Dummit and Foote Abridged

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Contents

0 Preliminaries

0.1 Basics

Proposition 1. Let $f\colon A \to B$.

1. The map f is injective if and only if f has a left inverse.
2. The map f is surjective if and onbly if f has a right inverse.
3. The map f is a bijection if and only if there exist $g\colon B \to A$ such that $f \circ g$ is the indentity map on B and $g \circ f$ is the identity map on A.
4. If A and B are finte sets with the same number of elements the $f\colon A \to B$ is bijective if and only if f is injective if and only if f is surjective.

Proposition 2. Let A be a nonempty set.

1. If $\sim$ defines an equivalence relation on A then the set of equivalence classes of $\sim$ form a partision of A.
2. If $\{A_i \mid i \in I\}$ is a parttion of A then there is an equivalence relation on A whose equivalence classes are precisely the sets $A_i, i \in I$

1 Group Theory

1.1 Basic Axioms and Examples

Definition.

1. A *binary operation* $\star$ on a set G is a function $\star \colon G \times G \to G$. For any $a, b \in G$ we shall write $a \star b$ for $\star(a, b)$.
2. A binary operation $\star$ on a set G is associative if for all $a, b, c \in G$ we have $a \star (b \star c) = (a \star b) \star c$.
3. If $\star$ is a binary operation on a set G we say elements a and b of G *commute* if $a \star b = b \star a$. We say $\star$ (or G) is *commutative* if for all $a, b \in G$, $a \star b = b \star a$.

Proposition 1. If G is a group under the operation $\cdot$, then

1. The identity of G is unique
2. for each $a \in G$, a^{-1} is uninuely determined
3. $(a^{-1})^{-1} = a$ for all $a \in G$
4. $(a \cdot b)^{-1} = (b^{-1}) \cdot (a^{-1})$
5. for any $a_q, a_2, \ldots, a_n \in G$ the value of $a_1 a_2 \cdots a_n$ is independent of how the expresion is bracketed

Proposition 2. Let G be a group and let $a, b \in G$. The equations $ax = b$ and $ya = b$ have unique solutions for $x, y \in G$. In particular, the left and right cancelation laws hold in G, i.e.,

1. if $au = av$, then $u = v$, and
2. if $ub = vb$, then $u = v$.

Definition. For G a group and $x \in G$ define the *order* of x to be the smallest positive integer n such that $x^n = 1$, denoted $|x|$. If there is no such integer than we define the order of x to be infinity.

1.6 Homomorphism and Isomorphisms

Definition. Let $(G, \star)$ and $(H, \diamond)$ be groups. A map $\phi \colon G \to H$ such that $\phi(x \star y) = \phi(x) \diamond \phi(y)$, for all $x, y \in G$ is called a *homomorphism.* Moreover, if ϕ is bijective it is called an *isomorphism* and we say that G and H are *isomorphic* or of the same *isomorphism type*, written $G \cong H$.

Note. If $\phi \colon G \to H$ is an isomorphism then

1. $|G| = |H|$
2. G is abelian if and only if H is abelian
3. for all $x \in G, |x| = |\phi(x)|$

1.7 Group Actions

Definition. A *group action* of a group G on a set A is a map from $G \times A$ to A (written as $g \cdot a$, for all $g \in G$ and $a \in A$) satisfying the following properties:

1. $g_1 \cdot (g_2 \cdot a) = (g_1 g_2) \cdot a$, for all $g_1, g_2 \in G, a \in A$, and
2. $1 \cdot a = a$ for all $a \in A$.

Note. Let the group G act on the set A. From each fixed $g \in G$ we get a map σ_g defined by

$$\sigma_g \colon A \to A$$
$$\sigma_g(a) = g \cdot a.$$

The following are true

1. for each fixed $g \in G$, σ_g is a permutation of A, and
2. the map from G to S_A defined bt $g \mapsto \sigma_g$ is a homomorphism. Moreover this map is called the *permuation representation* associated to the given action.

Note. As a consequence of the above remark, if $\phi \colon G \to S_A$ is a homomorphism (here S_A is the symmetric group on the set A), then the map from $G \times A$ to A defined by

$$g \cdot a = \phi(g)(a) \text{for all} g \in G, \text{ and all} a \in A$$

is a group action of G on A.

2 Subgoups

2.1 Definition and Examples

Definition. Let G be a group. The subset H of G is a *subgroup* of G if H is nonempty and H is closed under products and inverse (i.e, $x, y \in H$ implies $x \in H$ and $xy \in H$). If H is a subgroup of G we shall write $H \leq G$.

Proposition 1. (The Subgroup Criterion) A subset H of a group G is a subgroup if and only if

1. $H \neq \emptyset$, and
2. for all $x, y \in H, xy^{-1} \in H$

2.2 Centralizers and Nomalizers, Stabilizers and Kernels

Let G be a group and A a nonempty subset of G.

