# Flattening Irregular Nested Parallelism

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## Parallel Basic Blocks Recap

## Part I: Flattening Nested and Irregular Parallelism

- What is ”Flattening”? Recipe for Applying Flattening
- Several Re-Write Rules (inefficient for replicate & iota)
- Jagged (Irregular Multi-Dim) Array Representation
- Revisiting the Rewrites for Replicate & Iota Nested Inside Map
- Revisiting the Solution to Our Example

## Part II: Flattening Nested and Irregular Parallelism

- More Flattening Rules
- Flattening by Function Lifting
- Flattening Quicksort
- Flattening Prime-Number (Sieve) Computation

# Zip, Unzip, iota, replicate

- `zip` : $[n]\alpha_1 \to [n]\alpha_2 \to [n](\alpha_1, \alpha_2)$
- `zip` $[a_1, \ldots, a_n]$ $[b_1, \ldots, b_n] \equiv$ $[(a_1, b_1), \ldots, (a_n, b_n)]$,
- `unzip` : $[n](\alpha_1, \alpha_2) \to ([n]\alpha_1, [n]\alpha_2)$
- `unzip` $[(a_1, b_1), \ldots, (a_n, b_n)] \equiv ([a_1, \ldots, a_n], [b_1, \ldots, b_n])$,
- In some sense `zip`/`unzip` are syntactic sugar
- `replicate` : `(n:` `int)` $\to \alpha \to [n]\alpha$
- `replicate n a` $\equiv$ `[a, a,..., a]`,
- `iota` : `(n:` `int)` $\to$ `[n]int`
- `iota n` $\equiv$ `[0, 1,..., n-1]`

Note: in Haskell `zip` does not expect same-length arrays; in Futhark it does!

# Map, Reduce, and Scan Types and Semantics

- $[n]\alpha$ denotes the type of an array of $n$ elements of type $\alpha$.
- $\texttt{map} : (\alpha \to \beta) \to [n]\alpha \to [n]\beta$
  $\texttt{map f } [x_1, \ldots, x_n] = [\texttt{f } x_1, \ldots, \texttt{ f } x_n]$,
  i.e., $x_i : \alpha, \forall i$, and $\texttt{f} : \alpha \to \beta$.
- $\texttt{reduce} : (\alpha \to \alpha \to \alpha) \to \alpha \to [n]\alpha \to \alpha$
  $\texttt{reduce } \odot \texttt{ e } [x_1, x_2, \ldots, x_n] = \texttt{e} \odot x_1 \odot x_2 \odot \ldots \odot x_n$,
  i.e., $\texttt{e} : \alpha$, $x_i : \alpha, \forall i$, and $\odot : \alpha \to \alpha \to \alpha$.
- $\texttt{scan}^{exc} : (\alpha \to \alpha \to \alpha) \to \alpha \to [n]\alpha \to [n]\alpha$
  $\texttt{scan}^{exc} \odot \texttt{ e } [x_1, \ldots, x_n] = [\texttt{e}, \texttt{e} \odot x_1, .., \texttt{e} \odot x_1 \odot .. x_{n-1}]$
  i.e., $\texttt{e} : \alpha$, $x_i : \alpha, \forall i$, and $\odot : \alpha \to \alpha \to \alpha$.
- $\texttt{scan}^{inc} : (\alpha \to \alpha \to \alpha) \to \alpha \to [n]\alpha \to [n]\alpha$
  $\texttt{scan}^{inc} \odot \texttt{ e } [x_1, \ldots, x_n] = [\texttt{e} \odot x_1, \ldots, \texttt{e} \odot x_1 \odot \ldots x_n]$
  i.e., $\texttt{e} : \alpha$, $x_i : \alpha, \forall i$, and $\odot : \alpha \to \alpha \to \alpha$.

# Map2, Filter

- `map2:` $(\alpha_1 \to \alpha_2 \to \beta) \to$ `[n]`$\alpha_1 \to$ `[n]`$\alpha_2 \to$ `[n]`$\beta$
- `map2` $\odot$ `[a`$_1$`,...,a`$_n$`]` `[b`$_1$`,...,b`$_n$`]` $\equiv$ `[a`$_1 \odot$`b`$_1$`,...,a`$_n \odot$`b`$_n$`]`
- `map3 ...`
- `filter :` $(\alpha \to$ `Bool)` $\to$ `[n]`$\alpha \to$ `[m]`$\alpha$ (`m` $\leq$ `n`)
- `filter p [a`$_1$`, ..., a`$_n$`]` = `[a`$_{k_1}$`,..., a`$_{k_m}$`]` such that `k`$_1$ < `k`$_2$ < ... < `k`$_m$, and denoting $\overline{\text{k}}$ = k$_1$,..., k$_m$, we have `(p a`$_j$`==true)` $\forall\, j \in \overline{\text{k}}$, **and** `(p a`$_j$ `== false)` $\forall\, j \notin \overline{\text{k}}$.

Note: in Haskell `map2`, `map3` do not expect same-length arrays; in Futhark they do!

# Scatter: A Parallel Write Operator

Scatter updates in parallel a base array with a set of values at specified indices:

scatter : *[m]$\alpha \to$ [n]int $\to$ [n]$\alpha \to$ *[m]$\alpha$

A (data vector) =[b0, b1, b2, b3]
I (index vector) =[2, 4, 1, -1]
X (input array) =[a0, a1, a2, a3, a4, a5]
scatter X I A =[a0, b2, b0, a3, b1, a5]

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```
A (data vector)  =[b0, b1, b2, b3]
I (index vector)  =[2,  4,  1,  -1]
X (input array)  =[a0, a1, a2, a3, a4, a5]
scatter X I A     =[a0, b2, b0, a3, b1, a5]
```

scatter has $D(n) = \Theta(1)$ and $W(n) = \Theta(n)$,
i.e., requires n update operations (n is the size of I or A, not of X!).

1. Array X is consumed by `scatter`; following uses of X are illegal!
2. Similarly, X can alias neither I nor A!

In Futhark, `scatter` check and ignores the indices that are out of bounds (no update is performed on those). This is useful for padding the iteration space in order to obtain regular parallelism.

# Partition2/Filter Implementation

`partition2:` $(\alpha \to$`Bool`$) \to$ `[n]`$\alpha \to$ `(i32,[n]`$\alpha$`)`

In result, the elements satisfying the predicate occur before the others. Can be implemented by means of `map`, `scan`, `scatter`.

```
let partition2 't [n] (dummy: t)
      (cond: t -> bool) (X: [n]t) :
                        (i64, [n]t) =
 let cs = map cond X
 let tfs= map (\ f->if f then 1
                         else 0) cs
 let isT= scan (+) 0 tfs
 let i  = isT[n-1]

