Probability distributions EBP038A05
Study Guide

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INTRODUCTION

This study guide contains material organized per chapter of BH.

1. The simple questions and exercises are based on each section of BH and are meant to practice while you read. These questions are (often much) simpler than exam questions, but just help you to read and study well. It takes time and attention to understand definitions and notation. Mind that good notation and good understanding strongly correlate.

2. The part related to the obligatory exercises of BH provides motivational comments, hints and solutions.

3. The third part contains challenges. These problems are (quite a bit) above exam level, hence optional. However, if you like to be intellectually challenged, then you'll like these problems a lot.

In general, when working on the exercises, try first hard to find the answer and *write it down* on paper. Only after having written your answer on paper, meticulously compare your work with ours. Like this you' ll get a lot of feedback, and you'll see that it is quite hard to get the details right.

Finally, we included many old exam questions. Of course you are not expected to make them all. Instead do a just few until you feel comfortable with the level.

The selection of exercises in the table above are the bare minimum; I advice you to do more. To assure you, I found the problems quite hard at times; probability never 'comes for free'; not for you, not for me, not for anybody. You can expect to spend between 30 minutes (and sometimes more) per problem; if you are serious.

Here is a list of good, and important, advice when making the exercises. (As a student I did not always do this, partly because I was not aware about how useful this advice is. Hopefully you are smart enough to avoid making the same mistakes as I did as a student.)

- Read an example in the book. Close the book, and try to redo the example. When I try, I often fail. Why is that? Simple: I did not really think about the example while just reading it, I skimmed it. But you should get used to the fact that reading requires pen and paper.

- Before trying to solve an exercise, read all parts of it, i.e., part a, b, etc. Ensure you understand the problem.

- Before actually solving an exercise, *make a plan on how to solve it.* A first step is to look for simple corner cases (set things to zero, make certain probabilities equal to one, and so on), make extra assumptions that simplify the problem, and solve the problem under these simplifying (stronger) assumptions. Then drop an assumption, and try to generalize to a pattern or some property you expect to hold. You'll be astonished to see how many problems you can actually solve by following this strategy. And even if you cannot solve it with this approach, the corner cases help to check throughout whether you're still working in the right direction. Also, reduce the problem to simpler cases you do understand. Try to solve the simpler problem first, and then generalize.

- Carry out your plan, and *relax* if you cannot directly find the answer.
- Look back after solving the problem, and try to find a general pattern you used to solve the problem. Can you use this for other problems too?
- Look back again at the problem some time later. In other words, do not solve a problem just once, but also a few weeks later again. This is often very revealing.
- Work every day a reasonable amount of time. This is much more effective than working 10 h on one day, and not at all the next. The concept is often called 'Kaizen': try to improve every day a little bit. Over the course of time, you'll be amazed how much you can achieve.
- Finally, when I am stuck, this piece of advice of Jim Rohn (an author on personal development) helps: 'Don't wish it was easier, wish you were better.'

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CHAPTER 7: QUESTIONS AND REMARKS

7.1 SIMPLE QUESTIONS

Section 7.1

Ex 7.1.1. In your own words, explain what is

1. a joint PMF, PDF, CDF;
2. a conditional PMF, PDF, CDF;
3. a marginal PMF, PDF, CDF.

Ex 7.1.2. Suppose the probability of obtaining a head twice out of two coin flips is $\mathsf{P}\{X_1 = H, X_2 = H\}$. What has this to do with joint PMFs? Can you generalize this idea to other examples?

Ex 7.1.3. In the previous exercise, suppose the outcome of the second throw is always equal to that of the first. Specify the joint PMF.

Ex 7.1.4. We have the random vector $(X, Y) \in [0,1]^2$ (here $[0,1]^2 = [0,1] \times [0,1]$) consisting of the rvs X and Y with the joint PDF $f_{X,Y}(x, y) = 2\, I_{x \leq y}$.

1. Are X and Y independent?
2. Compute $F_{X,Y}(x, y)$.

Ex 7.1.5. We have two continuous rvs X, Y. Suppose the joint CDF factors into the product of the marginals, i.e., $F_{X,Y}(x, y) = F_X(x)F_Y(y)$. Can it still be possible in general that the joint PDF does not factor into a product of marginals PDFs of X and Y, i.e., $f_{X,Y}(x, y) \neq f_X(x)f_Y(y)$?

Ex 7.1.6. BH define the conditional CDF given an event A on page 416 as $F(y|A)$. Use this definition to write $F_{X,Y}(x, y)/F_X(x)$ as a conditional CDF. Is this equal to the conditional CDF of X and Y?

Ex 7.1.7. Let X be uniformly distributed on the set $\{0,1,2\}$ and let $Y \sim \text{Bern}(1/4)$; X and Y are independent.

1. Present a contingency table for X and Y.
2. What is the interpretation of the column sums of the table?
3. What is the interpretation of the row sums of the table?

4. Suppose you would change some of the entries in the table. Are X and Y still independent?

Ex 7.1.8. A machine makes items on a day. Some items, independent of the other items, are failed (i.e., do not meet the quality requirements). What are N and p in the context of the chicken-egg story of BH? What are the 'eggs' in this context, and what is the meaning of 'hatching'? What type of 'hatching' do we have here?

Ex 7.1.9. We have two rvs X and Y on $\mathbb{R}^+$. It is given that $F_{X,Y}(x,y) = F_X(x)F_Y(y)$ for $x, y \leq 1/3$. It is true that then X and Y are necessarily independent.

Ex 7.1.10. I select a random guy from the street, his height $X \sim \text{Norm}(1.8, 0.1)$, and I select a random woman from the street, her height is $Y \sim \text{Norm}(1.7, 0.08)$. I claim that since I selected the man and the woman independently, their heights are independent. Briefly comment on this claim.

Ex 7.1.11. For any two rvs X and Y on $\mathbb{R}^+$ with marginals F_X and F_Y, can it hold that $\mathsf{P}\left\{X \leq x, Y \leq y\right\} = F_X(x)F_Y(y)$?

Ex 7.1.12. Theorem 7.1.11. What is the meaning of the notation $X|N = n$?

Ex 7.1.13. Let X, Y be two discrete rvs with CDF $F_{X,Y}$. Can we compute the PDF as $\partial_x \partial_y F_{X,Y}(x,y)$?

Ex 7.1.14. Redo BH.7.1.24 with indicator functions and the fundamental bridge (recall, $\mathsf{P}\{A\} = \mathsf{E}[I_A]$ for an event A). (Indicators are often easy to use, and prevent many mistakes, as is demonstrated with this example.)

Section 7.2

Ex 7.1.15. BH.7.2.2. Write down the integral to compute $\mathsf{E}\left[(X-Y)^2\right]$, and solve it.

Ex 7.1.16. Explain that for a continuous r.v. X with CDF F and a and b (so it might be that $a > b$),

$$\mathsf{P}\{a < X < b\} = [F(b) - F(a)]^+. \tag{7.1.1}$$

Remark 7.1.17. If you are like me, you underestimate at first the importance of using indicator functions. In fact, they are extremely useful for several reasons.

1. They help to keep your formulas clean.

2. You can use them in computer code as logical conditions, or to help counting relevant events, something you need when numerically estimating multi-D integrals, for machine learning for instance.

3. Even though figures give geometrical insight into how to integrate over an 2D area, when it comes to reversing the sequence of integration, indicators are often easier to use.

4. In fact, *expectation is the fundamental concept in probability theory,* and the *probability of an event is defined* as
$$\mathsf{P}\{A\} := \mathsf{E}[I_A]. \tag{7.1.2}$$
Thus, the fundamental bridge is actually an application of LOTUS to indicator functions. Hence, reread BH.4.4!

Ex 7.1.18. What is $\int_{-\infty}^{\infty} I_{0\leq x\leq 3}\,\mathrm{d}x$?

Ex 7.1.19. What is
$$\int x\, I_{0\leq x\leq 4}\,\mathrm{d}x? \tag{7.1.3}$$

When we do an integral over a 2D surface we can first integrate over the x and then over the y, or the other way around, whatever is the most convenient. (There are conditions about how to handle multi-D integral, but for this course these are irrelevant.)

Ex 7.1.20. What is
$$\iint xy\, I_{0\leq x\leq 3}\, I_{0\leq y\leq 4}\,\mathrm{d}x\,\mathrm{d}y? \tag{7.1.4}$$

Ex 7.1.21. What is
$$\iint I_{0\leq x\leq 3}\, I_{0\leq y\leq 4}\, I_{x\leq y}\,\mathrm{d}x\,\mathrm{d}y? \tag{7.1.5}$$

Ex 7.1.22. Take $X \sim \text{Unif}([1,3])$, $Y \sim \text{Unif}([2,4])$ and independent. Compute
$$\mathsf{P}\{Y \leq 2X\}. \tag{7.1.6}$$

Section 7.3

Ex 7.1.23. Give a brief example of a situation where it might be more convenient to employ the correlation than the covariance. Explain why.

Ex 7.1.24. In queueing theory the concept of squared coefficient of variance SCV of a rv X is very important. It is defined as $C = \mathsf{V}[X]/(\mathsf{E}[X])^2$. Is the SCV of X equal to $\text{Corr}(X,X)$? Can it happen that $C > 1$?

Ex 7.1.25. Prove the key properties 1–5 of the covariance below BH.7.3.2.

Ex 7.1.26. Using the definition of Covariance (BH.7.3.1) derive the expression $\mathsf{Cov}[X,Y] = \mathsf{E}[XY] - \mathsf{E}[X]\,\mathsf{E}[Y]$. Use this to show why independence of X and Y implies their uncorrelatedness (Note that the converse does not hold).

Ex 7.1.27. Let U, V be two rvs and let $a, b \in \mathbb{R}$. Use the previous question to express $\mathsf{Cov}[a(U+V), b(U-V)]$ in terms of $\mathsf{V}[U]$, $\mathsf{V}[V]$ and $\mathsf{Cov}[U,V]$.

Ex 7.1.28. The solution of BH.7.3.6 is a somewhat tricky; I would have not found this trick myself. Here is an approach that is trick free.

Neglecting the event $\{X = Y\}$ as this has zero probability, we know that $M = X, L = Y$ or $M = Y, L = X$. Use this idea and the formula $\mathsf{Cov}[M,L] = \mathsf{E}[ML] - \mathsf{E}[M]\,\mathsf{E}[L]$ to derive the result of this example.

Section 7.4

Ex 7.1.29. Come up with a short illustrative example in which the random vector $\mathbf{X} = (X_1, \dots, X_6)$ follows a Multinomial Distribution with parameters $n = 10$ and $\mathbf{p} = (\frac{1}{6}, \dots, \frac{1}{6}) \in \mathbb{R}^6$.

Section 7.5

Ex 7.1.30. Is the following claim correct? If the rvs X, Y are both normally distributed, then (X, Y) follows a Bivariate Normal distribution.

Ex 7.1.31. Let X, Y, Z be iid $\mathcal{N}(0, 1)$. Determine whether or not the random vector

$$\mathbf{W} = (X + 2Y, 3X + 4Z, 5Y + 6Z, 2X - 4Y + Z, X - 9Z, 12X + \sqrt{3}Y - \pi Z + 18)$$

is Multivariate Normal. (Explain in words, don't do a lot of tedious math here!)

7.2 MEMORYLESS EXERCURSIONS: A CONFUSING PROBLEM WITH MEMORYLESS RVS

BY give a quick argument to compute $\mathsf{E}[M]$ and $\mathsf{E}[L]$ where $M = \max\{X, Y\}$ and $L = \min\{X, Y\}$ are the maximum and minimum of two iid exponential rvs X and Y. Since X and Y have the same distribution,

$$\mathsf{E}[L] + \mathsf{E}[M] = \mathsf{E}[L + M] = \mathsf{E}[X + Y] = 2\,\mathsf{E}[X].$$

Therefore,

$$\mathsf{E}[M] = 2\,\mathsf{E}[X] - \mathsf{E}[L]. \tag{7.2.1}$$

Next, by the fact that X and Y are memoryless,

$$\mathsf{E}[M] = \mathsf{E}[L] + \mathsf{E}[X]. \tag{7.2.2}$$

An interpretation can help to see this. There are two machines, each working on a job in parallel. Let X and Y be the production times at either machine. The time the first job finishes is evidently $L = \min\{X, Y\}$. Then, *due to memorylessness*, the service time of the remaining job 'restarts'; this takes an expected time $\mathsf{E}[X]$ to complete. Adding these two equations and noting that $\mathsf{E}[L]$ cancels we get $2\,\mathsf{E}[M] = 3\,\mathsf{E}[X]$, hence:

$$\mathsf{E}[M] = \frac{3}{2}\mathsf{E}[X], \qquad \mathsf{E}[L] = \mathsf{E}[M] - \mathsf{E}[X] = \frac{1}{2}\mathsf{E}[X]. \tag{7.2.3}$$

This argument seems general enough, so it must hold for discrete memoryless rvs too, i.e., when $X, Y \sim \mathrm{Geo}(p)$. But that is not the case: it is only true when $X, Y \sim \mathrm{Exp}(\lambda)$ and independent. To see what is wrong I tried as many different approaches to this problem I could think of, which resulted in this text.[1] In Section 7.2.1 we'll derive (7.2.1) for geometric rvs in

[1] Part of this material was born out of annoyance. The book uses one of those typical probability arguments: slick, half complete, and wrong as soon as one tries it in other situations. In other words, the type of argumnent beginner books should stay clear of. I admit that I was quite irritated about the argument offered by the book.

multiple different ways. Hence, the culprit must be (7.2.2). Then, in Section 7.2.2 we'll show that both equations *are true* for exponenial rvs. Finally, in Section 7.2.3 we find a formula that is similar to $\mathsf{E}[M] = \mathsf{E}[L] + \mathsf{E}[X]$ but that holds for both types of memoryless rvs, whether they are discrete or continuous.

THE ANALYSIS OF the above problem illustrates many general and useful probability concepts such as joint CDF, joint PMF/PDF, the fundamental bridge, integration over 2D areas, 2D LOTUS, conditional PMF/PDF, MGFs, and the change of variables formula. It pays off to do the exercises yourself and then study the hinta and solutions carefully.

YOU'LL NOTICE, hopefully, that I use many different methods to the same problem, and that I take pains to see how the answers of these methods relate. There are at least two reasons for this. Often, a problem can be solved in multiple ways, and one method is not necessarily better than another; better yet, different methods may augment the understanding of the problem. The second reason is that it is easy to make a mistake in probability. If different methods give the same answer, the probability of having made a mistake becomes smaller.

7.2.1 *Discrete memoryless rvs*

Before embarking on a problem, it often helps to refresh our memory. This is what we do first. Let X be $\sim \text{Geo}(p)$ and write $q = 1-p$.

Ex 7.2.1. What is the domain of X?

With some fun tricks with recursions it is possible to quickly derive the most important expressions for geometric rvs:

$$\begin{aligned}
\mathsf{P}\{X>0\} &= \mathsf{P}\{\text{failure}\} = q \\
\mathsf{P}\{X>j\} &= q\,\mathsf{P}\{X>j-1\} \implies \mathsf{P}\{X>j\} = q^j\,\mathsf{P}\{X>0\} = q^{j+1}. \\
\mathsf{P}\{X\geq j\} &= \mathsf{P}\{X>j-1\} = q^j. \\
\mathsf{P}\{X=j\} &= \mathsf{P}\{X>j-1\} - \mathsf{P}\{X>j\} = q^j - q^{j+1} = (1-q)q^j = pq^j. \\
\mathsf{E}[X] &= p\cdot 0 + q(1+\mathsf{E}[X]) \implies \mathsf{E}[X] = q/(1-q) = q/p.
\end{aligned}$$

Mind that, even though this is neat, it only work for geometric rvs.

Ex 7.2.2. Explain the above.

Clearly, such tricks are nice and quick, but they are not general. We should also practice with the general method.

Ex 7.2.3. Simplify $\mathsf{P}\{X>j\} = \sum_{i=j+1}^{\infty}\mathsf{P}\{X=i\}$ to see that this is equal to q^{j+1}. Realize that from this, $\mathsf{P}\{X\geq j\} = \mathsf{P}\{X>j-1\} = q^j$.

Ex 7.2.4. Use indicator variables show that

$$\mathsf{E}[X] = \sum_{i=0}^{\infty} i\,\mathsf{P}\{X=i\} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} I_{j<i}\,\mathsf{P}\{X=i\} = p/q.$$

Ex 7.2.5. Look up the definition of a memoryless rv, and check that X is memoryless.

WITH THIS REFRESHER, we can derive some useful properties of the minimum $L = \min\{X, Y\}$, where $Y \sim \text{Geo}(p)$ and independent of X. For this we use the fundamental bridge and 2D LOTUS, which in general read like

$$\mathsf{P}\left\{g(X,Y) \in A\right\} = \mathsf{E}\left[I_{g(X,Y)\in A}\right] = \sum_i \sum_j I_{g(i,j)\in A}\, \mathsf{P}\left\{X = i, Y = j\right\}.$$

Ex 7.2.6. What is the domain of L? Then, show that

$$\mathsf{P}\{L \geq i\} = q^{2i} \implies L \sim \text{Geo}(1-q^2).$$

Ex 7.2.7. Show that

$$\mathsf{E}[L] = q^2/(1-q^2).$$

Ex 7.2.8. Show that

$$\mathsf{E}[L] + \mathsf{E}[X] = \frac{q}{1-q}\frac{1+2q}{1+q}. \tag{7.2.4}$$

NOW WE CAN combine these facts with the properties of the maximum $M = \max\{X, Y\}$.

Ex 7.2.9. Show that

$$2\,\mathsf{E}[X] - \mathsf{E}[L] = \frac{q}{1-q}\frac{2+q}{1+q}.$$

Clearly, unless $q = 0$, $\mathsf{E}[L] + \mathsf{E}[X] \neq 2\,\mathsf{E}[X] - \mathsf{E}[L]$, hence, $\mathsf{E}[M]$ can only be one of the two and (7.2.1) and (7.2.2) cannot be both true.

To convince ourselves that [7.2.9], hence (7.2.1), is indeed true, we pursue three ideas.

HERE IS THE FIRST idea.

Ex 7.2.10. Show for the PMF of M that

$$p_M(k) = \mathsf{P}\{M = k\} = 2pq^k(1-q^k) + p^2q^{2k}.$$

Ex 7.2.11. With the previous exercise, show now that $p_M(k) = 2\,\mathsf{P}\{X = k\} - \mathsf{P}\{L = k\}$.

Ex 7.2.12. Finally, show that $\mathsf{E}[M] = 2\,\mathsf{E}[X] - \mathsf{E}[L]$.

THE SECOND IDEA.

Ex 7.2.13. First show that $\mathsf{P}\{M \leq k\} = (1-q^{k+1})^2$.

Ex 7.2.14. Simplify $\mathsf{P}\{M = k\} = \mathsf{P}\{M \leq k\} - \mathsf{P}\{M \leq k-1\}$ to see that $p_M(k) = 2\,\mathsf{P}\{X = k\} - \mathsf{P}\{L = k\}$.

AND HERE IS the third idea.

Ex 7.2.15. Explain that

$$\mathsf{P}\{L = i, M = k\} = 2p^2q^{i+k}\, I_{k>i} + p^2q^{2i}\, I_{i=k}).$$

Ex 7.2.16. Use [7.2.15] and marginalization to compute the marginal PMF $\mathsf{P}\{M = k\}$.

Ex 7.2.17. Use [7.2.15] to compute $\mathsf{P}\{L = i\}$.

IN CONCLUSION, we verified the correctness of $\mathsf{E}[M] = 2\,\mathsf{E}[X] - \mathsf{E}[L]$ in three different, and useful ways. Let us now focus on exponential rvs rather than geometric rvs.

7.2.2 *Continuous memoryless rvs*

In this section we analyze the correctness of (7.2.1) and (7.2.2) for continuous memoryless rvs, i.e., exponentially distributed rvs. I decided to analyze this in as much detail as I could think of, hoping that this would provide me with a lead to see how to generalize the equation $\mathsf{E}[M] = \mathsf{E}[L] + \mathsf{E}[X]$ such that it covers also the case with geometric rvs.

FIRST WE NEED to recall some basic facts about the exponential distribution.

Ex 7.2.18. Show that X is memoryless.

Ex 7.2.19. Show that $\mathsf{E}[X] = 1/\lambda$.

NOW WE CAN shift our attention to the rvs L and M.

Ex 7.2.20. Show that $F_L(x) = 1 - e^{-2\lambda x}$.

Clearly, this implies that $L \sim \text{Exp}(2\lambda)$ and $\mathsf{E}[L] = 1/(2\lambda) = \mathsf{E}[X]/2$. Hence, we see that (7.2.3) holds now. Moreover, with the same trick we see that the distribution function F_M for the maximum M is given by

$$F_M(v) = \mathsf{P}\{M \leq v\} = \mathsf{P}\{X \leq v, Y \leq v\} = (F_X(v))^2 = (1 - e^{-\lambda v})^2.$$

Of course this is a nice trick, but it is not a method that allows us to compute the distribution for more general functions of X and Y. For more general cases, we have to use the fundamental bridge and LOTUS, that is, for any set[2] A in the domain of $X \times Y$

$$\begin{aligned}\mathsf{P}\left\{g(X,Y) \in A\right\} &= \mathsf{E}\left[I_{g(X,Y)\in A}\right] = \iint I_{g(x,y)\in A} f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \iint_{g^{-1}(A)} f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{\{(x,y):g(x,y)\in A\}} f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

The jpoint CDF $F_{X,Y}$ then follows because $F_{X,Y}(x,y) = \mathsf{P}\left\{X \leq x, Y \leq y\right\} = \mathsf{E}\left[I_{X\leq x, Y\leq y}\right]$. A warning is in place: conceptually this approach is easy, but doing the integration can be very challenging (or impossible). However, this expression is very important as this is the preferred way to compute distributions by numerical methods and simulation.

Ex 7.2.21. Use the fundamental bridge to rederive the above expression for $F_M(v)$.

Ex 7.2.22. Show that the density of M has the form $f_M(v) = \partial_v F_M(v) = 2(1 - e^{-\lambda v})\lambda e^{-\lambda v}$.[3]

Ex 7.2.23. Use the density f_M to show that $\mathsf{E}[M] = 2\mathsf{E}[X] - \mathsf{E}[L]$.

Recalling that we already obtained $\mathsf{E}[L] = \mathsf{E}[X]/2$, we see that $\mathsf{E}[M] = 2\mathsf{E}[X] - \mathsf{E}[L] = 3\mathsf{E}[X]/2$, which settles the truth of (7.2.3).

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[2] If you like maths, you should be quite a bit more careful about what type of set A is acceptable. Here such matters are of no importance.

[3] We write ∂_v as a shorthand for $\mathrm{d}/\mathrm{d}v$ in the 1D case, and for ∂/∂_v the partial derivative in the 2D case.

WE CAN ALSO compute the densities $f_M(y)$ (and $f_L(x)$ by marginalizing the joint density $f_{L,M}(x,y)$. However, for this, we first need the joint distribution $F_{L,M}$, and then we can get $f_{L,M}$ by differentiation, i.e., $f_{X,Y} = \partial_x \partial_y F_{X,Y}$. Let us try this approach too.

Ex 7.2.24. Use the fundamental bridge to show that for $u \leq v$,

$$F_{L,M}(u,v) = \mathsf{P}\{L \leq u, M \leq v\} = 2\int_0^u (F_Y(v) - F_Y(x)) f_X(x)\,\mathrm{d}x.$$

Ex 7.2.25. Take partial derivatives to show that

$$f_{L,M}(u,v) = 2 f_X(u) f_Y(v)\, I_{u\leq v}.$$

Ex 7.2.26. In [7.2.25] marginalize out L to find f_M, and marginalize out M to find f_L.

WE DID A number of checks for the case X, Y, iid, $\sim \mathrm{Exp}(\lambda)$, but I have a second way to check the consistency of our results. For this I use the idea that the geometric distribution is the discrete analog of the exponential distribution. Now we study how this works, and that by taking proper limits we can obtain the results for the continuous setting from the discrete setting.

First, let's try to obtain an intuitive understanding of how $X \sim \mathrm{Geo}(\lambda/n)$ approaches $Y \sim \mathrm{Exp}(\lambda)$ as $n \to \infty$. For this, divide the interval $[0,\infty)$ into many small intervals of length $1/n$. Let $X \sim \mathrm{Geo}(\lambda/n)$ for some $\lambda > 0$ and $n \gg 0$. Then take some $x \geq 0$ and let i be such that $x \in [i/n, (i+1)/n)$.

Ex 7.2.27. Show that

$$\mathsf{P}\{X/n \approx x\} \approx \frac{\lambda}{n}\left(1 - \frac{\lambda}{n}\right)^{xn}. \tag{7.2.5}$$

Next, introduce the highly intuitive notation[4] $\mathrm{d}x = \lim_{n\to\infty} 1/n$, and use the standard limit[5] $(1-\lambda/n)^n \to e^{-\lambda}$ as $n \to \infty$ to see that (7.2.5) converges to

$$\mathsf{P}\{X/n \approx x\} \to \lambda e^{-\lambda x}\,\mathrm{d}x = f_X(x)\,\mathrm{d}x, \quad \text{as } n \to \infty.$$

If you don't like this trick with $\mathrm{d}x$, here is another method, based on with moment-generating functions.

Ex 7.2.28. Derive the moment-generating function $M_{X/n}(s)$ of X/n when $X \sim \mathrm{Geo}(p)$. Then, let $p = \lambda/n$, and show that $\lim_{n\to\infty} M_{X/n}(s) = M_Y(s)$, where $Y \sim \mathrm{Exp}(\lambda)$.

With these limits in place, we can relate the minimum $L = \min\{X, Y\}$ for the discrete and the continuous settings.

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4 In your math classes you learned that $\lim_{n\to\infty} 1/n = 0$. Doesn't this definition therefore imply that $\mathrm{d}x = 0$? Well, no, because $\mathrm{d}x$ is not a real number but an infinitesimal. Infinitesimals allow us to consider a quantity that is so small that it cannot be distinguished from 0 within the real numbers.

5 This is not entirely trivial to prove. If you like mathematics, check the neat proof in Rudin's Principles of mathematical analysis.

Ex 7.2.29. Suppose that $X, Y \sim \text{Geo}(\lambda/n)$, then check that $\lim_{n\to\infty} \mathsf{E}[L/n] = 1/2\lambda$.

Clearly, $1/2\lambda = \mathsf{E}[X]/2$ when $X \sim \text{Exp}(\lambda)$.
Here is yet another check on the correctness of $f_M(x)$.

Ex 7.2.30. Show that the PMF $\mathsf{P}\{M = k\}$ for the discrete M in [7.2.10] converges to $f_M(x)$ of [7.2.22] when $n \to \infty$. Take k suitable.

