# 4.5 THE DIMENSION OF A VECTOR SPACE.

If a vector space $V$ has a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then any set in $V$ containing more than $n$ vectors must be linearly dependent.

초과.

: basis $B=\{b_1 \cdots b_n\}$ 일때 V안에 어떤 set이 n개를 초과하면 linearly dependent.
(예시, 3차원공간에 4개의 벡터가 있는 경우)

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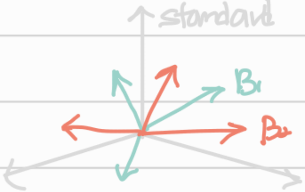
If a vector space $V$ has a basis of $n$ vectors, then every basis of $V$ must consist of exactly $n$ vectors.

: Vector space V의 basis가 n개의 vectors로 이루어져있다면, V의 다른 basis도 n개의 vector를 가져야한다.
(모든)

$$
\begin{array}{ccc}
 & \xrightarrow[\text{onto}]{1:1} & \mathbb{R}^n \text{ basis } B_1 \\
\text{Vector space } V & & \updownarrow \ 1:1.\ \text{onto} \quad p=n \\
 & \xrightarrow[\text{onto}]{1:1} & \mathbb{R}^p \text{ basis } B_2
\end{array}
$$

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If $V$ is spanned by a finite set, then $V$ is said to be **finite-dimensional**, and the **dimension** of $V$, written as dim $V$, is the number of vectors in a basis for $V$. The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be zero. If $V$ is not spanned by a finite set, then $V$ is said to be **infinite-dimensional**.

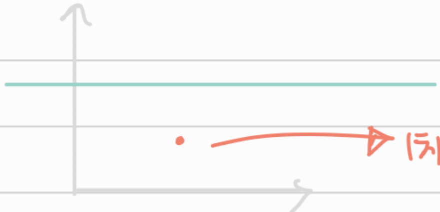
V가 유한한 set에대해 span 될때, V는 finite-dimensional 이라하고,
V의 dimension = V의 basis수.
만약 finit set에의해 span되지 않으면 infinite-dimensional 이라함

↳ basis의 개수 = vector space의 dimension.

Let $H$ be a subspace of a finite-dimensional vector space $V$. Any linearly independent set in $H$ can be expanded, if necessary, to a basis for $H$. Also, $H$ is finite-dimensional and

$$\dim H \leq \dim V$$

→ Vector space V의 subspace H의 차원은 $\dim H \leq \dim V$



1차원 subspace.

→ 1차원 subspace와 linear Independent하다면.

2차원 Vectorspace.

→ 1차원 subspace의 basis가 된다고 그럼 1차원 subspace는 2차원 subspace가 돼버리는...

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### The Basis Theorem

Let $V$ be a $p$-dimensional vector space, $p \geq 1$. Any linearly independent set of exactly $p$ elements in $V$ is automatically a basis for $V$. Any set of exactly $p$ elements that spans $V$ is automatically a basis for $V$.

: V에 속해있는 서로 linear Independent한 p개의 element들은 V의 basis가 됨.

: p개의 element set이 V를 span하고있다면 p개의 element들은 basis.

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The dimension of Nul $A$ is the number of free variables in the equation $A\mathbf{x} = \mathbf{0}$, and the dimension of Col $A$ is the number of pivot columns in $A$.

dim Nul A → Ax=0 의 Free variable 수.

dim Col A → A의 pivot column 수.

## 4.5 EXERCISES.

For each subspace in Exercises 1–8, (a) find a basis, and (b) state the dimension.

**5.** $\left\{ \begin{bmatrix} a-4b-2c \\ 2a+5b-4c \\ -a+2c \\ -3a+7b+6c \end{bmatrix} : a, b, c \text{ in } \mathbb{R} \right\}$

Sol.

$$\vec{x} = \begin{bmatrix} a-4b-2c \\ 2a+5b-4c \\ -a+2c \\ -3a+7a+6c \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ -1 \\ -3 \end{bmatrix} a + \begin{bmatrix} -4 \\ 5 \\ 0 \\ 7 \end{bmatrix} b + \begin{bmatrix} -2 \\ -4 \\ 2 \\ 6 \end{bmatrix} c$$

X 2으로 나타낼수있음

$$\therefore \text{Basis } B = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ 5 \\ 0 \\ 7 \end{bmatrix} \right\}$$

Basis가 2개이므로 2차원임.

# 4.6. RANK.

- Row Space. → row vector들의 linear combination으로 이루어진 set,,

  Row 어레이로써 나타내어짐.

  - m×n matrix에서 Row A는 $\mathbb{R}^n$ 의 Subspace.
  - $\text{Col } A^T = \text{row } A$.