 let ffs= map (\f->if f then 0
                        else 1) cs
 let isF= map (+i) <| scan (+) 0 ffs
 let inds=map (\(c,iT,iF) ->
                  if c then iT-1
                       else iF-1
              ) (zip3 cs isT isF)
 let tmp = replicate n dummy
 in (i, scatter tmp inds X)
```

Assume X = [5,4,2,3,7,8], and cond is T(rue) for even nums.

# Partition2/Filter Implementation

`partition2: (α →Bool) → [n]α → (i32,[n]α)`

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              ) (zip3 cs isT isF)
 let tmp = replicate n dummy
 in (i, scatter tmp inds X)
```

```
Assume X = [5,4,2,3,7,8], and
cond is T(rue) for even nums.
n    = 6
cs   = [F, T, T, F, F, T]
tfs  = [0, 1, 1, 0, 0, 1]

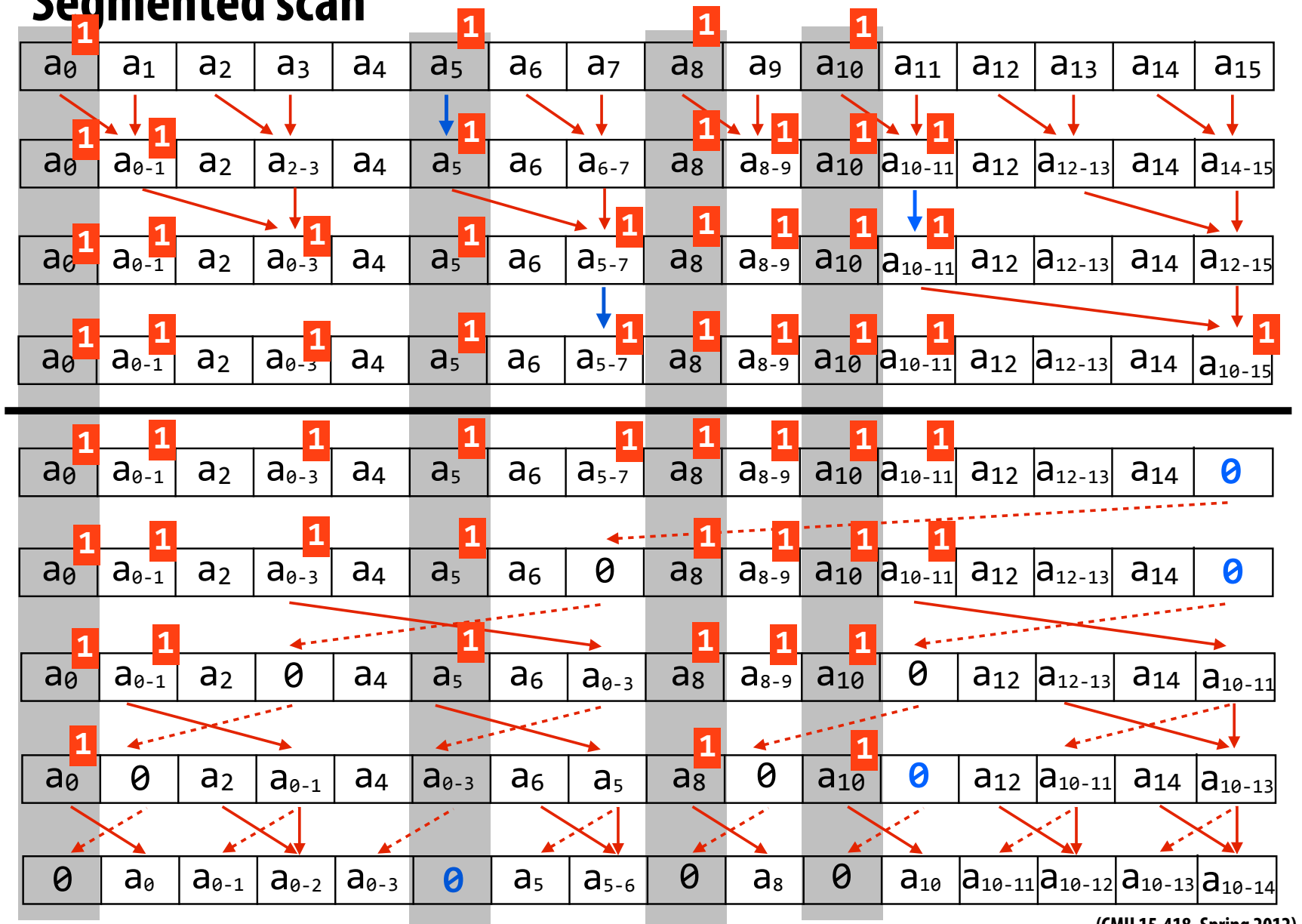
isT  = [0, 1, 2, 2, 2, 3]
i    = 3

ffs  = [1, 0, 0, 1, 1, 0]
isF  = [4, 4, 4, 5, 6, 6]

inds= [3, 0, 1, 4, 5, 2]


flags  = [3, 0, 0, 3, 0, 0]
Result = [4, 2, 8, 5, 3, 7]
```

# Segmented scan



# Segmented Scan Is a Sort of Scan

```
def sgmscan 't [n] (op: t->t->t) (ne: t)
               (flg : [n]bool) (arr : [n]t) : [n]t =
  let flgs_vals =
    zip flg arr |>
    scan ( \ (f1, x1) (f2, x2) ->
             let f = f1 || f2
             in  if f2 then (f, x2)
                       else (f, op x1 x2)
         ) (false,ne)
  let (_, vals) = unzip flgs_vals
  in vals
```

```
sgmscan (+) 0 [1,0,0,1,0, 0, 0]
              [1,2,3,4,5, 6, 7]
               = = = = =  =  =
              [1,3,6,4,9,15,22]
```

```
map (\ row -> scan (+) 0 row)
    [[1,2,3], [4,5, 6, 7]]
      = = =    = =  =  =
    [[1,3,6], [4,9,15,22]]
```

**Correctness Argument:**

# Segmented Scan Is a Sort of Scan

```
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         ) (false,ne)
  let (_, vals) = unzip flgs_vals
  in vals
```

```
sgmscan (+) 0 [1,0,0,1,0, 0, 0]
              [1,2,3,4,5, 6, 7]
               = = = = =  =  =
              [1,3,6,4,9,15,22]
```

```
map (\ row -> scan (+) 0 row)
    [[1,2,3], [4,5, 6, 7]]
      = = =    = =  =  =
    [[1,3,6], [4,9,15,22]]
```

**Correctness Argument:**

verify sequential semantics + associative operator $\Rightarrow$
parallel semantics also holds

Parallel Basic Blocks Recap

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# What is ”Flattening”?

A code transformation, attributed to Blelloch in the context of the NESL languages, that takes as input a nested parallel program—possibly involving recursion and irregular/jagged arrays—and produces a semantically-equivalent, flat-parallel programs that runs optimally on a PRAM machine.

**Meaning:** ***it is guaranteed to preserve the work and depth of the original nested-parallel program.***[**]

[**] As long as scan has $O(1)$ depth and concat has $O(1)$ work and . . .

# Flattening Pros and Cons

**Pros:**

+ clever code transformation
+ important as a programming technique as well (promotes parallel **thinking**)
+ perhaps the only way of mapping a set of challenging problems to capricious architectures such as GPUs (e.g., that do not supports dynamic scheduling of parallelism)

# Flattening Pros and Cons

**Pros:**

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+ perhaps the only way of mapping a set of challenging problems to capricious architectures such as GPUs (e.g., that do not supports dynamic scheduling of parallelism)

**Cons:**

- does not consider communication/locality and hardware gets more and more heterogeneous
- worse, it tends to destroy the available locality and may explode memory footprint
- useful to cover datasets that fall outside the "common case"
  **Demonstration at the end of the second Flattening lecture**

# Flattening: A Bird's Eye View

Incomplete recipe for flattening a nested-parallel program consisting of maps and scan/reduce/scatters at innermost level:

I. **Normalize the program (think 3-address form).**
   The easy way is to also eliminate free variables appearing in the current map if they are variant in an outer, enclosing map.

II. **Distribute the parallel context (perfect nest of maps) across the enclosed let-binding statements and handle recurrences by function lifting or map-loop interchange.** Systematic application results in a smallish number of code patterns.

III. **Apply a set of rewrite rules to flatten each pattern**, e.g., treating the cases of a reduce, scan, replicate, iota, scatter, array index which is perfectly nested inside the context.

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**Differences w.r.t. the PMPH material:**

1. **also optimize the number of accesses to global memory** flattening results in memory-bound performance behavior.
2. **cover more rewrite rules and more challenging problems** (e.g., divide-and-conquer recursion)

# PMPH Recap: A Simple Demonstration of How to Flatten

## Contrived Example:

```
let arr = [1, 2, 3, 4] in
map (\i -> map (+(i+1)) (iota i)) arr
-- Result: [[2],[3,4],[4,5,6],[5,6,7,8]]
```

# PMPH Recap: A Simple Demonstration of How to Flatten

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-- Result: [[2],[3,4],[4,5,6],[5,6,7,8]]
```

## I. Normalize the code:

```
map (\i -> let ip1 = i+1 in
           let iot = (iota i) in
           let ip1r = (replicate i ip1)
           in map2 (+) ip1r iot             ) arr
```

# PMPH Recap: A Simple Demonstration of How to Flatten

## Contrived Example:

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let arr = [1, 2, 3, 4] in
map (\i -> map (+(i+1)) (iota i)) arr
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```

## I. Normalize the code:

```
map (\i -> let ip1 = i+1 in
           let iot = (iota i) in
           let ip1r = (replicate i ip1)
           in map2 (+) ip1r iot            ) arr
```

## II. Distribute the map across every statement in the body

and adjust the inputs accordingly ($\mathcal{F}$ denotes the transformation)

```
F(map (\i -> map (+(i+1)) (iota i)) arr) ≡
1. let ip1s = map (\i -> i+1) arr in -- [2, 3, 4, 5]
2. let iots = F(map (\i -> (iota i)) arr) in
3. let ip1rs= F(map2 (\ i ip1 -> (replicate i ip1)) arr ip1s)
4. in  F(map2 (\ ip1r iot -> map2 (+) ip1r iot) ip1rs iots)
```

**For simplicity we assume `arr` contains strictly-positive integers.**

# PMPH Recap: A Simple Demonstration of How to Flatten

**According to inefficient rule "iota nested inside a map"**
(assuming `arr = [1,2,3,4]`):

```
2. let iots = ℱ(map (\i -> iota i) arr)

≡

inds = scan^exc (+) 0 arr                -- [0,1,3,6]
size = (last inds) + (last arr)          -- 6 + 4 = 10
flag = scatter (replicate size 0)        -- [1, 2, 0, 3, 0, 0, 4, 0, 0, 0]
               inds arr
tmp  = replicate size 1
iots = sgmScan^exc (+) 0 flag tmp        -- [0, 0, 1, 0, 1, 2, 0, 1, 2, 3]
```

# PMPH Recap: A Simple Demonstration of How to Flatten

## According to inefficient rule "replicate nested inside a map"

(assuming `arr = [1,2,3,4]`):

