Scalar and vector potentials, Helmholtz decomposition, and de Rham cohomology

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Outline

The objects

Beyond a doubt, among the "stars" of vector calculus we have the operators

- **grad**
- div
- **curl**

Aim of this talk is to understand better their properties and their connections with some topological concepts.

First results

First well-known results are (just compute...):

- $\mathbf{curl}\,\mathbf{grad}\,\psi = \mathbf{0}$ for each scalar function ψ
- $\operatorname{div}\mathbf{curl}\,\mathbf{H} = 0$ for each vector field $\mathbf{H}$.

We can thus write

Theorem (1)

If $\mathbf{H} = \mathbf{grad}\,\psi$, *then* $\mathbf{curl}\,\mathbf{H} = \mathbf{0}$ *(namely,* $\mathbf{H}$ *is curl-free).*

Theorem (2)

If $\mathbf{B} = \mathbf{curl}\,\mathbf{A}$, *then* $\operatorname{div}\mathbf{B} = 0$ *(namely,* $\mathbf{B}$ *is divergence-free).*

First results (cont'd)

The natural question is:

- are these conditions sufficient?

We will see that the answer depends on the geometry of the region Ω where we are working.

In the whole space...

Let us start from $\Omega = \mathbb{R}^3$. We need some tools. First of all we know [just compute...] that the function

$$K(\mathbf{x}, \mathbf{y}) = \frac{1}{4\pi} \frac{1}{|\mathbf{x} - \mathbf{y}|} \tag{1}$$

satisfies

$$-\Delta_x K(\mathbf{x}, \mathbf{y}) = 0 \qquad \text{for } \mathbf{x} \neq \mathbf{y}$$
$$\int_{\partial B} \mathbf{grad}_x K(\mathbf{x}, \mathbf{0}) \cdot \mathbf{n}(\mathbf{x}) \, dS_x = -1 \,,$$

where B is the ball of center $\mathbf{0}$ and radius 1, and $\mathbf{n}$ the unit outward normal on ∂B.

Dirac $\delta_{\mathbf{0}}$ distribution

[Indeed, in a more advanced mathematical language, the function $K(\mathbf{x},\mathbf{y})$ is the *fundamental solution* of the $-\Delta$ operator, namely, it satisfies $-\Delta_x K(\mathbf{x},\mathbf{y}) = \delta_{\mathbf{0}}(\mathbf{x}-\mathbf{y})$ in the distributional sense, $\delta_{\mathbf{0}}$ being the Dirac delta distribution centered at $\mathbf{0}$.

Roughly speaking, for each (suitable...) function f the Dirac delta distribution satisfies

$$\int_{\mathbb{R}^3} \delta_{\mathbf{0}}(\mathbf{x}-\mathbf{y}) f(\mathbf{y})\, d\mathbf{y} = f(\mathbf{x})\,.$$

We also know that the function

$$u(\mathbf{x}) = \int_{\mathbb{R}^3} K(\mathbf{x},\mathbf{y}) f(\mathbf{y})\, d\mathbf{y}$$

satisfies $-\Delta u = f$ in $\mathbb{R}^3$. In fact (formally...)

$$\begin{aligned} -\Delta u(\mathbf{x}) &= -\Delta_x [\textstyle\int_{\mathbb{R}^3} K(\mathbf{x},\mathbf{y}) f(\mathbf{y})\, d\mathbf{y}] = \textstyle\int_{\mathbb{R}^3} [-\Delta_x K(\mathbf{x},\mathbf{y})] f(\mathbf{y})\, d\mathbf{y} \\ &= \textstyle\int_{\mathbb{R}^3} \delta_{\mathbf{0}}(\mathbf{x}-\mathbf{y}) f(\mathbf{y})\, d\mathbf{y} = f(\mathbf{x})\,.] \end{aligned}$$

Scalar and vector potentials

Let us come to the determination of a scalar potential for a curl-free vector field $\mathbf{H}$ (namely, a scalar function ψ such that $\mathbf{grad}\,\psi = \mathbf{H}$) and of a vector potential $\mathbf{A}$ for a divergence-free vector field $\mathbf{B}$ (namely, a vector field $\mathbf{A}$ such that $\mathbf{curl}\,\mathbf{A} = \mathbf{B}$).

Consider a vector field $\mathbf{H}$ and define in $\mathbb{R}^3$ the function

$$\psi(\mathbf{x}) = -\int_{\mathbb{R}^3} K(\mathbf{x},\mathbf{y})\mathrm{div}\,\mathbf{H}(\mathbf{y})\,d\mathbf{y}\,. \qquad (2)$$

Consider a vector field $\mathbf{B}$ and define in $\mathbb{R}^3$ the vector field

$$\mathbf{A}(\mathbf{x}) = \int_{\mathbb{R}^3} K(\mathbf{x},\mathbf{y})\mathbf{curl}\,\mathbf{B}(\mathbf{y})\,d\mathbf{y}\,. \qquad (3)$$

Theorems

Theorem (3)

Assume that $\mathbf{H}$ *decays sufficiently fast at infinity and satisfies* $\mathbf{curl}\,\mathbf{H} = \mathbf{0}$ *in* $\mathbb{R}^3$. *The function* ψ *satisfies* $\mathbf{grad}\,\psi = \mathbf{H}$ *in* $\mathbb{R}^3$.

Proof. It is easily shown that

$$D_{x_i} K(\mathbf{x}, \mathbf{y}) = -\frac{1}{4\pi} \frac{x_i - y_i}{|\mathbf{x} - \mathbf{y}|^3} = -D_{y_i} K(\mathbf{x}, \mathbf{y}) \,,$$

hence (formally, and using that $D_i H_j = D_j H_i$...)

