수는 공간의 차원수와 일치해야한다.

② 3차원공간에서 basis가 3개일때
각 basis는 선형독립.



③ 3차원공간에서 basis가 3개이상일때
각 basis는 선형종속
(linear Independent)



4.3. EXERCISES.

11. Find a basis for the set of vectors in $\mathbb{R}^3$ in the plane $x + 2y + z = 0$. [*Hint:* Think of the equation as a "system" of homogeneous equations.]

Sol) $x + 2y + z = 0.$

Homogeneous eq $\Rightarrow Ax = 0.$

$$\Rightarrow \underbrace{\begin{bmatrix} 1 & 2 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{x} = 0.$$

Homogeneous eq 이니까, Null space를 구해서 basis를 계산해보면,

$$x = \begin{bmatrix} -2y + z \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} y + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} z.$$

basis는 $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

4.4. COORDINATE SYSTEM.

The Unique Representation Theorem

Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for a vector space V. Then for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, \dots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n \tag{1}$$

⇒ basis가 주어졌을때 특정벡터 x를 표현하는 스칼라 c는 유일하다.

Suppose $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is a basis for V and $\mathbf{x}$ is in V. The **coordinates of x relative to the basis $\mathcal{B}$** (or the **$\mathcal{B}$-coordinates of x**) are the weights $c_1, \dots, c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$.

If $c_1, \dots, c_n$ are the $\mathcal{B}$-coordinates of $\mathbf{x}$, then the vector in $\mathbb{R}^n$

$$\boxed{[\,\mathbf{x}\,]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}}$$

B에 대한 상대좌표,,

is the **coordinate vector of x (relative to $\mathcal{B}$)**, or the **$\mathcal{B}$-coordinate vector of x**. The mapping $\mathbf{x} \mapsto [\,\mathbf{x}\,]_{\mathcal{B}}$ is the **coordinate mapping (determined by $\mathcal{B}$)**.[1]

$$P_{\mathcal{B}} = [\,\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n\,]$$

좌표계 변환 행렬 (change of coordinates matrix)

$$\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n$$ 식은 $$\boxed{\mathbf{x} = P_{\mathcal{B}}[\,\mathbf{x}\,]_{\mathcal{B}}}$$ 나타낼수있음.

$[x]_B$: Basis B 상대좌표 (x는 standard 좌표)

↳ B상대좌표계 → 표준좌표 변환,,

$P_B^{-1} x = [x]_B$ ⇒ P_B의 inverse를 standard 좌표 x에 곱하게되면 표준좌표 → B상대좌표로 변환됨.

THEOREM 8 Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is a one-to-one linear transformation from V onto $\mathbb{R}^n$.

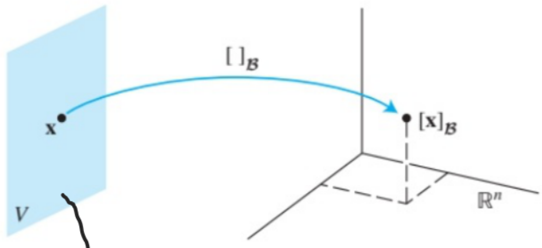


FIGURE 5 The coordinate mapping from V onto $\mathbb{R}^n$.

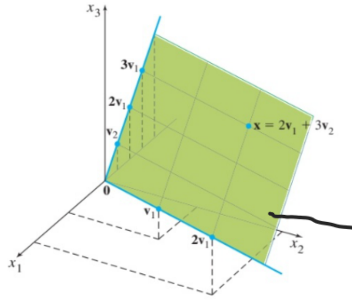


FIGURE 7 A coordinate system on a plane H in $\mathbb{R}^3$.

만약 n이 2라면, 2차원일테고, 이 2차원 공간은
Coordination mapping 하게되면
(one-to-one linear transformation)

아름답게변환이 될것.(onto).

(조금더 정리 하면)

basis set B는 linear independent 하고,
따라서 $X_1 \neq X_2$ 이다.
$X_1 \neq X_2$ 이면 $[X_1]_B \neq [X_2]_B$ 이므로 one-to-one.

$\Rightarrow$ 따라서 모든 $[X]_B \in \mathbb{R}^n$ 에 해당하는 $X = P_B[X]_B$가
존재하므로 onto.

- 벡터의 "좌표"만 변환한 것으로, 좌표계가 달라져도
벡터들 사이의 관계는 같다.

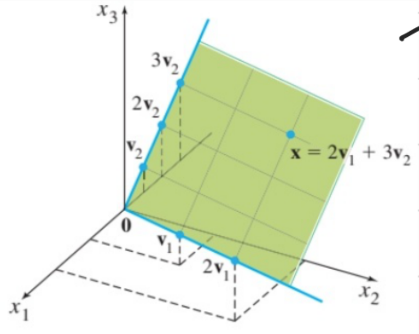


FIGURE 1 A coordinate system on a plane H in $\mathbb{R}^3$.

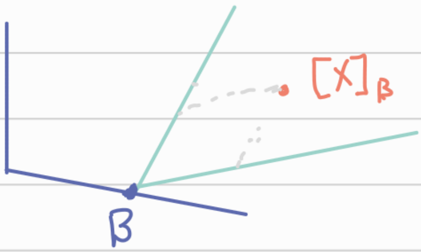
→ Basis vector (v_1, v_2)는 $\mathbb{R}^3$에 존재하지만, subspace가 $\mathbb{R}^2$에 존재하는것처럼 작용.
이는 subspace H는 $\mathbb{R}^2$에
Isomorphic 하다고 표현.

4.4. EXERCISES.

In Exercises 1–4, find the vector $\mathbf{x}$ determined by the given coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ and the given basis $\mathcal{B}$.

1. $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 6 \end{bmatrix} \right\}, [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

sol) $x = B[x]_B$



$$\Rightarrow \; = 5\begin{bmatrix} 3 \\ -5 \end{bmatrix} + 3\begin{bmatrix} -4 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 15 \\ -25 \end{bmatrix} + \begin{bmatrix} -12 \\ 18 \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \end{bmatrix}.$$