```
3. let ip1rs= ℱ(map2 (\ i ip1 -> replicate i ip1) arr ip1s)
≡
vals = scatter (replicate size 0) inds  ip1s -- [2,3,0,4,0,0,5,0,0,0]
ip1rs= sgmScan^inc (+) 0 flag vals              -- [2,3,3,4,4,4,5,5,5,5]
```

## According to rule "map nested inside a map"

```
ℱ(map2 (\ ip1r iot -> map2 (+) ip1r iot) ip1rs iots)
≡
4. result = map2 (+) ip1rs iots
-- [2, 3, 3, 4, 4, 4, 5, 5, 5, 5]
-- [0, 0, 1, 0, 1, 2, 0, 1, 2, 3]
--  +  +  +  +  +  +  +  +  +  +
---------------------------------
-- [2, 3, 4, 4, 5, 6, 5, 6, 7, 8] values
%-- [1, 2, 0, 3, 0, 0, 4, 0, 0, 0] flags
```

**At each step we also reason about the shape of the resulting array.**

**The shape of** the 2D jagged arrays `iots`, `ip1rs`, `result` **is** `arr`.

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# Nested *vs* Flattened Parallelism: Scan inside a Map

## (1) Scan nested inside a map:

```
res = map (\row->scan^inc (+) 0 row) [[1,3], [2,4,6]]
≡
res = [ scan^inc (+) 0 [1,3],    scan^inc (+) 0 [2,4,6] ]
≡
res = [ [ 1, 4],                 [2, 6, 12] ]
```

# Nested *vs* Flattened Parallelism: Scan inside a Map

**(1) Scan nested inside a map:**

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res = map (\row->scan^inc (+) 0 row) [[1,3], [2,4,6]]
≡
res = [ scan^inc (+) 0 [1,3],    scan^inc (+) 0 [2,4,6] ]
≡
res = [ [ 1, 4],                 [2, 6, 12] ]
```

**becomes a segmented scan**, which requires a flag array as arg:

```
sgmScan^inc (+) 0 [1, 0, 1, 0, 0] [1, 3, 2, 4, 6] ≡ [ 1, 4, 2, 6, 12 ]
```

Flattening a scan directly nested inside a map:

- $S^1_{arr}$, $F_{arr}$, $D_{arr}$ denote the shape, flag & flat data of input `arr`.
- The flat-data result is obtained by a segmented scan.
- The shape of the result array is the same as the input array.

$\mathcal{F}$(`res = map (\row -> scan (`$\odot$`) 0`$_\odot$ `row) arr`) $\Rightarrow$

$S^1_{res}$ = $S^1_{arr}$

$D_{res}$ = `sgmScan (`$\odot$`) 0`$_\odot$ $F_{arr}$ $D_{arr}$

# Nested *vs* Flattened Parallelism: Map inside a Map

## (2) Map nested inside a map:

```
res = map (\row->map f row) [[1,3], [2,4,6]]
≡
res = [ map f [1, 3],      map f [2, 4, 6] ]
≡
res = [ [f(1),f(3)], [f(2),f(4),f(6)] ]
```

# Nested *vs* Flattened Parallelism: Map inside a Map

## (2) Map nested inside a map:

```
res = map (\row->map f row) [[1,3], [2,4,6]]
≡
res = [ map f [1, 3],      map f [2, 4, 6] ]
≡
res = [ [f(1),f(3)], [f(2),f(4),f(6)] ]
```

Flattening a map directly nested inside a map:

- the flat-data array is obtained by a map on the flat input;
- the shape of the result array is the same as the input array.

$\mathcal{F}$(res = map (\row -> map f row) arr) $\Rightarrow$

$S^1_{res}$ = $S^1_{arr}$

$D_{res}$ = map f $D_{arr}$

# Nested *vs* Flattened Parallelism: Replicate inside Map

## (3) Replicate nested inside a map:

```
res = map2 (\ n m -> replicate n m) [1,0,3,2] [7,3,8,9] ≡
res = [ replicate 1 7, replicate 0 3, replicate 3 8, replicate 2 9 ] ≡
res = [ [7], [], [8,8,8], [9,9] ]
```

# Nested *vs* Flattened Parallelism: Replicate inside Map

## (3) Replicate nested inside a map:

```
res = map2 (\ n m -> replicate n m) [1,0,3,2] [7,3,8,9] ≡
res = [ replicate 1 7, replicate 0 3, replicate 3 8, replicate 2 9 ] ≡
res = [ [7], [], [8,8,8], [9,9] ]
```

```
res = map2(\n m-> replicate n m) ns ms
```

**becomes a scan-scatter composition:**

# Nested *vs* Flattened Parallelism: Replicate inside Map

## (3) Replicate nested inside a map:

```
res = map2 (\ n m -> replicate n m) [1,0,3,2] [7,3,8,9] ≡
res = [ replicate 1 7, replicate 0 3, replicate 3 8, replicate 2 9 ] ≡
res = [ [7], [], [8,8,8], [9,9] ]
```

```
res = map2(\n m-> replicate n m) ns ms
```

**becomes a scan-scatter composition:**

1. the shape of the result array is ns
2-3. builds the indices at which segment start (-1 for null shape)
4. get the size of the flat array (summing ns)
5-6. write the ms and ns values at the start of their segments
7. propagate the ms values throughout their segments.

```
ℱ(res = map2 (\n m -> replicate n m) ns ms) ⇒      -- ms = [7,3,8,9]
1. S^1_res = ns                                    -- ns = [1,0,3,2]
2. inds = scan^exc (+) 0 ns                        -- [0,1,1,4]
3.   |> map2 (\n i->if n>0 then i else -1) ns      -- inds = [0,-1,1,4]
4. size = (last inds) + (last ns)                  -- 4 + 2 = 6
5. vls = scatter (replicate size 0) inds ms        -- [7, 8, 0, 0, 9, 0]
6. F_arr = scatter (replicate size 0) inds ns      -- [1, 3, 0, 0, 2, 0]
6. D_res = sgmScan^inc (+) 0 F_res vls             -- [7, 8, 8, 8, 9, 9]
```

# Nested *vs* Flattened Parallelism: Iota inside Map

**(4) Iota nested inside a map** ((`iota n`)≡`[0,...,n-1]`):

```
res = map (\i -> iota i) [1,3,2] ≡
res = [ iota 1, iota 3, iota 2 ] ≡ [ [0], [0,1,2], [0,1] ]
```

# Nested *vs* Flattened Parallelism: Iota inside Map

**(4) Iota nested inside a map** (`(iota n)`$\equiv$`[0,...,n-1]`):

```
res = map (\i -> iota i) [1,3,2] ≡
res = [ iota 1, iota 3, iota 2 ] ≡ [ [0], [0,1,2], [0,1] ]
```

**boils down to a segmented scan applied to an array of ones:**

1. by definition of iota, `ns` contains the size of each subarray, hence the shape of the result is `ns`;

2-3. the flag-array of the result, $F_{res}$, is constructed from `ns`; (we will introduce function `mkFlagArray` a bit later).

4. the result is obtained by an exclusive segmented scan operation applied to an array of ones.

$\mathcal{F}$`(res = map (\n -> iota n) ns)` $\Rightarrow$

```
1. S^1_res = ns                                          -- ns = [1, 3, 2]
2. trues = replicate (length ns) true
3. (_, F_res) = mkFlagArray ns 0 trues                   -- F_res = [1, 1, 0, 0, 1, 0]
4. D_res = sgmScan^exc (+) 0 F_res (replicate flen_res 1) -- [0, 0, 1, 2, 0, 1]
```

Note 1: `iota n` $\equiv$ `scan`$^{exc}$ `(+) 0 (replicate n 1)`.

Note 2: `1` and `0` denote `true` and `false`; `flen`$_{res}$ is the sum of `ns`.