FINALLY, I HAVE a third way to check the above results, namely by verifying (7.2.2), i.e. $\mathsf{E}[M-L] = \mathsf{E}[X]$. For this, we compute the joint CDF $f_{L,M-L}(x,y)$. With this, you'll see directly how to compute $\mathsf{E}[M-L]$.

Ex 7.2.31. Use the fundamental bridge to obtain

$$F_{L,M-L}(x,y) = (1-e^{-2\lambda x})(1-e^{-\lambda y}) = F_L(x)F_Y(y).$$

Ex 7.2.32. Conclude that $M-L$ and L are independent, and $M-L \sim Y$.

By the above exercise, we find that $\mathsf{E}[M-L] = \mathsf{E}[Y] = \mathsf{E}[X]$, as X and Y are iid.

THIS MAKES ME wonder whether $M-L$ and L are also independent for the discrete case, i.e., when X, Y iid and $\sim \text{Geo}(p)$. Hence, we should check that for *all* i, j

$$\mathsf{P}\left\{L = i, M-L = j\right\} = \mathsf{P}\{L = i\}\,\mathsf{P}\left\{M-L = j\right\}. \tag{7.2.6}$$

Ex 7.2.33. Use [7.2.15] to see that

$$\mathsf{P}\left\{L = i, M-L = j\right\} = 2p^2q^{2i+j}\, I_{j\geq 1} + p^2q^{2i}\, I_{j=0}.$$

Now for the RHS.

Ex 7.2.34. Derive that

$$\mathsf{P}\left\{M-L = j\right\} = \frac{2p^2q^j}{1-q^2}\, I_{j\geq 1} + \frac{p^2q^j}{1-q^2}\, I_{j=0}.$$

Recalling that $\mathsf{P}\{L = i\} = (1-q^2)q^{2i}$, it follows right away from (7.2.6) that L and $M-L$ are independent. Interestingly, from [7.2.31] we see $M-L \sim Y$ for the continuous case. However, here, for the geometric case, $\mathsf{P}\left\{M-L = j\right\} \neq pq^j = \mathsf{P}\left\{Y = j\right\}$. This explains why $\mathsf{E}[M] \neq \mathsf{E}[L] + \mathsf{E}[X]$ for geometric rvs: we should be more carefull in how to split M in terms of L and X.

ALL IN ALL, we have checked and double checked all our expressions and limits for the geometric and exponential distribution. We had success too: the solution of the last exercise provides the key to understand why (7.2.1) and (7.2.2) are true for exponentially distributed rvs, but not for geometric random variables. In fact, in the solutions we can see the term corresponding to $X = Y = i$ for $X, Y \sim \text{Geo}(p)$ becomes negligibly small when $n \to 0$. In other words, $\mathsf{P}\{X = Y\} > 0$ when X and Y are discrete, but $\mathsf{P}\{X = Y\} = 0$ when X and Y are continuous. Moreover, by [7.2.31], $\mathsf{E}[M] = \mathsf{E}[L + M - L] = \mathsf{E}[L] + \mathsf{E}[M-L]$, but $\mathsf{E}[M-L] \neq \mathsf{E}[X]$. So, to resolve our leading problem we should reconsider $\mathsf{E}[M-L]$.

7.2.3 *The solution*

Let us now try to repair (7.2.2), i.e., $\mathsf{E}[M] = \mathsf{E}[L] + \mathsf{E}[X]$, for the case $X, Y \sim \text{Geo}(p)$. We should be careful about the non-negligible case that $M = L$, so we move, carefully, step by step.

Ex 7.2.35. Why is the following true:

$$\mathsf{E}[M] = \mathsf{E}[L] + \mathsf{E}[(M - L)\, I_{M>L}] = \mathsf{E}[L] + 2\,\mathsf{E}[(Y - X)\, I_{Y>X}]. \tag{7.2.7}$$

Ex 7.2.36. Show that

$$2\,\mathsf{E}[(Y - X)\, I_{Y>X}] = \frac{2q}{1 - q^2}.$$

Ex 7.2.37. Combine the above with the expression for $\mathsf{E}[L]$ of [7.2.7] to obtain [7.2.9] for $\mathsf{E}[M] = 2\,\mathsf{E}[X] - \mathsf{E}[L]$, thereby verifying the correctness of (7.2.7).

While (7.2.7) is correct, I am still not happy with the second part of (7.2.7) as I find it hard/unintuitive to interpret. Restarting again from scratch, here is another attempt to rewrite $\mathsf{E}[M]$ by using $Z \sim \text{FS}(p)$, i.e., Z has the first success distribution with parameter p, in other words, $Z \sim X + 1$ with $X \sim \text{Geo}(p)$.

Ex 7.2.38. Explain that

$$\mathsf{E}[M] = \mathsf{E}[L] + \mathsf{E}[Z\, I_{M>L}], \tag{7.2.8}$$

Ex 7.2.39. Show that

$$\mathsf{E}[Z\, I_{M>L}] = \frac{2q}{1 - q^2},$$

i.e., the same as [7.2.36], hence (7.2.8) is correct.

I AM NEARLY happy, but I want to see that (7.2.8), which is correct for discrete rvs, has also the correct limiting behavior.

Ex 7.2.40. Show that $\mathsf{E}[Z/n\, I_{M>L}] \to 1/\lambda$, which is the expectation of an $\text{Exp}(\lambda)$ rv!

Finally I understand why $\mathsf{E}[M] = \mathsf{E}[L] + \mathsf{E}[X]$ for $X, Y \sim \text{Exp}(\lambda)$ but not for when X, Y are discrete. For discrete rvs, L and M can be equal, while for continuous rvs, this is impossible[6] It took a long time, and a lot of work, to understand how to resolve the confusing problem, but I learned a lot. In particular, I find (7.2.8) a nice and revealing equation.

Finally, if you like to train with the tools you learned, you can try your hand at analyzing the same problem, but now for uniform X, Y.

[6] A bit more carefully formulated: the event $\{L = M\}$ has zero probability for continuous rvs.

7.3 EXERCISES ON 2D INTEGRATION

Here is some extra material for you practice on 2D integration, indicators and 2D LOTUS. These exercises are old exam questions, hence quite a bit harder than the above. They form important training.

Ex 7.3.1. Let X and Y be continuous random variables. Furthermore, $F(x,y)$ is the joint cumulative distribution function of X and Y. This function has the following properties.

1. $F(x,y) = \frac{1}{8}(x-1)^2(y-2)$ for $1 < x < 3$ and $2 < y < 4$,
2. $\frac{\partial F(x,y)}{\partial x} = 0$ for $x \notin (1,3)$,
3. $\frac{\partial F(x,y)}{\partial y} = 0$ for $y \notin (2,4)$.

Use these properties to answer the following questions.

1. What is $F(2,5)$?
2. Determine the joint probability density function of X and Y.
3. Determine $\mathrm{P}(2 < X < 3, 2 < Y < 4)$.
4. Determine the joint probability $\mathrm{P}(Y < 2X, 2X + Y > 6)$. Clearly draw the area over which you integrate.

Ex 7.3.2. Suppose X and V are independent, $X \sim \mathrm{Expo}(\lambda)$ and $V \sim \mathrm{Expo}(\mu)$. Define the ratio $R = X/V$ and derive the cumulative distribution function (CDF) of R. Provide at least two checks on the CDF to make sure that your result is indeed a valid CDF. Note: There is no need to derive the probability density function (PDF) of R.

Ex 7.3.3. Consider the following joint density function

$$f_{X,Y}(x,y) = \begin{cases} cxy & \text{for } 0 \le x < \frac{1}{2} \text{ and } 0 \le y \le x, \\ cxy & \text{for } \frac{1}{2} \le x \le 1 \text{ and } 0 \le y \le 1-x, \\ 0 & \text{otherwise.} \end{cases}$$

1. What is the correct value of the constant c?
2. Derive the conditional probability density function $f_{X|Y}(x|y)$. Verify that your result is indeed a valid density function.

Ex 7.3.4. Suppose the random variables X and Y have the joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} 2 & \text{if } 0 \le x < 1, \quad |y| < \frac{1}{2}(1-x), \\ 0 & \text{otherwise} \end{cases}$$

1. Calculate the marginal probability density functions $f_X(x)$ and $f_Y(y)$ and show that these are valid probability density functions.
2. Find the conditional expectation $E[X|Y = y]$. Provide at least one 'sanity check' that shows that your answer makes intuitive sense. If you did not find an answer to (a), you can use that

$$f_Y(y) = \begin{cases} 2(1-2|y|) & \text{if } -\frac{1}{2} < y < \frac{1}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

3. Calculate the joint cumulative distribution function $F_{X,Y}(x,y)$ for $x = \frac{1}{2}$ and $y = 3$.

Ex 7.3.5. Suppose the joint probability density function of X and Y is given by

$$f_{X,Y}(x,y) = \frac{c}{1-x}, \quad 0 < x+y < 1, \quad x > 0, \quad y > 0$$

and $f_{X,Y}(x,y) = 0$ otherwise.

1. For what value of the constant c is $f_{X,Y}(x,y)$ a joint probability density function?
2. What is the probability that $X + Y > \frac{1}{2}$?
3. What is the probability that both X and Y are smaller than $\frac{1}{2}$ given that $X + Y > \frac{1}{2}$?

Ex 7.3.6. X and Y be random variables with joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} \frac{3}{16}xy^2, & 0 \le x \le c \text{ and } 0 \le y \le c, \\ 0, & \text{elsewhere} \end{cases}$$

where $c > 0$ is a real number.

1. Show that $c = 2$.
2. Show that $P(X + Y > 2) = \frac{9}{10}$. Start by making a clear sketch of the area in the (x,y)-plane over which you take the required integral.
3. Calculate the conditional probability $P\left(Y < X^2 \,\middle|\, X + Y < 2\right)$.

7.4 BH EXERCISES: HINTS AND SOLUTIONS

Ex 7.4.1. BH.7.1. We simulate this in one of the assignments.

Ex 7.4.2. BH 7.9. We'll develop a simulation for this in the assignments.

Ex 7.4.3. BH.7.10.

Recall that a conditional CDF given an event A is defined as $F(y|A) = \mathsf{P}\{Y \le y|A\}$. Likewise, let us write here $F_T(t|x) = \mathsf{P}\{T \le t|X = x\}$. Just use this in your derivation. However, there is one problem with the fact that the event $\{X = x\}$ has probablity zero. In the solution I'll discuss how to around this.

Don't forget to compare this exercise to BH.7.9, which is the same but for discrete memoryless rvs.

Ex 7.4.4. BH.7.11

Ex 7.4.5. BH.7.13

Ex 7.4.6. BH7.15.

Ex 7.4.7. BH.7.24. In the assignments we'll develop a simulator.

Ex 7.4.8. BH.7.29

Ex 7.4.9. BH. 7.38. Besides the solution of BH, read our solution.

Ex 7.4.10. BH.7.53. We simulate this in one of the assignments. The ideas of this exercise find much use in finance, physics, and actuarial sciences. In particular, the expected time it takes the drunken person—It's not only guys that sometimes consume too much alcohol—to hit some boundary is interesting. The notation of the book is a bit clumsy. Here is better notation. Let X_i be the movement along the x-axis at step i, and Y_i along the y-axis. Then $S_n = \sum_{i=1}^n X_i$ and $T_n = \sum_{j=1}^n Y_j$, and $R_n^2 = S_n^2 + T_n^2$.

Ex 7.4.11. BH.7.58. This is a totally great exercise. First solve it yourself. In the solution, I'll explain why, in particular how to relate the concept of covariance to the determinant of a matrix.

Ex 7.4.12. BH.7.59. Read this exercise, then read (and do) BH.5.53 for some further background. You'll encounter these topics countless times in other courses! The final answer is really nice and intuitive.

Ex 7.4.13. BH.7.71.

Ex 7.4.14. BH.7.86. The concepts discussed here are a standard part of the education of GPs (i.e., medical doctors), and in data science in general.

7.5 CHALLENGE: A UNIQUENESS PROPERTY OF THE POISSON DISTRIBUTION

Consider the chicken-egg story (BH 7.1.9): A chicken lays a random number of eggs N an each egg independently hatches with probability p and fails to hatch with probability $q = 1 - p$. Formally, $X|N \sim \text{Bin}(N, p)$. Assume also that $X|N \sim \text{Bin}(N, p)$ and that $N - X$ is independent of X. For $N \sim \text{Pois}(\lambda)$ it is shown in BH 7.1.9 that X and Y are independent. This exercise asks for the converse: showing that the independence of X and Y implies that $N \sim \text{Pois}(\lambda)$ for some λ. Hence, the Poisson distribution is quite special: it is the only distribution for which the number of hatched eggs doesn't tell you anything about the number of unhatched eggs.

Let $0 < p < 1$. Let N be an rv. taking non-negative integer values with $P(N > 0) > 0$. Assume also that $X|N \sim \text{Bin}(N, p)$ and that $N - X$ is independent of X.

Ex 7.5.1. Use the assumption that $\mathsf{P}\{N > 0\} > 0$ to prove that N has support $\mathbb{N}$, i.e. $\mathsf{P}\{N = n\} > 0$ for all $n \in \mathbb{N}$. Note: $0 \in \mathbb{N}$.

Ex 7.5.2. Write $Y = N - X$. Prove that

$$\mathsf{P}\{X = x\}\,\mathsf{P}\left\{Y = y\right\} = \binom{x+y}{x} p^x (1-p)^y\, \mathsf{P}\left\{N = x + y\right\}. \tag{7.5.1}$$

Ex 7.5.3. Prove that N is Poisson distributed.

7.6 CHALLENGE: IMPROPER INTEGRALS AND THE CAUCHY DISTRIBUTION

This problem challenges your integration skills and lets you think about the subtleties of integrating a function over an infinite domain.[7]

Assume that X has the Cauchy distribution. Recall that $\mathsf{E}[X]$ does not exist (hence, it is not automatic that the expectation of a some arbitrary rv. exists).

Ex 7.6.1. Why does $\mathsf{E}\left[\frac{|X|}{X^2+1}\right]$ exist? Find its value. It is essential that you include your arguments.

Ex 7.6.2. Explain why the previous exercise implies that $\mathsf{E}\left[\frac{X}{X^2+1}\right]$ exists. Then find its value.

7.7 CHALLENGE: PROOF ABOUT INDEPENDENCE OF NORMAL RVS

Consider two iid rvs X, Y such that $X+Y$ and $X-Y$ are independent. In BH 7.5.8, it is claimed that this implies that X and Y are normally distributed.

This challenge asks to give a proof of this claim. Throughout this problem, you may assume that X and Y have a MGF that is defined for all $t \in \mathbb{R}$.[8] You may also use without proof the fact that MGFs that are defined everywhere are infinitely often continuously differentiable.

Ex 7.7.1. Let M_X be the MGF of X (and hence of Y).

1. Prove that $M_X(2t) = (M_X(t))^3(M_X(-t))$.
2. Define $f(t) = \log M_X(t)$. Prove that $8f'''(2t) = 3f'''(t) - f'''(-t)$.
3. Let $R > 0$ be arbitrary. Use Weierstrass' theorem to prove that f''' attains a minimum m and a maximum M on the interval $[-R, R]$, and then prove that $m = M = 0$.
4. Prove that X is normally distributed.

7.8 CHALLENGE: RECOURSE MODELS

This exercise will given an example of how probability theory can pop up in OR problems, in particular in linear programs. It introduces you to the concept of *recourse models*, which you will learn about in the master course Optimization Under Uncertainty. Disclaimer: the story is quite lengthy, but the concepts introduced and questions asked are in fact not very hard. We just added the story to make things more intuitive.

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[7] Such integrals are called improper Riemann integrals.

[8] This may seem like a big restriction, but this argument can easily be adapted to work with the *characteristic function* instead of the MGF, and the characteristic function does always exist. You will learn the characteristic function in the second year courses Statistical Inference and Linear Models in Statistics.

WE CONSIDER A pastry shop that only sells one product: chocolate muffins. Every morning at 5:00 a.m., the shop owner bakes a stock of fresh muffins, which he sells during the rest of the day. Making one muffin comes at a cost of $c = \$1$ per unit. Any leftover muffins must be discarded at the end of the day, so every morning he starts with an empty stock of muffins.

The owner has one question for you: determine the amount x of muffins that he should make in the morning to minimize his production cost. Note that the owner never wants to disappoint any customer, i.e., he requires that $x \geq d$, where d is the daily demand for muffins.

The problem can be formulated as a linear program (LP):

$$\min_{x \geq 0}\{cx \ : \ x \geq d\}. \tag{7.8.1}$$

For simplicity, we ignore the fact that x should be integer-valued.

Ex 7.8.1. Determine the optimal value x^* for x and the corresponding objective value in case d is deterministic.

Of course, in practice d is not deterministic. Instead, d is a random variable with some distribution. However, note that the LP above is ill-defined if d is a random variable. We cannot guarantee that $x \geq d$ if we do not know the value of d.

You explained the issue to the shop owner and he replies: "Of course, you're right! You know, whenever I've run out of muffins and a customer asks for one, I make one on the spot. I never disappoint a customer, you know! It does cost me 50% more money to produce them on the spot, though, you know."

Mathematically speaking, the shop owner just gave you all the (mathematical) ingredients to build a so-called *recourse model.* We introduce a *recourse variable* y in our model, representing the amount of muffins produced on the spot. Production comes at a unit cost of $q = 1.5c = \$1.5$. Assuming that we know the distribution of d, we can then minimize the *expected total cost*:

$$\min_{x \geq 0}\left\{cx + \mathsf{E}\left[v(d,x)\right]\right\}, \tag{7.8.2}$$

where $v(d,x)$ is the optimal value of another LP, namely the *recourse problem*:

$$v(d,x) := \min_{y \geq 0}\{qy \ : \ x + y \geq d\}, \tag{7.8.3}$$

for given values of d and x. The recourse problem can easily be solved explicitly: we get $y = d - x$ if $d \geq x$ and $y = 0$ if $d < x$. So we obtain

$$v(d,x) = q(d-x)^+, \tag{7.8.4}$$

where the operator $(\cdot)^+$ represents the *positive value* operator, defined as

$$(s)^+ = \begin{cases} s & \text{if } s \geq 0, \\ 0 & \text{if } s < 0. \end{cases} \tag{7.8.5}$$

Ex 7.8.2. To get some more insight into the model, suppose (for now) that $d \sim U\{10, 20\}$. Solve the model, i.e., find the optimal amount x^*.

Ex 7.8.3. What is the expected recourse cost (expected cost of on-the-spot production) at the optimal solution x^*, i.e., compute $\mathsf{E}[v(d, x^*)]$?

To solve the model correctly, we need the true distribution of d. We learn the following from the shop owner: "My granddaughter, who's always running around in my shop, is a bit data-crazy, you know, so she's been collecting some data. I remember her saying that 'the demand from male and female customers are both approximately normally distributed, with mean values both equal to 10 and standard deviations of 5'. She also mentioned something about correlation, but I don't remember exactly, you know. It was either almost 1 or almost −1. I hope this helps!"

Mathematically, we've learned that $d = d_m + d_f$, with $(d_m, d_f) \sim \mathcal{N}(\mu, \Sigma)$, where $\mu = (\mu_m, \mu_f) = (10, 10)$ and $\Sigma_{11} = \sigma_m^2 = \Sigma_{22} = \sigma_f^2 = 5^2 = 25$. Finally, $\Sigma_{12} = \Sigma_{21} = \mathsf{Cov}\left[d_m, d_f\right] = \rho\sigma_m\sigma_f = 25\rho$. Also, we know that either $\rho \approx 1$ or $\rho \approx -1$.

Ex 7.8.4. Calculate x^* and the corresponding objective value for the case $\rho = -1$. (Do not read $\rho = 1$, this case is not simple.)

Ex 7.8.5. Consider the two extreme cases $\rho = 1$ and $\rho = -1$. In which case will the shop owner have lower expected total costs? Provide a short, intuitive explanation.

8 CHAPTER 8: QUESTIONS AND REMARKS

8.1 SIMPLE QUESTIONS

Section 8.1

Ex 8.1.1. In probability theory we often want to study properties of functions of rvs. Provide an example for such a function.

Ex 8.1.2. Let the rv X be uniform on the set $\{0,1,2,3,4,5\}$. Derive the PMF and the CDF of $Z = 3X$. Explicitly specify the domain.

Ex 8.1.3. Suppose $y = g(x)$ for some differentiable function g. We like to express the PDF f_Y for $Y = g(X)$ in terms of the PDF f_X and g. This is easy when g is strictly increasing and has an inverse at y, because

$$F_Y(y) = \mathsf{P}\left\{Y \leq y\right\} = \mathsf{P}\left\{g(X) \leq y\right\} = \mathsf{P}\left\{X \leq g^{-1}(y)\right\} = F_X(g^{-1}(y)). \tag{8.1.1}$$

Now we take the derivative at the LHS and RHS to get with the chain rule

$$f_Y(y) = F_Y'(y) = \frac{\mathrm{d}}{\mathrm{d}y} F_X(g^{-1}(y)) = f_X(g^{-1}(y)) \frac{\mathrm{d}}{\mathrm{d}y} g^{-1}(y) = f_X(x) \frac{1}{g'(x)},$$

where we write $x = g^{-1}(y)$ in the last step. But why is the derivative of $g^{-1}(y)$ at y equal to $1/g'(x)$, with $x = g^{-1}(y)$?

Ex 8.1.4. When g is not strictly increasing everywhere, there can be no or multiple points x such that $g(x) = y$. Explain that in such cases it is much more difficult to express F_Y in terms of F_X than directly use the densities (assuming that g is differentiable). Extend your reasoning to 2D.

Ex 8.1.5. The general 1D change of variables formula is like this,

$$f_Y(y) = \sum_{x_i : g(x_i) = y} f_X(x_i) \frac{1}{|g'(x_i)|},$$

with some natural conditions on g. Apply this formula to the case $g(x) = x^2$.

Ex 8.1.6. If $X \sim \mathrm{Exp}(1)$, use the change-of-variables theorem to obtain the density of $Y = g(X) = \lambda X$. What is $\mathsf{E}[Y]$?

Ex 8.1.7. Show that the 1D change-of-variables formula relates directly to the substitution rule of integration theory to solve 1D integrals.

Ex 8.1.8. Use the change of variable formula to relate the Geo(p) and the FS(p) distributions.

Ex 8.1.9. BH.8.1.1 write that 'The support of Y is all $g(x)$ with x in the support of X.' Do they say that $\text{supp}(Y) = \{x : g(x) \in \text{supp}(X)$? BTW, what is the difference between $\text{supp}(X)$ and $\sup X$?

Ex 8.1.10. BH.8.1.3. Check how all moments were found.

Ex 8.1.11. Let $X \sim \text{Unif}(0,5)$. Using the one dimensional change of variables theorem (BH.8.1.1), derive the PDF and the CDF of $Z = 3X$. Explicitly specify the domain.

Ex 8.1.12. When $Z = X^3$ and $X \sim \text{Unif}(0,5)$, using the one dimensional change of variables theorem to derive the PDF and the CDF of Z. Specify the domain of Z.

Ex 8.1.13. Let $X \sim \text{Norm}(\mu, \sigma^2)$. Using the one dimensional change of variables theorem BH.8.1.1, show that $Z = \frac{X-\mu}{\sigma} \sim \text{Norm}(0,1)$.

Ex 8.1.14. Let $X \sim \text{Exp}(1)$. Derive the PDF of e^{-X}.

Ex 8.1.15. Let X, Y be iid standard normal. Using the n-dimensional change of variables theorem, derive the joint PDF of $(X+Y, X-Y)$.

Check your final answer using BH.7.5.8.

Ex 8.1.16. Specify the domain of the new random variable for the following transformations; this important aspect of the change of variables is often overlooked. Let U, V, W, X, X_1, X_2, Y and Z be rvs and let a, b and c be arbitrary constants.

1. $Z = Y^4$ for $Y \in (-\infty, \infty)$;
2. $Y = X^3 + a$ for $X \in (0,1)$;
3. $U = |V| + b$ for $V \in (-\infty, \infty)$;
4. $Y = e^{X^3}$ for $X \in (-\infty, \infty)$;
5. $V = U\, I_{U \le c}$ for $U \in (-\infty, \infty)$;
6. $Y = \sin(X)$ for $X \in (-\infty, \infty)$;
7. $Y = \frac{X_1}{X_1 + X_2}$ for $X_1 \in (0, \infty)$ and $X_2 \in (0, \infty)$;
8. $Z = \log(UV)$ for $U \in (0, \infty)$ and $V \in (0, \infty)$.

Ex 8.1.17. When adding a different equality, we need to be careful to not create a functional relationship between our two new variables U, V, for example $U = X + Y$ and $V = \sin(X+Y)$, or $U = \frac{X}{Y}$ and $V = \frac{Y}{X}$ for conforming X, Y. What would happen to the determinant of the Jacobian matrix if we did? Why would this happen? Explain in your own words.

Section 8.2

Ex 8.1.18. To find the distribution of a convolution through the change of variables formula, we seem to need to add a 'redundant' equality? But why is that? What would be the problem if we do not add this? Explain in your own words.

Ex 8.1.19. In this exercise, we combine what we learned in BH.8.1.4 and BH.8.1.9. Let S be the sum of two iid chi-square distributed variables (with one degree of freedom). Using just these two examples, show that $S \sim \text{Exp}(1/2)$.

Ex 8.1.20. A student has obtained an iid random sample of size 2 from a Cauchy distribution. Let the rvs X and Y model the values of the first and second sample. Since s/he does not know what the mean of a Cauchy distribution is, s/he wants to average the sample to obtain what she thinks is a good estimate of the true mean.

To find the distribution of this sample mean, we need to find an expression for $f_W(w)$, where $W = \frac{X+Y}{2}$.

1. Find an expression for $f_W(w)$ in the form of an integral, but do not solve it.
2. It turns out that if we solve the integral, we get that $f_W(w) = f_X(w)$. The distribution of our sample mean is still Cauchy; we did not obtain a better estimate of the Cauchy mean by calculating the sample mean!

   Explain (in your own words) why this makes sense.

Section 8.3

Ex 8.1.21. If $a = b = 1$, why is $\text{Beta}(a, b) = \text{Unif}([0, 1]])$?

Ex 8.1.22. If $a = b$, why is $\text{Beta}(a, b)$ symmetric?

Ex 8.1.23. If $a > b$ and $X \sim \text{Beta}(a, b)$, is $\mathsf{E}[X] > 1/2$?

Ex 8.1.24. BH.8.3.2, last equation. How do the authors get to the equation

$$\beta(a,b) = \frac{1}{(a+b-1)\binom{a+b-2}{a-1}}?$$

Ex 8.1.25. BH.8.3.3. The authors write that X is not marginally Binomial, but is conditionally Binomial. What is the difference?