$$\begin{aligned} D_i \psi(\mathbf{x}) &= -\int_{\mathbb{R}^3} D_{x_i} K(\mathbf{x}, \mathbf{y}) \mathrm{div}\, \mathbf{H}(\mathbf{y})\, d\mathbf{y} = \int_{\mathbb{R}^3} D_{y_i} K(\mathbf{x}, \mathbf{y}) \mathrm{div}\, \mathbf{H}(\mathbf{y})\, d\mathbf{y} \\ &= -\sum_j \int_{\mathbb{R}^3} D_{y_j} D_{y_i} K(\mathbf{x}, \mathbf{y}) H_j(\mathbf{y})\, d\mathbf{y} = \sum_j \int_{\mathbb{R}^3} D_{y_j} K(\mathbf{x}, \mathbf{y}) D_i H_j(\mathbf{y})\, d\mathbf{y} \\ &= \sum_j \int_{\mathbb{R}^3} D_{y_j} K(\mathbf{x}, \mathbf{y}) D_j H_i(\mathbf{y})\, d\mathbf{y} = -\int_{\mathbb{R}^3} \Delta_y K(\mathbf{x}, \mathbf{y}) H_i(\mathbf{y})\, d\mathbf{y} \\ &= \int_{\mathbb{R}^3} \delta_{\mathbf{0}}(\mathbf{x} - \mathbf{y}) H_i(\mathbf{y})\, d\mathbf{y} = H_i(\mathbf{x}) \,. \end{aligned}$$

$\square$

Theorems (cont'd)

Theorem (4)

Assume that $\mathbf{B}$ *decays sufficiently fast at infinity and satisfies* $\operatorname{div}\mathbf{B} = \mathbf{0}$ *in* $\mathbb{R}^3$*. The vector field* $\mathbf{A}$ *satisfies* $\mathbf{curl}\,\mathbf{A} = \mathbf{B}$ *(and* $\operatorname{div}\mathbf{A} = 0$*) in* $\mathbb{R}^3$ *.*

Proof. We have

$$\begin{aligned} D_1 A_2(\mathbf{x}) &= \int_{\mathbb{R}^3} D_{x_1} K(\mathbf{x},\mathbf{y})(D_3 B_1 - D_1 B_3)(\mathbf{y})\, d\mathbf{y} \\ &= -\int_{\mathbb{R}^3} D_{y_1} K(\mathbf{x},\mathbf{y})(D_3 B_1 - D_1 B_3)(\mathbf{y})\, d\mathbf{y} \\ &= \int_{\mathbb{R}^3} [-D_{y_1} D_{y_1} K(\mathbf{x},\mathbf{y}) B_3(\mathbf{y}) + D_{y_3} D_{y_1} K(\mathbf{x},\mathbf{y}) B_1(\mathbf{y})]\, d\mathbf{y} \\ &= \int_{\mathbb{R}^3} [-D_{y_1} D_{y_1} K(\mathbf{x},\mathbf{y}) B_3(\mathbf{y}) - D_{y_3} K(\mathbf{x},\mathbf{y}) D_1 B_1(\mathbf{y})]\, d\mathbf{y}\,. \end{aligned}$$

Similarly,

$$D_2 A_1(\mathbf{x}) = \int_{\mathbb{R}^3} [D_{y_2} D_{y_2} K(\mathbf{x},\mathbf{y}) B_3(\mathbf{y}) + D_{y_3} K(\mathbf{x},\mathbf{y}) D_2 B_2(\mathbf{y})]\, d\mathbf{y}\,.$$

Theorems (cont'd)

Since $D_1 B_1 + D_2 B_2 = -D_3 B_3$, we find

$$\begin{aligned} & - \int_{\mathbb{R}^3} D_{y_3} K(\mathbf{x}, \mathbf{y})[D_1 B_1(\mathbf{y}) + D_2 B_2(\mathbf{y})]\, d\mathbf{y} \\ & \quad = \int_{\mathbb{R}^3} D_{y_3} K(\mathbf{x}, \mathbf{y}) D_3 B_3(\mathbf{y})\, d\mathbf{y} = - \int_{\mathbb{R}^3} D_{y_3} D_{y_3} K(\mathbf{x}, \mathbf{y}) B_3(\mathbf{y})\, d\mathbf{y}\,, \end{aligned}$$

hence

$$\begin{aligned} D_1 A_2(\mathbf{x}) - D_2 A_1(\mathbf{x}) &= - \int_{\mathbb{R}^3} \Delta_y K(\mathbf{x}, \mathbf{y}) B_3(\mathbf{y})\, d\mathbf{y} \\ &= \int_{\mathbb{R}^3} \delta_{\mathbf{0}}(\mathbf{x} - \mathbf{y}) B_3(\mathbf{y})\, d\mathbf{y} = B_3(\mathbf{x})\,. \end{aligned}$$

Repeating the same computations for the other components, the first part of the thesis follows.

On the other hand

$$\begin{aligned} D_1 A_1(\mathbf{x}) &= - \int_{\mathbb{R}^3} D_{y_1} K(\mathbf{x}, \mathbf{y})(D_2 B_3 - D_3 B_2)(\mathbf{y})\, d\mathbf{y} \\ &= - \int_{\mathbb{R}^3} [D_{y_1} K(\mathbf{x}, \mathbf{y}) D_2 B_3(\mathbf{y}) - D_{y_3} K(\mathbf{x}, \mathbf{y}) D_1 B_2(\mathbf{y})]\, d\mathbf{y}\,, \end{aligned}$$

and, proceding similary for $D_2 A_2$ and $D_3 A_3$, the second part of the thesis is easily verified. $\square$

Leading idea

What has been the idea?

- If ψ satisfies $\mathbf{grad}\,\psi = \mathbf{H}$, then

$$-\mathrm{div}\,\mathbf{H} = -\mathrm{div}\,\mathbf{grad}\,\psi = -\Delta\psi\,,$$

  hence we can use the (scalar) integral representation formula in terms of the fundamental solution K;

- if $\mathbf{A}$ satisfies $\mathbf{curl}\,\mathbf{A} = \mathbf{B}$ (and $\mathrm{div}\,\mathbf{A} = 0$), then

$$\mathbf{curl}\,\mathbf{B} = \mathbf{curl}\,\mathbf{curl}\,\mathbf{A} = \mathbf{curl}\,\mathbf{curl}\,\mathbf{A} - \mathbf{grad}\,\mathrm{div}\,\mathbf{A} = -\Delta\mathbf{A}\,,$$

  hence we can use the (vector) integral representation formula in terms of the fundamental solution K.

Biot–Savart formulas

An alternative (and essentially equivalent) point of view is the one leading to the Biot–Savart formulas:

- Scalar potential: look for $\psi = \operatorname{div} \mathbf{grad}\, \varphi$.

