Dedução:

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Por Taylor:

$$p(x,t+\tau) = \overset{\to p(x,t)}{p} + \tau\frac{\partial p}{\partial t} + \frac{\tau^2}{2}\frac{\partial^2 p}{\partial t^2}$$

$$p(x,t+\tau) \simeq p + \tau\frac{\partial p}{\partial t}$$

(1) $$p + \tau\frac{\partial p}{\partial t} = \int_{-\infty}^{\infty} p(x-y,t)\varphi(y)\,dy$$

$$p(x,t) + \tau\frac{\partial p}{\partial t} = \int_{-\infty}^{\infty}\left[p(x,t) - y\frac{\partial p}{\partial x} + \frac{y^2}{2}\frac{\partial^2 p}{\partial x^2}\right]\varphi(y)\,dy$$

$$\cancel{p(x,t)} + \tau\frac{\partial p}{\partial t} \approx \cancel{p(x,t)} + \frac{\sigma^2}{2}\frac{\partial^2 p}{\partial x^2}$$

$$\frac{\partial p}{\partial t} = \frac{\sigma^2}{2\tau}\frac{\partial^2 p}{\partial x^2}$$

$$\boxed{\frac{\partial p}{\partial t} = D\frac{\partial^2 p}{\partial x^2}}$$

Probabilidade

$$\int_{-\infty}^{\infty} p(x-y,t)\,f(y)\,dy$$

Nesse caso,

$$\int_{-\infty}^{\infty} p(x-y,t)\varphi(q)\,dq$$

$$\int_{-\infty}^{\infty}\varphi(q)\,dq = 1$$

$$\int_{-\infty}^{\infty} q\varphi(q)\,dq = 0$$

$$\int_{\infty}^{\infty} q^2\varphi(q)\,dq = \sigma^2$$

Variança

Aplicando a transformada de Fourier:

$$\mathcal{F}\left[\frac{\partial p}{\partial t}\right] = \mathcal{F}\left[D\frac{\partial^2 p}{\partial x^2}\right] \rightsquigarrow \mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx = F(\alpha)$$

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} p(x,t)\, e^{-ikx}\, dx = D\int_{-\infty}^{\infty} \underbrace{\frac{\partial^2 p(x,t)}{\partial x^2}}_{v}\, \overbrace{e^{-ikx}}^{u}\, dx$$

$$\frac{1}{D}\frac{\partial}{\partial t}\tilde{p}(k,t) = \underbrace{\frac{\partial p}{\partial x}e^{-ikx}\Big|_{-\infty}^{\infty}}_{0} - \int_{-\infty}^{\infty}(-ik)\,e^{-ikx}\frac{\partial p}{\partial x}\,dx$$

$$\frac{1}{D}\frac{\partial \tilde{p}(k,t)}{\partial t} = ik\left(-\int_{-\infty}^{\infty}(-ik)\,e^{-ikx}\,p(x,t)\,dx\right)$$

$$-k^2\tilde{p}(k,t) = \frac{1}{D}\frac{\partial \tilde{p}(k,t)}{\partial t}$$

$$\ln(\tilde{p}) = -k^2 D\, dt$$

$$\tilde{p}(k,t) = e^{-k^2 D t}$$

Pela transformada de Fourier inversa:

$$p(x,t) = \frac{1}{2\pi}\int_{\infty}^{\infty} e^{-k^2 D t}\cdot e^{ikx}\, dk$$

$$p(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\beta(k-n_0)^2 - \frac{x^2}{4Dt}}\, dk$$

, com

$$\beta = -Dt \text{ e } n_0 = \frac{ix}{2Dt}$$

Completamento de quadrado!

$$-Dt\left(n - \frac{ix}{2Dt}\right)^2 - \frac{x^2}{4Dt}$$

$$\beta(n-n_0)^2 - \frac{x^2}{4Dt}$$

$$p(x,t) = \frac{e^{-\frac{x^2}{4Dt}}}{2\pi} \int_{-\infty}^{\infty} e^{-Dt\left(k - \frac{ix}{2Dt}\right)^2} dk$$

com $k - k_0 = z$

$dz = dk$

$$= \frac{e^{-\frac{x^2}{4Dt}}}{2\pi} \underbrace{\int_{-\infty}^{\infty} e^{-Dt(z)^2} dz}_{\text{Integral Gaussiana}}$$

$\rightarrow \int_{-\infty}^{\infty} e^{-ax^2} = \sqrt{\frac{\pi}{a}}$

$$p(x,t) = \sqrt{\frac{\pi}{Dt}} \, \frac{1}{2\pi} \, e^{-\frac{x^2}{4Dt}}$$

$$\boxed{p(x,t) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}}$$