Ex 8.1.26. BH.8.3.3. The authors use a smart trick to find an expression for the posterior distribution $f_{p|X=k}$ of p. Use this posterior to derive an expression for $\mathsf{P}\{X = k\}$ by using the fact that

$$\mathsf{P}\{X = k\} = \frac{\binom{n}{k} p^k (1-p)^{n-k} \text{Beta}(p; a, b)}{\text{Beta}(p; a+k, b+n-k)},$$

and simplyfing the RHS.

Ex 8.1.27. BH.8.3.3. Why does $a-1$ correspond to the number of prior successes, in other words, why is it not a, but $a-1$?

Ex 8.1.28. BH.8.3.4.b. Given that the first patient is cured, what is the probability that the rest of the patients, i.e., the other $n-1$, will also be cured?

Ex 8.1.29. Is this claim correct? Let T be the sum of two iid Unif$(0,1)$ rvs. Then there exist a, b such that $T \sim$ Beta(a, b). (You don't need to derive the distribution of T.)

Ex 8.1.30. Show that $\beta(1, b) = 1/b$ by integrating the PDF of the beta distribution for $a = 1$. (Do not use the results of BH 8.5 for this exercise.)

Ex 8.1.31. Let $a, b > 1$. Show that the PDF of the beta distribution attains a maximum at $x = \frac{a-1}{a+b-2}$. Explicitly indicate where the assumption that $a, b > 1$ is used.

Ex 8.1.32. Explain in your own words:

1. What is a prior?
2. What is a conjugate prior?

Ex 8.1.33.

1. Look up on the web: what is the conjugate prior of the multinomial distribution? Give a name and a formula.
2. Explain why the Beta distribution is a special case of this distribution.

Ex 8.1.34. You make a test with n multiple choice questions and you give the correct answer to each question independently with probability p. The teacher's prior belief about p is reflected by a uniform distribution: $p \sim$ Unif$(0,1)$. Let X be the number of correct answers you give. What is the teacher's posterior distribution $p|X = k$? (You don't have to do a lot of math here; simply use a result from the book.)

Ex 8.1.35. You find a coin on the street. Initially, you are rather confident that this should be (approximately) a fair coin. This is reflected in your prior belief of the probability p of heads: $p \sim$ Beta$(10, 10)$. Your friend is a bit more skeptical and assumes a uniform prior: $p \sim$ Unif$(0,1)$. You toss the coin 1000 times, and it comes up heads 900 times.

1. Determine your posterior distribution. (Again, use a result from the book)
2. Determine your friend's posterior distribution.
3. Compare the means of your posterior distribution and your friend's posterior distribution. Comment on the effect of the prior distribution if you have a lot of data.

Ex 8.1.36. We have an urn with 1000 coins. One of those is biased such that $\mathsf{P}\{H\} = 99/100 = 1 - \mathsf{P}\{T\}$, all others are fair. You select at random a coin, i.e., with probability $1/1000$ you select the biased one, and start throwing. You see 10 heads in row. What is the probability you picked the biased coin?

Ex 8.1.37. BH.8.3.5. write that X_j is an indicator of the jth throw beind made. Can this be the formal definition: $X_j = I_{N \geq j}$?

Ex 8.1.38. Use the pmf of the Beta-Binomial distribution to prove the following identity:

$$\sum_{k=0}^{n} \frac{\binom{n}{k}\binom{a+b-2}{a-1}(a+b-1)}{\binom{a+b+n-2}{a+k-1}(a+b+n-1)} = 1.$$

for all positive integers a, b, n.

Section 8.4

Ex 8.1.39. What is the SCV of Gamma(n, λ) distributed rv X?

Ex 8.1.40. We have a machine that has temperature $x_0 e^{-\alpha t}$ after a time t. We switch it on when q jobs arrive. Job interarrival times are Exp(λ). Why does the temperature at the moment the qth job arrives have the distribution $x_0 \exp -\alpha Y$, with $Y \sim$ Gamma$(q-1, \lambda)$?

Ex 8.1.41. Consider the chi-square distribution (with one degreee of freedom) from BH.8.1.4.
Starting from the expression $f_Y(y) = \varphi\left(\sqrt{y}\right) y^{-1/2}$ in this example, show that this chi-square distribution is a special case of the Gamma distribution and specify the corresponding values of the parameters a and λ.

Ex 8.1.42. Is the sum of any two Gamma distributions again Gamma?

Ex 8.1.43. Prove by induction that $\Gamma(n) = (n-1)!$ if n is a positive integer.

Ex 8.1.44. Is the Poisson distribution the conjugate prior of the Gamma distribution?

Ex 8.1.45. Let $X \sim$ Gamma$(4, 2)$ and $Y \sim$ Gamma$(7, 2)$ be independent rvs. What is the distribution of $X + Y$? What is the distribution of $\frac{X}{X+Y}$?

Section 8.6

Read the definition of an order statistic. Skip the rest of BH.8.6.

Ex 8.1.46. If you can answer this question, then you basically know everything you need to know about order statistics for the purpose of this course.)
Let $X_1, X_2, \ldots, X_9$ be a collection of random variables. Fill in the gaps (with just one word each time):

1. $X_{(1)}$ denotes the ... of $X_1, X_2, \ldots, X_9$.
2. $X_{(9)}$ denotes the ... of $X_1, X_2, \ldots, X_9$.
3. $X_{(5)}$ denotes the ... of $X_1, X_2, \ldots, X_9$.

8.2 BH EXERCISES: HINTS AND SOLUTIONS

Ex 8.2.1. BH.8.11. With convolution we know how to add and subtract independent rvs. Now we make a start with division. You'll see that this operator is not as simple as you always thought.

Before solving the problem, let's take a step back. You learned arithmetic at primary school. In all those problems, the numbers you had to add, subtract, etc. where supposed to be known precisely. At secondary school, you learned how to arithmetic with symbols. And now, at university, your next step is learn how to do arithmetic with rvs.

Here is an example to show you the relevance of this. In a paint factory at which a couple of my students did their master's thesis, the inventory level of dyes and other raw materials is often not known exactly. There are plenty of simple explanations for this. Raw materials are kept in big bags, and personnel uses shovels to take it out of the bags. Of course, occasionally, there is some spillage on the floor, and this extra 'demand' is not reported. The demand side is also not exact. A customer orders for example 500 kg of red paint. To make this, the operators follow a recipe, but dyes (in certain combinations) do not always give the same result. Therefore, the paint for each order is checked, and when it does not meet the quality level, the batch has to be adjusted by adding a bit more of certain dyes or solvents, or other chemical products.

When the planner has to make a decision on when to reorder a certain raw material, s/he divides the total amount of raw material by the average demand size. And this leads to occasional stock outs. When the stock level and the demands are treated as a rvs, such stock outs may be prevented, but this requires to be capable of determining the distribution of the something like Y/X.

Ex 8.2.2. BH.8.15. We'll use this exercise in a lecture to show how the normal distribution originates from astronomy (or dart throwing).

The notation is a bit clumsy for the angle coordinate. Write Θ for the rv and θ for its value.

Ex 8.2.3. BH. 8.18. Here we deal with division of rvs.

Ex 8.2.4. BH.8.23. We already analyzed how to handle addition, subtraction and division. It remains to deal with multiplication.

Ex 8.2.5. BH.8.31

Ex 8.2.6. BH.8.36.

Ex 8.2.7. BH.8.40. A nice question on the exam could be to take another prior, e.g., p uniform on $[1/3, 2/3]$. How would that affect the solution?

Ex 8.2.8. BH.8.52. The concepts discussed here are useful to better understand how to generate exponential random numbers.

Ex 8.2.9. BH.8.54. We tackle this also with simulation in an assignment.

I find it easier to consider $Y = pX$, rather than pX/q. Note that since $q = 1 - p \to 1$ as $p \to 0$, the factor $1/q$ is immaterial for the final result.

Read my solution too, as I develop some nice ideas in passing.

8.3 CHALLENGE: PING PONG BALLS INA BELUGA

This challenge is a continuation of the simulation we did for the Beluga case, and we discuss some ways to check whether $\mathsf{V}[N] \approx \mathsf{V}[V]\,\mathsf{V}[v]$ holds in general, and then we try to find a better approximation. We chopped up the challenge into many exercises, to help you organize the ideas.

Recall that earlier we have been a bit sloppy about the units, measuring the volumes of the airplane in m^3 and a ping pong ball in cm^3, so actually N is in millions of ping pong balls. Note that using different units can easily lead to confusion; as a take-away , choose one unit.

One way to check the correctness of $\mathsf{V}[N] \approx \mathsf{V}[V]\,\mathsf{V}[v]$ is to change the scale. In fact, memorize that changing scale is an easy way to check laws.

Ex 8.3.1. Suppose we instead measure the size of a ping pong ball in meters and the size of the airplane in hectometers. Explain that N is still in millions of ping pong balls. What happens to $\mathsf{V}[N]$ and what happens to $\mathsf{V}[V]\,\mathsf{V}[v]$ (theoretically)?

Another way to check a statement is to consider some extreme cases.

Ex 8.3.2. Suppose that we would know the size of a ping pong ball very accurately, i.e. we consider the extreme case where $\mathsf{V}[v] \to 0$. Explain that the approximation is not a good approximation in this limit.

Ex 8.3.3. Which of these two checks convinces you most that something is wrong with this approximation, and why?

We now turn to the task of trying to find a good approximation.

Ex 8.3.4. Assume that X and Y are independent. Show that

$$\mathsf{V}[XY] = \mathsf{V}[X]\,\mathsf{V}[Y] + \mathsf{V}[X]\,\mathsf{E}[Y]^2 + \mathsf{E}[X]^2\,\mathsf{V}[Y].$$

Ex 8.3.5. Assume in addition that we know at least one of X and Y quite precisely. Argue that the following is then a good approximation:

$$\mathsf{V}[XY] \approx \mathsf{V}[X]\,\mathsf{E}[Y]^2 + \mathsf{E}[X]^2\,\mathsf{V}[Y].$$

So far we have only considered the variance of a product, but we would like to know the variance of a ratio. For this we can use Taylor expansions to make accurate approximations.

Ex 8.3.6. Find the first order Taylor expansion of $\frac{1}{Z}$ around $a = \mathsf{E}[Z]$. By taking the expectation and the variance of this expansion, show that

$$\mathsf{E}\left[\frac{1}{Z}\right] \approx \frac{1}{\mathsf{E}[Z]}, \qquad \mathsf{V}\left[\frac{1}{Z}\right] \approx \frac{\mathsf{V}[Z]}{\mathsf{E}[Z]^4}.$$

Ex 8.3.7. Combine all of the above to derive the following approximation for the variance of the ratio of two independent random variables X and Z:

$$\mathsf{V}\left[\frac{X}{Z}\right] \approx \frac{\mathsf{V}[X]}{\mathsf{E}[Z]^2} + \mathsf{E}[X]^2\,\frac{\mathsf{V}[Z]}{\mathsf{E}[Z]^4}.$$

Ex 8.3.8. Check this approximation in the ways of the first two exercises.

After doing all this work, we would of course like to know how well this approximation does. When comparing the approximation to the sample standard deviation found in [7.1.22] for `num=500`, the result may be a bit disappointing. However, this is just because the sample standard deviation is also an estimate of the actual standard deviation of N, so by chance the result may be closer to $\mathsf{V}[V]\,\mathsf{V}[v]$ than to our new approximation.

In Chapter 10, you will learn something about the distribution of the sample variance. For now, just increase `num`. We know this decreases the variance of the sample mean and it also decreases the variance of the sample variance, so we get a more accurate estimate.

Ex 8.3.9. Use the result of the previous exercise to compute an approximation for $\mathsf{V}[N] = \mathsf{V}[V/v]$. Also use the code with a (much) higher value of `num`, to show that the approximation $\mathsf{V}[N] \approx \mathsf{V}[V]\,\mathsf{V}[v]$ is likely to be worse, even in the setting of [7.2.15] where it was quite good.

The following two exercises are really optional, but I found them very neat and insightful.

Ex 8.3.10. Recall that for a non-negative random variable X with finite variance, we define the squared coefficient of variation as $\text{SCV}(X) = \mathsf{V}[X]/\mathsf{E}[X]^2$. Using the SCV, show that the approximations of [8.3.5] and [8.3.6] can be rewritten in the following neat way:

$$\begin{aligned} \text{SCV}(XY) &\approx \text{SCV}(X) + \text{SCV}(Y). \\ \text{SCV}(1/Z) &\approx \text{SCV}(Z). \end{aligned}$$

In BH.10, you will learn Jensen's inequality, which implies that $\mathsf{E}\left[\frac{1}{Z}\right] \geq \frac{1}{\mathsf{E}[Z]}$ for all positive random variables Z. In the following exercise, we reflect on this by finding a more accurate approximation based on the second order Taylor expansion.

Ex 8.3.11. Find the second order Taylor expansion of $\frac{1}{Z}$ around $a = \mathsf{E}[Z]$. By taking the expectation, show that

$$\mathsf{E}\left[\frac{1}{Z}\right] \approx \frac{1}{\mathsf{E}[Z]} + \frac{2\mathsf{V}[Z]}{\mathsf{E}[Z]^3}.$$

Note that this is always at least $\dfrac{1}{\mathsf{E}[Z]}$.

8.4 CHALLENGE: BENFORD'S LAW

In this exercise, we discuss Benford's law. Recall that the first step towards this law was taken in Lecture 5, Exercise 3. In this exercise, we showed that if X, Y are iid uniform on $[1, 10)$, that then the density of $Z = XY$ is given by

$$f_Z(z) = \frac{\log(\min\{10, z\}) - \log(\max\{1, z/10\})}{81} I_{1 \leq z \leq 100}.$$

Note that log denotes the natural logarithm.
Benford's law is a statement about the distribution of the first digit of the product of sufficiently many variables that are iid uniform on $[1,10)$. We first consider the first digit of the product of two such variables, i.e. the first digit of Z.

Ex 8.4.1. Let K be the first digit of Z. Show that the PMF of K is given by

$$P(K=k)=\frac{9k\log(k)-9(k+1)\log(k+1)+9+10\log(10)}{81}$$

for $k \in \{1,2,\ldots,9\}$.

Ex 8.4.2. Check that $\sum_{k=1}^{9} P(K=k)=1$. This can be done nicely by recognizing a *telescoping sum*: many terms cancel because they appear once with a minus and once with a plus.

Another way to derive the first digit of Z is to first divide Z by 10 if $Z \geq 10$. This yields a random variable W with support $[1,10)$. Clearly, the division doesn't affect the first digit. The next exercise asks to derive the resulting density. This can be a bit tricky; you should check your answer by verifying that the distribution of the first digit of W matches the distribution of the first digit of Z.

Ex 8.4.3. Let $W=Z$ if $1 \leq Z < 10$ and $W=\frac{Z}{10}$ if $10 \leq Z < 100$. Derive the density of W.

We now turn to the product of more than two (independent) random variables. It would be very tedious to do this analytically, so we will instead use some code. However, to do this we have to approximate the continuous uniform variable by a discrete random variable. We use the discrete uniform distribution on $\{1+0.5\cdot\frac{9}{s},1+1.5\cdot\frac{9}{s},1+2.5\cdot\frac{9}{s},\ldots,1+(s-0.5)\cdot\frac{9}{s}\}$; in total this set has s elements. However, a product of two elements from this set may not again be an element of this set. To solve this, we identify all elements of the interval $\left(1+k\cdot\frac{9}{s},1+(k+1)\cdot\frac{9}{s}\right)$ with $1+(k+0.5)\cdot\frac{9}{s}$. We now use a loop to approximate the distribution of the product of $p+1$ random variables by looking at all possible values of the product of p random variables and one additional uniformly distributed random variable. Note that in the code, s is called `steps` and p is called `p_idx`.

Executing the code may take a while. If it takes more than 1 minute, you may decrease `steps`, but please do note that you did so.

Python Code

```python
import math

steps = 900
products = 15
p_unif = [1.0/steps] * steps
p_mat = [p_unif]

for p_idx in range(1, products):
    p_vec = [0] * steps
        for s1 in range(steps):
```

```
            for s2 in range(steps):
                product = (1 + (s1 + 0.5)*9/steps) * (1 + (s2 + 0.5)*9/steps)
                prod_probability = p_mat[p_idx - 1][s1] * 1/steps

                if product > 10:
                    product = product/10

                prod_idx = math.floor((product-1)/9 * steps)
                p_vec[prod_idx] += prod_probability

    p_mat.append(p_vec)


p_digits = []
for p_idx in range(products):
    vec = []
    for digit in range(1, 10):
        pd = sum(p_mat[p_idx][((digit-1)*steps//9):(digit*steps//9)])
        vec.append(round(pd, 6))
    p_digits.append(vec)

print(p_digits)
```

R Code

```
steps <- 900
products <- 15
p_unif <- rep(1/steps, steps)
p_mat <- matrix(0, nrow = steps, ncol = products)
p_mat[, 1] <- p_unif

for (p_idx in 2:products) {
  p_vec <- rep(0, steps)
  for (s1 in 1:steps) {
    for (s2 in 1:steps) {
      product <- (1 + (s1 - 0.5)*9/steps) * (1 + (s2 - 0.5)*9/steps)
      prod_probability <- p_mat[s1, p_idx - 1] *  1/steps

      if (product > 10) {
        product <- product/10
      }

      prod_idx <- ceiling((product-1)/9 * steps)
      p_vec[prod_idx] <- p_vec[prod_idx] + prod_probability
    }
  }
```

```
22   p_mat[, p_idx] = p_vec
23 }
24
25 p_digits <- matrix(0, nrow = 9, ncol = products)
26 for (p_idx in 1:products) {
27   for (digit in 1:9) {
28     pd = sum(p_mat[((digit-1)*(steps/9)+1):(digit*(steps/9)), p_idx])
29     p_digits[digit, p_idx] = round(pd, 6)
30   }
31 }
32 p_digits
```

Ex 8.4.4. Explain line P.13 (R.12) of the code.

Ex 8.4.5. Briefly comment on the results for $p = 2$ compared the exact result derived in the first exercise. Why is it important to make this comparison?

When looking at the results for larger p, it seems that the probabilities converge. The limit random variable B then satisfies the property that the first digit of B and the first digit of BU (where $U \sim \text{Unif}(1, 10)$) are identically distributed. Proving this is quite challenging (even for the challenge). In addition, we first need to know what the distribution of B is.

To guess the distribution of the first digit of B, we look at the results of our code and try some transformations to see if this yields familiar numbers. It turns out that the first digit M of B has the following distribution:

$$P(M = k) = \log_{10}\left(\frac{k+1}{k}\right),$$

for $k \in \{1, 2, \ldots, 9\}$.

Ex 8.4.6. Briefly comment on these exact values of $P(M = k)$ compared to the values for $p = 15$ that result from the code. Give two reasons why the code results are not exact. Which reason do you think is the most important?

Ex 8.4.7. Check that $\sum_{k=1}^{9} P(M = k) = 1$. You can again use a telescoping sum.

Besides the theoretical aspects covered in this challenge, Benford's law states that the first digit of numbers of naturally occurring sets that span several orders of magnitude, such as vote counts by county (or municipality), transaction sizes, etc., approximately follow this distribution. Initially this was just seen as an interesting curiosity of no practical value, but recently it has been used in fraud detection. If you're interested, you might check out this YouTube video by Numberphile: https://www.youtube.com/watch?v=XXjlR2OK1kM&ab_channel=Numberphile

9

CHAPTER 9: QUESTIONS AND REMARKS

9.1 SIMPLE QUESTIONS

Section 9.1

Remark 9.1.1. Skip BH.9.1.6.

Ex 9.1.2. BH.9.1.7. We will also simulate this in an assignment.

I like this example as it shows how to make optimal decisions under uncertainty, but I have to admit that I don't understand the reasoning, or the use of conditional probablity to solve this problem. Here is how I would solve the problem.

1. Explain that $X = (V - b)\, I_{b \geq \alpha V}$, with $\alpha = 2/3$, is the rv that models our payoff.
2. Why is this wrong: $\mathsf{E}[X] = \mathsf{E}[(V - b)\, I_{b\geq\alpha V}] = V\,\mathsf{E}[I_{b\geq\alpha V}] - b\,\mathsf{E}[I_{b\geq\alpha V}]$?
3. Compute $\mathsf{E}[X]$, and provide a bound on α to ensure that $\mathsf{E}[X] > 0$.

Ex 9.1.3. BH.9.1.7. Suppose $b < \alpha V$, for some $\alpha \in (0,1)$ the bid is rejected. By the solution of BH.9.1.7, if $\alpha = 2/3$, you should not enter this game. But is there another α (smaller or larger than $2/3$) at which entering the game is interesting?

Ex 9.1.4. BH.9.1.8. Apply the same type of argumentation to find $\mathsf{E}[X]$ when $X \sim \mathrm{FS}(p)$.

Ex 9.1.5. BH.9.1.8. Use first step analysis to find $N_r := \mathsf{E}[X]$ when $X \sim \mathrm{NBin}(r, p)$.

Ex 9.1.6. BH.9.1.9. I reason slightly differently here. Write N_r for the number of throws required to reach r heads in row. Then I need N_{r-1} throws in expectation to reach the state in which there are $r-1$ heads in row. Suppose now that we are in this state, i.e., there are $r-1$ heads in row. Then, if I throw heads, with probability p, I reach the state with r heads in row, and I am done. However, if I throw tails, with probability q, I have to start all over again. Use this argument to derive the recursion $N_r = N_{r-1} + p\cdot 1 + q(1 + N_r)$. Solve this to obtain

$$N_r = \sum_{i=1}^{r} 1/p^i. \tag{9.1.1}$$

Ex 9.1.7. Compute the expected outcome of a die throw (with a 6-sided die), given that the outcome is even. Introduce proper notation for random variables and events.

Ex 9.1.8. BH.9.1.10. Let p_i, $0 \leq i \leq b$ be the probability to hit b before 0.

1. Why is $p_0 = p_1/2$?

2. Why is $p_b = 1$?
3. Explain the recursion $p_i = p_{i-1}/2 + p_{i+1}/2$ for $0 < i < b$?
4. Show that point 3 implies that $p_i = \alpha i$ for $0 < i < b$ for any α we like.
5. Combine the fact that $p_b = 1$ with $p_i = \alpha i$ for all $0 < i < b$ to see that $\alpha = 1/b$.
6. Conclude that $p_0 = 1/2b$.

Section 9.2

Remark 9.1.9. About BH. 9.2.1. This definition is subtle, and it takes time to understand. Here is a slightly different explanation; perhaps it's useful for you.

Take some random variable X, say. Then, as in Chapter 7, we can be interested in $\mathsf{E}\left[g(X)\right]$, i.e., the expectation of the rv $g(X)$.

When Y is continuous we can compute $\mathsf{E}\left[Y \mid X = x\right]$ with the conditional CDF

$$\mathsf{E}\left[Y \mid X = x\right] = \int f_{Y|X}(y|x)\,\mathrm{d}y.$$

(For discrete rv., replace the integral by the PMF.) Observe that this is just a function of x; define this function as $g(x) = \int f_{Y|X}(y|x)\,\mathrm{d}y$. And now, as before, we consider the random variable $g(X)$, and we *call this rv the conditional expectation* of Y given X.

It is true that X plays some sort of double role—first we use it in the conditioning in the definition of the function g, and then we plug it into g again—and this is perhaps confusing. But I finally 'got it', when I understood that g can be interpreted as just some function of x. And then we compute $\mathsf{E}\left[g(X)\right]$, and so on.

Ex 9.1.10. BH.9.2.2. Is $\mathsf{E}\left[Y \mid I_A\right]$? a number or a rv?

Section 9.3

Remark 9.1.11. On BH.9.3.2 (Taking out what is known.) Perhaps it is easier to cristalize X into x. Then $g(x) = \mathsf{E}\left[h(x)Y|X\right] = h(x)\,\mathsf{E}\left[Y|X\right]$, because $h(x)$ is just a function. The rvs $\mathsf{E}\left[H(X)Y|X\right]$ and $h(X)\,\mathsf{E}\left[Y|X\right]$ are then both equal to $g(X)$.

Ex 9.1.12. On BH.9.3.9. Show that $\mathsf{Cov}\left[Y - \mathsf{E}\left[Y|X\right], \mathsf{E}\left[Y|X\right]\right] = 0$.

Section 9.4

Ex 9.1.13. Consider a casino where, for any $a > 0$, it is possible to pay a euro and get a chance of $\frac{1}{5}$ on receiving $4a$ euro and a chance of $\frac{4}{5}$ of receiving nothing. Adam enters the casino with b euros, and bets half of his money on this gamble. Let X be the amount of money he has after the gamble. After that, he again bets half of the money he then has (i.e. half of X) on this gamble. Let Y be the amount of money he has after the second gamble.

1. Compute $\mathsf{E}[X]$.
2. Compute $\mathsf{E}[Y|X]$.
3. Compute $\mathsf{E}[Y]$.

Explicitly mention the laws/rules you use.

Ex 9.1.14. Let $N \sim \text{Pois}(\lambda)$, and let $X|N \sim \text{Bin}(N,p)$, where $p \in (0,1)$ and $\lambda > 0$ are known constants. Compute $\mathsf{E}[X]$ using Adam's law. Check your answer using the chicken-egg story; with this story you can also obtain the distribution of X.

Ex 9.1.15. Correct? If A is an event and I_A is its indicator, then for all random variables X we have $\mathsf{E}[X \,|\, A] = \mathsf{E}[X \,|\, I_A]$.

Ex 9.1.16. Correct? If X and Y are independent, then $\mathsf{V}[\mathsf{E}[Y \,|\, X]] = 0$.

Ex 9.1.17. Let $X \sim \text{Exp}(\lambda)$, and let a be a constant.

1. Compute $\mathsf{E}[X \,|\, X \geq a]$ using an integral and an indicator.
2. Explain the answer using a property of the exponential distribution.

Ex 9.1.18. A hat contains 9 fair coins and one coin that lands heads with probability 0.8. You pick a coin from the hat uniformly at random and toss it 10 times. Let A be the event that you pick a fair coin, and let X be the number of heads. Let B be the event that the first four tosses all show heads.

1. Compute $\mathsf{E}[X \,|\, A]$.
2. Compute $\mathsf{E}[X \,|\, A^c]$.
3. Compute $\mathsf{E}[X]$.
4. Compute $\mathsf{P}\{B\}$.
5. Compute $\mathsf{P}\{A \,|\, B\}$.
6. Compute $\mathsf{E}[X \,|\, B]$.
7. Compute $\mathsf{E}[X \,|\, B^c]$. *Hint*: it is not necessary to compute $\mathsf{P}\left\{A \,|\, B^c\right\}$.

Ex 9.1.19. Consider random variables $X, Y \in [0,1]^2$ with joint PDF $f_{X,Y}(x,y) = 2\, I_{x \leq y}$. Determine $\mathsf{E}[Y \,|\, X]$ and $\mathsf{E}[X \,|\, Y]$.

Ex 9.1.20. Prove that $\mathsf{E}[X \,|\, X \geq a] > \mathsf{E}[X]$ for any a with $0 < \mathsf{P}\{X \geq a\} < 1$.