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- Flattening Quicksort
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# Shape-Based Representation

- Two dimensional arrays:

```
arr = [ [1,2,3], [4], [], [5,6] ]
```

$\Rightarrow$

$S^0_{arr}$ = [4]
$S^1_{arr}$ = [3, 1, 0, 2]
$D_{arr}$ = [1, 2, 3, 4, 5, 6]

# Shape-Based Representation

- Two dimensional arrays:

```
arr = [ [1,2,3], [4], [], [5,6] ]
```

$\Rightarrow$

$S^0_{arr}$ = [4]
$S^1_{arr}$ = [3, 1, 0, 2]
$D_{arr}$ = [1, 2, 3, 4, 5, 6]

- Three dimensional arrays:

```
arr = [ [], [ [1,2,3], [4], [], [5,6] ], [ [7], [], [8,9,10] ] ]
```

$\Rightarrow$

# Shape-Based Representation

- Two dimensional arrays:

```
arr = [ [1,2,3], [4], [], [5,6] ]
```

$\Rightarrow$

$S^0_{arr}$ = [4]
$S^1_{arr}$ = [3, 1, 0, 2]
$D_{arr}$ = [1, 2, 3, 4, 5, 6]

- Three dimensional arrays:

```
arr = [ [], [ [1,2,3], [4], [], [5,6] ], [ [7], [], [8,9,10] ] ]
```

$\Rightarrow$

$S^0_{arr}$ = [3]
$S^1_{arr}$ = [0, 4, 3]
$S^2_{arr}$ = [3, 1, 0, 2, 1, 0, 3]
$\texttt{flen}_{arr}$ = 10
$D_{arr}$ = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]

Assume a n-dimensional array; The following invariant holds:

$\texttt{length}\ S^i_{arr} = \textbf{reduce}\ (+)\ 0\ S^{i-1}_{arr}\ ,\ \forall 1 \leq i < n$
$\texttt{length}\ D_{arr} = \textbf{reduce}\ (+)\ 0\ S^{n-1}_{arr}$

# Flat Representation: Auxiliary Structures

```
arr = [ [], [ [1,2,3], [4], [], [5,6] ], [ [7], [], [8,9,10] ]
```

$\Rightarrow$

$S^0_{arr}$ = [3]
$S^1_{arr}$ = [0, 4, 3]
$S^2_{arr}$ = [3, 1, 0, 2, 1, 0, 3]
$D_{arr}$ = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]

- Offset Indices (B): segment-start offset in the flat data:

  $B^1_{arr}$ = [0, 0, 6]
  $B^2_{arr}$ = [0, 3, 4, 4, 6, 7, 7]

- Flag Array (F): start of a segment indicated by a `true` value (could also use `!=0` integrals), e.g., used for segmented scans:

  $F^1_{arr}$ = [1, 0, 0, 0, 0, 0, 1, 0, 0, 0]
  $F^2_{arr}$ = [1, 0, 0, 1, 1, 0, 1, 1, 0, 0]

- Segment and Inner indices (`II`):

  $II^1_{arr}$ = [1, 1, 1, 1, 1, 1, 2, 2, 2, 2]
  $II^2_{arr}$ = [0, 0, 0, 1, 3, 3, 0, 2, 2, 2]
  $II^3_{arr}$ = [0, 1, 2, 0, 0, 1, 0, 0, 1, 2]

# Auxiliary Structures: Intuitive Motivation

Auxiliary structures are useful to optimize the replication of values.

## Nested-Execution Example:

```
let xss = [ [1,2,3], [], [5,7] ]
let ys  = [ 4, 2, 1 ]
let rss = map2 (\ xs y -> map (+y) xs ) xss ys
```

⇒

```
rss = [ map (+4) [1,2,3], map (+2) [], map (+1) [5,7] ]
rss = [ [5,6,7], [], [6,8] ]
```

## Traditional flattening would replicate the values of y:

$$\texttt{let } (S^1_{yss}, D_{yss}) = \mathcal{F}(\texttt{map2 (\textbackslash xs y -> replicate (length xs) y) xss ys})$$

$$\texttt{let } D_{rss} = \texttt{map2 (\textbackslash x y -> x + y) } D_{xss}\ D_{yss}$$

⇒

$$D_{xss} = [1, 2, 3, 5, 7]$$
$$\phantom{D_{xss}} \quad + \quad + \quad + \quad + \quad +$$
$$D_{yss} = [4, 4, 4, 1, 1]$$
$$\phantom{D_{xss}} \quad = \quad = \quad = \quad = \quad =$$
$$D_{rss} = [5, 6, 7, 6, 8]$$

# Auxiliary Structures: Intuitive Motivation

Auxiliary structures are useful to optimize the replication of values.

## Nested-Execution Example:

```
let xss = [ [1,2,3], [], [5,7] ]
let ys  = [ 4, 2, 1 ]
let rss = map2 (\ xs y -> map (+y) xs ) xss ys
```

⇒

```
rss = [ map (+4) [1,2,3], map (+2) [], map (+1) [5,7] ]
rss = [ [5,6,7], [], [6,8] ]
```

Using the auxiliary structures we indirectly access other arrays:

```
let D_rss = map2 (\ x sgmind -> x + ys[sgmind]) D_xss II^1_rss
```

⇒

$S^1_{rss}$ = [3, 0, 2]
$II^1_{rss}$ = [0, 0, 0, 2, 2]
$D_{xss}$ = [1, 2, 3, 5, 7]
$D_{rss}$ = [1+4, 2+4, 3+4, 5+1, 7+1] = [5, 6, 7, 6, 8]

But what have we gained? Creating $II^1_{rss}$ is as expensive as `xss` (or better said the expanded `yss` from the other slide) ...

# Auxiliary Structures: Intuitive Motivation

**Auxiliary structures are useful to optimize replication:**

- they depend only on the shape of the result (created once)
- can indirectly access several lower-dimensional arrays, sharing parallel dimensions!

Nested-Execution Example:

```
let xss = [ [1,2,3], [], [5,7] ]
let ys  = [ 4, 2, 1 ]
let zs  = [ 1, 2, 3 ]
let rss = map3 (\ xs y z -> map (\x -> x*y + z ) xs ) xss ys zs
```

$\Rightarrow$

```
rss = [ [5,9,13], [], [8,10] ]
```

**Using the auxiliary structures we indirectly access other arrays**:

```
let D_rss = map2 (\ y sgmind -> x*ys[sgmind] + zs[sgmind]) D_xss II^1_rss
```

$\Rightarrow$

```
II^1_rss = [0, 0, 0, 2, 2]
D_xss = [1, 2, 3, 5, 7]
D_rss = [1*4+1, 2*4+1, 3*4+1, 5*1+3, 7*1+3] = [5, 9, 13, 8, 10]
```

**We build $II^1_{rss}$ once and reuse it twice. Also improves locality: `ys` and `zs` are much smaller than `xss`, hence reused from L1/2$.**

# Auxiliary Structures: Intuitive Motivation

## Nested-Execution Example:

```
let xss = [ [1,3], [2] ]
let yss = [ [2], [4,5] ]
let rss = map2 (\xs ys -> map (\x -> map (+x) ys ) xs ) xss yss
```

$\Rightarrow$

```
rss = [ [[3],[5]], [[6,7]] ]
```

## Using the auxiliary structures we indirectly access other arrays:

```
let D_rss = map3(\ s1 s2 s3 -> let ind_x = B^1_xss[s1] + s2
                               let ind_y = B^1_yss[s1] + s3
                               in  X[ind_x] + Y[ind_y]
                ) II^1_rss II^2_rss II^3_rss
```

$\Rightarrow$

$B^1_{xss}$ = [0, 2]
$B^1_{yss}$ = [0, 1]
$II^1_{rss}$ = [0, 0, 1, 1]
$II^2_{rss}$ = [0, 1, 0, 0]
$II^3_{rss}$ = [0, 0, 0, 1]
$D_{rss}$ = [ $D_{xss}[0+0]+D_{yss}[0+0]$, $D_{xss}[0+1]+D_{yss}[0+0]$
, $D_{xss}[2+0]+D_{yss}[1+0]$, $D_{xss}[2+0]+D_{yss}[1+1]$ ]
$D_{rss}$ = [ 1+2, 3+2, 2+4, 2+5] = [ 3, 5, 6, 7]

# Constructing the Offset Indices (B)

```
arr = [ [], [ [1,2,3], [4], [], [5,6] ], [ [7], [], [8,9,10] ]
```

$\Rightarrow$

$S^0_{arr}$ = [3]
$S^1_{arr}$ = [0, 4, 3]
$S^2_{arr}$ = [3, 1, 0, 2, 1, 0, 3]
$D_{arr}$ = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]

Offset Indices (B): segment-start offset in the flat data:

$B^1_{arr}$ = [0, 0, 6]
$B^2_{arr}$ = [0, 3, 4, 4, 6, 7, 7]

How to construct Offset Indices (B)?

# Constructing the Offset Indices (B)

```
arr = [ [], [ [1,2,3], [4], [], [5,6] ], [ [7], [], [8,9,10] ]
```

$\Rightarrow$

$S^0_{arr}$ = [3]
$S^1_{arr}$ = [0, 4, 3]
$S^2_{arr}$ = [3, 1, 0, 2, 1, 0, 3]
$D_{arr}$ = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]

Offset Indices (B): segment-start offset in the flat data:

$B^1_{arr}$ = [0, 0, 6]
$B^2_{arr}$ = [0, 3, 4, 4, 6, 7, 7]

How to construct Offset Indices (B)?