Definition. The *centralizer* of A in G is $C_G(A) = \{g \in G \mid gag^{-1} = a \text{ for all } a \in A\}$. Note that this is the set of elements of G which commute with every element of A. Note that $C_g(A) \leq G$.

Definition. The *center* of G is the set $Z(G) = \{g \in G \mid gx = xg \text{ for all } x \in G\}$. Note that, $Z(G) = C_G(G)$, thus $Z(G) \leq G$.

Definition. Define $gAg^{-1} = \{gag^{-1} \mid a \in A\}$. The *normalizer* of A in G is the set $N_G(A) = \{g \in G \mid gAg^{-1} = A\}$. Note that, $C_G(A) \leq N_G(A) \leq G$.

2.3 Cyclic Groups and Cyclic Subgroups

Definition. A group H is *cyclic* if H can be generated by a single element, i.e, there exist some $x \in H$ such that $H = \{x^n \mid n \in \mathbb{Z}\}$ when using multiplicative notation and $H = \{nx \mid n \in \mathbb{Z}\}$ when using additive notation. In either case we write $H = \langle x \rangle$.

Proposition 2. If $H = \langle x \rangle$, then $|H| = |x|$. Moreover,

1. if $|H| = n < \infty$, then $x^n = 1$ and $1, x, x^2, \dots, x^{n-1}$ are all distinct elements of H, and
2. if $|H| = \infty$, then $x^n \neq 1$ for all $n \neq 0$ and $x^a \neq x^b$ for all $a \neq b \in \mathbb{Z}$.

Proposition 3. Let G be an arbitrary group, $x \in G$ and let $m, n \in \mathbb{Z}$. If $x^n = 1$ and $x^m = 1$ then $x^d = 1$ where $d = (m, n)$. In particular, if $x^m = 1$ for some $m \in \mathbb{Z}$ then $|x|$ divides m.

Theorem 4. Any two cyclic groups of the same order are isomorphic. Moreover,

1. if $n \in \mathbb{Z}^+$ and $\langle x \rangle$ and $\langle y \rangle$ are both cyclic groups of orger n, then the map

$$\begin{aligned} \phi \colon \langle x \rangle &\to \langle y \rangle \\ x^k &\mapsto y^k \end{aligned}$$

   is well defined and is an isomorphism

2. if $\langle x \rangle$ is an infinite cyclic group, the map

$$\begin{aligned} \phi \colon \mathbb{Z} &\to \langle x \rangle \\ k &\mapsto x^k \end{aligned}$$

   is well defined and is an isomorphism

Proposition 5. Let G be a group, let $x \in G$ and let $a \in \mathbb{Z} - \{0\}$.

1. If $|x| = \infty$, then $|x^a| = \infty$.
2. If $|x| = n < \infty$, then $|x^a| = \frac{n}{(n,a)}$.
3. In particular, if $|x| = n < \infty$ and a is a postive integer dividing n, then $|x^a| = \frac{n}{a}$.

Proposition 6. Let $H = \langle x \rangle$.

1. Assume $|x| = \infty$. Then $H = \langle x^a \rangle$ if and only if $a = \pm 1$.
2. Assume $|x| = n < \infty$. Then $H = \langle x^a \rangle$ if and only if $(a, n) = 1$. In particular, the number of generators of H is $\phi(n)$ (where ϕ is Euler's ϕ-function)

Theorem 7. Let $H = \langle x \rangle$ be a cyclic group.

1. Every subgroup of H is cyclic. More precisely, if $K \leq H$, then either $K = \{1\}$ or $K = \langle x^d \rangle$, where d is the smallest positive integer such that $x^d \in K$.
2. If $|H| = \infty$, then for any distinct nonnegative integers a and b, $\langle x^a \rangle \neq \langle x^b \rangle$. Furthermore, for every integer m, $\langle x^m \rangle = \langle x^{|m|} \rangle$, where $|m|$ denotes the absolute value of m, so that the nontrival sungroups of H correspond bijectively with the integers $1, 2, 3, \dots$.
3. If $|H| = n < \infty$, then for each positive integer a dividing n there is a unique subgroup of H of order a. This subgroup is the cyclic group $\langle x^d \rangle$, where $d = \frac{n}{a}$. Furthermore, for every integer m, $\langle x^m \rangle = \langle x^{(n,m)} \rangle$, so that the subgroups of H correspond bijectively with the positive divisors of n.

2.4 Subgroups Generated by Subsets of a Group

Proposition 8. If $\mathcal{A}$ is any nonempty collection of subgroups of G, then the intersection of all members of $\mathcal{A}$ is also a subgroup of G.

