Tutorial 1

JV Aditya

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We have to prove that if A and B are square matrices then

$$|I - AB| \neq 0 \iff |I - BA| \neq 0$$

The backward direction will be proved here. Forward direction can be proved similarly just by swapping A with B everywhere in the proof.

Given: $I - BA$ is **invertible**.
To Prove: $I - AB$ is **invertible**.
Proof: We start with the following fact, which was provided as hint in the question

$$(I - BA)B = B(I - AB) \tag{1}$$

For the sake of simplicity, we use the shorthand X for $I - BA$ and the shorthand Y for $I - AB$. The hint now can be rewritten as

$$XB = BY \tag{2}$$

Our goal is to prove that Y is invertible. We prove this by contradiction. Assume that Y is non invertible. This means that the columns of Y are linearly dependent.Then, there exist $w_1, w_2, \ldots, w_n$, not all zero, such that

$$\sum_{i=1}^{n} w_i c_i = 0$$

where c_i's are the columns of Y. This is equivalent to saying that

$$Yw = 0 \tag{3}$$

for $w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$ $(w \neq 0)$

We now multiply both sides of equation 2 by w. This yields us with

$$\begin{aligned} XBw &= BYw \\ &= 0 \\ \implies XBw &= 0 \end{aligned}$$

Note that X is invertible, hence,

$$\begin{aligned} Xw' &= 0 \\ \implies X^{-1}Xw' &= 0 \\ \implies w' &= 0 \end{aligned}$$

Hence, it is easy to see that

$$Bw = 0 \tag{4}$$

since $X(Bw) = 0$ 3 is equivalent to $(I - AB)w = 0$

$$\begin{aligned} (I - AB)w &= 0 \\ \implies w - ABw &= 0 \\ \implies w &= 0 \\ \text{since Bw} &= 0 \text{ due to } 4 \end{aligned}$$

We get $w = 0$ which was contrary to our assumption that $w \neq 0$. Hence, we have proved by contradiction that $I - AB$ is invertible.