If ψ satisfies $\mathbf{grad}\, \psi = \mathbf{H}$, then

$$\mathbf{H} = \mathbf{grad} \operatorname{div} \mathbf{grad}\, \varphi = \Delta\, \mathbf{grad}\, \varphi \,,$$

hence

$$\mathbf{grad}\, \varphi = - \int_{\mathbb{R}^3} K(\mathbf{x}, \mathbf{y}) \mathbf{H}(\mathbf{y})\, d\mathbf{y}$$

and

$$\psi(\mathbf{x}) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|^3} \cdot \mathbf{H}(\mathbf{y})\, d\mathbf{y} \,. \qquad (4)$$

Biot–Savart formulas (cont'd)

- Vector potential: look for $\mathbf{A} = \mathbf{curl}\,\mathbf{Q}$ (with $\operatorname{div}\mathbf{Q} = 0$).

If $\mathbf{A}$ satisfies $\mathbf{curl}\,\mathbf{A} = \mathbf{B}$, then

$$\mathbf{B} = \mathbf{curl}\,\mathbf{curl}\,\mathbf{Q} = -\Delta\mathbf{Q}\,,$$

hence

$$\mathbf{Q}(\mathbf{x}) = \int_{\mathbb{R}^3} K(\mathbf{x},\mathbf{y})\mathbf{B}(\mathbf{y})\,d\mathbf{y}$$

and

$$\mathbf{A}(\mathbf{x}) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{x}-\mathbf{y}|^3} \times \mathbf{B}(\mathbf{y})\,d\mathbf{y}\,. \tag{5}$$

In a bounded domain...

The problem is more complicated in a bounded domain Ω. However, some well-known results are usually presented in any calculus course.

Theorem (5)

Assume that $\mathbf{H}$ *satisfies* $\mathbf{curl}\,\mathbf{H} = \mathbf{0}$ *in* Ω *and that any closed curve in* Ω *is the boundary of a suitable surface* $S \subset \Omega$*. Then there exists a scalar function* ψ *satisfying* $\mathbf{grad}\,\psi = \mathbf{H}$ *in* Ω*.*

Proof. Since the flux of $\mathbf{curl}\,\mathbf{H}$ is vanishing on each surface S, from the Stokes theorem the line integral of $\mathbf{H}$ on each closed curve in Ω is vanishing. $\square$

In a bounded domain... (cont'd)

- A domain Ω for which any closed curve $c \subset \Omega$ is the boundary of a surface $S \subset \Omega$ is called homologically trivial.

Many of you could have in mind the following definition: a domain Ω is said to be simply-connected if any closed curve $c \subset \Omega$ can be retracted in Ω to a point $\mathbf{p} \in \Omega$. [Using a different language, it is called homotopically trivial.]
The preceding theorem has clarified this fact: for establishing if a curl-free vector field is a gradient, the relevant geometrical property is related to homology, not to homotopy.

Question (left apart... but we will come back to it):

- A simply-connected domain is clearly homologically trivial. Do we have examples of homologically trivial domains that are not simply-connected?

In a bounded domain... (cont'd)

Concerning the vector potential, we have the (less known...) result:

Theorem (6)

Assume that $\mathbf{B}$ *satisfies* $\operatorname{div} \mathbf{B} = 0$ *in* Ω *and that any closed surface in* Ω *is the boundary of a suitable subdomain* $D \subset \Omega$*. Then there exists a scalar function* $\mathbf{A}$ *satisfying* $\mathbf{curl}\, \mathbf{A} = \mathbf{B}$ *in* Ω*.*

Proof. Since the integral of $\operatorname{div} \mathbf{B}$ is vanishing in each subdomain D, from the divergence theorem the flux of $\mathbf{B}$ on each closed surface in Ω is vanishing. This is enough to guarantee the existence of a vector potential $\mathbf{A}$ (more details later on...). $\square$

In a bounded domain... (cont'd)

Other simple results are the following:

Theorem (7)

Assume that $\mathbf{H}$ *satisfies* $\mathbf{curl}\,\mathbf{H} = \mathbf{0}$ *in* Ω *and* $\mathbf{H} \times \mathbf{n} = \mathbf{0}$ *on* $\partial\Omega$*. Then there exists a scalar function* ψ *satisfying* $\mathbf{grad}\,\psi = \mathbf{H}$ *in* Ω*.*

Proof. Extend $\mathbf{H}$ by $\mathbf{0}$ outside Ω; since $\mathbf{H} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$, the extension is still curl-free, therefore it is the gradient of a scalar potential in $\mathbb{R}^3$. $\square$

In a bounded domain... (cont'd)

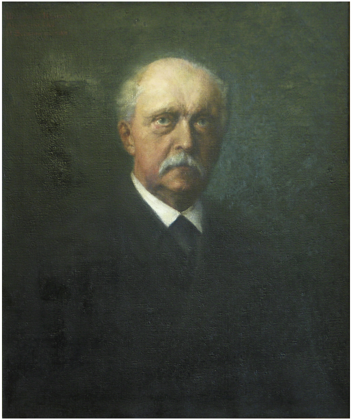
Theorem (8)

Assume that $\mathbf{B}$ *satisfies* $\operatorname{div} \mathbf{B} = 0$ *in* Ω *and* $\mathbf{B} \cdot \mathbf{n} = 0$ *on* $\partial\Omega$*. Then there exists a vector field* $\mathbf{A}$ *satisfying* $\mathbf{curl}\, \mathbf{A} = \mathbf{B}$ *in* Ω*.*

Proof. Extend $\mathbf{B}$ by $\mathbf{0}$ outside Ω; since $\mathbf{B} \cdot \mathbf{n} = 0$ on $\partial\Omega$, the extension is still divergence-free, therefore it is the curl of a vector potential in $\mathbb{R}^3$. $\square$

- But: can we find necessary and sufficient conditions?

Helmholtz



Hermann von Helmholtz (1821–1894),
in a painting by Hans Schadow (1891).

Harmonic fields

A way for finding the answer is the resort to the so-called Helmholtz decomposition: any vector field can be written as the sum of a gradient and a curl.

