1 Further Topics in Group Theory

1.1 p-Groups, Nilpotent Groups, and Solvable Groups

Definition. A *maximal subgroup* of a group G is a proper subgroup M of G such that there is no subgroups H of G with $M < H < G$.

Theorem 1. Let p be a prime and let P be a group of order p^a, $a \geq 1$. Then

1. The center of P is nontrivial: $Z(P) \neq 1$.
2. If H is a nontrivial normal subgroup of P then H contains a subgroup of order p^b that is normal in P for each divisor p^b of $|H|$. In particular, P has a normal subgroup of order p^b for every $b \in \{0, 1, \dots, a\}$.'
3. If $H < P$ then $H < N_P(H)$ (i.e., every proper subgroup of P is a proper subgroup of its normalizer in P).
4. Every maximal subgroup of P is of index p and is normal in P.

Definition.

1. For any (finite or infinite) group G define the following subgroups inductively:

$$Z_0(G) = 1 \qquad Z_1(G) = Z(G)$$

   and $Z_{i+1}(G)$ is the subgroup of G containing $Z_i(G)$ such that

$$Z_{i+1}(G)/Z_i(G) = Z(G/Z_i(G))$$

   (i.e., $Z_{i+1}(G)$ is the complete preimage in G of the center of $G/Z_i(G)$ under the natural projection). The chain of subgroups

$$Z_0(G) \leq Z_1(G) \leq Z_2(G) \leq \dots$$

   is called the *upper central series of* G. (The use of the term "upper" indicates that $Z_i(G) \leq Z_{i+1}(G)$.)

2. A group G is called *nilpotent* if $Z_c(G) = G$ for some $c \in \mathbb{Z}$. The smallest c is called the *nilpotence class* of G.

Note.

1. If G is abelian then it is nilpotent since $G = Z(G) = Z_1(G)$.
2. The following containments are proper

$$\text{cyclic groups} \subset \text{abelian groups} \subset \text{nilpotent groups} \subset \text{solvable groups} \subset \text{all groups}$$

3. For any finite group there must, by order considerations, be an integer n such that

$$Z_n(G) = Z_{n+1} = Z_{n+2} = \cdots .$$

4. For infinite groups G it may happen that all $Z_i(G)$ are proper subgroups of G (so G is not nilpotent) but
$$G = \bigcup_{i=0}^{\infty} Z_i(G).$$

Proposition 2. Let p be a prime and let P be a group of order p^a. Then P is nilpotent of nilpotence class at most $a-1$ for all $a \geq 2$ (and class equal to a when $a = 0$ or 1).

Theorem 3. Let G be a finite group, let $p_1, p_2, \ldots, p_s$ be the distinct primes dividing its order and let $P_i \in Syl_{p_i}(G), 1 \leq i \leq s$. Then the following are equivalent:

1. G is nilpotent
2. if $H < G$ then $H < N_G(H)$, i.e., every proper subgroup of G is a proper subgroup of its normalizer in G
3. $P_i \trianglelefteq G$ for $1 \leq i \leq s$, i.e., every Sylow subgroup is normal in G
4. $G \cong P_1 \times P_2 \times \cdots \times P_s$.

Corollary 4. A finite abelian group is the direct product of its Sylow subgroups.

Proposition 5. If G is a finite group such that for all positive integers n dividing its order, G contains at most n elements x satisfying $x^n = 1$, then G is cyclic.

Proposition 6. (Frattini's Argument) Let G be a finite group, let H be a normal subgroup of G and let P be a Sylow p-subgroup of H. Then $G = HN_G(P)$ and $|G : H|$ divides $|N_G(P)|$.

Proposition 7. A finite group is nilpotent if and only if every maximal subgroup is normal.