By exclusive scanning the corresponding shape and reindexing!

```
B²arr = scan^exc (+) 0 S²arr          -- [0, 3, 4, 4, 6, 7, 7]


B¹arr = scan^exc (+) 0 S¹arr          -- [0, 0, 4]
        |> map (\i -> B²arr[i])       -- [0, 0, 6]
```

# Constructing the Flag Array

## From now on, we discuss only TWO-dimensional irregular arrays!

```
def mkFlagArray 't [m]
          (aoa_shp: [m]u32) (zero: t)          -- aoa_shp=[0,3,1,0,4,2,0]
          (aoa_val: [m]t) : ([m]u32, []t) =    -- aoa_val=[1,1,1,1,1,1,1]
  let shp_rot = map (\i->if i==0 then 0        -- shp_rot=[0,0,3,1,0,4,2]
                         else aoa_shp[i-1]
                    ) (iota m)
  let shp_scn = scan (+) 0 shp_rot             -- shp_scn=[0,0,3,4,4,8,10
  let aoa_len = if m == 0 then 0i64            -- aoa_len = 10
                else i64.u32 <|
                     shp_scn[m-1]+aoa_shp[m-1]
  let shp_ind = map2 (\shp ind ->              -- shp_ind=
                        if shp==0 then -1i64   --  [-1,0,3,-1,4,8,-1]
                        else i64.u32 ind       -- scatter
                      ) aoa_shp shp_scn        --   [0,0,0,0,0,0,0,0,0,0]
  let r = scatter (replicate aoa_len zero)     --   [-1,0,3,-1,4,8,-1]
              shp_ind aoa_val                  --   [ 1,1,1, 1,1,1, 1]
  in (shp_scn, r)                              -- r=[1,0,0,1,1,0,0,0,1,0]
```

**Versatile:** computes $B^1$ and $F^1$ of a 2D jagged array of shape `aoa_shp`, with the start-segment values taken from `aoa_val`.

**Unless you have a good reason, `F` should be a `bool` array (to reduce memory traffic).**

# Constructing the Segment and Inner Indices

**From now on, we discuss only TWO-dimensional irregular arrays!**

```
arr = [ [1,2,3], [4], [], [5,6], [7], [], [8,9,10] ]
```

$\Rightarrow$

$S^0_{arr}$ = [7]
$S^1_{arr}$ = [3, 1, 0, 2, 1, 0, 3]
$flen_{arr}$ = **reduce** (+) 0 $S^1_{arr}$ = 10
$D_{arr}$ = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]

## Segment and Inner indices (II):

$II^1_{arr}$ = [0, 0, 0, 1, 3, 3, 4, 6, 6, 6]
$II^2_{arr}$ = [0, 1, 2, 0, 0, 1, 0, 0, 1, 2]

## Constructing Segment and Inner indices (II):

```
(B¹arr , Farr) = mkFlagArray S¹arr 0 (iota (length S¹arr))
        -- ([0, 3, 4, 4, 6, 7, 7], [0, 0, 0, 1, 3, 0, 4, 6, 0, 0])
```

$II^1_{arr}$ = ???
$II^2_{arr}$ = ???

# Constructing the Segment and Inner Indices

I need to get this:

$II^1_{arr}$ = [0, 0, 0, 1, 3, 3, 4, 6, 6, 6]

from this:

```
(_, F_arr) = mkFlagArray S^1_arr 0 (iota (length S^1_arr))
        --  [0, 0, 0, 1, 3, 0, 4, 6, 0, 0]
```

How ?

# Constructing the Segment and Inner Indices

I need to get this:

$II^2_{arr}$ = [0, 1, 2, 0, 0, 1, 0, 0, 1, 2]

from these:

$B_{arr}$ = [0, 3, 4, 4, 6, 7, 7]
$II^1_{arr}$ = [0, 0, 0, 1, 3, 3, 4, 6, 6, 6]
**iota** 10 = [0, 1, 2, 3, 4, 5, 6, 7, 8, 9]

$II^1_{arr}$ and $II^2_{arr}$ have the same length as flat `arr`, in our case 10.

How?

We can also construct it by binary searching $B_{arr}$ or by means of a segmented scan.

# Constructing the Segment and Inner Indices

## From now on, we discuss only TWO-dimensional irregular arrays!

```
arr = [ [1,2,3], [4], [], [5,6], [7], [], [8,9,10] ]
```

$\Rightarrow$

$S^0_{arr}$ = [7]
$S^1_{arr}$ = [3, 1, 0, 2, 1, 0, 3]
$flen_{arr}$ = **reduce** (+) 0 $S^1_{arr}$ = 10
$D_{arr}$ = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]

## Segment and Inner indices (II):

$II^1_{arr}$ = [0, 0, 0, 1, 3, 3, 4, 6, 6, 6]
$II^2_{arr}$ = [0, 1, 2, 0, 0, 1, 0, 0, 1, 2]

## Constructing Segment and Inner indices (II):

```
(B^1_arr , F'_arr) = mkFlagArray S^1_arr 0 (iota (length S^1_arr))
        -- ([0, 3, 4, 4, 6, 7, 7], [0, 0, 0, 1, 3, 0, 4, 6, 0, 0])
F_arr = map bool.u32 F'_arr

II^1_arr = sgmScan^inc (+) 0 F_arr F'_arr

II^2_arr = map2 (\ i sgm -> i - B^1_arr[sgm] ) (iota flen_arr) II^1_arr
-- ^ this fuses better & performs less memory traffic than the below:
II^2_arr = sgmScan^inc (+) 0 F_arr (replicate flen 1) |> map (-1)
```

# $B^{inc}$ and $II^1$ Are the Important Ones

**Because you can deduce the other arrays by means of simple maps, that fuse better and generate less traffic.**

```
arr = [ [1,2,3], [4], [], [5,6], [7], [], [8,9,10] ]
```

$\Rightarrow$

$S^0_{arr}$ = [7]
$S^1_{arr}$ = [3, 1, 0, 2, 1, 0, 3]
$flen_{arr}$ = **reduce** (+) 0 $S^1_{arr}$ = 10
$D_{arr}$ = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]

## If we know the Segment Offsets ($B^{inc}_{arr}$) and Indices ($II^1_{arr}$):

$B^{inc}_{arr}$ = [3, 4, 4, 6, 7, 7, 10] -- *inclusive scan of* $S^1_{arr}$ *and*
$II^1_{arr}$ = [0, 0, 0, 1, 3, 3, 4, 6, 6, 6]

## We can efficiently compute the $S^1_{arr}$ and $F_{arr}$ arrays (also $II^2_{arr}$) by:

```
S¹_arr = iota (length B^inc_arr)
   |> map (\i -> if i == 0 then B^inc_arr[i] else B^inc_arr[i] - B^inc_arr[i-1])

F_arr = iota flen_arr -- flen_arr is the length of II¹_arr
   |> map (\i -> if i == 0 then true else II¹_arr[i] != II¹_arr[i-1])
```

**Note: $B^{inc}_{arr}$ different than $B^1_{arr}$: inclusive *vs* exclusive scan of shape!**

Parallel Basic Blocks Recap

Part I: Flattening Nested and Irregular Parallelism

- What is ”Flattening”? Recipe for Applying Flattening
- Several Re-Write Rules (inefficient for replicate & iota)
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- Revisiting the Rewrites for Replicate & Iota Nested Inside Map
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# Revisiting Replicate inside Map

## (3) Replicate nested inside a map:

```
res = map2 (\ n m -> replicate n m) [1,0,3,2] [7,3,8,9] ≡
res = [ replicate 1 7, replicate 0 3, replicate 3 8, replicate 2 9 ] ≡
res = [ [7], [], [8,8,8], [9,9] ]
```

```
res = map2(\n m-> replicate n m) ns ms
```

## becomes a very simple gather operation:

1. the shape of the result array is `ns`
2. build $\texttt{II}^1$ of a jagged array of shape `ns`

# Revisiting Replicate inside Map

## (3) Replicate nested inside a map:

```
res = map2 (\ n m -> replicate n m) [1,0,3,2] [7,3,8,9] ≡
res = [ replicate 1 7, replicate 0 3, replicate 3 8, replicate 2 9 ] ≡
res = [ [7], [], [8,8,8], [9,9] ]
```

```
res = map2(\n m-> replicate n m) ns ms
```

## becomes a very simple gather operation:

1. the shape of the result array is `ns`
2. build $\texttt{II}^1$ of a jagged array of shape `ns`
3. gather the corresponding values from `ms` by indexing through $\texttt{II}^1$

```
ℱ(res = map2 (\n m -> replicate n m) ns ms) ⇒      -- ms = [7,3,8,9]
1. S¹_res = ns                                     -- ns = [1,0,3,2]
2. II¹_res = ... -- construct II¹ for a jagged array of shape ns
                                                   -- [0, 2,2,2, 3,3]
3. D_res = map (\ sgm -> ms[sgm] ) II¹_res         -- [7, 8,8,8, 9,9]
```

# Nested *vs* Flattened Parallelism: Iota inside Map

**(4) Iota nested inside a map** (`(iota n)`$\equiv$`[0,...,n-1]`):

```
res = map (\i -> iota i) [1,3,2] ≡
res = [ iota 1, iota 3, iota 2 ] ≡ [ [0], [0,1,2], [0,1] ]
```

```
res = map (\n-> iota n) ns
```

**The result is exactly the $II^2$ array of a jagged array of shape** `ns`

$\mathcal{F}$`(res = map (\n -> iota n) ns)` $\Rightarrow$

1. $S^1_{res}$ `= ns` `-- ns = [1, 3, 2]`
2. $II^2_{res}$ `= ...` `-- construct II^2 of a jagged array of shape ns`
4. $D_{res}$ `=` $II^2_{res}$

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# Revisiting Our Demonstration of How to Flatten

## Contrived Example:

```
let arr = [1, 2, 3, 4] in
map (\i -> map (+(i+1)) (iota i)) arr
-- Result: [[2],[3,4],[4,5,6],[5,6,7,8]]
```

## I. Normalize the code:

```
map (\i -> let ip1 = i+1 in
           let iot = (iota i) in
           let ip1r = (replicate i ip1)
           in map2 (+) ip1r iot            ) arr
```

## II. Distribute the map across every statement in the body

and adjust the inputs accordingly ($\mathcal{F}$ denotes the transformation)

```
ℱ(map (\i -> map (+(i+1)) (iota i)) arr) ≡
1. let ip1s = map (\i -> i+1) arr in -- [2, 3, 4, 5]
2. let iots = ℱ(map (\i -> (iota i)) arr) in
3. let ip1rs= ℱ(map2 (\ i ip1 -> (replicate i ip1)) arr ip1s)
4. in  ℱ(map2 (\ ip1r iot -> map2 (+) ip1r iot) ip1rs iots)
```

We do **not** assume that `arr` contains strictly-positive integers.