For stating the precise results we need the definitions of the spaces of harmonic fields:

$$\mathcal{H}(m;\Omega) = \{\mathbf{w} \in (L^2(\Omega))^3 \,|\, \mathbf{curl}\,\mathbf{w} = \mathbf{0}, \operatorname{div}\mathbf{w} = 0, \mathbf{w}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\}$$

$$\mathcal{H}(e;\Omega) = \{\mathbf{w} \in (L^2(\Omega))^3 \,|\, \mathbf{curl}\,\mathbf{w} = \mathbf{0}, \operatorname{div}\mathbf{w} = 0, \mathbf{w}\times\mathbf{n} = \mathbf{0} \text{ on } \partial\Omega\}\,.$$

Note that an element of $\mathcal{H}(m;\Omega)$ can be written as the curl of a vector potential, an element of $\mathcal{H}(e;\Omega)$ can be written as the gradient of a scalar potential.

Harmonic fields (cont'd)

A preliminary remark about the structure of these spaces: they are "reading" some topological properties of the domain Ω. In fact:

Theorem (9)

Let Ω be topologically equivalent to a ball. Then $\mathcal{H}(m;\Omega) = \{\mathbf{0}\}$ and $\mathcal{H}(e;\Omega) = \{\mathbf{0}\}$.

Proof. Let $\mathbf{w} \in \mathcal{H}(m;\Omega)$ or $\mathbf{w} \in \mathcal{H}(e;\Omega)$. Since a ball is homologically trivial, we have $\mathbf{w} = \mathbf{grad}\,\psi$, where ψ satisfies $\Delta\psi = 0$ in Ω. When $\mathbf{w} \in \mathcal{H}(m;\Omega)$ we also have $\mathbf{grad}\,\psi \cdot \mathbf{n} = 0$ on $\partial\Omega$. Well-known results on the Neumann problem furnish $\mathbf{grad}\,\psi = \mathbf{0}$ in Ω. When $\mathbf{w} \in \mathcal{H}(e;\Omega)$, we also have $\mathbf{grad}\,\psi \times \mathbf{n} = 0$ on $\partial\Omega$. Since $\partial\Omega$ is connected, we obtain $\psi = \text{const}$ on $\partial\Omega$. Well-known results on the Dirichlet boundary value problem for the Laplace operator give $\psi = \text{const}$ and $\mathbf{grad}\,\psi = \mathbf{0}$ in Ω. $\square$

Harmonic fields (cont'd)

A doubt:

- Have we examples of non-trivial harmonic fields?

The answer is "yes".

- Take the magnetic field generated in the vacuum by a current of constant intensity I^0 passing along the x_3-axis: as it is well-known, for $x_1^2 + x_2^2 > 0$ it is given by

$$\mathbf{H}(x_1, x_2, x_3) = \frac{I^0}{2\pi} \left(-\frac{x_2}{x_1^2 + x_2^2}, \frac{x_1}{x_1^2 + x_2^2}, 0 \right) .$$

As Maxwell equations require, one sees that $\mathbf{curl}\, \mathbf{H} = \mathbf{0}$ and $\mathrm{div}\, \mathbf{H} = 0$. Consider the torus T obtained by rotating a disk (contained in the plane $\{x_2 = 0\}$) around the x_3-axis: it is easily checked that $\mathbf{H} \cdot \mathbf{n} = 0$ on ∂T. Hence we have found a non-trivial harmonic field $\mathbf{H} \in \mathcal{H}(m; T)$.

Harmonic fields (cont'd)

- Consider the electric field generated in the vacuum by a pointwise charge ρ_0 placed at the origin. For $\mathbf{x} \neq \mathbf{0}$ it is given by

$$\mathbf{E}(x_1, x_2, x_3) = \frac{\rho_0}{4\pi\varepsilon_0} \frac{\mathbf{x}}{|\mathbf{x}|^3} ,$$

where ε_0 is the electric permittivity of the vacuum. It satisfies $\mathrm{div}\, \mathbf{E} = 0$ and $\mathbf{curl}\, \mathbf{E} = \mathbf{0}$, and moreover $\mathbf{E} \times \mathbf{n} = \mathbf{0}$ on the boundary of $C = B_{R_2} \setminus \overline{B_{R_1}}$ (here $0 < R_1 < R_2$, and B_R is the ball of centre $\mathbf{0}$ and radius R). We have thus found a non-trivial harmonic field $\mathbf{E} \in \mathcal{H}(e; C)$.

Helmholtz decomposition

Theorem (10)

Any vector function $\mathbf{v} \in (L^2(\Omega))^3$ *can be decomposed into the following sum*

$$\mathbf{v} = \mathbf{curl}\,\mathbf{Q} + \mathbf{grad}\,\psi + \boldsymbol{\rho}\,, \tag{6}$$

where $\boldsymbol{\rho} \in \mathcal{H}(m;\Omega)$ *(hence it can be written as the curl of a vector potential), and each term of the decomposition is orthogonal to the others.*

Moreover, if $\mathbf{curl}\,\mathbf{v} = \mathbf{0}$ *in* Ω *it follows* $\mathbf{Q} = \mathbf{0}$*, if* $\operatorname{div}\mathbf{v} = 0$ *in* Ω *and* $\mathbf{v} \cdot \mathbf{n} = 0$ *on* $\partial\Omega$ *one has* $\mathbf{grad}\,\psi = \mathbf{0}$*, and if* $\mathbf{v} \perp \mathcal{H}(m;\Omega)$ *one finds* $\boldsymbol{\rho} = \mathbf{0}$.

Helmholtz decomposition (cont'd)

Proof. Take: the vector field $\mathbf{Q}$ solution to

$$\begin{cases} \mathbf{curl}\,\mathbf{curl}\,\mathbf{Q} = \mathbf{curl}\,\mathbf{v} & \text{in } \Omega \\ \operatorname{div}\mathbf{Q} = 0 & \text{in } \Omega \\ \mathbf{Q} \times \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega \\ \mathbf{Q} \perp \mathcal{H}(e;\Omega)\,; \end{cases}$$

the scalar function ψ solution to

$$\begin{cases} \Delta\psi = \operatorname{div}\mathbf{v} & \text{in } \Omega \\ \mathbf{grad}\,\psi \cdot \mathbf{n} = \mathbf{v} \cdot \mathbf{n} & \text{on } \partial\Omega\,; \end{cases}$$

the vector field $\boldsymbol{\rho}$, orthogonal projection of $\mathbf{v}$ on $\mathcal{H}(m;\Omega)$.