Definition. If A is any subset of the group G define

$$\langle A \rangle = \bigcap_{\substack{A \subseteq H \\ H \leq G}} H.$$

This is called the *subgroup of G generated by A*.

Note. $\langle A \rangle = \{a_1^{\epsilon_1} a_2^{\epsilon_2} \dots a_n^{\epsilon_n} \mid n \in \mathbb{Z}, n \geq 0 \text{ and } a_i \in A, \epsilon_i = \pm 1 \text{ for each } i\}$.

3 Quotient Groups and Homomorphisms

3.1 Definitions and Examples

Definition. If ϕ is a homomorphism $\phi\colon G \to H$, the *kernel* of ϕ is the set

$$\{g \in G \mid \phi(g) = 1\}$$

and will be denoted by kerϕ (here 1 is the identity of H).

Proposition 1. Let G and H be groups and let $\phi\colon H \to H$ be a homomorphism.

1. $\phi(1_G) = 1_H$, where 1_G and 1_H are the identities of G and H, respectively.
2. $\phi(g^{-1}) = \phi(g)^{-1}$ for all $g \in G$.
3. $\phi(g^n) = \phi(g)^n$ for all $n \in \mathbb{Z}$.
4. kerϕ is a subgroup of G.
5. imϕ, the image of G uner ϕ, is a subgorup of H.

Definition. Let $\phi\colon G \to H$ be a homomorphism with kernel K. The *quotient group* or *factor group*, G/K (read *G modulo K* or simply *G mod K*), is the group whose elements are the fibers of ϕ with the following group operation: If X is the fiber above a and Y is the fiber above b then the product XY in G/K is defined to be the fiber above the product ab in G.

Proposition 2. Let $\phi\colon G \to H$ be a homomorphism with kernel K. Let $X \in G/K$ be the fiber above a, i.e., $X = \phi^{-1}(a)$. Then

1. For any $u \in X$, $X = \{uk \mid k \in K\}$
2. For any $u \in X$, $X = \{ku \mid k \in K\}$

Definition. For any $N \leq G$ and any $g \in G$ let

$$gN = \{gn \mid n \in N\} \text{ and } Ng = \{ng \mid n \in N\}$$

called respectively a *left coset* and a *right coset* of N in G. Any element of a coset is called a *representative* for the coset.

Theorem 3. Let G be a group and let K be the kernel of some homomorphism from G to another group. Then the set of whose elements are ;eft coeset of K in G with operation defined by

$$uK \circ vK = (uv)K$$

forms a group, G/K. This operation is well defined and does not depend on the choice of representatives.

Proposition 4. Let N be any subgroup of the group G. The set of left cosets of N in G form a partition of G. Furthermore, for all $u, v \in G, uN = vN$ if and only if $v^{-1}u \in N$ and in particular, $uN = vN$ if and only if u and v are representatives of the same coset.

Proposition 5. Let G be a group and let N be a subgroup of G.

1. The operation on the set of left cosets of N in G described by

$$uN \cdot vN = (uv)N$$

   is well defined if and only if gng^{-1} for all $g \in G$ and all $n \in N$.

2. If the above operation is well defined, then it makes the set of left cosets of N in G into a group. In particular the identity of this group is the coset $1N$ and the inverse of gN is the coset g^{-1}, i.e, $(gN)^{-1} = g^{-1}N$.

Definition. The element gng^{-1} is called the *conjugate* of $n \in N$ by g. The set $gNg^{-1} = \{gng^{-1} \mid n \in N\}$ is called the *conjugate* of N by g. The element g is said to *normalize* N if $gNg^{-1} = N$. A subgroup N of a group G is called *normal* if every element of G normalizes N, i.e., if $gNg^{-1} = N$ for all $g \in G$. If N is a normal subgoup of G we shall write $N \trianglelefteq G$.

Theorem 6. Let N be a subgroup of the group G. The following are equivalent:

1. $N \trianglelefteq G$
2. $N_G(N) = G$ (recall $N_G(N)$ is the normalizer in G of N)
3. $gN = Ng$ for all $g \in G$
4. the operation on left cosets of N in G described in Proposition 5 makes the set of left cosets into a group
5. $gNg^{-1} \subseteq N$ for all $g \in G$.

Proposition 7. A subgroup N of the group G is normal if and only if it is the kernel of some homomorphism.

Definition. Let $N \trianglelefteq G$. The homomorphism $\pi\colon G \to G/N$ defined by $\pi(g) = gN$ is called the *natural projection (homomorphism)* of G onto G/N. If $\overline{H} \leq G/N$ is a subgroup of G/N, the *complete preimage* of $\overline{H}$ in G is the preimage of $\overline{H}$ under the natural projection homomorphism.

3.2 More on Cosets and Lagrange's Thoerem