Ex 9.1.21. Let $N \sim \text{Pois}(\lambda)$ and let $X|N \sim \text{Bin}(N,p)$, where $p \in (0,1)$ and $\lambda > 0$ are known constants. Find $\mathsf{E}[N \,|\, X]$.

Section 9.5

Ex 9.1.22. BH.9.5.1. Is $\mathsf{V}[Y|X] = \mathsf{V}[\mathsf{E}[Y|X]]$?

Ex 9.1.23. Use Eve's law to show that $\mathsf{V}[Y] \geq \mathsf{V}[\mathsf{E}[Y \mid X]]$.

Ex 9.1.24. Let $Z \sim \mathcal{N}(\mu, \sigma^2)$ and $Y = \sqrt{Z} + Z^2$. Find $\mathsf{V}[Y|Z]$.

Ex 9.1.25. Correct? $\mathsf{V}[Y] = \mathsf{V}[Y|A]\,\mathsf{P}\{A\} + \mathsf{V}[Y|A^c]\,\mathsf{P}\{A^c\}$ for any random variable Y and event A.

Ex 9.1.26. Let X, Y be random variables. Explain the difference between $\mathsf{V}[Y|X]$ and $\mathsf{V}[Y|X = x]$.

Ex 9.1.27. Show that $\mathsf{E}\left[(Y - \mathsf{E}[Y|X])^2|X\right] = \mathsf{E}\left[Y^2|X\right] - (\mathsf{E}[Y|X])^2$.

Ex 9.1.28. Let $X \sim \mathcal{N}(\mu, \sigma^2)$ and $W|X \sim \mathcal{N}(0, X^2)$. Find $\mathsf{V}[W]$.

9.2 BH EXERCISES: HINTS AND SOLUTIONS

Ex 9.2.1. BH.9.1. It is easy to associate an indicator with the route chosen: let $R \in \{1,2,3\}$ be the rv corresponding to the route.

We tackle this problem also in an assignment.

Ex 9.2.2. BH.9.25. We tackle this problem also in an assignment with simulation. Check out https://en.wikipedia.org/wiki/Kelly_criterion you're interested.

Ex 9.2.3. BH.9.28.

Ex 9.2.4. BH.9.32. The results of this exercise are (or should be) used by nearly all software packages to control inventory levels of companies such as supermarkets and bol.com.

Ex 9.2.5. BH.9.37.

Bootstrapping is used in statistics to, for instance, construct confidence intervals. It is a much used and intuitive technique.

Extra exercise to help you recall some ideas of Ch 1. How many different bootstrap samples are possible?

I used some extra ideas to save some time. We say that the rvs $\{X_i\}$ are independent and distributed as the common rv X when $X_i \sim F_X$ where F_X is the CDF of the rv X. Then $\mathsf{E}[X_i] = \mathsf{E}[X]$, and so on. Next, I prefer to write $Y_j = X_j^*$, as this writes (and types) faster. Finally, it is easy to define $Y_j = \sum_{i=1}^n X_i I_{S_j=i}$, where $S_j \sim \text{DUnif}(\{1,\dots,n\})$ is the jth sample of the $\{X_i\}$.

Ex 9.2.6. BH.9.39. There are numerous examples of rvs with non-zero kurtosis, for instance, claim sizes of car accidents, the time patients spend in hospital beds, finance. This exercise helps to understand how a positive kurtosis may originate.

Ex 9.2.7. BH.9.50. We will also simulate this in an assignment.

Ex 9.2.8. BH.9.52

Ex 9.2.9. BH.9.55. Suppose first you draw just one number per day, what is then the recursion? Then suppose you draw 2 numbers per day.

An interesting variation is to find a recursion for the number of *draws* instead of *days* are needed until all numbers have been seen.

Ex 9.2.10. BH.9.56.

Ex 9.2.11. BH.9.57

Ex 9.2.12. BH.9.58. In part c. the prior is the uniform distribution. What would happen if you would take the prior of part b, i.e., a out of j wins?

9.3 CHALLENGE: BETTING

Consider the setting of BH.9.25, which you also studied in the coding section. We use the notation from that exercise. In this exercise we will discuss how to set f, the betting fraction. In particular, we will discuss the *Kelly criterion*, which states that the betting fraction should be $f = 2p - 1$ if $p > \frac{1}{2}$ is the winning probability.

We discuss its relationship to expected utility theory, which you will also study in Introduction to Mathematical Economics. Expected utility theory states that bets should be chosen to maximize expected utility. So we solve $\max_{0 \le f \le 1} \mathsf{E}[U(X_{n+1}) \mid X_n]$.

Ex 9.3.1. Show that solving the maximization problem for the utility function $U(x) = \log(x)$ yields the betting fractions from the Kelly criterion, $f = 2p - 1$ if $p > \frac{1}{2}$ and $f = 0$ if $p \le \frac{1}{2}$.

Other people may have a different utility function, which yields a different betting fraction.

Ex 9.3.2. Calculate the utility maximizing betting fraction f if $U(x) = \sqrt{x}$.

Note that for both of these utility functions, the betting fraction f does not depend on the wealth X_n before the gamble, but in general f does depend on X_n.

Now that we have two different betting fractions, we compare them. For that, we first need the following result:

Ex 9.3.3. Assume that f does not depend on X_n. Let $x_0 = 1$. Show that there exist constants a, b such that $\log(X_n) = aW + b$ and $W \sim \text{Bin}(n, p)$, and determine a and b in terms of f.

Theorem 10.3.6. states that (for sufficiently large n) we can approximate a random variable with the binomial distribution $W \sim \text{Bin}(n, p)$ by a random variable with the normal distribution $\text{Norm}(np, np(1-p))$. While you will only learn about the proof of this next week, we are already going to use this approximation here.

Ex 9.3.4. Two people (Carl and Daria) participate in n rounds of this betting game. Their games are independent. Carl's initial wealth is $x_0 = 1$ and Daria's initial wealth is $y_0 = 1$. We denote Carl's wealth after n rounds by X_n and Daria's wealth after n rounds by Y_n. Carl chooses f according to the Kelly criterion, i.e. $f = 2p - 1$. Daria chooses f to be the utility maximizing betting fraction for $U(x) = \sqrt{x}$. Use the previously mentioned normal approximation to derive an approximation for the difference $\log(Y_n) - \log(X_n)$.

Kelly's criterion does not mention utility functions, it just recommends to set $f = 2p - 1$ regardless of one's utility function. The next exercise is meant to give some insight why.

Ex 9.3.5. Use `pnorm` in R, or `norm.cdf` in Python, to approximate $P(X_n > Y_n)$ for some chosen values for n and p. What do you think that happens if $n \to \infty$ for a fixed p? Explain why this is an argument to use the Kelly criterion regardless of one's utility function. Also, explain why maximizing utility suggests a different f in spite of this result.

10

CHAPTER 10: QUESTIONS AND REMARKS

10.1 SIMPLE QUESTIONS

Section 10.1

Ex 10.1.1. On BH.10.1.1: here is perhaps simpler proof of the Cauchy-Schwarz inequality. Define $f(t) = \mathsf{E}\left[(Y - tX)^2\right]$.

1. Explain that $f(t) \geq 0$.
2. Write $f(t) = \mathsf{E}\left[(Y - tX)^2\right]$ as a polynomial of the second degree, i.e., in the form $f(t) = at^2 + bt + c$ (Hint, see the proof of BH.10.1.1).
3. Since $f(t) \geq 0$, how many (real) roots can it have at most?
4. What are the implications of this for the discriminant $D = b^2 - 4ac$?
5. Show that the Cauchy-Schwarz inequality directly follows from this restriction on D.

Remark 10.1.2. I find it easier to remember the Cauchy-Schwarz inequality in the form $(\mathsf{E}[XY])^2 \leq \mathsf{E}\left[X^2\right] \mathsf{E}\left[Y^2\right]$; like this there are squares on both sides.

Ex 10.1.3. On BH.10.1.3. How do they get from $\mathsf{P}\{X > 0\}$ to the inequality for $\mathsf{P}\{X = 0\}$? (Provide the details.)

Ex 10.1.4. On BH.10.1.3. Do the algebra to show that $\mathsf{P}\{X = 0\} = 1/(\mu + 1)$.

Ex 10.1.5. On BH.10.1.3. Explain that we actually use Markov's inequality.

Ex 10.1.6. On BH.10.1.3. What is the probability of two people with birthdays 2 days apart?

Remark 10.1.7. I often forget the direction in Jensen's inequality. To check, the following reasoning works for me: I know that $\mathsf{V}[X] \geq 0$, but $\mathsf{V}[X] = \mathsf{E}\left[X^2\right] - (\mathsf{E}[X])^2 = \mathsf{E}\left[g(X)\right] - g(\mathsf{E}[X])$ with $g(x) = x^2$. Then, from the graph of the parabola, i.e., the graph of g, I know that g is convex.

Ex 10.1.8. In Jensen's inequality, when does equality hold? Can you explain (in terms of convexity and concavity) why equality holds for only this type of functions?

Remark 10.1.9. Skip BH.10.1.7, 10.1.8, 10.1.9

Ex 10.1.10. When X is a non-negative rv, prove the simplest form of Markov's inequality: $\mathsf{P}\{X \geq a\} \leq \mathsf{E}[X]/a$ for $a \geq 0$. Then show that BH.10.1.10 follows from this.

Ex 10.1.11. Which of the following are equivalent to Chebyshev's inequality? Show why or why not.

1. $P(|X - \mathsf{E}[X]| \geq a) \leq \frac{\mathsf{V}[X]}{a^2}$ for all $a > 0$

2. $P(|X - \mathsf{E}[X]| < a) > \frac{\mathsf{V}[X]}{a^2}$ for all $a > 0$

3. $P(|X - \mathsf{E}[X]| < a) \geq 1 - \frac{\mathsf{V}[X]}{a^2}$ for all $a > 0$

4. $P(|X - \mathsf{E}[X]| \geq c\sigma) \leq \frac{1}{c^2}$ for all $c > 0$ and $\sigma^2 = \mathsf{V}[X]$.

Ex 10.1.12. On BH.10.1.11. Why is Chebyshev's inequality of no use if we try to plug in values for $0 < a \leq \sqrt{\mathsf{V}[X]}$?

Ex 10.1.13. BH.10.1.13 shows that Chernoff's inequality is a very strict bound. Is Chernoff's inequality always the tightest bound (out of the ones you know)? What about the case where X is defined as follows

$$\mathsf{P}\{X = 0\} = \frac{3}{4}, \qquad \mathsf{P}\{X = 2\} = \frac{1}{4}.$$

Section 10.2

Ex 10.1.14. Is the following statement equivalent to the strong or the weak law of large numbers? Fix $\epsilon > 0$. For all $\delta > 0$, there is an n so large that $\mathsf{P}\left\{|\bar{X}_n - \mu| > \epsilon\right\} \leq \delta$.

Ex 10.1.15. Which of the two following statements correctly represents the strong law: $\lim_{n\to\infty} \mathsf{P}\left\{|X_n - \mu| > \epsilon\right.$
1 for all $\epsilon > 0$, or $\mathsf{P}\left\{\lim_{n\to\infty} |X_n - \mu| = 0\right\} = 1$.

Ex 10.1.16. On BH.10.2.5. Where have we applied this idea earlier?

Section 10.3

Ex 10.1.17. On BH.10.3.1. I prefer to write $\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}$. Why could that be?

Ex 10.1.18. On BH.10.3.7. $\mathsf{E}\left[\log Y_n\right] = \log 100 - 0.081n$. Explain the 0.081. Think also about the paradoxical outcome. (Once again, probability theory *is* hard.)

Ex 10.1.19. On BH.10.3.7. The stock rises α % and decreases β %. Find a relation between α, β such that $\lim Y_n \geq 1$.

Ex 10.1.20. On BH.10.3.7. Note that $\mathsf{E}\left[\log Y_n\right] \sim -0.081n$, i.e., has has negative drift, but $\log \mathsf{E}[Y_n] \sim n \log 1.1$. Check that this is not in conflict with Jensen's inequality.

Section 10.4

Ex 10.1.21. On BH.10.4.3. Why is $n\bar{Z}_n^2 \sim \chi_1^2$?

Ex 10.1.22. On BH.10.4.3. Show that $\sum_j^n (Z_j - \bar{Z}_n)^2$ and $\bar{Z}_n^2$ are independent.

Ex 10.1.23. A fair coin is tossed 100 times. We are interested in the probability that the number of heads that turn up is at most 40. What is the tightest upper bound on this probability that we can find by using Chebyshev's inequality? Hint: use a symmetry argument.

Ex 10.1.24. Here is inequality from which all inequalities in BH 10.1.3 immediately follow. It's worth memorizing. Take any rv X and a function f that is non-negative and non-decreasing.

1. Why is this true for any a: $f(a)\, I_{X\geq a} \leq f(X)\, I_{X\geq a} \leq f(X)$?
2. Take expectations in the inequality of the previous step and use the fundamental bridge to show that $\mathsf{P}\{X \geq a\} \leq \mathsf{E}\left[f(X)\right]/f(a)$.
3. What part of the proof goes wrong if f can also be negative?
4. Show that Markov's inequality follows by taking $Y = |X|$ and $f(x) = x$. Why don't we take $f(x) = |x|$?
5. Show that Chebyshev's inequality follows by taking $Y = |X - \mu|$ and $f(x) = x^2$.
6. Show that Chernoff's inequality follows by taking $f(x) = e^x$.

Ex 10.1.25. Let the set of r.v.s $\{X_k, k \geq 1\}$ be the outcomes of throws of a biased coin. We take $X_j = 1$ if the outcome is heads, and $X_j = 0$ if tails. Suppose $\mathsf{E}[X_k] = \mu$ and $\mathsf{V}[X_k] = \sigma^2$. Let $Y_j = \sum_{i=nj+1}^{(n+1)j} X_i/n$, i.e., Y_j is the sample mean of the j batch of throws. Since $\{Y_j, j \geq 1\}$ are iid, take Y as the common r.v., i.e., $Y_j \sim Y$. What is a (frequentist) explanation of the statement $\mathsf{P}\left\{|Y - \mu| > \epsilon\right\} \leq \sigma^2/n\epsilon$?

Ex 10.1.26. Interpret the WLLN in terms of the previous exercise.

Ex 10.1.27. In the setting of [10.1.25], the probability of a sequence of outcomes like this: $H, T, H, T, H, T, \ldots$, i.e., a sequence in which the heads and tails alternate, has probability zero. However, $\sum_{i=1}^n I_{X_i=H}/n \to 1/2$. So, we have a sequence that occurs with probability zero, but still the average along the sequence has the proper limit. Doesn't this violate the SLLN?

10.2 BH EXERCISES: HINTS AND SOLUTIONS

Ex 10.2.1. BH.10.2

Ex 10.2.2. BH.10.3 This is just a funny exercise, but I wonder whether it has a practical value.

Ex 10.2.3. BH.10.6

Ex 10.2.4. B.10.9

Ex 10.2.5. BH.10.23.

Ex 10.2.6. BH.10.26.

Ex 10.2.7. BH.10.28. Note that standardized version of a rv X is $Y = (X - \mu)/\sigma$ where $\mathsf{E}[X] = \mu$ and $\mathsf{V}[X] = \sigma$.

Ex 10.2.8. BH..10.30. The problem demonstrates a simple investment strategy. If you plan to work as a quant in finance or as an actuary, or if you play poker, or some similar game, such strategies should interest you naturally.

Ex 10.2.9. BH.10.36.

Ex 10.2.10. BH.10.39.

10.3 CHALLENGE: RECORDS

In BH.7.48 you looked at the number of records in high jumping. Let X_j be how high the jth jumper jumped. As in that exercise, we assume that $X_1, X_2, \ldots$ are iid. with a continuous distribution and say that the jth jumper sets a record if X_j is larger than X_i for all $1 \le i \le j-1$. Let X_i^* denote the ith record, i.e., the height of highest jump for the first i jumps. We write f_X and F_X for the PDF and CDF of the iid jumping heights, and X for a random variable with density f_X. Finally, we write G_X for the survivor function of X, i.e. $G_X(x) = 1 - F_X(x)$.

It is not necessary to know the solution of BH.7.48 to do the challenge. In the challenge, we instead look at the distribution of the ith record, the expectation of the ith record and the expected improvement of the record: $\mathsf{E}\left[X_{n+1}^* - X_n^*\right]$.

Ex 10.3.1. Let $f_{X_{i+1}^*, X_i^*}$ be the joint PDF of the $(i+1)$th and the ith record. Prove that

$$f_{X_{i+1}^*, X_i^*}(u, v) = \frac{f_X(u)}{G_X(v)} f_{X_i^*}(v)\, I_{u>v}.$$

Note that $X_1^* = X_1$. We now derive the density of X_2^*, and then proceed with the general case. These are challenging problems, be sure to check out the hints if you are stuck.

Ex 10.3.2. Prove that

$$f_{X_2^*}(u) = -f_X(u) \log(G_X(u)).$$

Ex 10.3.3. Prove that

$$f_{X_n^*}(u) = f_X(u) \cdot \frac{(-\log(G_X(u)))^{n-1}}{\Gamma(n)}.$$

In general case, it is hard to compute the integral of $u f_{X_n^*}(u)$ which is required to compute $\mathsf{E}\left[X_n^*\right]$. For the exponential distribution however, it is still possible to find the result analytically.

Ex 10.3.4. Assume that $X \sim \text{Exp}(\lambda)$. Determine $\mathsf{E}\left[X_n^*\right]$, and hence the expected improvement of the record: $\mathsf{E}\left[X_{n+1}^* - X_n^*\right]$, using the PDF from the previous exercise.

When X is exponentially distributed, we can also use directly use the properties of the exponential distribution to determine $\mathsf{E}\left[X_{n+1}^* - X_n^*\right]$.

Ex 10.3.5. In Exercise 6 of Assignment 5 you found an expression for $\mathsf{E}[X \mid X \geq a]$ if $X \sim \text{Exp}(\lambda)$. Use this expression and Adam's law to determine $\mathsf{E}\left[X_{n+1}^* - X_n^*\right]$ if $X \sim \text{Exp}(\lambda)$.

Ex 10.3.6. Is the exponential distribution a realistic model for record improvements? Why (not)? If not, why is it still good to look at this case? Explain briefly.

We now consider this model for other distributions as well. Although for many distributions finding analytical results is very difficult or impossible, it can still be interesting to make a plot for the record improvements for other distributions.

Ex 10.3.7. Assume that X has PDF $f_X(x) = 2xe^{-x^2}$ for $x > 0$, and $f_X(x) = 0$ otherwise. Plot $\mathsf{E}\left[X_{n+1}^* - X_n^*\right]$ as a function of n. You may use code for computing the survival function and the expectation, although it is possible (but not recommended) to do it analytically.

11

OLD EXAM QUESTIONS

QUESTION

Ex 11.0.1. Inspired by BH.7.3.6, compute $\mathsf{V}[M]$ and $\mathsf{V}[L]$ by first computing $\mathsf{E}\left[M^2\right]$ and $\mathsf{E}\left[L^2\right]$.

QUESTION

Ex 11.0.2. Let (X, Y) follow a Bivariate Normal distribution, with X and Y marginally following $\mathcal{N}(\mu, \sigma^2)$ and with correlation ρ between X and Y.

1. Use the definition of a Multivariate Normal Distribution to show that $(X+Y, X-Y)$ is also Bivariate Normal.
2. Find the marginal distributions of $X+Y$ and $X-Y$.
3. Compute $\mathsf{Cov}[X+Y, X-Y]$. Then, write down the expression for the joint PDF of $(X+Y, X-Y)$.

QUESTION

Ex 11.0.3. This is about the simplest model for an insurance company that I can think of. We start with an initial capital $I_0 = 2$. The company receives claims and contributions every period, a week say. In the ith period, we receive a contribution X_i uniform on the set $\{1, 2, \ldots, 10\}$ and a claim C_i uniform on $\{0, 1, \ldots 8\}$.

1. What is the meaning of $I_1 = I_0 + X_1 - C_1$?
2. What is the meaning of $I_2 = I_1 + X_2 - C_2$?
3. What is the interpretation of $I_1' = \max\{I_0 - C_1, 0\} + X_1$?
4. What is the interpretation of $I_2' = \max\{I_1' - C_2, 0\} + X_2$?
5. What is the interpretation of $\bar{I}_n = \min\{I_i : 0 \leq i \leq n\}$?
6. What is $\mathsf{P}\{I_1 < 0\}$?
7. What is $\mathsf{P}\left\{I_1' < 0\right\}$?
8. What is $\mathsf{P}\{I_2 < 0\}$?
9. What is $\mathsf{P}\left\{I_2' < 0\right\}$?

10. Provide an interpretation in terms of the inventory of rice, say, at a supermarket for I_1 and I_1'.

QUESTION

Ex 11.0.4. Take $X \sim \text{Unif}(\{-2,-1,1,2\})$ and $Y = X^2$. What is the correlation coefficient of X and Y? If we would consider another distribution for X, would that change the correlation?

QUESTION

Ex 11.0.5. We have a machine that consists of two components. The machine works as long as both components have not failed (in other words, the machine fails when one of the two components fails). Let X_i be the lifetime of component i.

1. What is the interpretation of $\min\{X_1, X_2\}$?

2. If X_1, X_2 iid $\sim \text{Exp}(10)$ (in hours), what is the probability that the machine is still 'up' (i.e., not failed) at time T?

3. Use the previous result to determine the distribution of $\min\{X_1, X_2\}$.

4. What is the expected time until the machine fails?

QUESTION

Ex 11.0.6. We have two rvs X and Y with the joint PDF $f_{X,Y}(x,y) = \frac{6}{7}(x+y)^2$ for $x, y \in (0,1)$ and 0 else. Also we consider the two rvs U and V with the joint PDF $f_{U,V}(u,v) = 2$ for $u, v \in [0,1]$, $u+v \leq 1$ and 0 else.

1. Compute $\mathsf{P}\{X+Y>1\}$.

2. Compute $\mathsf{Cov}[U,V]$.

(Hint: first draw the area over which you want to integrate, if this does not help check out the discussion board post on exercise 7.13a from the first Tutorial)

QUESTION

Ex 11.0.7. Let $U = X+Y$ and $V = X-Y$ where $X, Y \sim U[0,1]$ and independent. Show that

$$f_{U,V}(u,v) = \frac{1}{2} I_{|v| \leq u \leq 2-|v|}.$$

QUESTION

Ex 11.0.8. Let X and Y have PDF $f_{X,Y}$. Take $g(x,y) = (\min\{x,y\}, \max\{x,y\})$. Why is

$$f_{U,V}(u,v) = f_{X,Y}(u,v) + f_{X,Y}(v,u)?$$

Simplify to $f_{U,V}(u,v) = 2f(u)f(v)$ for the case X, Y iid with common PDF f.

QUESTION

Ex 11.0.9. Let X, Y be continuous rvs with CDF $F_{X,Y}(x,y) = (x-1)^2(y-2)/8$ for $x \in (1,3)$.

a. Explain that $y \in (2,4)$ for F to be a proper CDF.

b. What is $F(3,7)$?

c. Determine the PDF.

d. Compute $\mathsf{P}\{2 < X < 3\}$

e. Compute $\mathsf{P}\{2 < X < 3, 2 < Y < 3\}$.

f. Compute $\mathsf{P}\{Y < 2X\}$.

g. Compute $\mathsf{P}\{Y \leq 2X\}$.

h. Compute $\mathsf{P}\{Y < 2X, Y + 2X > 6\}$.

QUESTION

Ex 11.0.10. Consider the general case where we are given the relationship $U = V^4$ between the random variables U and V for $V \in (-3,2)$. Explain why we cannot simply invoke the change of variables theorem.

Now imagine V following a uniform distribution on the given interval. Consider the given transformation on the intervals $(-3,0)$ and $(0,2)$ separately. Explain why this allows you to employ the change of variables theorem and find the distribution of U on these intervals. Finally combine these results (using indicator functions) and state the PDF of U (remember to adjust the domain for the indicator functions according to the transformations).

QUESTION

Ex 11.0.11. Let $U \sim \text{Unif}(0,\pi)$. Use BH.8.1.9 to show that $X = \tan(U)$ has the Cauchy distribution. Compare this exercise to BH.8.1.5.

QUESTION

You walk into a bar and you find two people, Amy and Bob, playing a game of darts. Their game consists of several rounds, called *legs*, and the first person to win 7 legs wins the match. You have never seen Amy or Bob play before, so you don't know their strength. Denoting by p the probability that Amy wins a leg, your prior belief can be modeled by a uniform distribution: $p \sim \text{Unif}(0,1)$. (Note: we assume that p remains constant during the entire match; even though your *beliefs* about p might change.)

Denoting by A a leg won by Amy and by B a leg won by Bob, the result of the first 10 legs is: $AAABBAABAB$. Your friend Charles is very confident that Amy will win the match and he offers you a bet: if Amy wins the match, you must pay €10 to Charles; if Bob wins the match, he must pay you €300. You are tempted to take the bet, but you want to do some calculations first.

Ex 11.0.12. Is the order in which Amy and Bob won their respective legs relevant for your posterior probability that Bob will win the match?

Ex 11.0.13. Let A_n denote the number of legs that Amy won out of a total of n legs. Express the result of the first 10 legs in terms of A_n

Ex 11.0.14. What is the distribution of $A_n|p$ (i.e., the distribution of A_n given the value of p)?

Ex 11.0.15. Find the posterior distribution of p after observing $A_n = k$.

Ex 11.0.16. According to your posterior belief about p, what is the probability that Bob wins the match? Express your answer in terms of beta functions. (Hint: Use the law of total probability.)

Ex 11.0.17. Using the expression

$$\beta(a,b) = \frac{(a-1)!(b-1)!}{(a+b-1)!} \tag{11.0.1}$$

for every positive integers a, b, compute the probabilty from the previous question as a number.

Ex 11.0.18. Assuming that you want to maximize your expected outcome, should you take the bet?

QUESTION

Ex 11.0.19. On BH.9.5.4. With Z and W as given in the example, show that $\mathsf{E}[Z|W] = \rho$ and $\mathsf{V}[Z|W] = 1 - \rho^2$.

Ex 11.0.20. Prove that

$$\mathsf{E}\left[(Y - \mathsf{E}[Y|X] - h(X))^2\right] = \mathsf{E}\left[(Y - \mathsf{E}[Y|X])^2\right] + \mathsf{E}\left[(h(X))^2\right] \tag{11.0.2}$$

for all random variables X, Y and all functions h. Explain why this result implies that $\mathsf{E}[Y|X]$ is the best predictor of Y based on X.

QUESTION

In the next couple of problems we derive Eve's law in a slightly different way then BH.

Define $\hat{X} = \mathsf{E}[X \mid Y]$ as an *estimator* of X and $\tilde{X} = X - \hat{X}$ as the estimation error.

