1 Direct and Semidirect Products and Abelian Groups

1.1 Direct Products

Definition.

1. The *direct product* $G_1 \times G_2 \times \cdots \times G_n$ of the groups $G_1, G_2, \ldots, G_n$ with operations $\star_1, \star_2, \ldots, \star_n$, respectively, is the set of n-tuples $(g_1, g_2, \ldots, g_n)$ where $g_i \in G_i$ with the operation defined componentwise:

$$(g_1, g_2, \ldots, g_n) \star (h_1, h_2, \ldots, h_n) = (g_1 \star_1 h_1, g_2 \star_2 h_2. \ldots g_n \star_n h_n).$$

2. Similarly, the *direct product* $G_1 \times G_2 \times \cdots$ of the groups $G_1, G_2, \ldots$ with operations $\star_1, \star_2, \ldots$, respectively, is the set of sequences $(g_1, g_2, \ldots)$ where $g_i \in G_i$ with the operation defined componentwise:

$$(g_1, g_2, \ldots) \star (h_1, h_2, \ldots) = (g_1 \star_1 h_1, g_2 \star_2 h_2 \ldots).$$

Proposition 1. If $G_1, \ldots, G_n$ are groups, their direct product is a group of order $|G_1||G_2| \cdots |G_n|$ (if any G_i is infinite, so is the direct product).

Proposition 2. Let $G_1, G_2, \ldots, G_n$ be group and let $G = G_1 \times G_2 \times \cdots \times G_n$ be their direct product.

1. For each fixed i the set of elements of G which have the identity of G_j in the j^{th} position for all $j \neq i$ and arbitrary elements of G_i in position i is a subgroup of G isomorphic G_i:
$$G_i \cong \{(1, 1, \ldots, 1, g_i, 1, \ldots, 1) \mid g_i \in G_i\},$$
(here g_i appears in the i^{th} position). If we identity G_i with this subgroup, then $G_i \trianglelefteq G$ and
$$G/G_i \cong G_1 \times \cdots \times G_{i-1} \times G_{i+1} \times \cdots \times G_n.$$

2. For each fixed i define $\pi_i \colon G \to G_i$ by

$$\pi_i((g_1, g_2, \ldots, g_n)) = g_i.$$

Then π_i is a surjective homomorphism with

$$\begin{aligned} \ker \pi_i &= \{(g_1, g_2, \ldots, g_{i-1}, 1, g_{i+1}) \mid g_j \in G_j \text{ for all } j \neq i\} \\ &\cong G_1 \times \cdots \times G_{i-1} \times G_{i+1} \times \cdots \times G_n \end{aligned}$$

(here 1 appears in position i).

3. Under the identifications in part 1, if $x \in G_i$ and $y \in G_j$ for some $i \neq j$, then $xy = yx$.