

# Dummit and Foote Abridged

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## 0 Preliminaries

### 0.1 Basics

**Proposition 1.** Let  $f: A \rightarrow B$ .

1. The map  $f$  is injective if and only if  $f$  has a left inverse.
2. The map  $f$  is surjective if and only if  $f$  has a right inverse.
3. The map  $f$  is a bijection if and only if there exist  $g: B \rightarrow A$  such that  $f \circ g$  is the identity map on  $B$  and  $g \circ f$  is the identity map on  $A$ .
4. If  $A$  and  $B$  are finite sets with the same number of elements the  $f: A \rightarrow B$  is bijective if and only if  $f$  is injective if and only if  $f$  is surjective.

**Proposition 2.** Let  $A$  be a nonempty set.

1. If  $\sim$  defines an equivalence relation on  $A$  then the set of equivalence classes of  $\sim$  form a partition of  $A$ .
2. If  $\{A_i \mid i \in I\}$  is a partition of  $A$  then there is an equivalence relation on  $A$  whose equivalence classes are precisely the sets  $A_i, i \in I$

# 1 Group Theory

## 1.1 Basic Axioms and Examples

**Proposition 1.** If  $G$  is a group under the operation  $\cdot$ , then

1. The identity of  $G$  is unique
2. for each  $a \in G$ ,  $a^{-1}$  is uniquely determined
3.  $(a^{-1})^{-1} = a$  for all  $a \in G$
4.  $(a \cdot b)^{-1} = (b^{-1}) \cdot (a^{-1})$
5. for any  $a_1, a_2, \dots, a_n \in G$  the value of  $a_1 a_2 \cdots a_n$  is independent of how the expression is bracketed

**Proposition 2.** Let  $G$  be a group and let  $a, b \in G$ . The equations  $ax = b$  and  $ya = b$  have unique solutions for  $x, y \in G$ . In particular, the left and right cancellation laws hold in  $G$ , i.e.,

1. if  $au = av$ , then  $u = v$ , and
2. if  $ub = vb$ , then  $u = v$ .

## 2 Subgroups

### 2.1 Definition and Examples

**Proposition 1.** (The Subgroup Criterion) A subset  $H$  of a group  $G$  is a subgroup if and only if

1.  $H \neq \emptyset$ , and
2. for all  $x, y \in H$ ,  $xy^{-1} \in H$

### 2.3 Cyclic Groups and Cyclic Subgroups

**Proposition 2.** If  $H = \langle x \rangle$ , then  $|H| = |x|$ . Moreover,

1. if  $|H| = n < \infty$ , then  $x^n = 1$  and  $1, x, x^2, \dots, x^{n-1}$  are all distinct elements of  $H$ , and
2. if  $|H| = \infty$ , then  $x^n \neq 1$  for all  $n \neq 0$  and  $x^a \neq x^b$  for all  $a \neq b \in \mathbb{Z}$ .

**Proposition 3.** Let  $G$  be an arbitrary group,  $x \in G$  and let  $m, n \in \mathbb{Z}$ . If  $x^n = 1$  and  $x^m = 1$  then  $x^d = 1$  where  $d = (m, n)$ . In particular, if  $x^m = 1$  for some  $m \in \mathbb{Z}$  then  $|x|$  divides  $m$ .

**Theorem 4.** Any two cyclic groups of the same order are isomorphic. Moreover,

1. if  $n \in \mathbb{Z}^+$  and  $\langle x \rangle$  and  $\langle y \rangle$  are both cyclic groups of order  $n$ , then the map

$$\begin{aligned} \phi: \langle x \rangle &\rightarrow \langle y \rangle \\ x^k &\mapsto y^k \end{aligned}$$

is well defined and is an isomorphism

2. if  $\langle x \rangle$  is an infinite cyclic group, the map

$$\begin{aligned}\phi: \mathbb{Z} &\rightarrow \langle x \rangle \\ k &\mapsto x^k\end{aligned}$$

is well defined and is an isomorphism

**Proposition 5.** Let  $G$  be a group, let  $x \in G$  and let  $a \in \mathbb{Z} - \{0\}$ .

1. If  $|x| = \infty$ , then  $|x^a| = \infty$ .
2. If  $|x| = n < \infty$ , then  $|x^a| = \frac{n}{(n,a)}$ .
3. In particular, if  $|x| = n < \infty$  and  $a$  is a positive integer dividing  $n$ , then  $|x^a| = \frac{n}{a}$ .

**Proposition 6.** Let  $H = \langle x \rangle$ .

1. Assume  $|x| = \infty$ . Then  $H = \langle x^a \rangle$  if and only if  $a = \pm 1$ .
2. Assume  $|x| = n < \infty$ . Then  $H = \langle x^a \rangle$  if and only if  $(a, n) = 1$ . In particular, the number of generators of  $H$  is  $\phi(n)$  (where  $\phi$  is Euler's  $\phi$ -function)

**Theorem 7.** Let  $H = \langle x \rangle$  be a cyclic group.

1. Every subgroup of  $H$  is cyclic. More precisely, if  $K \leq H$ , then either  $K = \{1\}$  or  $K = \langle x^d \rangle$ , where  $d$  is the smallest positive integer such that  $x^d \in K$ .
2. If  $|H| = \infty$ , then for any distinct nonnegative integers  $a$  and  $b$ ,  $\langle x^a \rangle \neq \langle x^b \rangle$ . Furthermore, for every integer  $m$ ,  $\langle x^m \rangle = \langle x^{|m|} \rangle$ , where  $|m|$  denotes the absolute value of  $m$ , so that the nontrivial subgroups of  $H$  correspond bijectively with the integers  $1, 2, 3, \dots$
3. If  $|H| = n < \infty$ , then for each positive integer  $a$  dividing  $n$  there is a unique subgroup of  $H$  of order  $a$ . This subgroup is the cyclic group  $\langle x^d \rangle$ , where  $d = \frac{n}{a}$ . Furthermore, for every integer  $m$ ,  $\langle x^m \rangle = \langle x^{(n,m)} \rangle$ , so that the subgroups of  $H$  correspond bijectively with the positive divisors of  $n$ .

## 2.4 Subgroups Generated by Subsets of a Group

**Proposition 8.** If  $\mathcal{A}$  is any nonempty collection of subgroups of  $G$ , then the intersection of all members of  $\mathcal{A}$  is also a subgroup of  $G$ .

**Proposition 9.**  $\overline{A} = \langle A \rangle$ .

## 3 Quotient Groups and Homomorphisms

### 3.1 Definitions and Examples

**Proposition 1.**