Ex 11.0.21. Show that $\mathsf{E}\left[\tilde{X} \,\middle|\, Y\right] = 0$.

Ex 11.0.22. Prove that $\mathsf{E}\left[\tilde{X}\right] = 0$. What does it mean that $\mathsf{E}\left[\tilde{X}\right] = 0$?

Ex 11.0.23. Prove that $\mathsf{E}\left[\tilde{X}\hat{X}\right] = 0$.

Ex 11.0.24. Show that $\mathsf{Cov}\left[\hat{X}, \tilde{X}\right] = 0$. Conclude that

$$\mathsf{V}[X] = \mathsf{V}\left[\hat{X} + \tilde{X}\right] = \mathsf{V}\left[\hat{X}\right] + \mathsf{V}\left[\tilde{X}\right]. \tag{11.0.3}$$

Ex 11.0.25. Prove that $\mathsf{V}\left[\tilde{X}\right] = \mathsf{E}[\mathsf{V}[X \mid Y]]$. Conclude Eve's law.

CHEBYSHEV'S INEQUALITY IS useful in proving notions of convergence in probability, which you will see repeatedly in later courses. We say X_n converges in probability to the random variable Z if

$$\lim_{n\to\infty} P(|X_n - Z| \geq \varepsilon) = 0 \quad \forall \varepsilon > 0$$

Note that in the above definition setting $Z = a$ for some constant a would still be valid, as technically the constant a is a random variable.

Ex 11.0.26. Let Y_n denote the number of heads obtained from throwing a fair coin n times. Then $\frac{Y_n}{n}$ clearly is the proportion of heads in the sample. Find the expectation of this proportion, and show that it converges in probability to its mean. This is denoted as $\frac{Y_n}{n} \xrightarrow{p} \mathsf{E}\left[\frac{Y_n}{n}\right]$ and is known as the weak law of large numbers.

Ex 11.0.27. Where would this proof break down if we try to apply it to e.g. the Cauchy distribution?

QUESTION

Ex 11.0.28. Let $Z \sim \text{Norm}(0,1)$. In this exercise, we try to find yet another bound for $\mathsf{P}\{|Z| > 3\}$ not yet presented in BH Example 10.1.3.

1. Prove that $\mathsf{P}\{|Z| > 3\} \leq e^{-9t}\,\mathsf{E}\left[e^{tZ^2}\right]$.
2. If $X \sim \text{Gamma}(n, \lambda)$ then the MGF of X is given by $M_X(t) = (1 - t/\lambda)^{-n}$.

   Use this to derive the MGF of the chi-square distribution with 1 degree of freedom.
3. Determine which t yields the best upper bound of $\mathsf{P}\{|Z| > 3\}$, and give this bound.

4. Argue that $\mathsf{P}\{|Z| > 3\} = \mathsf{P}\{Z^4 > 81\}$, and use this to prove that $\mathsf{P}\{|Z| > 3\} \leq \frac{1}{27}$.
5. Recall from BH Example 6.5.2. that $\mathsf{E}\left[Z^{2n}\right] = \frac{(2n)!}{2^n n!}$ for all positive integers n.

   Use this (in a way similar to the previous part) to give yet another bound for $\mathsf{P}\{|Z| > 3\}$.

   For what value(s) of n do we obtain the strongest bound for $\mathsf{P}\{|Z| > 3\}$?

QUESTION

An insurance company offers a theft insurance for electric bikes. When a claim is filed, the insurer pays out the size of the claim, with a maximum of 1000 euros. So a claim of 500 euros is paid out completely, while a claim of 1500 euros yields a payout of 1000 euros.

Let X denote the size of the claim in thousands of euros. We assume that X has the following pdf:

$$f_X(x) = \frac{3}{4}x(2-x), \quad 0 \leq x \leq 2. \tag{11.0.4}$$

Let Y denote the size of the payout. Note that $Y = \min\{X, 1\}$.

Ex 11.0.29 (1)**.** What is the probability that the claim is at most 1000 euros?

Ex 11.0.30 (1.5)**.** What is the expected payout given the information that the claim is at most 1000 euros?

Ex 11.0.31 (1.5)**.** What is the (unconditional) expected payout?

The time T (in hours) it takes for the company to process a payout of size $Y = y$ is uniformly distributed on the interval $[y, 2y]$.

Ex 11.0.32 (1)**.** Compute the (unconditional) expected value of T.

QUESTION

Suppose it is now Sinterklaas and everyone in your family writes and reads poems for each other for celebration. You are in a family of 5 (including you) and you have already heard 3 poems, which means there are still 2 poems left. Let X_1, X_2, X_3 be the time (in minutes) spent on each of the first 3 poems, and X_4, X_5 be that of the remaining poems. Assume that $X_i \sim \text{Norm}(3, 1)$ for $i = 1, \ldots, 5$.

Ex 11.0.33 (1.5)**.** First assume that the times spent on each poem are all independent. What is the expected number of remaining poems that take more time to read than each of the 3 poems you have already heard?

For the next two exercises, suppose that $(X_1, \ldots, X_5)$ is now Multivariate Normal distributed with $\mathsf{Corr}\left[X_1, X_j\right] = \frac{1}{2}$ for $j = 4, 5$

Ex 11.0.34 (2.5)**.** On average, how many of the remaining poems take at least 1 minutes more to read compared to the 1st poem?

Ex 11.0.35 (1)**.** Show that there exists a constant c such that $X_1 - cX_4$ and X_4 are independent, and determine the value of c.

Remarks and grading scheme:

1. Ex 3.1: Many students assume that $X_i > X_1$ and $X_i > X_2$ is independent. This is not the case. In fact, If $X_i > X_1$, then it's more likely that X_i is large. In consequence, it is also more likely that $X_i > X_2$.

2. Ex 3.1: 0.5 point for multiply your probability with 2 (even if it is calculated wrongly). Full point(1.5) for correct answer.

3. Ex 3.2: 0.5 point for mentioning that $X_4 - X_1$ is normally distributed ,0.5 point for correctly calculated $\mathsf{E}[X_4 - X_1]$ and 0.5 point for correctly calculated $Var(X_4 - X_1)$.

4. Ex 3.3: 0.5 point for writing out the formula for covariance.

QUESTION

Let X and Y be independent and exponentially distributed with rates $\lambda_1 = 1$ and $\lambda_2 = 2$ respectively.

Ex 11.0.36 (0.5)**.** Find the joint PDF $f(x, y)$ of X and Y.

Ex 11.0.37 (0.5)**.** Show that $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, \mathrm{d}x\, \mathrm{d}y = 1$.

Ex 11.0.38 (3)**.** Find $\mathsf{E}[|X - Y|]$, i.e., the expected distance between X and Y.

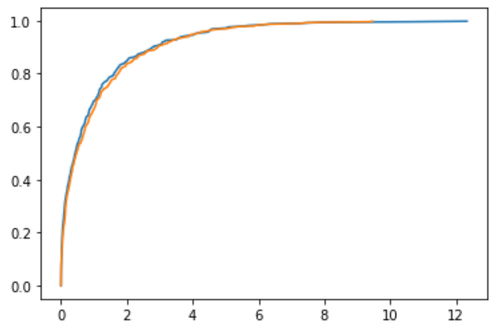
Consider the following code:

Python Code

```python
import numpy as np
np.random.seed(3)

num = 500

x = np.random.normal(loc = 50, scale = 200, size = num)

result2 = np.zeros(num)
for i in range(0,num):
    result2[i] = ((x[i]-50)/200)**2

probs = np.arange(0,num)/num
result2 = np.sort(result2)
y = np.random.chisquare(df = 1, size = num)
```

```
y = np.sort(y)
plt.plot(result2, probs)
plt.plot(y, probs)
plt.show()
```

Ex 11.0.39 (1)**.** What does the code above do and why would you expect to get the graph below as output?



QUESTION

Ex 11.0.40 (1)**.** Let $U_1, U_2 \sim \text{Unif}(0,1)$. Find the PDF of $X_1 = (U_1)^{1/a}$ and then immediately give the PDF of $X_2 = (U_2)^{1/b}$ for $a, b > 0$.

Ex 11.0.41 (0.5)**.** What distributions do X_1 and X_2 have? Also give the corresponding parameters.

Ex 11.0.42 (2)**.** Let $B \sim \text{Beta}(p,q)$ for some $p, q > 0$. Show that $1 - B \sim \text{Beta}(q,p)$.

Ex 11.0.43 (1.5)**.** Let Z be a random variable on $(0,1)$. The PDF of Z is given by

$$f_Z(z) = \begin{cases} f_{X_1}(z) & \text{if } z \in (0, \frac{1}{2}] \\ f_{1-X_2}(z) & \text{if } z \in (\frac{1}{2}, 1) \\ 0 & \text{elsewhere.} \end{cases}$$

(i) Does there exist *more than one* combination of $a, b > 0$ (and $a, b \in \mathbf{R}$) such that this is a valid PDF?

(ii) Does there exist *at least one* combination of a, b as above and $a = b$ such that Z follows a Beta distribution?

Explain your answers clearly.

QUESTION

Let X be Unif(1,3) distributed and Y be exponentially distributed with rate $\lambda = 2$. X and Y are independent.

Ex 11.0.44 (1)**.** Find the joint PDF $f(x, y)$ of X and Y.

Ex 11.0.45 (2)**.** Find $\mathsf{P}\{X \leq Y\}$.

Consider the following code:

Python Code

```python
import numpy as np
np.random.seed(3)

num = 10000

# Lambda = 1/1 = 1
x = np.random.exponential(scale = 1, size = num)
# Lambda = 1/2
y = np.random.exponential(scale = 2, size = num)

result = np.zeros(num)
for i in range(0, len(result)):
    result[i] = min(x[i],y[i])

print(np.mean(result))
```

R Code

```r
set.seed(3)

num = 10000

# Lambda= = 1
x = rexp(num, 1)
# Lambda = 1/2
y = rexp(num,0.5)

result = rep(0, num)
for (i in 1:length(result)) {
  result[i] = min(x[i], y[i])
}

print(mean(result))
```

Ex 11.0.46 (2)**.** The output of the code above is approximately $\frac{2}{3}$. Why would you expect to get this output? Explicitly mention which convergence result you are using in your reasoning.

QUESTION

Tom and Jerry are the only two clerks at a local bank. Tom serves N_1 customers per hour, $N_1 \sim \text{Poisson}(\lambda_1)$; Jerry serves N_2 customers per hour, $N_2 \sim \text{Poisson}(\lambda_2)$ such that $\lambda_1 > \lambda_2 > 0$. Each customer that gets served has a probability p of applying for a credit card, independently. Let X be the number of customers that apply for credit cards per hour.

Ex 11.0.47 (1)**.** Show that $2N_1 + N_2$ and $2N_1 - N_2$ are **not** independent of each other.

Ex 11.0.48 (1)**.** Let $N = N_1 + N_2$. Suppose N_1 and N_2 are independent, what is the distribution of N? What is the distribution of $X|N$? What is the distribution of X?

Ex 11.0.49 (2)**.** Calculate $\rho_{X,N}$, the correlation between X and N.

Consider the following codes:

R Code

```r
library(mvtnorm)
set.seed(444)
A<-diag(x=1,nrow = 3)
B<-rep(0,3)
X<-rmvnorm(100,mean=B,sigma=A)
output=cov(X[,-3])
```

Python Code

```python
import random
import numpy as np
random.seed(444)
A = np.diag([1,1,1])
B = np.zeros(3)
X = np.random.multivariate_normal(B, A, size = 100)
output = np.cov(X, rowvar = False)[0:2,0:2]
```

Ex 11.0.50 (1)**.** Explain in detail the purpose of each line of the above codes.

QUESTION

Let X and Y be independent and $\mathcal{N}(0,1)$ distributed.

Ex 11.0.51 (1)**.** Show that $X - Y = \sqrt{2}Z$, where $Z \sim \mathcal{N}(0,1)$.

Ex 11.0.52 (2)**.** Consider the expectation $E|X-Y|$. Show that

$$E|X-Y| = 2\sqrt{2}\int_0^\infty z\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\,dz.$$

You may use the result in the previous exercise and the fact that by the Fundamental Theorem of Calculus, $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$, if $b > a$.

Ex 11.0.53 (1)**.** Solve the integral in the previous question. *Hint: use integration by substitution*

Consider the following code:

Python Code

```python
import numpy as np
np.random.seed(3)

num = 10000

y = np.random.normal(loc = 1, scale = np.sqrt(2), size = num)

result = np.zeros(num)
for i in range(0, len(result)):
    result[i] = np.exp(y[i])

print(np.mean(result))
```

R Code

```r
set.seed(3)

num = 10000

y = rnorm(num, mean = 1, sd = sqrt(2))

result = rep(0, num)
for (i in 1:length(result)) {
  result[i] = exp(y[i])
}

print(mean(result))
```

Ex 11.0.54 (1)**.** What does the code above do?

QUESTION

A random point (X, Y) is chosen in the following square:

$$\{(x, y) : x^2 < \pi^2, y^2 < \pi^2\}$$

All points are equally likely to be chosen. Let N be the scaled Euclidean norm of (X, Y). So $N = c\sqrt{X^2 + Y^2}$, where $c > 0$.

Ex 11.0.55 (1)**.** Find the joint PDF $f(x, y)$ of X and Y.

Ex 11.0.56 (3)**.** Find the value for c such that $\mathsf{E}\left[N^2\right] = 1$.

Consider the following code:

Python Code

```python
import math
from scipy.integrate import quad

def f(x):
    return 1/(math.pi*(1+x**2))

print(quad(f, -math.inf, math.inf))
```

R Code

```r
f = function(x){
  return(1/(pi*(1+x^2)))
}
integrate(f, -Inf, Inf)
```

Ex 11.0.57 (1)**.** What will the code above return? You may use the fact that the pdf of a Cauchy random variable is given by

$$f(x) = \frac{1}{\pi(1+x^2)}, \quad x \in (-\infty, \infty).$$

QUESTION

Ex 11.0.58 (2)**.** Let $\lambda > 0$ be given. Let $X_1, X_2, \ldots, X_n \sim \text{Expo}(\lambda)$ be independent. Let $Y_i = 2\lambda X_i$ for $i = 1, 2, \ldots, n$. Using theorems, not results (so show all calculations), what is the PDF of $S = Y_1 + Y_2$?

Ex 11.0.59 (1)**.** What is the difference between the PDF of $\sum_{i=1}^{n} X_i$ and that of nX_i? Why are they different? What distributions do they follow? You can use results from the book here, so keep it brief.

Ex 11.0.60 (2)**.** Let $Z \sim \chi^2(2n)$ and let S be as in part (a). Assume that Z and S are independent. Showing all calculations, what is the PDF of $W = S + Z$? What is its distribution?

QUESTION

Ex 11.0.61 (1.5)**.** Let $U \sim \text{Unif}(-1, 1)$. Find the PDF of $B = |U|$. What is its distribution? What is $\mathsf{E}[B]$?

Ex 11.0.62 (1.5)**.** Let X be a continuous random variable such that $M_X(t) = e^t M_X(-t)$. What is $\mathsf{E}[X]$? Can you conclude that X is distributed in the same manner as B?

Ex 11.0.63 (1)**.** Let B be as you found it in part (a). Find the CDF of $X = \kappa + \lambda \ln\left(\frac{B}{1-B}\right)$.

Ex 11.0.64 (1)**.** Let $\kappa = 0$, $\lambda = 1$. The quantile function $Q_X(\cdot)$ is defined to be the function such that $Q_X(F_X(x)) = x$. Find $Q_X(\cdot)$ for the random variable X as in part (c). You may assume $F_X(x)$ to be strictly increasing without proof. This function Q_X is known as the 'log-odds', or 'logit' function, and is used often in regression analysis to model a binary random variable.

QUESTION

Consider the following code:

Python Code

```python
import numpy as np
from scipy.stats import expon
np.random.seed(42)

n = 600
N = 250

X = expon(scale = 1/4).rvs([n, N])
Y = X.mean(axis = 1)

mu = 1/4
sigma = 1/4
Z = np.sqrt(N) * (Y - mu)/sigma

print((Z ** 53).mean())
```

R Code

```r
set.seed(42)

n <- 600
N <- 250

X <- matrix(rexp(N * n, rate = 4), nrow = n, ncol = N)
Y <- rowMeans(X)

mu <- 1/4
sigma <- 1/4
Z <- sqrt(N) * (Y - mu)/sigma

print(mean(Z^53))
```

Ex 11.0.65 (1.5)**.** Y is a vector of i.i.d. random variables.

(i) What is the length ℓ of Y?

(ii) Each element of Y is a mean of k i.i.d. $\text{Exp}(\lambda)$ r.v.s. What are k and λ?

(iii) What are the (theoretical) expectation and variance of an element of Y?

Recall that each element of Y is the mean of k i.i.d. $\text{Exp}(\lambda)$ r.v.s.

Ex 11.0.66 (1)**.** What is the exact distribution of an element of Y?
Give its name and its parameters, and explain the answer.

Let $(Y_1, \dots, Y_\ell)$ be the elements of Y and let $(Z_1, \dots, Z_\ell)$ be the elements of Z. Recall that each Z_i depends on k because Y_i is the mean of k i.i.d. $\text{Exp}(\lambda)$ r.v.s. Let T be the random variable to which Z_1 converges in the limit $k \to \infty$.

Ex 11.0.67 (1)**.** What is the distribution of T, and why?
Hence, what is an approximate distribution of an element of Y (e.g. Y_1) and why?

Ex 11.0.68 (0.5)**.** In the last line of the code, we compute $S = \frac{1}{\ell}\sum_{i=1}^{\ell} Z_i^{53}$.
If $k \to \infty$ (for fixed ℓ, e.g. $\ell = 3$), does S converge to a constant?
You do not have to give formal proofs in this subquestion, but you should give clear explanations using the appropriate theorem(s).

Ex 11.0.69 (1)**.** In the last line of the code, we compute $S = \frac{1}{\ell}\sum_{i=1}^{\ell} Z_i^{53}$.
If $\ell \to \infty$ (for fixed k), does S converge to a constant? If so, does it converge to $\mathsf{E}\left[T^{53}\right]$?
You do not have to give formal proofs in this subquestion, but you should give clear explanations using the appropriate theorem(s).

QUESTION

We have a queue of people served by a single server. Let $L(t)$ be the number of people present in the system at time t. For any time $t \geq 0$ the time to the next arriving person is $X \sim \text{Exp}(\lambda)$ and, when $L(t) > 0$, the time to the next departing customer is $S \sim \text{Exp}(\mu)$. Assume that $\lambda < \mu$.
Suppose $L(0) = n \geq 0$. Then let T be the first time until the system becomes empty, i.e., $T = \inf\{t \geq 0 : L(t) = 0\}$.

Ex 11.0.70 (1)**.** Explain that $\lambda/(\lambda + \mu)$ is the probability an arrival occurs before a departure.

For the moment, assume that $\mathsf{E}[T] < \infty$.

Ex 11.0.71 (1)**.** Explain that for $n > 0$:

$$\mathsf{E}[T|L(0) = n] = \mathsf{E}[T|L(0) = n+1]\frac{\lambda}{\lambda+\mu} + \mathsf{E}[T|L(0) = n-1]\frac{\mu}{\lambda+\mu} + \frac{1}{\lambda+\mu}. \tag{11.0.5}$$

Ex 11.0.72 (1)**.** Show that $\mathsf{E}[T|L(0) = n] = n/(\mu-\lambda)$.

Define $\rho = \lambda/\mu$. Assume that $L(0) \sim \text{Geo}(1-\rho)$.

Ex 11.0.73 (1)**.** Find a simple expression for $\mathsf{E}[T]$.

Ex 11.0.74 (1)**.** Up to now we simply assumed that $\mathsf{E}[T] < \infty$. Motivate intuitively that the condition $\lambda < \mu$ ensures that $\mathsf{E}[T] < \infty$.

QUESTION

Bob and his father argue about tomorrow's weather. Bob thinks it will rain, but his father doesn't agree. They make the following deal. Bob will put down a glass in the back yard. At the end of the day, dad will give Bob one euro for every inch of water in the glass. To be safe, dad gives Bob a shot glass with a height of only one inch to put down in the back yard.

Let X denote the amount of rainfall tomorrow (in inches) and let Y denote the amount of rain collected in the shot glass (in inches). We assume that X has the following pdf:

$$f_X(x) = \frac{3}{4}x(2-x), \quad 0 \le x \le 2. \tag{11.0.6}$$

Ex 11.0.75 (1)**.** Compute the probability that the amount of rainfall tomorrow is at most one inch.

Ex 11.0.76 (1.5)**.** Determine the expected amount of rain collected in the shot glass conditional on the amount of rainfall X being at most one inch.

Ex 11.0.77 (1.5)**.** What is the (unconditional) expected amount of rain collected in the shot glass?

To make things more interesting, dad decides to randomize the amount of euros he will give to Bob. Given an amount $Y = y$ collected in the cup, he will pay Bob an amount Z that is uniformly distributed on $[y, 2y]$.

Ex 11.0.78 (1)**.** What is the (unconditional) expected value of the payout Z?

QUESTION

Consider the following code:

Python Code

```python
import numpy as np
from scipy.stats import expon
np.random.seed(42)

n = 200
N = 500

X = expon(scale = 2).rvs([N, n])
Y = X.mean(axis = 1)

mu = 2
sigma = 2
Z = np.sqrt(n) * (Y - mu)/sigma

print((Z ** 29).mean())
```

R Code

```r
set.seed(42)

n <- 200
N <- 500

X <- matrix(rexp(n * N, rate = 1/2), nrow = N, ncol = n)
Y <- rowMeans(X)

mu <- 2
sigma <- 2
Z <- sqrt(n) * (Y - mu)/sigma

print(mean(Z^29))
```

Ex 11.0.79 (1.5)**.** Y is a vector of i.i.d. random variables.

(i) What is the length ℓ of Y?

(ii) Each element of Y is a mean of k i.i.d. Exp(λ) r.v.s. What are k and λ?

(iii) What are the (theoretical) expectation and variance of an element of Y?

Recall that each element of Y is the mean of k i.i.d. Exp(λ) r.v.s.

Ex 11.0.80 (1)**.** What is the exact distribution of an element of Y?

Give its name and its parameters, and explain the answer.

Let $(Y_1,\dots,Y_\ell)$ be the elements of Y and let $(Z_1,\dots,Z_\ell)$ be the elements of Z. Recall that each Z_i depends on k because Y_i is the mean of k i.i.d. Exp(λ) r.v.s. Let T be the random variable to which Z_1 converges in the limit $k\to\infty$.

Ex 11.0.81 (1)**.** What is the distribution of T, and why?

Hence, what is an approximate distribution of an element of Y (e.g. Y_1) and why?

Ex 11.0.82 (0.5)**.** In the last line of the code, we compute $S=\frac{1}{\ell}\sum_{i=1}^{\ell}Z_i^{29}$.

If $k\to\infty$ (for fixed ℓ, e.g. $\ell=3$), does S converge to a constant?

You do not have to give formal proofs in this subquestion, but you should give clear explanations using the appropriate theorem(s).

Ex 11.0.83 (1)**.** In the last line of the code, we compute $S=\frac{1}{\ell}\sum_{i=1}^{\ell}Z_i^{29}$.

If $\ell\to\infty$ (for fixed k), does S converge to a constant? If so, does it converge to $\mathsf{E}\left[T^{29}\right]$?

You do not have to give formal proofs in this subquestion, but you should give clear explanations using the appropriate theorem(s).

QUESTION

There are three cheese shops in town, shop A, shop B and shop C. N_1 customers enter shop A per hour, $N_1\sim$ Poisson(λ_1); N_2 customers enter shop B per hour, $N_2\sim$ Poisson(λ_2); N_3 customers enter shop C per hour, $N_3\sim$ Poisson(λ_3). N_1,N_2,N_3 are independent of each other. Each customer that enters in any of the three shops, buys cheese with probability p, independently. Let X be the total number of customers that buys cheese per hour.

Ex 11.0.84 (1)**.** Show that N_1+N_2 and N_2-2N_3 are **not** independent of each other.

Ex 11.0.85 (1)**.** Let $N=N_1+N_2+N_3$. What is the distribution of N? What is the distribution of $X|N$? What is the distribution of X?

Ex 11.0.86 (2)**.** Calculate $\rho_{X,N}$, the correlation between X and N.

Consider the following codes:

R Code

```r
library(mvtnorm)
set.seed(777)
A<-matrix(c(1,2,2,4),nrow = 2)
```

```
B<-c(1,2)
X<-rmvnorm(50,mean=B,sigma=A)
output=cor(X[3,],X[41,])
```

Python Code

```
import random
import numpy as np
random.seed(777)
A = [[1,2],[2,4]]
B = [1,2]
X = np.random.multivariate_normal(B, A, size = 50)
output = np.corrcoef(X[[3,41],:].reshape(4))
```

Ex 11.0.87 (1)**.** Explain in detail the purpose of **each line** of the above codes.

QUESTION

A server spends a random amount T on a job. While the server works on the job, it sometimes gets interrupted to do other tasks $\{R_i\}$. Such tasks need to be repaired before the machine can continue working again. It is clear that such interruptions can only occur when the server is working. Assume that the interruptions arrive according to a Poisson process with rate λ, i.e., the number of failures $N(t) \sim \text{Pois}(\lambda t)$. The interruptions $\{R_i\}$ are independent of N and of T and form an iid sequence with common mean $\mathsf{E}[R]$ and variance $\mathsf{V}[R]$. The duration S of a job is its own service time T plus all interruptions, hence $S = T + \sum_{i=1}^{N(T)} R_i$.

Ex 11.0.88 (1.5)**.** The computation below consists of a number of steps, a, b, ... Explain for each step which property is used to ensure the step is true.

$$\mathsf{E}[S \mid T = t] \stackrel{a}{=} \mathsf{E}\left[T + \sum_{i=1}^{N(T)} R_i \,\middle|\, T = t\right] \tag{11.0.7}$$

$$\stackrel{b}{=} \mathsf{E}[T \mid T = t] + \mathsf{E}\left[\sum_{i=1}^{N(T)} R_i \,\middle|\, T = t\right] \tag{11.0.8}$$

$$\stackrel{c}{=} \mathsf{E}[t \mid T = t] + \mathsf{E}\left[\sum_{i=1}^{N(T)} R_i \,\middle|\, T = t\right] \tag{11.0.9}$$

$$\stackrel{d}{=} t + \mathsf{E}\left[\sum_{i=1}^{N(t)} R_i\right]. \tag{11.0.10}$$

Ex 11.0.89 (2)**.** Suppose $R \sim \text{Exp}(\mu)$ and $\mathsf{P}\{T = t\} = 1$, compute $\mathsf{E}[S]$.

Ex 11.0.90 (1.5)**.** Explain what this code computes.