Helmholtz decomposition (cont'd)

The orthogonality is easily checked:

$$\int_\Omega \mathbf{curl}\,\mathbf{Q} \cdot \mathbf{grad}\,\psi = \int_\Omega \mathbf{Q} \cdot \mathbf{curl}\,\mathbf{grad}\,\psi + \int_{\partial\Omega} \mathbf{n} \times \mathbf{Q} \cdot \mathbf{grad}\,\psi = 0$$

$$\int_\Omega \mathbf{curl}\,\mathbf{Q} \cdot \boldsymbol{\rho} = \int_\Omega \mathbf{Q} \cdot \mathbf{curl}\,\boldsymbol{\rho} + \int_{\partial\Omega} \mathbf{n} \times \mathbf{Q} \cdot \boldsymbol{\rho} = 0\,,$$

$$\int_\Omega \mathbf{grad}\,\psi \cdot \boldsymbol{\rho} = -\int_\Omega \psi \,\mathrm{div}\,\boldsymbol{\rho} + \int_{\partial\Omega} \psi\,\mathbf{n} \cdot \boldsymbol{\rho} = 0\,.$$

Moreover we have

$$\begin{cases} \mathbf{curl}\,(\mathbf{v} - \mathbf{curl}\,\mathbf{Q} - \mathbf{grad}\,\psi - \boldsymbol{\rho}) = \mathbf{0} & \text{in } \Omega \\ \mathrm{div}\,(\mathbf{v} - \mathbf{curl}\,\mathbf{Q} - \mathbf{grad}\,\psi - \boldsymbol{\rho}) = 0 & \text{in } \Omega \\ (\mathbf{v} - \mathbf{curl}\,\mathbf{Q} - \mathbf{grad}\,\psi - \boldsymbol{\rho}) \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \end{cases}$$

(recall that $\mathbf{Q} \times \mathbf{n} = \mathbf{0}$ gives $\mathbf{curl}\,\mathbf{Q} \cdot \mathbf{n} = 0$ on $\partial\Omega$).

Helmholtz decomposition (cont'd)

Hence we have found

$$(\mathbf{v} - \mathbf{curl}\,\mathbf{Q} - \mathbf{grad}\,\psi - \boldsymbol{\rho}) \in \mathcal{H}(m;\Omega)\,,$$

but we also have

$$(\mathbf{v} - \mathbf{curl}\,\mathbf{Q} - \mathbf{grad}\,\psi - \boldsymbol{\rho}) \perp \mathcal{H}(m;\Omega)\,,$$

therefore $\mathbf{v} = \mathbf{curl}\,\mathbf{Q} + \mathbf{grad}\,\psi + \boldsymbol{\rho}$. $\square$

The characterization theorem for scalar potentials

We can conclude with

Theorem (11)

The following statements are equivalent:

- *there exists a scalar function* φ *such that* $\mathbf{v} = \mathbf{grad}\,\varphi$ *in* Ω
- $\mathbf{curl}\,\mathbf{v} = \mathbf{0}$ *in* Ω *and* $\mathbf{v} \perp \mathcal{H}(m;\Omega)$.

Proof. We have only to check that $\mathbf{grad}\,\varphi \perp \mathcal{H}(m;\Omega)$. Taking $\boldsymbol{\rho} \in \mathcal{H}(m;\Omega)$ we have

$$\int_\Omega \mathbf{grad}\,\varphi \cdot \boldsymbol{\rho} = -\int_\Omega \varphi \,\mathrm{div}\,\boldsymbol{\rho} + \int_{\partial\Omega} \varphi\,\mathbf{n} \cdot \boldsymbol{\rho} = 0\,. \qquad \square$$

Helmholtz decomposition

Theorem (12)

Any vector function $\mathbf{v} \in (L^2(\Omega))^3$ *can be decomposed into the following sum*

$$\mathbf{v} = \mathbf{curl}\,\mathbf{A} + \mathbf{grad}\,\chi + \boldsymbol{\eta}\,, \tag{7}$$

where $\boldsymbol{\eta} \in \mathcal{H}(e;\Omega)$ *(hence it can be written as the gradient of a scalar potential), and each term of the decomposition is orthogonal to the others.*
Moreover, if $\mathbf{curl}\,\mathbf{v} = \mathbf{0}$ *in* Ω *and* $\mathbf{v} \times \mathbf{n} = \mathbf{0}$ *on* $\partial\Omega$ *it follows* $\mathbf{A} = \mathbf{0}$*, if* $\mathrm{div}\,\mathbf{v} = 0$ *in* Ω *one has* $\mathbf{grad}\,\chi = \mathbf{0}$*, and if* $\mathbf{v} \perp \mathcal{H}(e;\Omega)$ *one finds* $\boldsymbol{\eta} = \mathbf{0}$.

Helmholtz decomposition (cont'd)

Proof. Take: the vector field $\mathbf{A}$ solution to

$$\begin{cases} \mathbf{curl}\,\mathbf{curl}\,\mathbf{A} = \mathbf{curl}\,\mathbf{v} & \text{in } \Omega \\ \operatorname{div}\mathbf{A} = 0 & \text{in } \Omega \\ \mathbf{A}\cdot\mathbf{n} = 0 & \text{on } \partial\Omega \\ \mathbf{curl}\,\mathbf{A}\times\mathbf{n} = \mathbf{v}\times\mathbf{n} & \text{on } \partial\Omega \\ \mathbf{Q}\perp\mathcal{H}(m;\Omega)\,; \end{cases}$$

the scalar function χ solution to

$$\begin{cases} \Delta\chi = \operatorname{div}\mathbf{v} & \text{in } \Omega \\ \chi = 0 & \text{on } \partial\Omega\,; \end{cases}$$

the vector field $\boldsymbol{\eta}$, orthogonal projection of $\mathbf{v}$ on $\mathcal{H}(e;\Omega)$.

