1 Polynomial Rings

In this chapter the ring R will always be a commutative ring with identity $1 \neq 0$.

1.1 Definitions and Basic Properties

Proposition 1. Let R be an integral domain. Then

1. degree $p(x)q(x) =$ degree $p(x)$ + degree $q(x)$ if $p(x), q(x)$ are nonzero
2. the units of $R[x]$ are just the units of R
3. $R[x]$ is an integral domain.

Proposition 2. Let I be an ideal of the ring R and let $(I) = I[x]$ denote the ideal of $R[x]$ generated by I (the set of polynomials with coefficients in I). Then

$$R[x]/(I) \cong (R/I)[x].$$

In particular, if I is a prime ideal of R then (I) is a prime ideal of $R[x]$

Definition. The *polynomial ring in variables $x_1, x_2, \dots, x_n$ with coefficients in R*, denoted $R[x_1, x_2, \dots, x_n]$ is defined inductively by

$$R[x_1, x_2, \dots, x_n] = R[x_1, x_2, \dots, x_{n-1}][x_n].$$

1.2 Polynomial Rings over Fields I

Theorem 3. Let F be a field. The polynomial ring $F[x]$ is a Euclidean Domain. Specifically, if $a(x)$ and $b(x)$ are two polynomials in $F[x]$ with $b(x)$ nonzero, then there are unique $q(x)$ and $r(x)$ in $F[x]$ such that

$$a(x) = q(x)b(x) + r(x) \qquad \text{with } r(x) = 0 \text{ or degree } r(x) < \text{ degree } b(x).$$

Corollary 4. If F is a field, then $F[x]$ is a Principal Ideal Domain and a Unique Factorization Domain.

1.3 Polynomial Rings that are Unique Factorization Domains