Python Code

```
import numpy as np

```

```python
3  labda = 0.5
4  size = 10
5  num_runs = 50
6
7
8  def do_run():
9      T = np.random.uniform(0, 20)
10     N = np.random.poisson(labda * T)
11     R = np.random.uniform(1, 5, size=N)
12     S = T + R.sum()
13     return S
14
15
16 samples = np.zeros(num_runs)
17 for i in range(num_runs):
18     samples[i] = do_run()
19
20
21 print(samples[samples > 4].mean())
```

R Code

```r
1  labda <- 0.5
2  size <- 10
3  num_runs <- 50
4
5  do_run <- function() {
6    bigT <- runif(n = 1, min = 0, max = 20)
7    N <- rpois(n = 1, labda * bigT)
8    R <- runif(n = N, min = 1, max = 5)
9    S <- bigT + sum(R)
10   return(S)
11 }
12
13 samples <- rep(0, num_runs)
14 for (i in 1:num_runs) {
15   samples[i] <- do_run()
16 }
17
18 print(mean(samples[samples > 4]))
```

Hint, you should know that in P.21 (R18) the string `samples > 4` collects only the samples with value larger than 4.

QUESTION

Let $Y \sim \text{Norm}(0,1)$. In this exercise, we find an upper bound for $\mathsf{P}\{|Y| > 3\}$.

Ex 11.0.91 (1.5)**.** Let g be a positive and increasing function, and let Z be a r.v. Consider the following inequality:

$$\mathsf{P}\{Z \geq a\} = \mathsf{P}\left\{g(Z) \geq g(a)\right\} \leq \frac{\mathsf{E}\left[g(Z)\right]}{g(a)}.$$

(i) Explain why $\mathsf{P}\{Z \geq a\} = \mathsf{P}\left\{g(Z) \geq g(a)\right\}$ holds.

(ii) Explain why $\mathsf{P}\left\{g(Z) \geq g(a)\right\} \leq \frac{\mathsf{E}[g(Z)]}{g(a)}$ holds.

Make sure to clearly indicate where you use that g is positive and increasing.

Ex 11.0.92 (1)**.** Prove that $\mathsf{P}\{|Y| > 3\} \leq e^{-9t}\,\mathsf{E}\left[e^{tY^2}\right]$ for $t > 0$.

Ex 11.0.93 (2.5)**.** For which t do we find the best upper bound for $\mathsf{P}\{|Y| > 3\}$? Also calculate the upper bound for this value of t.

Hint 1. You may use that if $X \sim \chi_1^2$, then the MGF of X is given by $M_X(t) = (1-2t)^{-1/2}$ for $t < 1/2$. However, you should explain clearly how you use this fact.

Hint 2. Do not forget to check the second order condition of minimization.

QUESTION

John likes to spend his time watching trains pass by on the railway near his house. John is interested in the *interarrival times* of the trains: the time between the arrivals of two subsequent trains. John knows that the interarrival times X_i, $i = 1, \dots, n$, are i.i.d. Exponentially distributed with a rate parameter λ. Hence, given the value of λ, the pdf of interarrival time X_i is

$$f_{X_i|\lambda}(x|\lambda) = \lambda e^{-\lambda x}, \qquad x > 0. \tag{11.0.11}$$

John is interested in the value of λ. His prior belief about the distribution of λ is that it follows a Gamma(a, b) distribution with some particular choices for a and b (the exact values of a and b are not relevant for this question).

Ex 11.0.94 (2.5)**.** Suppose that John observes a first interarrival time of $X_1 = x_1$. Derive John's *posterior* distribution of λ.

Ex 11.0.95 (1)**.** Is John's prior distribution a *conjugate* prior?

Ex 11.0.96 (1.5)**.** Suppose John observes the first n interarrival times, with values $X_1 = x_1, \dots, X_n = x_n$. What is John's posterior distribution after these observations?
Hint: you don't need to redo all the math here!

QUESTION

A mouse is trapped in a pit with three tunnels. When the mouse takes tunnel A, the time to get out of the pit is 2 minutes. Tunnel B leads back to the pit (in other words, the mouse cannot escape when it takes tunnel B) and takes 3 minutes. Tunnel C leads also back to the pit, and takes 4 minutes. Every time the mouse is in the pit, it selects a tunnel at random with equal probability. (This mouse much dumber than a real mouse.) Write X for the tunnel selected by the mouse, and let T be the time until the mouse escapes. The travel times of the tunnels are constant.

For the moment, assume that $\mathsf{E}[T] < \infty$.

Ex 11.0.97 (1)**.** Explain that $\mathsf{E}[T|X = B] = 3 + \mathsf{E}[T]$.

Ex 11.0.98 (1)**.** Compute $\mathsf{E}[T]$.

Ex 11.0.99 (2)**.** Compute $\mathsf{V}[T]$.

Ex 11.0.100 (1)**.** Why was it actually allowed to assume that $\mathsf{E}[T] < \infty$?

QUESTION

A server spends a random amount T on a job. While the server works on the job, it sometimes gets interrupted to do other tasks $\{R_i\}$. Such tasks need to be repaired before the machine can continue working again. It is clear that such interruptions can only occur when the server is working. Assume that the interruptions arrive according to a Poisson process with rate λ, i.e., the number of failures $N(t) \sim \text{Pois}(\lambda t)$. The interruptions $\{R_i\}$ are independent of N and of T and form an iid sequence with common mean $\mathsf{E}[R]$ and variance $\mathsf{V}[R]$. The duration S of a job is its own service time T plus all interruptions, hence $S = T + \sum_{i=1}^{N(T)} R_i$.

Ex 11.0.101 (1.5)**.** In the computation of $\mathsf{V}[S]$ we encounter the following steps.

$$\mathsf{V}\left[\sum_{i=1}^{N(t)} R_i\right] = \mathsf{E}\left[\mathsf{V}\left[\sum_{i=1}^{N(t)} R_i \,\middle|\, N(t)\right]\right] + \mathsf{V}\left[\mathsf{E}\left[\sum_{i=1}^{N(t)} R_i \,\middle|\, N(t)\right]\right]. \tag{11.0.12}$$

The computation below consists of a number of steps, a, b, ... Explain for each step which property is used to ensure the step is true.

$$\mathsf{V}\left[\mathsf{E}\left[\sum_{i=1}^{N(t)} R_i \,\middle|\, N(t)\right]\right] \stackrel{a}{=} \mathsf{V}[N(t)\,\mathsf{E}[R]] \tag{11.0.13}$$

$$\stackrel{b}{=} (\mathsf{E}[R])^2\,\mathsf{V}[N(t)] \tag{11.0.14}$$

$$\stackrel{c}{=} (\mathsf{E}[R])^2 \lambda t. \tag{11.0.15}$$

Ex 11.0.102 (2)**.** Suppose $\mathsf{P}\{R = r\} = \mathsf{P}\{T = t\} = 1$, compute $\mathsf{E}[S]$.

Ex 11.0.103 (1.5)**.** Explain what this code computes.

Python Code

```
1  import numpy as np
2
3  labda = 0.5
4  size = 10
5  num_runs = 50
6
7
8  def do_run():
9      T = np.random.uniform(0, 20)
10     N = np.random.poisson(labda * T)
11     R = np.random.uniform(1, 5, size=N)
12     S = T + R.sum()
13     return S
14
15
16 samples = np.zeros(num_runs)
17 for i in range(num_runs):
18     samples[i] = do_run()
19
20
21 print((samples > 8).mean())
```

R Code

```
1  labda <- 0.5
2  size <- 10
3  num_runs <- 50
4
5  do_run <- function() {
6    bigT <- runif(n = 1, min = 0, max = 20)
7    N <- rpois(n = 1, labda * bigT)
8    R <- runif(n = N, min = 1, max = 5)
9    S <- bigT + sum(R)
10   return(S)
11 }
12
13 samples <- rep(0, num_runs)
14 for (i in 1:num_runs) {
15   samples[i] <- do_run()
16 }
17
18 print(mean(samples > 8))
```

Hint, you should know that in P.21 (R18) the string `samples > 8` collects only the samples with value larger than 8.

QUESTION

Ex 11.0.104 (1.5)**.** Let $X \sim \mathcal{N}(\mu, \mu^2)$ and let $Y = e^X$. Showing your work, find the PDF of Y.

Ex 11.0.105 (1)**.** Consider now the independent random variables $X_1, X_2 \sim \mathcal{N}(\mu, \sigma^2)$. Let $Y_1 = e^{X_1}$ and $Y_2 = e^{X_2}$. Are $Y_1 Y_2$ and $\frac{Y_1}{Y_2}$ independent? You can use results from the book here.

Ex 11.0.106 (2.5)**.** Find the joint PDF of $U = Y_1 Y_2$ and $V = \frac{Y_1}{Y_2}$.

QUESTION

Ex 11.0.107 (1)**.** Let $U \sim \text{Unif}\{-n, -n+1, \dots, n-1, n\}$ for some $n \in \mathbb{N}$. Find the PMF of $B = |U|$. What is $\mathsf{E}[B]$?

Ex 11.0.108 (2)**.** Now, consider a random variable X distributed according to a $\text{Beta}(p, q)$ distribution. Since this distribution is only defined on $(0, 1)$, we will transform it to be more general. Consider the random variable $Z = bX + a(1 - X)$, for some $a, b \in \mathbb{R}$ such that $a < b$, and find its PDF.

Ex 11.0.109 (2)**.** Consider again Z as in the previous exercise. Assume that $a = -b$, that $b > 0$, and that $p = q = 2$. What is the PDF of $|Z|$?

QUESTION

Consider the following code:

Python Code

```python
import numpy as np
from scipy.stats import expon
np.random.seed(42)

n = 750
N = 300

X = expon(scale = 3).rvs([n, N])
Y = X.mean(axis = 1)

mu = 3
sigma = 3
Z = np.sqrt(N) * (Y - mu)/sigma

print((Z ** 71).mean())
```

R Code

```r
set.seed(42)

```

```
n <- 750
N <- 300

X <- matrix(rexp(n * N, rate = 1/3), nrow = n, ncol = N)
Y <- rowMeans(X)

mu <- 3
sigma <- 3
Z <- sqrt(N) * (Y - mu)/sigma

print(mean(Z^71))
```

Ex 11.0.110 (1.5)**.** Y is a vector of i.i.d. random variables.

(i) What is the length ℓ of Y?

(ii) Each element of Y is a mean of k i.i.d. Exp(λ) r.v.s. What are k and λ?

(iii) What are the (theoretical) expectation and variance of an element of Y?

Recall that each element of Y is the mean of k i.i.d. Exp(λ) r.v.s.

Ex 11.0.111 (1)**.** What is the exact distribution of an element of Y?
Give its name and its parameters, and explain the answer.

Let $(Y_1,\ldots,Y_\ell)$ be the elements of Y and let $(Z_1,\ldots,Z_\ell)$ be the elements of Z. Recall that each Z_i depends on k because Y_i is the mean of k i.i.d. Exp(λ) r.v.s. Let T be the random variable to which Z_1 converges in the limit $k\to\infty$.

Ex 11.0.112 (1)**.** What is the distribution of T, and why?
Hence, what is an approximate distribution of an element of Y (e.g. Y_1) and why?

Ex 11.0.113 (0.5)**.** In the last line of the code, we compute $S=\frac{1}{\ell}\sum_{i=1}^{\ell} Z_i^{71}$.
If $k\to\infty$ (for fixed ℓ, e.g. $\ell=4$), does S converge to a constant?
You do not have to give formal proofs in this subquestion, but you should give clear explanations using the appropriate theorem(s).

Ex 11.0.114 (1)**.** In the last line of the code, we compute $S=\frac{1}{\ell}\sum_{i=1}^{\ell} Z_i^{71}$.
If $\ell\to\infty$ (for fixed k), does S converge to a constant? If so, does it converge to $\mathsf{E}\left[T^{71}\right]$?
You do not have to give formal proofs in this subquestion, but you should give clear explanations using the appropriate theorem(s).

QUESTION

John is an archer who likes to shoot at small targets. To find out his skill level, John plays the following game. He shoots arrows at a target and counts the number of times he successfully hits the target. He keeps counting until he has missed r times, at which moment the current round of the game stops. His score for the round is the total number of successful shots in the round. John plays n rounds in total and we assume that all shots are independent and have the same (unknown) success probability p. John is interested in finding out his skill level. That is, he is interested in learning the value of p.

Given the value of p, John's score Y_i for the ith round of the game follows a negative binomial distribution with parameters r and p. That is, for every $i = 1, \ldots, n$, we have that $Y_i | p \sim \mathrm{NB}(r, p)$, with a corresponding pmf defined by

$$\mathsf{P}\left\{Y_i = y_i | p\right\} = \binom{y_i + r - 1}{y_i} (1-p)^r p^{y_i}, \tag{11.0.16}$$

for $y_i = 0, 1, 2, \ldots$. John's prior belief about the distribution of p is that it follows a Beta(a, b) distribution with given values for a and b (the exact values of a and b are not relevant for this question).

Ex 11.0.115 (2.5)**.** In the first round, John gets a score of $Y_1 = y_1$. What is John's *posterior* distribution of p, given this first observation?

Ex 11.0.116 (1)**.** Is John's prior distribution a *conjugate* prior?

Ex 11.0.117 (1.5)**.** Suppose John plays n rounds and observes the scores $Y_1 = y_1, \ldots, Y_n = y_n$. What is his posterior distribution after these observations?
Hint: you don't need to do a lot of math here! Instead, use the result of the first question above and a logical argument.

QUESTION

Let $Y \sim \mathrm{Norm}(0, 1)$. In this exercise, we find an upper bound for $\mathsf{P}\{|Y| > 4\}$.

Ex 11.0.118 (1.5)**.** If $X \sim \mathrm{Gamma}(a, \lambda)$ then the rth moment of X is given by $\frac{\Gamma(a+r)}{\lambda^r \Gamma(a)}$. Use this to prove that $\mathsf{E}\left[Y^{2n+2}\right] = (2n+1)\,\mathsf{E}\left[Y^{2n}\right]$ for all positive integers n.

Hint. Use the chi-square distribution.

Ex 11.0.119 (1)**.** Use the previous exercise to calculate $\mathsf{E}\left[Y^4\right]$ and $\mathsf{E}\left[Y^8\right]$.

Ex 11.0.120 (1)**.** We now provide a bound for $\mathsf{P}\{|Y| > 4\}$.

(i) Prove that $\mathsf{P}\{|Y| > 4\} = \mathsf{P}\left\{Y^4 > 256\right\}$.

(ii) Use this to prove that $\mathsf{P}\{|Y| > 4\} \leq \frac{3}{256}$.

Ex 11.0.121 (1.5)**.** Prove that $\mathsf{P}\{|Y| > 4\} \leq \frac{\mathsf{E}[Y^{2n}]}{16^n}$ for all $n \in \mathbb{N}$.
For what value(s) of n do we obtain the strongest bound for $\mathsf{P}\{|Y| > 4\}$?

QUESTION

We have a queue of people served by a potentially infinite number of servers. Let $L(t)$ be the number of people present in the system at time t. For any time $t \geq 0$ the time to the next arriving person is $X \sim \text{Exp}(\lambda)$, and given $L(t) = n$ customers in the system at time t, the time to the next departing customer is $S \sim \text{Exp}(\mu n)$. The rvs S and X are independent, and $\lambda, \mu > 0$. Write $B(h)$ for the number of arrivals during an interval of length h, and $D(h)$ for the number of departures. (Hint: recall the relation between the Poisson distribution and the exponential distribution.)

In the sequel, take h positive, but very, very small, i.e, $h \ll 1$. With this, we use the shorthand $o(h)$ to capture all terms of a polynomial in h with a power higher than 1, for instance,

$$2h + 3h^2 + 44h^{21} = 2h + o(h). \tag{11.0.17}$$

Like this we can hide all nonlinear terms of a polynomial in the $o(h)$ function. This is easy when we want to take limits, for example,

$$\lim_{h\to 0} \frac{2h + 3h^2 + 44h^{21}}{h} = 2 + \lim_{h\to 0} \frac{o(h)}{h} = 2 + 0. \tag{11.0.18}$$

In other words, when computing this limit for $h \to 0$, we don't care about the details in $o(h)$ because $o(h)/h \to 0$ anyway.

Ex 11.0.122 (1)**.** Explain that

$$\mathsf{P}\{B(h) = 1, D(h) = 0 | L(0) = n\} = \lambda h e^{-\lambda h} e^{-\mu n h}. \tag{11.0.19}$$

Ex 11.0.123 (1)**.** Use the first degree Taylor's expansion, $f(h) \approx f(0) + hf'(0) + o(h)$, to motivate that

$$\mathsf{P}\{B(h) = 0, D(h) = 1 | L(0) = n\} = n\mu h + o(h). \tag{11.0.20}$$

Ex 11.0.124 (2)**.** Explain that

$$\mathsf{E}[L(t+h) | L(t) = n] = n + (\lambda - \mu n)h + o(h). \tag{11.0.21}$$

Write $M(t) = \mathsf{E}[L(t)]$.

Ex 11.0.125 (1)**.** Derive that

$$M(t+h) = M(t) + (\lambda - \mu M(t))h + o(h). \tag{11.0.22}$$

QUESTION

Ex 11.0.126 (0.5)**.** Consider a random variable $X \sim \mathcal{N}(\mu, \sigma^2)$. Write down the PDF of the random variable $Y = e^X$. You do not have to elaborate on your answer, but make sure to get everything correct.

Ex 11.0.127 (1.5)**.** Consider now the random variable $W_k = \frac{k}{5Y^2}$. What is the distribution of W_k? You can use results from the book here.

Ex 11.0.128 (1)**.** Calculate $\mathsf{P}\left\{\frac{W_k}{W_l} = \frac{k}{l}\right\}$ for some $l > k > 0$. Are $W_k W_l$ and $\frac{W_k}{W_l}$ independent?

Ex 11.0.129 (2)**.** Let $X_1, X_2 \sim X$ be IID random variables, where X has PDF

$$f_X(x) = \frac{1}{\sqrt{2\pi x}} \exp\left(-\frac{x}{2}\right)$$

for $x > 0$. Find the joint PDF of the random variables $U = X_1 + X_2$ and $V = X_1 - X_2$.

QUESTION

A random point (X, Y) is chosen in the following square:

$$\{(x, y) : -\sqrt{\pi} < x < \sqrt{\pi}, -\sqrt{\pi} < y < \sqrt{\pi}\}.$$

All points are equally likely to be chosen. Let R be its distance from the origin.

Ex 11.0.130 (0.5)**.** Find the joint PDF $f(x, y)$ of X and Y.

Ex 11.0.131 (0.5)**.** Show that $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1$

Ex 11.0.132 (3)**.** Find the expectation of R^2, i.e., the expected squared difference from the origin.

Consider the following code:

Python Code

```python
import numpy as np
np.random.seed(3)

num = 100000

x = np.random.normal(loc = 50, scale = 200, size = num)
y = np.random.normal(loc = 20, scale = 100, size = num)

result = np.zeros(num)
for i in range(0,num):
    result[i] = x[i]*y[i]

print(np.mean(result))
```

Ex 11.0.133 (0.5)**.** What does the code above do?

Ex 11.0.134 (0.5)**.** The code gives as output 1008.99966.
Explain why you would expect to get this output from the code.

QUESTION

Catherine and Denny are playing a game. Each player throws a single (fair) die. The die has three pairs of identical sides: a pair of ones, a pair of twos and a pair of threes. Let X and Y denote the outcome of Catherine and Denny's throw, respectively. The person that throws the highest number wins. If both throw the same number, they have a draw. The final score C of Catherine is determined as follows:

- If Catherine loses, then she gets zero points.
- If Catherine and Denny draw, then she gets 0.5 point.
- If Catherine wins, then her score is the difference $X - Y$ in the numbers they threw.

The final score D for Denny is determined analogously. Assume that the dice are fair and that all throws are independent.

Ex 11.0.135 (1)**.** Find the joint distribution of X and Y conditional on Catherine winning.

Ex 11.0.136 (1.5)**.** What is Catherine's expected score conditional on Catherine winning?

Ex 11.0.137 (1.5)**.** Determine Catherine's (unconditional) expected score.

To spice things up, Catherine and Denny decide to play for money. After playing the dice game and scoring C points, Catherine receives an amount of S euros, where S is determined randomly. Here, conditional on the outcome of C, S follows a uniform distribution on $[C, 2C]$.

Ex 11.0.138 (1)**.** What is the (unconditional) expected reward for Catherine? That is, compute $\mathsf{E}[S]$.

QUESTION

A random point (X, Y) is chosen in the following square:

$$\{(x, y) : x^2 < e, y^2 < e\}$$

All points are equally likely to be chosen. Let S be the Euclidean norm of (X, Y).

Ex 11.0.139 (1)**.** Find the joint PDF $f(x, y)$ of X and Y.

Ex 11.0.140 (3)**.** Find the expectation of S^2, i.e., the squared norm of (X, Y).

Consider the following code:

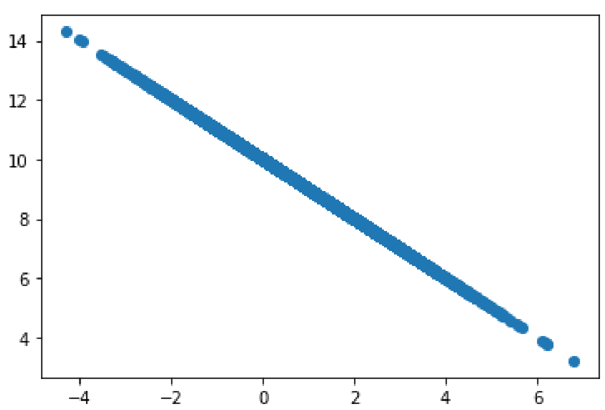
Python Code

```
import numpy as np
import matplotlib.pyplot as plt
np.random.seed(3)

num = 10000

x = np.random.normal(loc = 1, scale = np.sqrt(2), size = num)
y = 10 - x

print(np.corrcoef(x,y))

plt.scatter(x,y)
```

R Code

```
set.seed(3)

num = 10000

x = rnorm(num, mean = 1, sd = sqrt(2))
y = 10 - x

print(cor(x,y))

plot(x,y)
```

Ex 11.0.141 (1)**.** This code gives the value -1 and the following graph.



Explain, what the relationship is between the numerical and graphical output and why the output is -1.

QUESTION

N number of employees will participate in the pension system. There are currently two types of pension schemes, Plan A and Plan B. Each employee independently chooses Plan A with probability p, and Plan B with probability $1-p$.

Ex 11.0.142 (2)**.** Suppose $N \sim \text{Pois}(\lambda)$. Let X_A be the number of people that choose Plan A and $X_B = N - X_A$ be the number of people that choose Plan B. Find $Var(X_A - X_B)$ and $\rho_{X_B,N}$.

Ex 11.0.143 (2)**.** Suppose $N = 500$. Two new pension schemes are now introduced, called Plan C and Plan D. Each of the 500 employees now independently chooses one of the four pension schemes with equal probabilities $\frac{1}{4}$. Let X_i be the number of employees that choose Plan i, i=A,B,C,D, $\sum_i X_i = N = 500$. Find $\text{Cov}[X_B, X_C]$ and ρ_{X_B,X_C}.

Consider the following codes:

R Code

```
1 library(mvtnorm)
2 set.seed(999)
3 A<-c(1,2)
4 B<-c(2,3)
5 C<-A+B
6 D<-cbind(A,B)
7 X<-rmvnorm(200,mean=C,sigma=D)
8 output<-colMeans(X)
```

Python Code

```
1 import random
2 import numpy as np
3 random.seed(999)
4 A = np.array([1,2])
5 B = np.array([2,3])
6 C = A+B
7 D = np.transpose([A,B])
8 X = np.random.multivariate_normal(C, D, size = 200)
9 output = X.mean(axis=0)
```

Ex 11.0.144 (1)**.** Explain in detail the purpose of **Line 1, 2, 5, 6, 7, 8** of the above codes.

QUESTION

An investor wants to keep track of the daily return of his portfolio. Let X_t be the portfolio daily return on day t, where $X_1, X_2, \ldots$ are i.i.d. r.v.s from a continuous distribution. We say that day t hits a *record low* if the return on day t is lower than on all previous $t-1$ days. Let A_t be the event that day t hits a *record low*, and let I_t be the indicator r.v. that is 1 if day t hits a *record low* and 0 otherwise.

Ex 11.0.145 (0.5)**.** Find $\mathsf{P}\{A_t\}$, the probability that day t hits the *record low*.

Ex 11.0.146 (1)**.** Find $\mathsf{P}\{A_t \cap A_{t+1}\}$, the probability that the *record low* is hit on both day t and day $t+1$. Are A_t and A_{t+1} independent?

Ex 11.0.147 (1.5)**.** Show that A_s and A_t are independent if $s < t$. This means that whether day s hits a *record low* does not influence whether day t hits a *record low*, with $s < t$.

Ex 11.0.148 (2)**.** Let N be the number of *record low* days from day 1 up to t. Find $\mathsf{Cov}[N, I_t]$.

Remarks and grading scheme:

1. Ex 3.2: only full points if the permutation is well explained. 0.5 point for no or bad explanation.

2. Ex 3.4: 0.5 point for writing out the formula for covariance. 0.5 point for correctly calculated $\mathsf{E}[N]$, $\mathsf{E}[I_t]$.

QUESTION

X number of people will get vaccinated for Covid-19. There are currently two types of vaccines, Vaccine A and Vaccine B. Each person independently choose vaccine A with probability p, and vaccine B with probability $1-p$.

Ex 11.0.149 (2)**.** Suppose $X \sim \text{Poi}(\lambda)$. Let X_A be the number of people that choose Vaccine A and $X_B = X - X_A$ be the number of people that choose Vaccine B. Find $Var(X_A - X_B)$ and $\rho_{X_A,X}$.

Ex 11.0.150 (2)**.** Suppose $X = 1000$. A new vaccine is now available, called Vaccine C. Each of the 1000 people now independently chooses one of the three vaccines with equal probabilities $\frac{1}{3}$. Let X_A be the number of people that choose Vaccine i, i=A,B,C and $\sum_i X_i = X = 1000$. Calculate $\mathsf{Cov}[X_A, X_C]$ and ρ_{X_A,X_C}.

Consider the following codes:

R Code

```r
library(mvtnorm)
set.seed(888)
A<-c(1,2,1)
B<-c(2,3,1)
C<-c(1,1,8)
D<-cbind(A,B,C)
X<-rmvnorm(200,mean=A,sigma=D)
output=cor(X[,1]+X[,2],X[,3])
```

Python Code

```
1 import random
2 import numpy as np
3 random.seed(888)
4 A = [1,2,1]
5 B = [2,3,1]
6 C = [1,1,8]
7 D = np.transpose([A,B,C])
8 X = np.random.multivariate_normal(A, D, size = 200)
9 output = np.corrcoef(X[:,0]+X[:,1]+X[:,2])
```

Ex 11.0.151 (1)**.** Explain in detail the purpose of **Line 1,2, 6,7,8** of the above codes.