# Revisiting Our Example: Think Like a Compiler

```
ℱ(map (\i -> map (+(i+1)) (iota i)) arr) ≡
1. let ip1s = map (\i -> i+1) arr in -- [2, 3, 4, 5]
2. let iots = ℱ(map (\i -> (iota i)) arr) in
3. let ip1rs= ℱ(map2 (\ i ip1 -> (replicate i ip1)) arr ip1s)
4. in  ℱ(map2 (\ ip1r iot -> map2 (+) ip1r iot) ip1rs iots)
```

## Applying the new rules results in:

```
1. S^1_res = arr                                          -- arr = [1, 2, 3, 4]
2. (B^1_res, F'_res) = mkFlagArray arr 0 (iota (length arr))
                                                          -- B^1_res = [0, 1, 3, 6]
3. F_res = map bool.u32 F'_res
4. II^1_res = sgmScan^inc (+) 0 F_res F'_res              -- [0, 1,1, 2,2,2, 3,3,3,3]
5. ip1s = map (\ i -> i+1 ) arr                           -- [2, 3, 4, 5]
6. iots = map2 (\ ind sgm -> ind - B^1_res[sgm] ) (iota flen_res) II^1_res
                                                  -- = II^2_arr = [0, 0,1, 0,1,2, 0,1,2,3]
7. ip1rs= map (\ sgm -> ip1s[sgm] ) II^1_res              -- [2, 3,3, 4,4,4, 5,5,5,5]
8. in map2 (+) ip1rs iots                                 -- [2, 3,4, 4,5,6, 5,6,7,8]
```

Lines 6 – 8 are trivially fusable ⇒ the `iots` and `ip1rs` arrays are not manifested in memory.

# Revisiting Our Example: Think Like a Human

## I. Normalize the code:

```
map (\i -> let ip1 = i+1 in
           let iot = (iota i) in
           let ip1r = (replicate i ip1)
           in map2 (+) ip1r iot            ) arr
```

## Using the new intuition results in:

1. $S^1_{res}$ = arr -- *arr = [1, 2, 3, 4]*
2. ($B^1_{res}$, $F'_{res}$) = mkFlagArray arr 0 (**iota** (length arr))
3. $F_{res}$ = **map bool**.u32 $F'_{res}$
4. $II^1_{res}$ = $sgmScan^{inc}$ (+) 0 $F_{res}$ $F'_{res}$
5. **in map2** (\ sgm ind -> **let** ip1 = arr[sgm] + 1
6. **let** iot_el = ind - $B^1_{res}$[sgm]
7. **in** ip1 + iot_el
8. ) $II^1_{res}$ (**iota** (length $II^1_{res}$))

Have done a tiny bit better job than the compiler, as array `ip1s` is not manifested either.

# Demonstrating Performance of New *vs* Old Rules

**PERFORMANCE**

**DEMONSTRATION**

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# Nested *vs* Flattened Parallelism: Reduce Inside Map

## (5) Reduce Inside a Map or Segmented Reduce:

```
let arr = [[1, 3, 4], [6, 7]] in
let res = map (\x -> reduce (+) 0 x) arr
-- should result in [8, 13]
```

# Nested *vs* Flattened Parallelism: Reduce Inside Map

## (5) Reduce Inside a Map or Segmented Reduce:

```
let arr = [[1, 3, 4], [6, 7]] in
let res = map (\x -> reduce (+) 0 x) arr
-- should result in [8, 13]
```

## translates to a scan-pack composition:

1. the length of `res` equals the number of subarrays of `arr`;
2. the shape of `arr` is scanned: the result records the position of the last element in a segment plus one;
3. segmented scan is applied on the input array: the last elem in a segment holds the reduced value of the segment;
4. segment's last element is extracted by a `map` operation.

$\mathcal{F}$(`res = map (\row -> reduce` $\odot$ $0_{\odot}$ `row) arr`) $\Rightarrow$

*-- $S^0_{arr}$ = [2], $S^1_{arr}$ = [3,2], $F_{arr}$ = [1,0,0,1,0], $D_{arr}$ = [1,3,4,6,7]*

1. $S^0_{res}$ = $S^0_{arr}$ *-- $S^0_{res}$ = [2]*
2. `indsp1 = scan (+) 0` $S^1_{arr}$ *-- indsp1 = [3, 5]*
3. `tmp = sgmScan` ($\odot$) $0_{\odot}$ $F_{arr}$ $D_{arr}$ *-- tmp = [1, 4, 8, 6, 13]*
4. $D_{res}$ = **map2**`(\s ip1 -> if s≤0 then` $0_{\odot}$ **else** `tmp[ip1-1])` $S^1_{arr}$ `indsp1` *-- $D_{res}$ = [8, 13]*

# Nested *vs* Flattened Parallelism: Reduce Inside Map

## (5) Reduce Inside a Map or Segmented Reduce:

```
let arr = [[1, 3, 4], [6, 7]] in
let res = map (\x -> reduce (+) 0 x) arr
-- should result in [8, 13]
```

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$\mathcal{F}$(res = map (\row -> reduce $\odot$ $0_{\odot}$ row) arr) $\Rightarrow$

*-- $S^0_{arr}$ = [2], $S^1_{arr}$ = [3,2], $F_{arr}$ = [1,0,0,1,0], $D_{arr}$ = [1,3,4,6,7]*

1. $S^0_{res}$ = $S^0_{arr}$ *-- $S^0_{res}$ = [2]*
2. $D_{res}$ = hist ($\odot$) $0_{\odot}$ ($S^0_{arr}$[0]) $II^1_{arr}$ $D_{arr}$

**How else can I try to optimize this code by hand?**

# Nested *vs* Flattened Parallelism: Reduce Inside Map

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1. $S^0_{res}$ = $S^0_{arr}$ *-- $S^0_{res}$ = [2]*
2. $D_{res}$ = hist ($\odot$) $0_{\odot}$ ($S^0_{arr}$[0]) $II^1_{arr}$ $D_{arr}$

### How else can I try to optimize this code by hand?

- practical performance refers to how many global-memory accesses you perform
- accessing $II^1_{arr}$ from memory has significant cost
- in some practical cases, it might be more efficient to not manifest $II^1_{arr}$, but instead to compute its elements by binary searching the $B^1_{arr}$ array.

# Flattening Scatter and Histogram

How does one flattens a scatter perfectly nested inside a map?

How does one flattens a histogram perfectly nested inside a map?

# Flattening Scatter and Histogram

How does one flattens a scatter perfectly nested inside a map?

How does one flattens a histogram perfectly nested inside a map?