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \quad \text{and} \quad \begin{aligned} \mathbf{r}_1 &= (-2, -5, 8, 0, -17) \\ \mathbf{r}_2 &= (1, 3, -5, 1, 5) \\ \mathbf{r}_3 &= (3, 11, -19, 7, 1) \\ \mathbf{r}_4 &= (1, 7, -13, 5, -3) \end{aligned}$$

$\Rightarrow \text{Row } A = \text{Span}\{r_1, r_2, r_3, r_4\}$

- Row operation은 row간의 linear relationship을 바꿀수있음.

  ex) row1 = row2 + row4 였는데 이 관계성이 깨질수도.

If two matrices $A$ and $B$ are row equivalent, then their row spaces are the same. If $B$ is in echelon form, the nonzero rows of $B$ form a basis for the row space of $A$ as well as for that of $B$.

: A와 B가 row equivalent 하면 A와 B의 row space는 같다

: B가 echelon form 일때, B의 none zero rows는 row space A와 B의 basis를 이룸.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \end{bmatrix}$$

echelon form에서 none zero row는 그아래의 none zero row 들로 만들수없음

→ linear Independent → row A basis.

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \qquad A \sim B = \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

· A ~ B (echelon form) 일때,

**Row Space**

① A와 B의 row vector들 사이의 관계는 다르다.

② A와 B는 row space가 같으므로 basis도 공유한다.

**Column space.**

① A와 B의 Colum vector들 사이의 관계는 같다.

② A와 B는 Column space가 다르므로 B의 pivot column에 해당하는 A의 column들이 ColA의 basis이다.

[RANK].

The **rank** of $A$ is the dimension of the column space of $A$.

: rank는 A의 column space의 dimension.

= A의 Colum space 의 basis vector 개수 //.

**The Rank Theorem**

The dimensions of the column space and the row space of an $m \times n$ matrix $A$ are equal. This common dimension, the rank of $A$, also equals the number of pivot positions in $A$ and satisfies the equation

$$\text{rank } A + \dim \text{Nul } A = n$$

pivot column = dim col A = rank A

non pivot column. = dim Nul A.

- Row 와 Nul A 는 zero vector 만 공통적으로 가지고, 다른 vector 들은 서로 "perpendicular" 함.

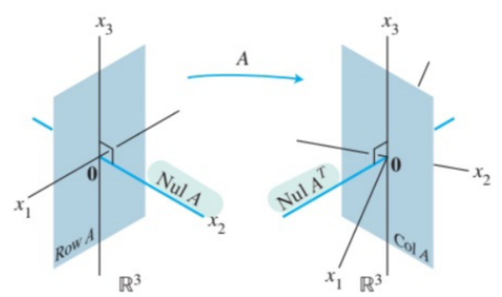


**FIGURE 1** Subspaces determined by a matrix $A$.

(역행렬정리, 추가버전)

- **The Invertible Matrix Theorem (continued)**

  Let $A$ be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that $A$ is an invertible matrix.

  m. The columns of $A$ form a basis of $\mathbb{R}^n$.
  n. $\text{Col } A = \mathbb{R}^n$
  o. $\dim \text{Col } A = n$
  p. $\text{rank } A = n$
  q. $\text{Nul } A = \{\mathbf{0}\}$
  r. $\dim \text{Nul } A = 0$

  역행렬이 가능하려면 Free variable이 없고 pivot 수 = n 이니까,,

4.6. EXERCISES.

**5.** If a $3 \times 8$ matrix $A$ has rank 3, find $\dim \text{Nul } A$, $\dim \text{Row } A$, and $\text{rank } A^T$.

sol. rank A = dim Col A = 3.

3행 8열 matrix에서 dim col A = 3은 pivot col = 3개
nonpivot col = 5개.

dim Nul A = nonpivot col 개수 = 5.

또, row space의 basis = pivot = 3 = dim Row A.

Col $A^T$ = row A ⇒ rank $A^T$ = dim Col $A^T$ = dim row A
= 3.

# 4.7. CHANGE OF BASIS.

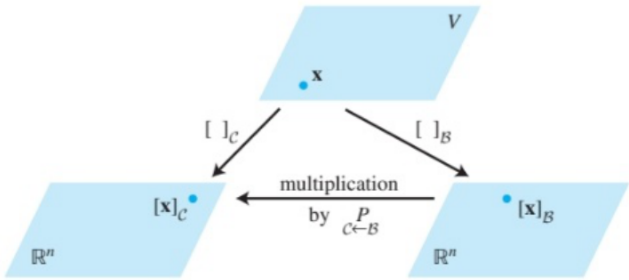


**FIGURE 2** Two coordinate systems for $V$.

Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ be bases of a vector space $V$. Then there is a unique $n \times n$ matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ such that

$$[\mathbf{x}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\mathbf{x}]_{\mathcal{B}} \tag{4}$$

The columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ are the $\mathcal{C}$-coordinate vectors of the vectors in the basis $\mathcal{B}$. That is,

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} & \cdots & [\mathbf{b}_n]_{\mathcal{C}} \end{bmatrix} \tag{5}$$

Change of coordinates matrix from B to C.

$$(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P})^{-1}[\mathbf{x}]_{\mathcal{C}} = [\mathbf{x}]_{\mathcal{B}} \Rightarrow (\underset{\mathcal{C}\leftarrow\mathcal{B}}{P})^{-1} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$$

Invertible이 성립하는 이유,

$\Rightarrow P_B = [b_1 \cdots b_n]$ $(m \times n)$

$P_C = [c_1 \cdots c_n]$ $(m \times n)$ $(m \geq n)$

$\{b_1 \cdots b_n\}$과 $\{c_1 \cdots c_n\}$의 각 vector $\rightarrow$ L·I

$P_B x = [b_1 \cdots b_n] x = 0. \Rightarrow x = 0.$

$b_i = P_C [b_i]_C$ 라 할때

$P_B x = [P_C[b_1]_C \cdots P_C[b_n]_C] x = 0.$

$= P_C \underline{[[b_1]_C \cdots [b_n]_C] x} = 0$

$\Rightarrow$ y라고하면

$= P_C y = 0. \Rightarrow y = 0.$

$y = [[b_1]_C \cdots [b_n]_C] x = 0.$ $\leftarrow$ $x = 0$

$\therefore [[b_1]_C \cdots [b_n]_C]$는 L.I.

- Standard basis $\{e_1, \cdots e_n\}$ 에대해서 $[b_i]_{\varepsilon} = b_i$.

- How to Find the change of coordinates matrix from B to C.

(basis)

$\Rightarrow$ $B = \{b_1\ b_2\}$, $C = \{c_1, c_2\}$ 가 주어지고,

$$\underset{C \leftarrow B}{P} = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$$ 라고할때,

$$\parallel$$

$$\underset{C \leftarrow B}{P} = [\,[\mathbf{b}_1]_C \quad [\mathbf{b}_2]_C\,]$$

$\Rightarrow$ $[\mathbf{b}_1]_C = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $[\mathbf{b}_2]_C = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ 이렇게도 표현가능하고,

이때, $\underset{C \leftarrow B}{P}$ 는 $[\,c_1\ c_2 \mid b_1\ b_2\,]$ 의 Row reducing 을 통하여 구할수있다.

$$\boxed{[\,\mathbf{c}_1 \quad \mathbf{c}_2 \mid \mathbf{b}_1 \quad \mathbf{b}_2\,] \sim [\,I \mid \underset{C \leftarrow B}{P}\,]}$$

4.7. EXERCISES.

**2.** Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ be bases for a vector space $V$, and suppose $\mathbf{b}_1 = -\mathbf{c}_1 + 4\mathbf{c}_2$ and $\mathbf{b}_2 = 5\mathbf{c}_1 - 3\mathbf{c}_2$.

a. Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

b. Find $[\mathbf{x}]_{\mathcal{C}}$ for $\mathbf{x} = 5\mathbf{b}_1 + 3\mathbf{b}_2$.

sol. ⓐ

$$\underset{C\leftarrow B}{P} = [\,[b_1]_c \ \ [b_2]_c\,] = \quad [c_1 \ c_2][b_1]_c = b_1$$

$$[c_1 \ c_2][b_2]_c = b_2.$$

$$b_1 = [c_1 \ c_2]\begin{bmatrix} -1 \\ 4 \end{bmatrix}, \quad b_2 = [c_1 \ c_2]\begin{bmatrix} 5 \\ -3 \end{bmatrix}.$$

$$\therefore \underset{C\leftarrow B}{P} = \begin{bmatrix} -1 & 5 \\ 4 & -3 \end{bmatrix}.$$

ⓑ $$[x]_c = \underset{C\leftarrow B}{P}[x]_B,$$

$$x = P_B[x]_B$$
$$= 5b_1 + 3b_2$$
$$= [b_1 \ b_2]\begin{bmatrix} 5 \\ 3 \end{bmatrix}.$$

$$\therefore [x]_B = \begin{bmatrix} 5 \\ 3 \end{bmatrix}.$$

$$(P_B = \underset{B\leftarrow E}{P}).$$

$$= \begin{bmatrix} -1 & 5 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} (-1\times 5) + (5\times 3) \\ (4\times 5) + (-3\times 3) \end{bmatrix}$$

$$= \begin{bmatrix} 10 \\ 11 \end{bmatrix}$$