Helmholtz decomposition (cont'd)

The orthogonality is easily checked:

$$\int_\Omega \mathbf{curl}\,\mathbf{A}\cdot\mathbf{grad}\,\chi = \int_\Omega \mathbf{A}\cdot\mathbf{curl}\,\mathbf{grad}\,\chi - \int_{\partial\Omega}\mathbf{n}\times\mathbf{grad}\,\chi\cdot\mathbf{A} = 0$$

$$\int_\Omega \mathbf{curl}\,\mathbf{A}\cdot\boldsymbol{\eta} = \int_\Omega \mathbf{A}\cdot\mathbf{curl}\,\boldsymbol{\eta} - \int_{\partial\Omega}\mathbf{n}\times\boldsymbol{\eta}\cdot\mathbf{A} = 0\,,$$

$$\int_\Omega \mathbf{grad}\,\chi\cdot\boldsymbol{\eta} = -\int_\Omega \chi\,\mathrm{div}\,\boldsymbol{\eta} + \int_{\partial\Omega}\chi\,\mathbf{n}\cdot\boldsymbol{\eta} = 0\,.$$

Moreover we have

$$\begin{cases} \mathbf{curl}\,(\mathbf{v}-\mathbf{curl}\,\mathbf{A}-\mathbf{grad}\,\chi-\boldsymbol{\eta}) = \mathbf{0} & \text{in } \Omega \\ \mathrm{div}\,(\mathbf{v}-\mathbf{curl}\,\mathbf{A}-\mathbf{grad}\,\chi-\boldsymbol{\eta}) = 0 & \text{in } \Omega \\ (\mathbf{v}-\mathbf{curl}\,\mathbf{A}-\mathbf{grad}\,\chi-\boldsymbol{\eta})\times\mathbf{n} = \mathbf{0} & \text{on } \partial\Omega\,. \end{cases}$$

Helmholtz decomposition (cont'd)

Hence we have found

$$(\mathbf{v} - \mathbf{curl}\,\mathbf{A} - \mathbf{grad}\,\chi - \boldsymbol{\eta}) \in \mathcal{H}(e;\Omega)\,,$$

but we also have

$$(\mathbf{v} - \mathbf{curl}\,\mathbf{A} - \mathbf{grad}\,\chi - \boldsymbol{\eta}) \perp \mathcal{H}(e;\Omega)\,,$$

therefore $\mathbf{v} = \mathbf{curl}\,\mathbf{A} + \mathbf{grad}\,\chi + \boldsymbol{\eta}$. $\square$

The characterization theorem for vector potentials

We can conclude with

Theorem (13)

The following statements are equivalent:

- *there exists a vector field* $\mathbf{w}$ *such that* $\mathbf{v} = \mathbf{curl}\,\mathbf{w}$ *in* Ω
- $\operatorname{div} \mathbf{v} = \mathbf{0}$ *in* Ω *and* $\mathbf{v} \perp \mathcal{H}(e;\Omega)$.

Proof. We have only to check that $\mathbf{curl}\,\mathbf{w} \perp \mathcal{H}(e;\Omega)$. Taking $\boldsymbol{\eta} \in \mathcal{H}(e;\Omega)$ we have

$$\int_\Omega \mathbf{curl}\,\mathbf{w} \cdot \boldsymbol{\eta} = \int_\Omega \mathbf{w} \cdot \mathbf{curl}\,\boldsymbol{\eta} - \int_{\partial\Omega} \mathbf{n} \times \boldsymbol{\eta} \cdot \mathbf{w} = 0\,. \qquad \square$$

Back to the harmonic fields

It is now useful trying to characterize the spaces of harmonic fields $\mathcal{H}(m;\Omega)$ and $\mathcal{H}(e;\Omega)$.
Let us start from the latter. Denote by $(\partial\Omega)_j$, $j = 0, 1, \dots, p$, the connected components of $\partial\Omega$ ($(\partial\Omega)_0$ being the external one).

Theorem (14)

The space $\mathcal{H}(e;\Omega)$ is finite dimensional. Its dimension is p (one less than the number of the connected components of $\partial\Omega$). A basis is given by $\mathbf{grad}\, w_j$, $j = 1, \dots, p$, where w_j is the solution of

$$\begin{cases} \Delta w_j = 0 & \textit{in } \Omega \\ w_j = 0 & \textit{on } \partial\Omega \setminus (\partial\Omega)_j \\ w_j = 1 & \textit{on } (\partial\Omega)_j \,. \end{cases}$$

Back to the harmonic fields (cont'd)

Proof. Clearly, $\mathbf{grad}\, w_j \in \mathcal{H}(e;\Omega)$. It is enough to show that they give a basis. From $\sum_{j=1}^{p} \alpha_j \mathbf{grad}\, w_j = \mathbf{0}$ we find $\sum_{j=1}^{p} \alpha_j w_j = \text{const}$. Since all the w_j are 0 on $(\partial\Omega)_0$, it follows $\sum_{j=1}^{p} \alpha_j w_j = 0$ in $\overline{\Omega}$. But $\sum_{j=1}^{p} \alpha_j w_j = \alpha_k$ on $(\partial\Omega)_k$, hence $\alpha_k = 0$ for each $k = 1, \ldots, p$, and $\mathbf{grad}\, w_j$ are linearly independent.
Take now $\boldsymbol{\eta} \in \mathcal{H}(e;\Omega)$. We already know that there exists q such that $\mathbf{grad}\, q = \boldsymbol{\eta}$. Due to the boundary condition $\mathbf{grad}\, q \times \mathbf{n} = \mathbf{0}$ we know that q is constant on each connected component $(\partial\Omega)_j$, $j = 0, 1, \ldots, p$ (and on $(\partial\Omega)_0$ we can suppose that it is vanishing).
Define $\beta_j = q_{|(\partial\Omega)_j}$, $j = 1, \ldots, p$, and consider $z = q - \sum_{j=1}^{p} \beta_j w_j$.
We have $\Delta z = 0$ in Ω and $z = 0$ on $(\partial\Omega)_k$, $k = 0, 1, \ldots, p$.
Therefore we obtain $z = 0$, hence $q = \sum_{j=1}^{p} \beta_j w_j$ in Ω, and $\mathbf{grad}\, w_j$ are generators. $\square$

Back to the characterization theorem for vector potentials

This characterization of $\mathcal{H}(e;\Omega)$ permits to rephrase the main theorem on vector potentials.

Theorem (15)

The following statements are equivalent:

- *there exists a vector field* $\mathbf{w}$ *such that* $\mathbf{v} = \mathbf{curl}\,\mathbf{w}$ *in* Ω
- $\operatorname{div}\mathbf{v} = \mathbf{0}$ *in* Ω *and* $\int_{(\partial\Omega)_j} \mathbf{v}\cdot\mathbf{n} = 0$ *for each* $j = 1, \ldots, p$.