You will have to answer it yourselves as part of the third weekly assignment :)

# Treating a Scalar Variant to the Outer Map

## (6) The inner construct uses a scalar variant to the outer map:

```
let res = map2 (\x ys -> map (+x) ys) [1,3] [[4,5,6], [9,7]] ≡
let res = [map (+1) [4,5,6], map (+3) [9,7]]
let res = [ [5,6,7], [12,10] ]
```

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let res = [ [5,6,7], [12,10] ]
```

Traditionally, this is handled by expanding (replicating) each x across the whole segment

```
let D_xss = [1, 1, 1, 3, 3]
let res = map2 (+) [1, 1, 1, 3,  3 ]
                   [4, 5, 6, 9,  7 ]
                    =  =  =  =   =
                   [5, 6, 7, 12, 10]
```

Instead, we use $\mathtt{II}^1_{arr}$ to indirectly access in the `xs` array:

$\mathcal{F}$(`res = map2 (\x ys -> map (f x) ys) xs yss`) $\Rightarrow$

*-- xs = [1,3], $S^1_{yss}$ = [3,2], $F_{yss}$ = [1,0,0,1,0], $D_{yss}$ = [4,5,6,9,7]*

1. $S^1_{res} = S^1_{yss}$
2. $D_{res}$ = **map2** (\y sgmind -> f xs[sgmind] y ) $D_{yss}$ $II^1_{yss}$

*-- $II^1_{yss}$ = [0,0,0,1,1], $D_{res}$ = [5,6,7,12,10]*

# Treating Indexing Variant to the Outer Map

## (7) Indexing Operations Variant to the Outer Map:

```
let res = map2 (\i xs -> xs[i]) [2,0] [[4,5,6], [9,7]] ≡
let res = [ 6, 9 ]
```

# Treating Indexing Variant to the Outer Map

## (7) Indexing Operations Variant to the Outer Map:

```
let res = map2 (\i xs -> xs[i]) [2,0] [[4,5,6], [9,7]] ≡
let res = [ 6, 9 ]
```

To corresponding flat index in $D_{yss}$ is obtained by summing up

- the start offset of every segment, which we get from $B^1_{yss}$, and
- the index inside the segment, which we get from `is`

$\mathcal{F}$`(res = map2 (\i xs -> xs[i]) is xss)` $\Rightarrow$

*-- is = [2,0], $S^1_{xss}$ = [3,2], $B^1_{xss}$ = [0,3], $D_{xss}$ = [4,5,6,9,7]*

1. $S^0_{res}$ = $S^0_{is}$ *-- = $S^0_{is}$ = [2]*
2. $D_{res}$ = **map2** (\ off i -> $D_{xss}$[off + i]) $B^1_{xss}$ is *-- $D_{res}$ = [6, 9]*

# Nested *vs* Flattened Parallelism: If Inside a Map 2D Case

## (8) If-Then-Else with inner parallelism nested inside a map:

```
bs  = [F,T,F,T]
xss = [[1,2,3],[4,5,6,7],[8,9],[10]]
res = map(\b xs -> if b  then map (+1) xs  else map (*2) xs) bs xss
res = [ map(*2)[1,2,3], map(+1)[4,5,6,7], map(*2)[8,9], map(+1)[10] ]
res = [ [2,4,6], [5,6,7,8], [16,18], [11] ]
```

# Nested *vs* Flattened Parallelism: If Inside a Map 2D Case

**(8) If-Then-Else with inner parallelism nested inside a map:**

```
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res = [ map(*2)[1,2,3], map(+1)[4,5,6,7], map(*2)[8,9], map(+1)[10] ]
res = [ [2,4,6], [5,6,7,8], [16,18], [11] ]
```

**translates to a scatter-map-gather composition**. Intuition:

1. compute `iinds`, the permutation of segments w.r.t. `bs`;

2-3. partition the `xss` array based on `bs`;

4-5. **flatten outer map and/on top of the parallel code of the `then` and `else` branches;**

6. inverse permute the resulted segments according to `iinds`.

```
1. iinds = partition2 (λi -> bs[i]) (iota (length b)) -- [1,3,0,2]
2. xss_then = gatherThen iinds xss -- ([4,1], [4,5,6,7, 10])
3. xss_else = gatherElse iinds xss -- ([3,2], [1,2,3,  8,9])
-- Recursively Flatten the Then and Else Branches!
4. res_then = ℱ(map (map (+1)) xss_then) -- ([4,1], [5,6,7,8, 11])
5. res_else = ℱ(map (map (*2)) xss_else) -- ([3,2], [2,4,6,16,18])
6. res = inversePermute iinds (res_then++res_else)
-- ([3,4,2,1], [2,4,6, 5,6,7,8, 16,18, 11])
```

# Nested *vs* Flattened Parallelism: If Inside a Map 2D Case

## (8) If-Then-Else with inner parallelism nested inside a map:

bs = [F,T,F,T], xss = [[1,2,3],[4,5,6,7],[8,9],[10]], $S^1_{xss}$=[3,4,2,1], f=map (+1), g=map (*2)

$\mathcal{F}$(res = map2 (\b xs -> if b **then** f xs **else** g xs) bs xss) $\Rightarrow$
(spl, iinds) = partition2 bs (**iota** (length bs)) -- (2, [1,3,0,2])
($S^1_{xss_{then}}$, $S^1_{xss_{else}}$) = split spl (**map** (\ii -> $S^1_{xss}$[ii]) iinds) --([4,1],[3,2])
$mask_{xss}$ = **map** (\sgmind -> bs[sgmind]) $II^1_{xss}$ -- [F,F,F,T,T,T,T,F,F,T]
(brk, $D^p_{xss}$) = partition2 $mask_{xss}$ $D_{xss}$
($D_{xss_{then}}$, $D_{xss_{else}}$) = split brk $D^p_{xss}$ -- ([4,5,6,7,10],[1,2,3,8,9])
($S^1_{res_{then}}$, $D_{res_{then}}$) = $\mathcal{F}$(**map** f) ($S^1_{xss_{then}}$, $D_{xss_{then}}$) -- ([4,1], [5,6,7,8,11])
($S^1_{res_{else}}$, $D_{res_{else}}$) = $\mathcal{F}$(**map** g) ($S^1_{xss_{else}}$, $D_{xss_{else}}$) -- ([3,2], [2,4,6,16,18])
$S^{1P}_{res}$ = $S^1_{res_{then}}$ ++ $S^1_{res_{else}}$ -- [4,1,3,2]
$S^1_{res}$ = **scatter** (**replicate** (length bs) 0) iinds $S^{1P}_{res}$ -- [3,4,2,1]
$B^1_{res}$ = **scan**$^{exc}$ (+) 0 $S^1_{res}$ -- [0,3,7,9]
$F^P_{res}$ = mkFlagArray $S^{1P}_{res}$ 0 (**map** (+1) iinds) -- [2,0,0,0,4,1,0,0,3,0]
$II^{1P}_{res}$ = sgmscan (+) 0 $F^P_{res}$ $F^P_{res}$|> **map**(\x -> x-1) -- [1,1,1,1,3,0,0,0,2,2]
$II^{2P}_{res}$ = $II^2_{res_{then}}$ ++ $II^2_{res_{else}}$ -- [0,1,2,3,0, 0,1,2,0,1]
$sinds_{res}$ = **map2** (\sgm iin -> $B^1_{res}$[sgm] + iin) $II^{1P}_{res}$ $II^{2P}_{res}$
--[3+0,3+1,3+2,3+3, 9+0, 0+0,0+1,0+2, 7+0,7+1]=[3,4,5,6,9,0,1,2,7,8]
$D_{res}$ = **scatter** (**replicate** $flen_{res}$ 0) $sinds_{res}$ ($D_{res_{then}}$ ++ $D_{res_{else}}$)
-- [2,4,6, 5,6,7,8, 16,18, 11]
($S^1_{res}$, $D_{res}$)

# Nested *vs* Flattened Parallelism: Do Loop Inside a Map

## (9) Flattening a Do Loop Nested Inside a Map:

- compute the maximal loop count $n_{max}$
- interchange the loop and the map:
  - loop count becomes $n_{max}$
  - the loop body is wrapped inside a `if i<n` condition, and
  - the new loop body is flattened!

$\mathcal{F}$(`res = map2 (\n xs -> loop (xs) for i < n do f xs) ns xss`) $\Rightarrow$

1. $n_{max}$ = **reduce** max 0i32 ns
2. g i m arr = **if** i < m **then** f arr **else** arr
3. **loop** ($S^1_{xss}$, $D_{xss}$) **for** i < $n_{max}$ **do**
4. $\mathcal{F}$(**map2** (g i)) ns ($S^1_{xss}$, $D_{xss}$)
5. -- $(g\ i)^L$ ns ($S^1_{xss}$, $D_{xss}$)

But this treatment does not necessarily preserve the work asymptotic ... what to do?

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But this treatment does not necessarily preserve the work asymptotic ... what to do?

If the size of the result can be deduced/inferred:

- allocate the flat-result array before the loop
- filter out the empty segments, and make the loop iterate until the shape is empty
- each time a segment finish execution (i) it is scattered into the result, and (2) it is filtered out from the running set of segments.

Parallel Basic Blocks Recap

Part I: Flattening Nested and Irregular Parallelism

- What is ”Flattening”? Recipe for Applying Flattening
- Several Re-Write Rules (inefficient for replicate & iota)
- Jagged (Irregular Multi-Dim) Array Representation
- Revisiting the Rewrites for Replicate & Iota Nested Inside Map
- Revisiting the Solution to Our Example

Part II: Flattening Nested and Irregular Parallelism

- More Flattening Rules
- Flattening by Function Lifting
- Flattening Quicksort
- Flattening Prime-Number (Sieve) Computation

# Flattening by Function Lifting: Basic Idea

Assume a simple function `f`:

```
let f (x: i32) : i32 = x + 1
```

`f` lifted, denoted $f^L$ semantically corresponds to `map f`, where the arguments have been expanded to an extra array dimension, and the inner operators/functions have also been lifted:

```
let +^L [n] (as: [n]i32) (bs: [n]i32) : [n]i32 =
    map2 (+) as bs

let f^L [n] (xs: [n]i32) : [n]i32 =
    xs +^L (replicate n 1)
```

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    xs +^L (replicate n 1)
```

- Locals such as x $\Rightarrow$ left alone
- Global such as + $\Rightarrow$ lifted ($+^L$)
- Constants such as k $\Rightarrow$ `replicate (length xs) k`
  - good for vectorization, bad for locality, asymptotics
  - for GPU better to indirectly index into a smaller array, rather than replicate.

