1 Group Actions

1.1 Group Actions and Permutation Representations

Definition. Let G be a group acting on a set A

1. The *kernel* of the action is the set of elements of G that act trivially on every element of A: $\{g \in G \mid g \cdot a = a \text{ for all } a \in A\}$.
2. For each $a \in A$ the *stabilizer* of a in G is the set of elements of G that fix the element a: $\{g \in G \mid g \cdot a = a\}$ and is denoted by G_a.
3. An action is *faithful* if its kernel is the identity.

Note. The kernel pf an action is precisely the same as the kernel of the associated permutation representation as defined in the note in section 1.7 and is rephrased below.

Proposition 1. For any group G and any nonempty set A there is a bijection between the actions of G on A and the homomorphisms of G into S_A.

Definition. If G is a group a *permutation representation* of G into the symmetric group S_A for some nonempty set A. We shall say a given action of G on A *affords* or *induces* the associated representation of G.

Proposition 2. Let G be a group acting on the nonempty set A. the relation on A defined by

$$a \sim b \text{ if and only if } a = g \cdot b \text{ for some } g \in G$$

is an equivalence relation. For each $a \in A$, the number of elements in the equivalence class containing a is $|G : G_a|$, the index of the stabilizer of a.

Definition. Let G be a group acting on the set A.

1. The equivalence class $\{g \cdot \mid g \in G\}$ is called the *orbit* of G containing a.
2. The action of G on A is called *transitive* if there is only one orbit, i.e., given any two elements $a, b \in A$ there is some $g \in G$ such that $a = g \cdot b$.

Note.

1. Every element of S_n has a unique cycle decomposition
2. Subgroups of symmetric groups are called *permutation groups*.
3. The orbits of a permutation group will refer to its orbits on $\{1, 2, \ldots, n\}$
4. The orbits of an element $\sigma \in S_n$ will refer to the orbits of the group $\langle \sigma \rangle$.

1.2 Group Acting on Themselves by Left Multiplication - Cayley's Theorem

Theorem 3.