QUESTION

A random point (X, Y) is chosen in the following square:

$$\{(x, y) : x^2 < 7, y^2 < 7\}$$

All points are equally likely to be chosen. Let S be the squared norm of (X, Y).

Ex 11.0.152 (0.5)**.** Find the joint PDF $f(x, y)$ of X and Y.

Ex 11.0.153 (0.5)**.** Show that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1$

Ex 11.0.154 (3)**.** Find the expectation of S, i.e., the squared norm of (X, Y).

Consider the following code:

Python Code

```
import numpy as np
import math
np.random.seed(3)

num = 100000
distances = np.zeros(num)
for i in range(0,num):
    angle = 2*math.pi*np.random.uniform(0,1,1)
    position = np.sqrt(np.random.uniform(0,1,1))
    x = np.cos(angle)*position
    y = np.sin(angle)*position
    distances[i] = np.sqrt(x**2 + y**2)

print(np.mean(distances))
```

Ex 11.0.155 (0.5)**.** What does the code above do? *Hint:* unit circle.

Ex 11.0.156 (0.5)**.** The output of the code is 0.66629. Explain this result.

QUESTION

A portfolio manager wants to investigate the monthly return of a particular portfolio, starting from an arbitrary month in history. Let X_t be the portfolio monthly return on month t, with $X_1, X_2, \ldots$ i.i.d. We say that month t is *worst in a year* if the return in month t is lower than all previous 11 months. Let A_t be the event that month t is the *worst in a year*, and let I_t be the indicator r.v. that is 1 if month t hits a *worst in a year* an 0 otherwise.

Ex 11.0.157 (0.5)**.** Find $\mathsf{P}\{A_t\}$, the probability that month t is the *worst in a year*, $t \geq 12$.

Ex 11.0.158 (2)**.** Let N be the number of *worst in a year* months from the month 12 to month t. Find $\mathsf{E}[N]$ and $\mathsf{Cov}[N, I_t]$.

Ex 11.0.159 (1.5)**.** Find $\mathsf{P}\{A_t \cap A_{t+1}\}$, the probability that two consecutive months are both *worst in a year*. Are A_t and A_{t+1} independent?

Ex 11.0.160 (1)**.** Let B_t be the event that return in month t is lower than all previous months. Find $\mathsf{P}\{B_t \cap B_{t+1}\}$.

Remarks and grading scheme:

1. First notice that A_t and A_{t+1} are not independent. This is illustrated in Exercise 3.3. Many students wrongly assumed their independence.
2. Ex 3.2: 0.5 point for correctly calculated $\mathsf{E}[N], \mathsf{E}[I_t]$. 0.5 point for writing out the formula for covariance.
3. Ex 3.3: No point if you assume them to be independent before solving the question. Even though you might have also got the same answer in the end by accident.

QUESTION

Suppose you received a collection of books as your birthday gift. You already read 2 of them and there are still 4 books left. Let X_1, X_2 be the number of pages (in hundreds of pages) of the first 2 books you read, and let $X_3, \ldots, X_6$ be the number of pages (in hundreds of pages) of the remaining books. Assume that $X_i \sim \mathrm{Norm}(4, 1)$ for $i = 1, \ldots, 6$.

Ex 11.0.161 (1.5)**.** First assume that the number of pages of the books are all independent. What is the expected number of remaining books that have more pages than each of the 2 books you have already read?

For the next two exercises, suppose that $(X_1, \ldots, X_6)$ is now Multivariate Normal distributed with $\mathsf{Corr}\left[X_1, X_j\right] = \frac{1}{2}$ for $3 \leq j \leq 6$.

Ex 11.0.162 (2.5)**.** On average, how many of the remaining books are at least 100 pages longer than the first book you read?

Ex 11.0.163 (1)**.** Show that there exists a constant c such that $X_1 - cX_3$ and X_3 are independent, and determine the value of c.

Remarks and grading scheme:

1. Ex 3.1: Many students assume that $X_i > X_1$ and $X_i > X_2$ is independent. This is not the case. In fact, If $X_i > X_1$, then it's more likely that X_i is large. In consequence, it is also more likely that $X_i > X_2$.
2. Ex 3.1: 0.5 point for multiply your probability with 4 (even if it is calculated wrongly). Full point(1.5) for correct answer.
3. Ex 3.2: 0.5 point for mentioning that $X_3 - X_1$ is normally distributed ,0.5 point for correctly calculated $\mathsf{E}[X_3 - X_1]$ and 0.5 point for correctly calculated $Var(X_3 - X_1)$.
4. Ex 3.3: 0.5 point for writing out the formula for covariance.

QUESTION

A server spends a random amount T on a job. While the server works on the job, it sometimes gets interrupted to do other tasks $\{R_i\}$. Such tasks need to be repaired before the machine can continue working again. It is clear that such interruptions can only occur when the server is working. Assume that the interruptions arrive according to a Poisson process with rate λ, i.e., the number of failures $N(t) \sim \text{Pois}(\lambda t)$. The interruptions $\{R_i\}$ are independent of N and of T and form an iid sequence with common mean $\mathsf{E}[R]$ and variance $\mathsf{V}[R]$. The duration S of a job is its own service time T plus all interruptions, hence $S = T + \sum_{i=1}^{N(T)} R_i$.

Ex 11.0.164 (1.5)**.** In the computation of $\mathsf{V}[S]$ we encounter the following steps.

$$\mathsf{V}\left[\sum_{i=1}^{N(t)} R_i\right] = \mathsf{E}\left[\mathsf{V}\left[\sum_{i=1}^{N(t)} R_i \,\middle|\, N(t)\right]\right] + \mathsf{V}\left[\mathsf{E}\left[\sum_{i=1}^{N(t)} R_i \,\middle|\, N(t)\right]\right]. \tag{11.0.23}$$

The computation below consists of a number of steps, a, b, ... Explain for each step which property is used to ensure the step is true.

$$\mathsf{V}\left[\sum_{i=1}^{N(t)} R_i \,\middle|\, N(t) = n\right] \stackrel{a}{=} \mathsf{V}\left[\sum_{i=1}^{n} R_i\right] \tag{11.0.24}$$

$$\stackrel{b}{=} n\mathsf{V}[R] \tag{11.0.25}$$

$$\mathsf{E}\left[\mathsf{V}\left[\sum_{i=1}^{N(t)} R_i \,\middle|\, N(t)\right]\right] \stackrel{c}{=} \mathsf{E}[N(t)\mathsf{V}[R]] \tag{11.0.26}$$

$$\stackrel{d}{=} \lambda t\mathsf{V}[R]. \tag{11.0.27}$$

Ex 11.0.165 (2)**.** Suppose R is equal to the constant r and $T \sim \text{Unif}([0, a])$, compute $\mathsf{E}[S]$.

Ex 11.0.166 (1.5)**.** Explain what this code computes.

Python Code

```python
1  import numpy as np
2
3  labda = 0.5
4  size = 10
5  num_runs = 50
6
7
8  def do_run():
9      T = np.random.uniform(0, 20)
10     N = np.random.poisson(labda * T)
11     R = np.random.uniform(1, 5, size=N)
12     S = T + R.sum()
13     return S
14
15
16 samples = np.zeros(num_runs)
17 for i in range(num_runs):
18     samples[i] = do_run()
19
20
21 print((samples > 8).sum())
```

R Code

```r
1  labda <- 0.5
2  size <- 10
3  num_runs <- 50
4
5  do_run <- function() {
6    bigT <- runif(n = 1, min = 0, max = 20)
7    N <- rpois(n = 1, labda * bigT)
8    R <- runif(n = N, min = 1, max = 5)
9    S <- bigT + sum(R)
10   return(S)
11 }
12
13 samples <- rep(0, num_runs)
14 for (i in 1:num_runs) {
15   samples[i] <- do_run()
16 }
17
18 print(sum(samples > 8))
```

Hint, you should know that in P.21 (R18) the string `samples > 8` collects only the samples with value larger than 8.

QUESTION

Consider the following code:

Python Code

```python
import numpy as np
from scipy.stats import expon
np.random.seed(42)

n = 100
N = 1000

X = expon(scale = 1/2).rvs([N, n])
Y = X.mean(axis = 1)

mu = 1/2
sigma = 1/2
Z = np.sqrt(n) * (Y - mu)/sigma

print((Z ** 37).mean())
```

R Code

```r
set.seed(42)

n <- 100
N <- 1000

X <- matrix(rexp(N * n, rate = 2), nrow = N, ncol = n)
Y <- rowMeans(X)

mu <- 1/2
sigma <- 1/2
Z <- sqrt(n) * (Y - mu)/sigma

print(mean(Z^37))
```

Ex 11.0.167 (1.5)**.** Y is a vector of i.i.d. random variables.

(i) What is the length ℓ of Y?

(ii) Each element of Y is a mean of k i.i.d. Exp(λ) r.v.s. What are k and λ?

(iii) What are the (theoretical) expectation and variance of an element of Y?

Recall that each element of Y is the mean of k i.i.d. $\text{Exp}(\lambda)$ r.v.s.

Ex 11.0.168 (1)**.** What is the exact distribution of an element of Y?
Give its name and its parameters, and explain the answer.

Let $(Y_1, \ldots, Y_\ell)$ be the elements of Y and let $(Z_1, \ldots, Z_\ell)$ be the elements of Z. Recall that each Z_i depends on k because Y_i is the mean of k i.i.d. $\text{Exp}(\lambda)$ r.v.s. Let T be the random variable to which Z_1 converges in the limit $k \to \infty$.

Ex 11.0.169 (1)**.** What is the distribution of T, and why?
Hence, what is an approximate distribution of an element of Y (e.g. Y_1) and why?

Ex 11.0.170 (0.5)**.** In the last line of the code, we compute $S = \frac{1}{\ell}\sum_{i=1}^{\ell} Z_i^{37}$.
If $k \to \infty$ (for fixed ℓ, e.g. $\ell = 3$), does S converge to a constant?
You do not have to give formal proofs in this subquestion, but you should give clear explanations using the appropriate theorem(s).

Ex 11.0.171 (1)**.** In the last line of the code, we compute $S = \frac{1}{\ell}\sum_{i=1}^{\ell} Z_i^{37}$.
If $\ell \to \infty$ (for fixed k), does S converge to a constant? If so, does it converge to $\mathsf{E}\left[T^{37}\right]$?
You do not have to give formal proofs in this subquestion, but you should give clear explanations using the appropriate theorem(s).

QUESTION

Let $Z \sim \text{Norm}(0,1)$. In this exercise, we find an upper bound for $\mathsf{P}\{|Z| > 2\}$.

Ex 11.0.172 (1.5)**.** Let f be a positive and increasing function, and let X be a r.v.
Consider the following inequality:

$$\mathsf{P}\{X \geq a\} = \mathsf{P}\left\{f(X) \geq f(a)\right\} \leq \frac{\mathsf{E}\left[f(X)\right]}{f(a)}.$$

(i) Explain why $\mathsf{P}\{X \geq a\} = \mathsf{P}\left\{f(X) \geq f(a)\right\}$ holds.

(ii) Explain why $\mathsf{P}\left\{f(X) \geq f(a)\right\} \leq \frac{\mathsf{E}[f(X)]}{f(a)}$ holds.

Make sure to clearly indicate where you use that f is positive and increasing.

Ex 11.0.173 (1)**.** Prove that $\mathsf{P}\{|Z| > 2\} \le e^{-4t}\,\mathsf{E}\left[e^{tZ^2}\right]$ for $t > 0$.

Ex 11.0.174 (2.5)**.** For which t do we find the best upper bound for $\mathsf{P}\{|Z| > 2\}$? Also calculate the upper bound for this value of t.

Hint 1. You may use that if $Y \sim \chi_1^2$, then the MGF of Y is given by $M_Y(t) = (1-2t)^{-1/2}$ for $t < 1/2$. However, you should explain clearly how you use this fact.

Hint 2. Do not forget to check the second order condition of minimization.

QUESTION

Amy and Bob are playing a dice game. Every (fair) die has three pairs of identical sides: a pair of ones, a pair of twos and a pair of threes. Each player throws a single die. Let X and Y denote the outcome of Amy and Bob's throw, respectively. The person that throws the highest number wins. If both throw the same number, they have a draw. The final score A of Amy is determined as follows:

- If Amy loses, then she gets zero points.
- If Amy and Bob draw, then she gets 0.5 point.
- If Amy wins, then her score is the difference $X - Y$ in the numbers they threw.

The final score B for Bob is determined analogously. Assume that the dice are fair and that all throws are independent.

Ex 11.0.175 (1)**.** Determine the joint distribution of X and Y conditional on Amy winning.

Ex 11.0.176 (1.5)**.** Find Amy's expected score conditional on Amy winning.

Ex 11.0.177 (1.5)**.** Find Amy's (unconditional) expected score.

To make the game more interesting, Amy and Bob decide to play for money. After playing the dice game and scoring A points, Amy receives an amount of T euros, where T is determined randomly. Here, conditional on the outcome of A, T follows a uniform distribution on $[A, 2A]$.

Ex 11.0.178 (1)**.** What is the (unconditional) expected reward for Amy? That is, compute $\mathsf{E}[T]$.

QUESTION

Let $Z \sim \mathrm{Norm}(0,1)$. In this exercise, we find an upper bound for $\mathsf{P}\{|Z| > 3\}$.

Ex 11.0.179 (1.5)**.** If $X \sim \mathrm{Gamma}(a, \lambda)$ then the rth moment of X is given by $\frac{\Gamma(a+r)}{\lambda^r \Gamma(a)}$. Use this to prove that $\mathsf{E}\left[Z^{2n+2}\right] = (2n+1)\,\mathsf{E}\left[Z^{2n}\right]$ for all positive integers n.

Hint. Use the chi-square distribution.

Ex 11.0.180 (0.5)**.** Use the previous exercise to calculate $\mathsf{E}\left[Z^4\right]$.

Remarks and grading scheme:

- If the exercise explicitly asks to use the previous exercise, don't do it in a different way.
- You should really know that $\mathsf{E}\left[Z^2\right] = 1$ for $Z \sim \text{Norm}(0,1)$.
- Grading: 0.5 for a correct solution.

Ex 11.0.181 (1)**.** We now provide a bound for $\mathsf{P}\{|Z| > 3\}$.

(i) Prove that $\mathsf{P}\{|Z| > 3\} = \mathsf{P}\left\{Z^4 > 81\right\}$.

(ii) Use this to prove that $\mathsf{P}\{|Z| > 3\} \leq \frac{1}{27}$.

Ex 11.0.182 (2)**.** Prove that $\mathsf{P}\{|Z| > 3\} \leq \frac{\mathsf{E}[Z^{2n}]}{9^n}$ for all $n \in \mathbb{N}$. For what value(s) of n do we obtain the strongest bound for $\mathsf{P}\{|Z| > 3\}$? Also provide this upper bound.

QUESTION

A server spends a random amount T on a job. While the server works on the job, it sometimes gets interrupted to do other tasks $\{R_i\}$. Such tasks need to be repaired before the machine can continue working again. It is clear that such interruptions can only occur when the server is working. Assume that the interruptions arrive according to a Poisson process with rate λ, i.e., the number of failures $N(t) \sim \text{Pois}(\lambda t)$. The interruptions $\{R_i\}$ are independent of N and of T and form an iid sequence with common mean $\mathsf{E}[R]$ and variance $\mathsf{V}[R]$. The duration S of a job is its own service time T plus all interruptions, hence $S = T + \sum_{i=1}^{N(T)} R_i$.

Ex 11.0.183 (1.5)**.** The computation below consists of a number of steps, a, b, ... Explain for each step which property is used to ensure the step is true.

$$\mathsf{E}\left[\sum_{i=1}^{N(t)} R_i \,\middle|\, N(t) = n\right] \stackrel{a}{=} \mathsf{E}\left[\sum_{i=1}^{n} R_i\right] \tag{11.0.28}$$

$$\stackrel{b}{=} n\,\mathsf{E}[R] \tag{11.0.29}$$

$$\mathsf{E}\left[\sum_{i=1}^{N(t)} R_i \,\middle|\, N(t)\right] \stackrel{c}{=} N(t)\,\mathsf{E}[R] \tag{11.0.30}$$

Ex 11.0.184 (2)**.** Suppose R is equal to the constant r and $T \sim \text{Exp}(\mu)$, compute $\mathsf{E}[S]$.

Ex 11.0.185 (1.5)**.** Explain what this code computes.

Python Code

```python
import numpy as np

```

```
3  labda = 0.5
4  size = 10
5  num_runs = 50
6
7
8  def do_run():
9      T = np.random.uniform(0, 20)
10     N = np.random.poisson(labda * T)
11     R = np.random.uniform(1, 5, size=N)
12     S = T + R.sum()
13     return S
14
15
16 samples = np.zeros(num_runs)
17 for i in range(num_runs):
18     samples[i] = do_run()
19
20
21 print(samples[samples > 4].var())
```

R Code

```
1  labda <- 0.5
2  size <- 10
3  num_runs <- 50
4
5  do_run <- function() {
6    bigT <- runif(n = 1, min = 0, max = 20)
7    N <- rpois(n = 1, labda * bigT)
8    R <- runif(n = N, min = 1, max = 5)
9    S <- bigT + sum(R)
10   return(S)
11 }
12
13 samples <- rep(0, num_runs)
14 for (i in 1:num_runs) {
15   samples[i] <- do_run()
16 }
17
18 print(var(samples[samples > 4]))
```

Hint, you should know that in P.21 (R18) the string `samples > 4` collects only the samples with value larger than 4.

QUESTION

We have a population of $X(t)$ individuals at time t. At time t, the time to the next birth is $Z \sim \text{Exp}(\lambda X(t)+\theta)$, and the time to the next death is $Y \sim \text{Exp}(\mu X(t))$; $\lambda, \mu, \theta \geq 0$, and rvs Y and Z are independent. Write $B(h)$ for the number of births during an interval of length h, and $D(h)$ for the number of deaths. (Hint, recall the relation between the Poisson distribution and the exponential distribution.)

In the sequel, take h positive, but very, very small, i.e, $h \ll 1$. With this, we use the shorthand $o(h)$ to capture all terms of a polynomial in h with a power higher than 1, for instance,

$$2h + 3h^2 + 44h^{21} = 2h + o(h). \tag{11.0.31}$$

Like this we can hide all nonlinear terms of a polynomial in the $o(h)$ function. This is easy when we want to take limits, for example,

$$\lim_{h\to 0} \frac{2h + 3h^2 + 44h^{21}}{h} = 2 + \lim_{h\to 0} \frac{o(h)}{h} = 2 + 0. \tag{11.0.32}$$

In other words, when computing this limit for $h \to 0$, we don't care about the details in $o(h)$ because $o(h)/h \to 0$ anyway.

Ex 11.0.186 (1)**.** Provide intuitive motivation about the correctness of the following equality:

$$\mathsf{P}\{B(h) = 1, D(h) = 0 | X(0) = n\} = (\lambda n + \theta) h e^{-(\lambda n+\theta)h} e^{-\mu n h} + o(h). \tag{11.0.33}$$

The $o(h)$ here is a subtlety to get the mathematics correct, but you don't have to explain why this term is necessary.

Ex 11.0.187 (1)**.** Use the first degree Taylor's expansion, $f(h) \approx f(0) + hf'(0) + o(h)$, to show that

$$\mathsf{P}\{B(h) = 0, D(h) = 1 | X(0) = n\} = n\mu h + o(h). \tag{11.0.34}$$

Ex 11.0.188 (2)**.** Explain that

$$\mathsf{E}[X(t+h) | X(t) = n] = n + (\lambda n + \theta - \mu n) h + o(h). \tag{11.0.35}$$

Write $M(t) = \mathsf{E}[X(t)]$.

Ex 11.0.189 (1)**.** Derive that

$$M(t+h) = M(t) + (\lambda - \mu) M(t) h + \theta h + o(h). \tag{11.0.36}$$

QUESTION

Let X and Y be i.i.d and Unif(1,3) distributed.

Ex 11.0.190 (0.5)**.** Find the joint PDF $f(x, y)$ of X and Y.

Ex 11.0.191 (0.5)**.** Show that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

Ex 11.0.192 (3)**.** Find $SD(|X-Y|)$, the standard deviation of the distance between X and Y.

Consider the following code:

Python Code

```
import numpy as np
np.random.seed(3)

num = 100000

x = np.random.normal(loc = 50, scale = 200, size = num)

result1 = np.zeros(num)
for i in range(0,num):
    result1[i] = abs(x[i]-50)<2*200

print(np.sum(result1)/num)
```

Ex 11.0.193 (0.5)**.** What does the code above do?

Ex 11.0.194 (0.5)**.** The code gives as output 0.95429. Explain why you would expect to get this output from the code. *Hint:* use Theorem 5.4.5 in the book.

Let $\{X_k\}$ be a set of iid demands at a shop, distributed as the common rv X, with mean $\mu = \mathsf{E}[X]$ and std $\sigma = \sqrt{\mathsf{V}[X]}$. A random number N of people visit the shop on some day. Let $D_N = \sum_{i=1}^{N} X_k$.

Ex 11.0.195 (1)**.** When the rvs X and Y are independent, then $\mathsf{E}[Y|X] = \mathsf{E}[Y]$. It is clear that $\mathsf{E}[Y]$ is a number. Now BH say that $\mathsf{E}[Y|X]$ is a rv. But a rv is not a number. Explain how it can be that $\mathsf{E}[Y|X] = \mathsf{E}[Y]$.

Ex 11.0.196. Use the definition of $\mathsf{E}[Y \mid A]$ to show that $\mathsf{E}[X \mid X = x] = x$.

We break a stick of length 1 at a uniformly distributed point X. Then we choose the smallest of the two parts. Let the length of this be Y.

Ex 11.0.197 (1)**.** Find an expression for $\mathsf{E}[Y|X = x]$.

Ex 11.0.198 (0.5)**.** Find an expression for $\mathsf{E}[Y|X]$.

Ex 11.0.199 (1.5)**.** Compute $\mathsf{V}[Y]$?

Ex 11.0.200 (1)**.** What does the code below compute?

Python Code

```python
import random

random.seed(3)

num = 5
tot = 0
for i in range(num):
    X = random.uniform(0, 1)
    Y = random.uniform(0, 1)
    tot += abs(Y - X)

print(tot / num)
```

R Code

```r
x = 3
```

QUESTION

Ex 11.0.201 (0.5). Let $\lambda > 0$ be some parameter. Let $X_1, X_2, \ldots, X_n \sim \text{Expo}(\lambda)$ be independent. Find the distribution of $\min\{X_1, X_2, \ldots, X_n\}$. You can use results from the book here.

Ex 11.0.202 (2). From here on, consider the random variables $X, Y \sim \text{Expo}(\lambda)$, again for $\lambda > 0$. Assume X, Y are independent. We will in steps show the distribution of $|X - Y|$.
To start, consider the PDF of a random variable W, which is as follows:

$$f_W(w) = \frac{\lambda}{2} e^{-\lambda|w|},$$

for $w \in \mathbf{R}$. Find the moment-generating function of W. As a hint, be careful of what assumptions are necessary to make sure the required integral(s) converge.

Ex 11.0.203 (1). Show that $X - Y \sim W$. You may use any known results from *previous* courses.

Ex 11.0.204 (1.5). Finally, calculate the MGF of the random variable $|X - Y|$. Do you recognize it?

QUESTION

Amy is playing a game. She throws a basketball at a hoop and counts the number of times she successfully throws the ball through the hoop. She keeps counting until she has missed r times, at which moment the current round of the game stops. Her score for the round is the total number of successful throws in the round. Amy plays n rounds in total. We assume that all throws are independent and have the same (unknown) success probability p. Amy is interested in finding out her skill level. That is, she is interested in the value of p.

Given the value of p, Amy's score X_i for the ith round of the game follows a negative binomial distribution with parameters r and p. That is, for every $i = 1,\dots,n$, we have that $X_i|p \sim \mathrm{NB}(r,p)$, with a corresponding pmf defined by

$$\mathsf{P}\{X_i = x_i|p\} = \binom{x_i + r - 1}{x_i}(1-p)^r p^{x_i}, \tag{11.0.37}$$

for $x_i = 0,1,2,\dots$. Amy's prior belief about the distribution of p is that it follows a Beta(a,b) distribution with given values for a and b (the exact values of a and b are not relevant for this question).

Ex 11.0.205 (2.5)**.** In the first round, Amy gets a score of $X_1 = x_1$. Find Amy's *posterior* distribution of p, given this observation.

Ex 11.0.206 (1)**.** Is Amy's prior distribution a *conjugate* prior?

Ex 11.0.207 (1.5)**.** Suppose Amy plays n rounds and observes the scores $X_1 = x_1,\dots,X_n = x_n$. What is Amy's posterior distribution after these observations?
Hint: you don't need to do a lot of math here! Instead, use the result of the first question above and a logical argument.

QUESTION

Ex 11.0.208 (1)**.** Let X follow the student's t distribution with ν degrees of freedom. Consider the random variable $Y = \frac{1}{X}$. Find the CDF of Y, $F_Y(y)$, in terms of $\mathsf{P}\left\{X \le \frac{1}{y}\right\}$.

Ex 11.0.209 (1)**.** Show that $f_Y(y) = \frac{1}{y^2} f_X(\frac{1}{y})$ for all $y \neq 0$.

Ex 11.0.210 (1)**.** Let Y be as in the previous question. What distribution does Y follow when $\nu = 1$?

Ex 11.0.211 (0.5)**.** What happens to the t distribution when $\nu \to \infty$?

Ex 11.0.212 (1.5)**.** Let $\nu > 1$. For what value(s) of y is $f_Y(y)$ maximal? You may neglect the possibility that $y = 0$.

QUESTION

Denise is the proud owner of a small supermarket. In order to gain some insight into the behavior of her customers, she analyzes their arrival times. In particular, she is interested in the customers' *interarrival times*. Denise knows that the interarrival times Y_i, $i = 1,\dots,n$, are i.i.d. Exponentially distributed with a rate parameter λ (i.e., with a mean value of $1/\lambda$). However, Denise does not know the value of λ. Her prior belief about λ is captured by a Gamma(a,b) distribution, with some particular values of $a, b > 0$.

Ex 11.0.213 (2.5)**.** Denise starts observing the customers' interarrival times. For the first customer she observes $Y_1 = y_1$. What is Denise's *posterior* distribution of λ after this observation?

Ex 11.0.214 (1.5)**.** After an hour Denise has observed n interarrival times $Y_1 = y_1, \ldots, Y_N = y_n$. Without redoing all the math, determine Denise's posterior distribution.

Ex 11.0.215 (1)**.** Does Denise have a *conjugate* prior?