# Flattening by Function Lifting: Key Insight!

```
let f (xs: []i32) : []i32 = map g xs -- = g^L xs
let f^L (xss: [][]i32) : [][]i32 = (g^L)^L -- ???
```

How do we stop lifting? g and $g^L$ are enough: no need for $(g^L)^L$!

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```

How do we stop lifting? g and $g^L$ are enough: no need for $(g^L)^L$!

```
let f (xs: []i32) : []i32 = map g xs -- = g^L xs
-- in nested parallel form
let f^L (xss: [][]i32) : [][]i32 =
    segment xss (g^L (concat xss))
-- in flatten form
let f^L (S^1_xss: []i32, D_xss: []i32) : ([]i32, []i32) =
    (S^1_xss, g^L D_xss)
```

In Haskell Notation:

```
concat  :: [[a]]  ->  [a]
segment :: [[a]]  ->  [b]     ->   [[b]]
           shape     flat data    nested data
```

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# Recounting Quicksort

## Recount the classic nested-parallel definition:

```
let quicksort [n] (arr : [n]f32) : [n]f32 =
  if n < 2 then arr else
  let i = getRand (0, (length arr) - 1)
  let a = arr[i]
  let s1 = filter (< a ) arr
  let s2 = filter (== a) arr
  let s3 = filter (>  a) arr
  in  (quicksort s1) ++ s2 ++ (quicksort s3)
  -- can be re-written as:
  -- rs = map nestedQuicksort [s1, s3]
  -- in (rs[0]) ++ s2 ++ (rs[1])
```

Note: Futhark does not support recursive calls, hence not valid code!

# Nested-Parallel Quicksort Simplified

**For simplicity we will rewrite it in terms of `partition2`:**

```
let isSorted [n] (as: [n]f32) : bool =
    map (\i -> if i==0 then true else as[i-1] < as[i]) (iota n)
    |> reduce (&&) true

let quicksort [n] (arr: [n]f32) : [n]f32 =
  if isSorted arr then arr else
  let i = getRand (0, (length arr) - 1)
  let a = arr[i]
  let bs = map (< a) arr
  let (q, arr') = partition2 bs 0.0f32 arr
  let (arr<, arr≥) = split q arr'
  in  concat <| map quicksort [arr<, arr≥]
```

Note: Futhark does not support recursive calls, irregular `map` operation, or `concat`!

# Partition2

Reorders the elements of an array such that those that correspond to a true mask come before those corresponding to false.

```
let partition2 [n] 't (conds: [n]bool) (dummy: t) (arr: [n]t)
        : (i32, [n]t) =
  let tflgs = map (\ c -> if c then 1 else 0) conds
  let fflgs = map (\ b -> 1 - b) tflgs

  let indsT = scan (+) 0 tflgs
  let tmp   = scan (+) 0 fflgs
  let lst   = if n > 0 then indsT[n-1] else -1
  let indsF = map (+lst) tmp

  let inds  = map3 (\ c indT indF -> if c then indT-1 else indF-1)
                   conds indsT indsF

  let fltarr= scatter (replicate n dummy) inds arr
  in  (lst, fltarr)
```

For example:

```
conds = [F,T,F,T,F,F,T]
xss =   [1,2,3,4,5,6,7]
partition2 conds 0 xss => (3, [2,4,7,1,3,5,6])
```

# Lifting Quicksort

**Key Idea: write a function with the semantics of**
`map nestedQuicksort`, i.e., it operates on array of arrays.

```
let isSorted [n] (as: [n]f32) : bool =
    map (\i -> if i==0 then true else as[i-1] < as[i]) (iota n)
    |> reduce (&&) true

let quicksort^L (xss: [][]f32) : [][]f32 =
  map (\xs ->
        if isSorted xs then xs else  -- oh, boy, not if-then-else!
        let i = getRand (0, (length xs) - 1)
        let a = xs[i]
        let bs = map (< a) xs
        let (q, xs^p) = partition2 bs 0.0f32 xs
        let (xs_<, xs_≥) = split q xs^p
        in  concat <| map quicksort [xs_<, xs_≥]
  ) xss
```

Important observations:

- `map quicksort` $\equiv$ `quicksort`$^L$
- `map(map quicksort)`$\equiv$`quicksort`$^{L^L}\equiv$`segment o quicksort`$^L$`o concat`
- the flat data of $[xs_<, xs_\geq] \equiv xs^p$, the result of `partition2`

# Lifting Quicksort

Let us treat the last three lines from the previous implem:.

```
let quicksort^L (S^1_xss :[]i32 , D_xss :[]f32): ([]i32 ,[]f32) = --(xss: [][]f32)
  if isSorted D_xss then (S^1_xss :[]i32 , D_xss :[]f32) else -- big cheat!
  let (S^1_bss , D_bss) = F (
      map (\xs ->
             let i = getRand (0, (length xs) - 1)
             let a = xs[i]
             let bs = map (< a) xs
             in  bs
          ) xss
    )
  let (ps, (S^1_xssp ,D_xssp )) = partition2^L D_bss 0.0f32 (S^1_xss , D_xss)
  -- Invariant: S^1_xssp == S^1_bss == S^1_xss
  let S^1_[xss<,xss≥] = filter (!=0) <| flatten <|
        map2 (λ p s -> if s==0 then [0,0] else [p,s-p]) ps S^1_xss
  in  quicksort^L (S^1_[xss<,xss≥], D_xssp)
```

- $S^1_{[xss_{<a}, xss_{\geq a}]}$ is the shape of $[xs_<, xs_\geq]$
- $(\texttt{concat} <| \texttt{quicksort}^L)^L \texttt{xsss} \equiv \texttt{concat} <| \texttt{segment xsss} <| \texttt{quicksort}^L (\texttt{concat xsss}) \equiv \texttt{quicksort}^L (\texttt{concat xsss})$
- The function looks tail recursive now: let's replace it with a loop!

# Lifting Quicksort: Final Implementation

```
let quicksort^L [m][n] (S^1_xss :[m]i32, D_xss :[n]f32): [n]f32 =
  let (stop, count) = (isSorted D_xss, 0i32)
  let (_,res,_,_) =
    loop(S^1_xss, D_xss, stop, count) while (!stop) do
      -- compute helper-representation structures
      let B^1_xss = scan^exc (+) 0 S^1_xss
      let F^1_xss = mkFlagArray S^1_xss 0i32 <| map (+1) <| iota m
      let II^1_xss = sgmscan (+) 0 F^1_xss <|
                     map (\f -> if f==0 then 0 else f-1) F^1_xss
      -- flattening quicksort:
      let rL = map (\u -> randomInd (0,u-1) count) S^1_xss
      let aL = map3(\r l i-> if l <= 0 then 0.0 else D_xss[B^1_xss[i]+r]
                    ) rL S^1_xss (iota m)
      let D_bss= map2 (\x sgmind -> aL[sgmind] > x ) D_xss II^1_xss
      let (ps, (S^1_xss^p ,D^per_xss)) = partition2^L D_bss 0.0f32 (S^1_xss, D_xss)
      let S^1_[xss<,xss≥] = filter (!=0) <| flatten <|
            map2 (\ p s -> if s==0 then [0,0] else [p,s-p]) ps S^1_xss
      in  (S^1_[xss<,xss≥], D^per_xss, isSorted D^per_xss, count+1)
  in  res
```

PFP Weekly 2 Exercise: Implement `partition2`$^L$

Parallel Basic Blocks Recap

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# How Does One Flattens Prime Numbers?

## The important bit with nested parallelism:

```
sqrt_primes = primesOpt (sqrt (fromIntegral n))
nested = map (\p -> let m = (n `div` p)
                        in  map (\j -> j*p) [2..m]
             ) sqrt_primes
not_primes  = reduce (++) [] nested
```

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not_primes  = reduce (++) [] nested
```

## Normalize the nested map:

```
sqrt_primes = primesOpt (sqrt (fromIntegral n))
nested = map (\p ->
                  let m   = n `div` p        in      -- distribute map
                  let mm1 = m - 1            in      -- distribute map
                  let iot = iota mm1         in          -- F rule 4
                  let twom= map (+2) iot     in          -- F rule 2
                  let rp  = replicate mm1 p  in          -- F rule 3
                  in  map (\(j,p) -> j*p) (zip twom rp) -- F rule 2
             ) sqrt_primes
not_primes  = reduce (++) [] nested       -- ignore, already flat
```

Flattening PrimeOpt was part of PMPH's Weekly Assignment 2!