Proof. We have only to check the meaning of the condition $\mathbf{v} \perp \mathcal{H}(e;\Omega)$. We find, by integration by parts

$$
\begin{aligned}
0 &= \int_\Omega \mathbf{v}\cdot\mathbf{grad}\,w_j = -\int_\Omega \operatorname{div}\mathbf{v}\,w_j + \int_{\partial\Omega}\mathbf{v}\cdot\mathbf{n}\,w_j \\
&= \int_{(\partial\Omega)_j}\mathbf{v}\cdot\mathbf{n}\,.
\end{aligned}
$$

$\square$

Another theorem for vector potentials (rephrased)

We are now in a condition to prove a result we stated before.

Theorem (16)

Assume that a divergence-free vector field $\mathbf{B}$ has vanishing flux on each closed surface in Ω. Then there exists of a vector field $\mathbf{A}$ such that $\mathbf{curl}\,\mathbf{A} = \mathbf{B}$.

Proof. Nothing has to be proved if the boundary of Ω is connected. If it is not connected, slightly "inflating" a connected component $(\partial\Omega)_j$ we find a closed surface S_j in Ω. From the divergence theorem, the flux of $\mathbf{B}$ on S_j is equal to the flux of $\mathbf{B}$ on $(\partial\Omega)_j$, hence the latter is vanishing. $\square$

Homology and de Rham cohomology

The characterization of $\mathcal{H}(m;\Omega)$ is less straightforward, and needs a (very) brief dive in the theory of algebraic topology. First of all, two definitions:

- the first homology group is given, roughly speaking, by the quotient between the cycles and the bounding cycles in $\overline{\Omega}$.
- the first de Rham cohomology group is given by the quotient between the curl-free vector fields and the gradients defined in Ω.

de Rham

Georges de Rham (1903–1990).

[Thanks to Oscar Burlet, Souvenirs de Georges de Rham, 2004, for this picture and the following ones.]

de Rham (cont'd)

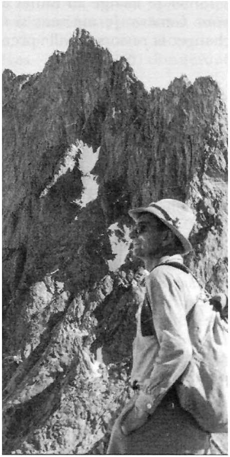
A parenthesis, on a different topic:

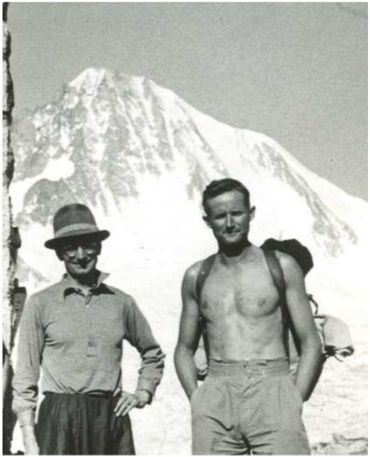
Claire-Eliane
Engel
Storia
dell'alpinismo

Fra la folla degli alpinisti svizzeri figurano arrampicatori di gran classe: André Roch, Georges de Rham, E.-R. Blanchet, morto qualche anno addietro, René Dittert e altri ancora.

La tecnica dei chiodi ha reso accessibile la cresta più difficile, quella di Furggen, con l'immenso strapiombo. Nel 1941 Alfred Perina, Luigi Carrel e Jacques Chiara risalgono completamente il Grand Ressaut in scalata artificiale. Evidentemente è il modo di risolvere quella scalata asperrima. Da allora la via è stata rifatta dal professor G. de Rham e Alfred Tissière di Losanna, poi da Lionel Terray e Louis Lachenal nel luglio 1947.

de Rham (cont'd)





Furggen ridge: second climbing

Homology and de Rham cohomology (cont'd)

Theorem (de Rham)

The first homology group and the first de Rham cohomology group are finitely generated, and have the same rank, that is given by g, the first Betti number of $\overline{\Omega}$.

In other words, the first homology group is generated by g independent (classes of equivalence of) non-bounding cycles in $\overline{\Omega}$, and the first de Rham cohomology group is generated by g independent (classes of equivalence of) loop fields in Ω (namely, curl-free vector fields that cannot be represented as gradients in Ω).

Let us denote by $\{\sigma_k\}_{k=1,\ldots,g}$, a set of cycles such that their classes of equivalence $\{[\sigma_k]\}_{k=1,\ldots,g}$ are generators of the first homology group.

Homology and de Rham cohomology (cont'd)

Theorem (17)

A set of generators of the first de Rham cohomology group is given by the classes of equivalence of g loop fields $\widehat{\boldsymbol{\rho}}_k$ such that

$$\oint_{\sigma_k} \widehat{\boldsymbol{\rho}}_k \cdot d\mathbf{s} = 1 \quad , \quad \oint_{\sigma_l} \widehat{\boldsymbol{\rho}}_k \cdot d\mathbf{s} = 0 \quad \textit{for } l \neq k \, .$$

Proof. It is enough to show that $[\widehat{\boldsymbol{\rho}}_k]$ are linearly independent. If $\sum_k \alpha_k [\widehat{\boldsymbol{\rho}}_k] = \mathbf{0}$ (namely, if $\sum_k \alpha_k \widehat{\boldsymbol{\rho}}_k = \mathbf{grad}\, \chi$), integrating on σ_l we have

$$0 = \oint_{\sigma_l} \sum_k \alpha_k \widehat{\boldsymbol{\rho}}_k \cdot d\mathbf{s} = \alpha_l \, .$$

□

Back to the harmonic fields

Denote by ω_k the solution of the Neumann problem

$$\begin{cases} \Delta\omega_k = \operatorname{div}\widehat{\boldsymbol{\rho}}_k & \text{in } \Omega \\ \mathbf{grad}\,\omega_k \cdot \mathbf{n} = \widehat{\boldsymbol{\rho}}_k \cdot \mathbf{n} & \text{on } \partial\Omega\,. \end{cases}$$

We have

Theorem (18)

The space $\mathcal{H}(m;\Omega)$ is finite dimensional. Its dimension is g, the first Betti number of $\overline{\Omega}$. A basis is given by $\boldsymbol{\rho}_k = \widehat{\boldsymbol{\rho}}_k - \mathbf{grad}\,\omega_k$, $k = 1,\dots,g$.

