Tutorial 2

JV Aditya

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We first prove that if we have a set of m(each of dimension nx1) linearly independent vectors $v_1, v_2 \ldots v_m$ and we represent this set of vectors as a matrix

$$A = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} \tag{1}$$

(A is an mxn matrix) then the row vectors remain linearly independent even after applying any number of row operations. We will be using the fact **one** row operation leaves the rows linearly independent when operated on a matrix with linearly independent rows(Proof of this: check for yourself).

Assumption: A has linearly independent rows.
Proof: We prove this by induction on the number of elementary row operations 'p'.

Base case: p = 1. Use the fact mentioned above

Now, assume that the hypothesis is true for p = k. That is $E_k E_{k-1} \cdots E_1 A$ has linearly independent rows. We would now have to prove the hypothesis for p=$k+1$.

Observe that $E_{k+1} E_k E_{k-1} \cdots E_1 A = E_{k+1}(E_k E_{k-1} \cdots E_1 A) = E_{k+1} B \, (say)$
We know that B has linearly independent rows due to the inductive assumption for p=k. This also means that $E_{k+1}B$ has linearly independent rows(due to the 'fact' :). Thus, we have proved that the hypothesis is true for p=$k+1$. We can now say that by the principle of mathematical induction, the hypothesis is true for any **p**.
Now, we come to the actual question which is to prove that a square matrix has linearly independent columns if and only if it is invertible.

The reverse direction is fairly straightforward; it is left as an exercise to you to prove it(take help from tutorial 1's solution to a similar question :). The forward direction of the bi-implication will be proved here.

Given: Square matrix A of size nxn has linearly independent columns.
To Prove: A is invertible

Proof: Observe that the columns of A are the same as the rows of A^T. It is given that the columns of A are linearly independent, hence, the rows of A^T are linearly independent. Our focus hereon will be on the matrix A^T.

The **Row Canonical Form**(RCF) of a matrix is obtained by performing a sequence of **row operations** on the matrix. So, if the original matrix had linearly independent rows, then the RCF will have linearly independent rows as well(follows from the lemma proved above). We will prove by contradiction that the RCF of A^T has no rows that are null vectors.
If the RCF of A^T has a row with no non-zero elements, then the RCF would look like

$$RCF(A^T) = \begin{bmatrix} w_1 \\ \vdots \\ w_j \\ \vdots \\ w_n \end{bmatrix}$$

where w_i's are row vectors and $w_j = 0$ for some j. Consider the set of constants $\{\lambda_i\}$ where $\lambda_k = 1$ when $k = j$ and 0 otherwise. Observe that $\sum_{k=1}^{n} \lambda_k w_k = 0$. Thus, the null vector can be formed by a non trivial linear combination(since not all the λ_k's are 0). This implies that the vectors $\{w_i\}$ are linearly dependent which is contrary to our assumptions that the rows of RCF(A^T) are linearly independent.

So, by contradiction, we have the the RCF of A^T has no rows that are identically zero. This means that there has to be one non zero row in each row in the RCF. We have n pivots and we also have n rows. So, every row has a pivot. Let $\{(A^T)_{ij_i}\}$ be the pivots. By the definition of RCF, we know that

$$j_1 < j_2 < \cdots j_n$$

and we also, know that

$$\forall k \ 1 \leq j_k \leq n$$

Putting both these together, we get

$$\forall k \ j_k = k$$

Also, by the definition of RCF, all the elements above a pivot are 0

$$\implies \forall p < i \ (RCF(A^T))_{pj_i} = 0$$
$$\implies \forall p < i \ (RCF(A^T))_{pi} = 0$$

(since $j_i = i$)
Since the RCF is also a **Row Echelon Form**, it also has to satisfy the condition that $\forall p > i \ (RCF(A^T))_{pj_i} = 0$. This is equivalent to

$$\forall p > i \ (RCF(A^T))_{pi} = 0$$

(since $j_i = i$). Note that we are writing $\forall p$ in the above equations because there is a pivot in each row in $RCF(A^T)$.
The above inequalities

$$\forall p < i \ (RCF(A^T))_{pi} = 0$$

$$\forall p > i \ (RCF(A^T))_{pi} = 0$$

together imply that RCF(A^T) $= I$.
Since we get the RCF through a sequence of row operations, let the elementary row matrices be $E_1, E_2, \ldots E_l$ corresponding to the row operations, in that order. Since RCF(A^T) is I, we can write that

$$E_l E_{l-1} \ldots E_1 A^T = I$$

We know that the elementary row matrices are invertible. So, by pre multiplying their inverses in the right order, we get

$$A^T = E_l^{-1} E_{l-1}^{-1} \cdots E_1^{-1} = (E_1 E_2 \ldots E_l)^{-1}$$

Taking transpose on both sides and rewriting the equation, we get

$$A = ((E_1 E_2 \ldots E_l)^{-1})^T = ((E_1 E_2 \ldots E_l)^T)^{-1}$$

The RHS of the above equation is invertible, hence, the LHS is invertible too. We now take inverse on both sides to get

$$A^{-1} = (E_1 E_2 \ldots E_l)^T$$

Hence, we have proved that A is invertible