Probability distributions EBP038A05
Lecture slides

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CONTENTS

1 LECTURE 2

Ex 1.1. Let $L = \min\{X, Y\}$, where $X, Y \sim \text{Geo}(p)$ and independent. What is the domain of L? Then, use the fundamental bridge and 2D LOTUS to show that

$$\mathsf{P}\{L \geq i\} = q^{2i} \implies L \sim \text{Geo}(1 - q^2).$$

h.1.1. The fundamental bridge and 2D LOTUS have the general form

$$\mathsf{P}\{g(X,Y) \in A\} = \mathsf{E}\left[I_{g(X,Y)\in A}\right] = \sum_i \sum_j I_{g(i,j)\in A}\, \mathsf{P}\{X = i, Y = j\}.$$

Take $g(i,j) = \min\{i,j\}$.

s.1.1. With the hint,

$$\begin{aligned}
\mathsf{P}\{L \geq k\} &= \sum_i \sum_j I_{\min\{i,j\}\geq k}\, \mathsf{P}\{X = i, Y = j\} \\
&= \sum_{i\geq k} \sum_{j\geq k} \mathsf{P}\{X = i\}\, \mathsf{P}\{Y = j\} \\
&= \mathsf{P}\{X \geq k\}\, \mathsf{P}\{Y \geq k\} = q^k q^k = q^{2k}.
\end{aligned}$$

$\mathsf{P}\{L > i\}$ has the same form as $\mathsf{P}\{X > i\}$, but now with q^{2i} rather than q^i.

Ex 1.2. Let $M = \max\{X,Y\}$, where $X,Y \sim \text{Geo}(p)$ and independent. Show that

$$\mathsf{P}\{M=k\} = 2pq^k(1-q^k) + p^2q^{2k}.$$

h.1.2. Use 2D LOTUS on $g(x,y) = I_{\max\{x,y\}=k}$.

s.1.2.

$$\begin{aligned}
\mathsf{P}\{M=k\} &= \mathsf{P}\{\max\{X,Y\}=k\} \\
&= p^2 \sum_{ij} I_{\max\{i,j\}=k}\, q^i q^j \\
&= 2p^2 \sum_{ij} I_{i=k} I_{j<k}\, q^i q^j + p^2 \sum_{ij} I_{i=j=k}\, q^i q^j \\
&= 2p^2 q^k \sum_{j<k} q^j + p^2 q^{2k} \\
&= 2p^2 q^k \frac{1-q^k}{1-q} + p^2 q^{2k}
\end{aligned}$$

Ex 1.3. Explain that

$$\mathsf{P}\{L = i, M = k\} = 2p^2 q^{i+k} I_{k>i} + p^2 q^{2i} I_{i=k}).$$

s.1.3.

$$\begin{aligned}\mathsf{P}\{L = i, M = k\} &= 2\,\mathsf{P}\{X = i, Y = k\}\, I_{k>i} + \mathsf{P}\{X = Y = i\}\, I_{i=k} \\ &= 2p^2 q^{i+k} I_{k>i} + p^2 q^{2i} I_{i=k}.\end{aligned}$$

Ex 1.4. With the previous exercise, use marginalization to compute the marginal PMF $\mathsf{P}\{M = k\}$.x

s.1.4.

$$\begin{aligned}
\mathsf{P}\{M = k\} &= \sum_i \mathsf{P}\{L = i, M = k\} \\
&= \sum_i (2p^2 q^{i+k} I_{k>i} + p^2 q^{2i} I_{i=k}) \\
&= 2p^2 q^k \sum_{i=0}^{k-1} q^i + p^2 q^{2k} \\
&= 2pq^k(1-q^k) + p^2 q^{2k} \\
&= 2pq^k + (p^2 - 2p)q^{2k},
\end{aligned}$$

Ex 1.5. Now take X, Y iid and $\sim \text{Exp}(\lambda)$. Use the fundamental bridge to show that for $u \le v$, the joint CDF has the form

$$F_{L,M}(u,v) = \mathsf{P}\{L \le u, M \le v\} = 2\int_0^u (F_Y(v) - F_Y(x)) f_X(x)\,\mathrm{d}x.$$

s.1.5. First the joint distribution. With $u \le v$,

$$\begin{aligned}
F_{L,M}(u,v) &= \mathsf{P}\{L \le u, M \le v\} \\
&= 2\iint I_{x\le u, y\le v, x\le y} f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= 2\int_0^u \int_x^v f_Y(y)\,\mathrm{d}y f_X(x)\,\mathrm{d}x && \text{independence} \\
&= 2\int_0^u (F_Y(v) - F_Y(x)) f_X(x)\,\mathrm{d}x.
\end{aligned}$$

Ex 1.6. Take partial derivatives to show that for the joint PDF,

$$f_{L,M}(u,v) = 2f_X(u)f_Y(v)I_{u\le v}.$$

s.1.6. Taking partial derivatives,

$$\begin{aligned}
f_{L,M}(u,v) &= \partial_v \partial_u F_{L,M}(u,v) \\
&= 2\partial_v \partial_u \int_0^u (F_Y(v) - F_Y(x))f_X(x)\,\mathrm{d}x \\
&= 2\partial_v \{(F_Y(v) - F_Y(u))f_X(u)\} \\
&= 2f_X(u)\partial_v F_Y(v) \\
&= 2f_X(u)f_Y(v).
\end{aligned}$$