Back to the harmonic fields (cont'd)

Proof. The $\boldsymbol{\rho}_k$ are linearly independent, as, from $\sum_k \alpha_k \boldsymbol{\rho}_k = \mathbf{0}$, integrating on σ_l we find

$$\begin{aligned} 0 &= \oint_{\sigma_l} \sum_k \alpha_k \boldsymbol{\rho}_k \cdot d\mathbf{s} = \oint_{\sigma_l} \sum_k \alpha_k [\widehat{\boldsymbol{\rho}}_k - \mathbf{grad}\, \omega_k] \cdot d\mathbf{s} \\ &= \oint_{\sigma_l} \sum_k \alpha_k \widehat{\boldsymbol{\rho}}_k \cdot d\mathbf{s} = \alpha_l \,. \end{aligned}$$

Let $\boldsymbol{\rho} \in \mathcal{H}(m; \Omega)$. Its class of equivalence $[\boldsymbol{\rho}]$ is an element of the first de Rham cohomology group, hence we can write

$$[\boldsymbol{\rho}] = \sum_k \beta_k [\widehat{\boldsymbol{\rho}}_k] \quad , \quad \boldsymbol{\rho} = \sum_k \beta_k \widehat{\boldsymbol{\rho}}_k + \mathbf{grad}\, \chi \,,$$

and clearly χ satisfies

$$\begin{cases} \Delta\chi = -\sum_k \beta_k \mathrm{div}\, \widehat{\boldsymbol{\rho}}_k = -\sum_k \beta_k \Delta\omega_k & \text{in } \Omega \\ \mathbf{grad}\, \chi \cdot \mathbf{n} = -\sum_k \beta_k \widehat{\boldsymbol{\rho}}_k \cdot \mathbf{n} = -\sum_k \beta_k \mathbf{grad}\, \omega_k \cdot \mathbf{n} & \text{on } \partial\Omega \,. \end{cases}$$

Hence $\mathbf{grad}\, \chi = -\sum_k \beta_k \mathbf{grad}\, \omega_k$ and $\boldsymbol{\rho} = \sum_k \beta_k \boldsymbol{\rho}_k$. □

Back to the characterization theorem for scalar potentials

This characterization of $\mathcal{H}(m;\Omega)$ permits to rephrase the main theorem on scalar potentials.

Theorem (19)

The following statements are equivalent:

- *there exists a scalar function φ such that $\mathbf{v} = \mathbf{grad}\,\varphi$ in Ω*
- $\mathbf{curl}\,\mathbf{v} = \mathbf{0}$ *in Ω and $\oint_{\sigma_k} \mathbf{v} \cdot d\mathbf{s} = 0$ for each $k = 1, \ldots, g$.*

Proof. It is enough to show that a curl-free vector field $\mathbf{v}$ with $\oint_{\sigma_k} \mathbf{v} \cdot d\mathbf{s} = 0$ for each $k = 1, \ldots, g$ can be written as a gradient. First of all, since it is curl-free, from (6) we know that $\mathbf{v} = \mathbf{grad}\,\psi + \boldsymbol{\rho}$, and $\boldsymbol{\rho} = \sum_k \beta_k \boldsymbol{\rho}_k$. Integrating on σ_l it follows $0 = \oint_{\sigma_l} \mathbf{v} \cdot d\mathbf{s} = \oint_{\sigma_l} \sum_k \beta_k \boldsymbol{\rho}_k \cdot d\mathbf{s} = \beta_l$, hence $\mathbf{v} = \mathbf{grad}\,\psi$. $\square$

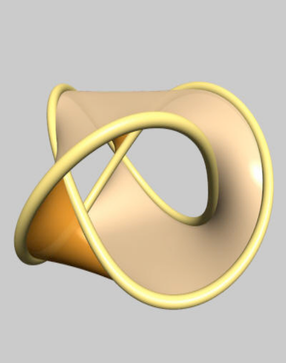
Homotopy or homology?

A question was left apart: do we have examples of homologically trivial domains that are not simply-connected?
Let us recall the definitions:

- a domain Ω is said simply-connected (or homotopically trivial) if any cycle c can be retracted in Ω to a point $\mathbf{p} \in \Omega$
- a domain Ω is said homologically trivial if any cycle c is the boundary of a surface $S \subset \Omega$.

Clearly, if a cycle can be retracted to a point, it is also the boundary of a surface. Hence, a simply-connected domain is homologically trivial.
However, there are cycles that are the boundary of a surface, but that cannot be retracted (consider the complement in a box of the trefoil knot, and take a cycle... as explained in the picture).

The trefoil knot



The trefoil knot and its Seifert surface.

[Image produced with SeifertView, Jarke J. van Wijk, Technische Universiteit Eindhoven.]

Homotopy or homology? (cont'd)

This seems to suggest that there exist homologically trivial domains that are not simply-connected. However, what we have seen is not an example of this fact, as, if we look at the previous picture, in the complement of the trefoil knot there is another cycle that is not bounding a surface.

So, try to refine the analysis: it is worth noting that, in the engineering literature, this example is indeed the basis for the statement that "homologically trivial" is a less restrictive than "simply-connected". The reason is that, considering the complement in a box of the trefoil knot together with its Seifert surface, we have cut the latter cycle (the one that is non-bounding), without cutting the former (the one that is bounding but cannot be retracted).

Homotopy or homology? (cont'd)

Let us jump to the conclusion:

Theorem (Borsuk, Benedetti–Frigerio–Ghiloni)

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary. Then it is simply-connected if and only if it is homologically trivial.

And what we have told just above? There is a subtle mistake in the argument: we have not cut the former cycle, but we have cut the surface of which it was the boundary!

It can be easily checked that now it is not a bounding cycle (in the electrical engineering language, it links the current running along the Seifert surface).

Homotopy or homology? (cont'd)

Conclusion: you are lucky.

It is true that, speaking about the operators **grad**, div, **curl**, homology (and not homotopy) is the right concept. But we can still use the words "a simply-connected domain", as it has the same meaning of "a homologically trivial domain".