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HINTS

h.7.1.16. Recall that $F \in [0,1]$.

h.7.1.28. Realize that $\mathsf{E}[ML] = \mathsf{E}[XY]$.

h.7.2.6. For 2D LOTUS, take $g(i,j) = \min\{i,j\}$.

h.7.2.7. $X \sim \text{Geo}(1-q) \implies \mathsf{E}[X] = q/(1-q)$. Now use that $L \sim \text{Geo}(1-q^2)$.

h.7.2.8. Use that $1-q^2 = (1-q)(1+q)$.

h.7.2.10. Use 2D LOTUS on $g(x,y) = I_{\max\{x,y\}=k}$.

h.7.2.12.

h.7.2.16. Marginalize out L.

h.7.2.17. Marginalize out M.

h.7.2.19. Use partial integration.

h.7.2.20. Using independence and the specific property of the r.v. L that $\{L > x\} \iff \{X > x, Y > x\}$:

h.7.2.28. If you recall the Poisson distribution, you know that $e^\lambda = \sum_{i=0}^\infty \lambda^i/i!$. In fact, this is precisely Taylor's expansion of e^λ.

h.7.2.29. Use [7.2.7].

h.7.2.31. The idea is not difficult, but the technical details require attention, in particular the limits in the integrations.

h.7.2.33. Realize that $\mathsf{P}\{L = i, M - L = j\} = \mathsf{P}\{L = i, M = i + j\}$. Now fill in the formala of [7.2.15].

h.7.2.36. Reread BH.7.2.2. to realize that $\mathsf{E}[M-L] = \mathsf{E}[|X-Y|]$. Relate the latter expectation to the expression in the problem.

h.7.2.37.

h.7.2.39. Observe that Z is independent from X and Y, hence from M and L

h.7.4.1. Check BH 7.2.2. Bigger hint: Let A the arrival time of Alice, and B the time of Bob. Then we want to compute $\mathsf{P}\{|A-B| \leq 1/4\}$. (15 minutes is 1/4 hour.) Why is $f_{A,B}(a,b) = I_{a\in[0,1]}\, I_{b\in[0,1]}$? Now apply 2D-LOTUS to the function $g(a,b) = I_{|a-b|\leq 1/4}$.

h.7.4.2. a. $\mathsf{P}\left\{X = i, Y = j, N = n\right\} = \mathsf{P}\left\{X = i, Y = j\right\} I_{i+j=n}$.
c. $\mathsf{P}\{X = i \,|\, N = n\} = 1/(n+1)$. Why is this uniform?

h.7.4.4. a. First find $f_{Y|X}$ and $f Z|X$. Then, given X, Z and Y are iid. Hence $f_{X,Y,Z} = f_{Y,Z|X} f_X$. Use independence to split $f_{Y,Z|X}$ into a product. b. Suppose that Y is really big. Since Y is dependent on X, X must be dependent on Y. But Z is in turn dependent on X. What are the consequences?

h.7.4.5. Section 7.2

h.7.4.6. Make a drawing.

h.7.4.7. Check BH.7.1.24 and BH.7.1.25 First draw the area over which we have to integrate. Then use an indicator function over which to integrate. What is the joint PDF $f_{Y_1,Y-2}$?

h.7.4.8. Section 7.2

h.7.4.10. Use the hint of the book and independence to see that $\mathsf{E}\left[S_n^2 T_n^2\right] = \mathsf{E}\left[S_n^2\right] \mathsf{E}\left[T_n^2\right]$. Then try to simplify.

b. It is immediate that $\mathsf{E}[S_n] = 0$. Hence, focus on $\mathsf{E}[S_n T_n]$. Expand the sums of $\mathsf{E}[S_n T_n]$, and consider the individual terms $\mathsf{E}\left[X_i Y_j\right]$. When $i \neq j$, are X_i and Y_j independent? What if $i = j$?

c. It is clear that $R_n^2 = S_n^2 + T_n^2$. Now use linearity to split $\mathsf{E}\left[R_n^2\right]$. Finally, realize that $\mathsf{E}[S_n] = 0$, hence $\mathsf{E}\left[S_n^2\right] = \mathsf{V}[S_n]$. But then we can use the formula of the variance of a sum to split it up into a sum of variances plus covariances.

h.7.4.11. a. Expand the brackets in the expression for the sample variance r to see that

$$r = 1/n \sum_i x_i y_i - \bar{x}\bar{y}.$$

Next, we choose with probability $1/n$ one the points (x_i, y_i). Under this probability, $\mathsf{E}[XY] = 1/n \sum_i x_i y_i$, $\mathsf{E}[X] = \bar{x}$, $\mathsf{E}[Y] = \bar{y}$. So, how do $\mathsf{Cov}[X,Y]$ and r relate?

b. Expand the brackets and use iid and linearity properties to show that the expected area spanned by two random points (X,Y) and $(\tilde{X},\tilde{Y})$ satisfies

$$\mathsf{E}\left[(X-\tilde{X})(Y-\tilde{Y})\right] = 2\,\mathsf{Cov}[X,Y].$$

h.7.4.12. a. Use that expectation is linear.

b. Read the entire exercise in its entirety before trying to solve it. In this case trying to solve c. seems simpler because of the extra iid assumption. You might want to use this to formulate some simple guesses.

Thus first part c. It is given that the X_i and Y_j are iid. Then, if I could improve the estimator $\hat{\theta}$ by splitting the measurements into two sets X_i and Y_j, then I would certainly do that. And not only I would do that; anybody in his right mind would do that. But, I never heard of this idea, and I am sure you have neither, so this must be impossible (because if it would, people would have been using this trick for ages.) Hence, we can place this in the context of the maxim: 'we cannot obtain information for free'. For this case, this must imply that splitting iid measurements into smaller sets cannot help with improving the estimator. What does this idea imply for the weights?

Part b, continued. I always try to solve the problem myself without a hint. This lead to the following considerations, which gave me quite a bit of extra understanding beyond the problem itself. As a next piece of advice, before doing hard work, I prefer to look at some corner cases to acquire some intuitive understanding. I also use the rvs of Part c.

Suppose that $v_2 := \mathsf{V}\left[Y_j\right] = 0$, but $v_1 := \mathsf{V}\left[X_i\right] > 0$. (For instance, Y_j is the jth measurement of a perfect machine and X_j of an imperfect machine.) Then we know that the set $\{Y_j\}$ forms a set of perfect measurements. But then I am not interested in the $\{X_i\}$ measurements anymore; why should I as I have the perfect measurements $\{Y_j\}$ at my disposal. So, then I put $w_1 = 0$, because I don't want the $\{X_i\}$ measurements to pollute my estimator. In other words, the final result should be such that $v_2 = 0 \implies w_1 = 0$, and vice versa.

More generally, I learned from this corner case that I want this for the final result: when $v_2 < v_1 \implies w_1 < w_2$, and vice versa.

How would you choose the weights such that this requirement is satisfied, but also the condition imposed by Part c.?

h.7.4.13. b. The people in the sample of size n with an A is $X_1 + X_2$. But this is the same as $n - X_3$. Hence, what is $\mathsf{P}\{X_3 = n - i\}$?

c. I found this a hard problem. Here is my hint based on recursion. Let S_n be the number of As in n individuals. We want to know $f_n(i) = \mathsf{P}\{S_n = i\}$. A simple recursive idea, i.e., one-step analysis by conditioning on the phenotype of the nth person, gives that

$$f_n(i) = f_{n-1}(i-2)p^2 + f_{n-1}(i-1)2pq + f_{n-1}(i)q^2,$$

with $q = 1 - p$ as always. Now I was a bit stuck, but just to try to see whether I could see some structure, I tried a simpler case, namely, a recursion for the binomial distribution. Derive this, and then use this to solve the problem.

d. It is easiest to work with $f(p) = \log \mathsf{P}\{X_1 = k, X_2 = l, X_3 = m\}$, where $\mathsf{P}\{X_1 = k, X_2 = l, X_3 = m\}$ follows from a., and then differentiate with respect to p.

e. Follow the same scheme as for d.

h.7.4.14. The challenge for you is to try to understand the mathematics behind these concepts. Read the exercise a number of times. I found it quite difficult to capture the concepts in formulas. (I solved it once. After two weeks, I tried to solve it again, and found it just as hard as the first time.) Once you have the model, the technical part itself is simple.

h.7.5.1. In this exercise we want to prove that N is Poisson distributed. So you cannot assume this in your solution.

h.7.5.3. Use the relation of the previous exercise to show that

$$P(N=n+1) = \frac{\lambda}{1+n} P(N=n). \tag{12.0.1}$$

Bigger hint: Fill in $y=0$ in the LHS and RHS of (7.5.1); call this expression 1. Then fill in $y=1$ to a obtain a second expression. Divide these two expressions and note that $\mathsf{P}\{X=x\}$ cancels. Finally, define

$$\lambda = \frac{\mathsf{P}\{Y=1\}}{(1-p)\,\mathsf{P}\{Y=0\}}. \tag{12.0.2}$$

h.7.8.2. First, compute the value of $\mathsf{E}[v(d,x)]$ as a function of x. Then find the optimal value of x.

h.7.8.5. You don't have to compute x^* for the case where $\rho=1$; this is not easy!

h.8.1.8.

h.8.1.9.

h.8.1.10.

h.8.1.19. Let X, Y be iid standard normal. Since the square of a standard normal r.v. is chi-square distributed, we can write S as $S = X^2 + Y^2$ (here we use BH.8.1.4).

h.8.1.21.

h.8.1.22.

h.8.1.23.

h.8.1.24.

h.8.1.25.

h.8.1.26.

h.8.1.27.

h.8.1.28.

h.8.1.36.

h.8.1.37.

h.8.2.1. Start with the case $v = 0$. Use the proof of BH.8.1.1. Reason carefully; corner cases as simple to miss.

Then, make a graph of the two branches of the hyperbola's $1/t$, one branch for $t > 0$, the other for $t < 0$. Then draw a horizontal line to indicate the level $V = v$; this shows with part(s) of the hyperbola's lie below v. Then compute the probability for each branch. This will give the answer of the book immediately.

h.8.2.2. a. See BH.8.1.9.

b. If (X, Y) are uniform on the disk, then the function $g(x, y)$ must be constant on this disk. Use an indicator to ensure that $X^2 + Y^2 \leq 1$. Finally, normalize.

c. What are the densities of X and Y when they are $N(0,1)$?

h.8.2.3. We can make a transform T, U such that $T = X/Y$ and $U = X$ to use a 2D transformation. Compute x and y as functions of t and u. Then the Jacobian.

h.8.2.4. You might want to follow the approach of BH.8.18.

h.8.2.5. Use the bank-post office Story 8.5.1 to see that T and W are independent.

h.8.2.6. a. See BH.8.5.1. The exponential is a special case of the gamma distribution. See also BH.8.34.c. T_1/T_2 is a function of $T_1/(T_1 + T_2)$.

b. This can be solved with a joint distribution function and integration over the event $\{T_1 < T_2\}$. However, we can use Exercise BH.7.10 or BH.7.1.24.

c. First she has to wait for the first server to become free. This is the minimum of the two exponentials. With $\mathsf{P}\{T_1 < T_2\}$ server 1 is the first. What is the probability that the other server is empty first? Then, once she is at a server, what is her expected service time? The total time in the system is the time in queue plus the service time.

h.8.2.7. Apply beta-binomial conjugacy.

h.8.2.8. a.

$$\mathsf{P}\left\{X_j \leq c\right\} = \mathsf{P}\left\{\log U_j \geq -c\right\} = \mathsf{P}\left\{U_j \geq e^{-c}\right\} = \mathsf{P}\left\{1 - U_j \leq 1 - e^{-c}\right\}.$$

What is the distribution of $1 - U_j$?

b. $\log \Pi_{j=1}^n U_j = \sum_{j=1}^n \log U_j = \sum_{j=1}^n (-X_j)$. But $-X_j \sim \text{Exp}(1)$, hence the sum is just a sum of iid Exp rvs. What is the distribution of this sum?

h.8.2.9. Use BH.4.3.9. Then, start with a geometic rv, then extend to a negative binomial rv.

h.9.1.4.

h.9.1.10.

h.9.1.21. For a smart argument, use the chicken-egg story. Recall that the number of hatched eggs and the number of unhatched eggs are independent (since $N \sim \text{Pois}(\lambda)$); i.e. $N - X$ and X are independent.

h.9.1.22.

h.9.2.1. Then, $\mathsf{E}[T \mid R] = \sum_j \mathsf{E}\left[R_j\right] I_{R=j}$, where R_j is the time of route j. We know that $\mathsf{E}\left[R_j\right] = \mu_j$. Now apply the LOTP.

For b. realize that $\mathsf{V}[T] = \mathsf{E}\left[T^2\right] - (\mathsf{E}[T])^2$.

h.9.2.2. Use Adam's law to express $\mathsf{E}[X_{n+1}]$ in terms of $\mathsf{E}[X_n]$, then use recursion.

h.9.2.4. a. Let Y be the amount purchased by the first customer that comes along, let P be the rv that is 1 if the customer does indeed purchase, and 0 otherwise, and let X be the size of the purchase. Why is $Y = XP$? What is $\mathsf{E}[P]$? What is $\mathsf{E}[Y \mid P]$? What is $\mathsf{E}\left[Y^2 \mid P\right]$. You might want to use BH.9.1.

b. Let $N \sim \text{Pois}(8\lambda)$ be the number of customers that pass by. Given $N = n$, what is $\mathsf{E}[S \mid N]$, where $S = \sum_{i=1}^N X_i P_i$ is the total sales. Now use the law of total expectation. What is $\mathsf{V}[S \mid N]$? Use Eve's law to compute $\mathsf{V}[S]$. Bigger hint, read Example 9.6.1.

h.9.2.7. a. $N|\lambda \sim \text{Pois}(\lambda)$.

b. Analogous to BH.9.6.1

c. and d. See BH.8.4.5.

h.9.2.10. Refresh your knowledge of the Beta distributions.

a. Since we include the win, the number of games $T|p$ (since we assume p given) must be $\sim \text{FS}(p)$. Hence, $\mathsf{E}\left[T|p\right] = 1/p$

To get $\mathsf{E}[T]$ use Adam's law. Realize that you have to take the integral with respect to p!

b. $1 + \mathsf{E}[G]$ is smaller than the expected time as computed in a. Why is this so?

c. The number of wins, conditional on p, out of n is $X|p \sim \text{Bin}(n, p)$. Then use Beta-Binomial conjugacy.

BTW, I find it easier to think about $f(p, X = k)$ instead of $f(p|X = k)$, since on the event $(p, X = k)$.

$$f(p, X = k) \propto p^{a-1} q^{b-1} \binom{n}{k} p^k q^{n-k} \propto p^{a-1+k} q^{b-1+n-k}.$$

Then, as $f(p|X = k) = f(p, X = k)/\mathsf{P}\{X = k\} \sim f(p, X = k)$ (because $\mathsf{P}\{X = k\}$ is just a constant) we get the same result op to a scaling factor. But we can use the reasoning of BH.8.3.3 to get the correct constant.

h.9.2.11. a. The prior of p is uniform on $[0, 1]$. But this is equal to Beta$(1, 1)$. Now use Beta-Binomial conjugacy.

b. Write $S_n = \sum_{i=1}^n X_i$. What are $\mathsf{P}\left\{X_{n+1} = 1|p\right\}$ and $\mathsf{P}\left\{S_n = k|p\right\}$?

h.9.2.12. a. Recall that the uniform distribution on $[0, 1]$ is Beta(a, b) with $a = b = 1$. I prefer to write $S_n = \sum_{j=1}^n X_j$. First compute $\mathsf{E}\left[S_n \mid p\right]$. Then compute $\mathsf{E}\left[\mathsf{E}\left[S_n|p\right]\right]$. Note that the outer expectation is an integral with respect to p and the density of Beta$(1, 1)$.

For the variance, use Eve's law.

b. Use Beta-Binomial conjugacy. Or use the insights of BH.9.56 and BH.9.57.

c. Bayes Billiards.

h.10.1.3.

h.10.1.4.

h.10.1.5.

h.10.1.6.

h.10.1.10. Use that $X \geq X\, I_{X \geq a} \geq a\, I_{X \geq a}$.

h.10.1.13. Consider $\mathsf{P}\{X \geq 2\}$ and compare Chernoff's inequality to Markov's inequality.

h.10.1.14.

h.10.1.15.

h.10.1.16.

h.10.1.18.

h.10.1.19.

h.10.1.20.

h.10.1.21.

h.10.1.22.

h.10.2.2. First check the assumption that $Y \neq aX$, for some $a > 0$; why is it there? Then, take a suitable g in Jensen's inequality. Bigger hint: $g(x) = 1/x$.

In the solution guide, the authors do not explain the $>$, while in Jensen's inequality there is a $\leq$. To see why the $>$ is allowed here, rethink the assumption in the exercise, and reread Theorem 10.1.5.

Finally, at what p is $p(1-p)$ maximal?

h.10.2.3. Apple the idea of BH.10.1.3 to $W = (X - \mu)^2$.

h.10.2.4. a. Jensen's inequality, $g(x) = e^x$

b. Use symmetry: X and Y are iid.

c. Which set of events is larger?

d. Use Jensen's inequality and Cauchy-Schwarz.

e. Eve's law.

f. Use Markov's inequality and the triangle inequality

h.10.2.7. The idea is to prove that the MGF of X_n converges to the MGF of a $N(\mu, \sigma^2)$ rv as $n \to \infty$. Thus, read and follow the proof of the CTL, BH.10.3.1.

What are $\mathsf{E}[X_n]$ and $\mathsf{V}[X_n]$ if $X \sim \text{Pois}(n)$? Once you know that, explain that the MGF of the standardized version of X_n is equal to $\exp\{-n + s\sqrt{n} + ne^{-s/\sqrt{n}}\}$.

Perhaps you should do BH.10.27 first.

h.10.2.8. a. See BH.10.3.7. Try to convert the recursion for Y_n to a form as in that example.

b. Just substitute α in the relevant formula of part a.

h.10.3.2. Use a substitution. What is the derivative of the survivor function?

h.10.3.3. If you need to prove something for all natural numbers n, it is always good to try using mathematical induction, especially since we already know something relating X_{n+1}^* and X_n^*. Note that you can again use the same substitution as in the previous exercise. After that, you need another substitution.

h.10.3.4. Do you recognize the PDF of X_n^* for $X \sim \text{Exp}(\lambda)$?

h.11.0.1. By the previous exercise, realize that $L \sim \text{Exp}(2\lambda)$. Use these properties.

h.11.0.10. What is the domain of V on each of the intervals $(-3, 0)$ and $[0, 2)$? For the final part, combining the results into one PDF: Use LOTP, conditioning on $U \geq 0$.

h.11.0.19.

h.11.0.22. Use [11.0.21]

h.11.0.23. Use [11.0.22] and the definitions.

h.11.0.26. Use Chebyshev's inequality; then take the limit on both sides.

13

SOLUTIONS

s.7.1.1. Check the definitions of the book.

Mistake: To say that $\mathsf{P}\{X = x\}$ is the PMF for a continuous random variable is wrong, because $\mathsf{P}\{X = x\} = 0$ when X is continuous.

Why is $\mathsf{P}\{1 < x \leq 4\}$ wrong notation? hint: X should be a capital. What is the difference between X and x?

s.7.1.2. This example shows why joint distributions are important! In any experiment that involves a sequence of measurements, such as multiple throws of a coin, or the weighing of a bunch of chimpanzees, we have to deal with joint CDFs and PMFs.

s.7.1.3. Here, we deal with two rvs, and we have to specify how they depend. In the present case $\mathsf{P}\{X_1 = H, X_2 = H\} = \mathsf{P}\{X_1 = H\}$ and $\mathsf{P}\{X_1 = T, X_2 = T\} = \mathsf{P}\{X_1 = T\}$, $\mathsf{P}\{X_1 = H, X_2 = T\} = \mathsf{P}\{X_1 = T, X_2 = H\} = 0$. Note that with this, we specified the joint PMF on all possible outcomes.

s.7.1.4.

$$f_X(x) = \int_0^1 f_{X,Y}(x,y)\,\mathrm{d}y = 2\int_0^1 I_{x\leq y}\,\mathrm{d}y = 2\int_x^1 \mathrm{d}y = 2(1-x) \tag{13.0.1}$$

$$f_Y(y) = \int_0^1 f_{X,Y}(x,y)\,\mathrm{d}x = 2\int_0^1 I_{x\leq y}\,\mathrm{d}x = 2\int_0^y \mathrm{d}y = 2y. \tag{13.0.2}$$

But $f_{X,Y}(x,y) \neq f_X(x) f_Y(y)$, hence X, Y are dependent.

$$\begin{aligned}
F_{X,Y}(x,y) &= \int_0^x \int_0^y f_{X,Y}(u,v)\,\mathrm{d}v\,\mathrm{d}u && (13.0.3)\\
&= 2\int_0^x \int_0^y I_{u\leq v}\,\mathrm{d}v\,\mathrm{d}u && (13.0.4)\\
&= 2\int_0^x \int I_{u\leq v} I_{0\leq v\leq y}\,\mathrm{d}v\,\mathrm{d}u && (13.0.5)\\
&= 2\int_0^x \int I_{u\leq v\leq y}\,\mathrm{d}v\,\mathrm{d}u && (13.0.6)\\
&= 2\int_0^x [y-u]^+\,\mathrm{d}u, && (13.0.7)
\end{aligned}$$

because $u > y \implies I_{u\leq v\leq y} = 0$. Now, if $y > x$,

$$2\int_0^x [y-u]^+\,\mathrm{d}u = 2\int_0^x (y-u)\,\mathrm{d}u = 2yx - x^2, \tag{13.0.8}$$

while if $y \le x$,

$$2\int_0^x [y-u]^+ \,\mathrm{d}u = 2\int_0^y (y-u)\,\mathrm{d}u = 2y^2 - y^2 = y^2 \tag{13.0.9}$$

Make a drawing of the support of $f_{X,Y}$ to help to understand this better.

s.7.1.5.

$$\partial_x \partial_y F_{X,Y}(x,y) = \partial_x \partial_y F_X(x) F_Y(y) = \partial_x F_X(x) \partial_y F_Y(y) = f_X(x) f_Y(y).$$

s.7.1.6.

$$\frac{F_{X,Y}(x,y)}{F_X(x)} = \frac{\mathsf{P}\left\{X \le x, Y \le y\right\}}{\mathsf{P}\left\{X \le x\right\}} = \mathsf{P}\left\{Y \le y, X \le x | X \le x\right\} = \mathsf{P}\left\{Y \le y | X \le x\right\}. \tag{13.0.10}$$

It is a big mistake to write $F_{X,Y}(x,y) = \mathsf{P}\left\{X = x, Y = y\right\}$. If you wrote this, recheck the definitions of BH.

s.7.1.7. $\mathsf{P}\{X=0, Y=0\} = 1/3 \cdot 3/4$, $\mathsf{P}\{X=0, Y=1\} = 1/3 \cdot 1/4$, and so on.

If we have one column with $Y=0$ and the other with $Y=1$, then the sum over the columns are $\mathsf{P}\{Y=0\}$ and $\mathsf{P}\{Y=1\}$. The row sum for row i are $\mathsf{P}\{X=i\}$.

Changing the values will (most of the time) make X and Y dependent. But, what if we changes the values such that $\mathsf{P}\{X=0, Y=0\} = 1$? Are X and Y then again independent? Check the conditions again.

s.7.1.8. The number of produced items (laid eggs) is N. The probability of hatching is p, that is, an item is ok. The hatched eggs are the good items.

s.7.1.9. For X, Y to be independent, it is necessary that $F_{X,Y}(x,y) = F_X(x)F_Y(y)$ for all x, y, not just one particular choice. (This is an example that satisfying a necessary condition is not necessarily sufficient.)

s.7.1.10. Many answers are possible here, depending on extra assumptions you make. Here is one. Suppose, just by change, the fraction of taller guys in the street is a bit higher than the population fraction. Assuming that taller (shorter) people prefer taller (shorter) spouses, there must be a dependence between the height of the men and the women. This is because when selecting a man, I can also select his wife.

From this exercise you should memorize that *independence is a property of the joint CDF, not of the rvs.*

Mistake: $\mathsf{P}\{Y\}$ is wrong notation wrong because we can only compute the probability of an event, such as $\{Y \le y\}$. But Y itself is not an event.

s.7.1.11. Only when X, Y are independent.

Mistake: independence of X and Y is not the same as the linear independence. Don't confuse these two types of dependene.

s.7.1.12. Given $N = n$, the random variable X has a certain distribution, here binomial.

s.7.1.13. This claim is incorrect, because X, Y are discrete, hence they have a PMF, not a PDF.

Mistake: Someone said that $\partial_x \partial_y$ is not correct notation; however, it is correct! It's a (much used) abbreviation of the much heaver $\partial^2/\partial x \partial y$. Next, the derivative of the PMF is not well-defined (at least, not within this course. If you object, ok, but then show that you passed a decent course on measure theory.)

s.7.1.14.

$$
\begin{aligned}
\mathsf{P}\{T_1 < T_2\} = \mathsf{E}\left[I_{T_1<T_2}\right] = &= \int_0^\infty \int_0^\infty I_{t_1<t_2} f_{T_1,T_2}(t_1,t_2)\,\mathrm{d}t_1\,\mathrm{d}t_2 \\
&= \int_0^\infty \int_{t_1}^\infty \lambda_1 e^{-\lambda_1 t_1} \lambda_2 e^{-\lambda_2 t_2}\,\mathrm{d}t_2\,\mathrm{d}t_1 \\
&= \int_0^\infty \lambda_1 e^{-\lambda_1 t_1} \lambda_2 \int_{t_1}^\infty e^{-\lambda_2 t_2}\,\mathrm{d}t_2\,\mathrm{d}t_1 \\
&= \int_0^\infty \lambda_1 e^{-\lambda_1 t_1} e^{-\lambda_2 t_1}\,\mathrm{d}t_1 \\
&= \int_0^\infty \lambda_1 e^{-\lambda_1 t_1 - \lambda_2 t_1}\,\mathrm{d}t_1 \\
&= \frac{\lambda_1}{\lambda_1+\lambda_2}.
\end{aligned}
$$

s.7.1.15. We have

$$
\begin{aligned}
\mathsf{E}\left[(X-Y)^2\right] &= \int_{-\infty}^\infty \int_{-\infty}^\infty (x-y)^2 f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y && (13.0.11) \\
&= \int_0^1 \int_0^1 (x-y)^2\,\mathrm{d}x\,\mathrm{d}y && (13.0.12) \\
&= \int_0^1 \int_0^1 (x^2 - 2xy + y^2)\,\mathrm{d}x\,\mathrm{d}y && (13.0.13) \\
&= \int_0^1 \int_0^1 x^2\,\mathrm{d}x\,\mathrm{d}y - 2\int_0^1 \int_0^1 xy\,\mathrm{d}x\,\mathrm{d}y + \int_0^1 \int_0^1 y^2\,\mathrm{d}x\,\mathrm{d}y && (13.0.14) \\
&= \int_0^1 x^2\,\mathrm{d}x - 2\int_0^1 \int_0^1 xy\,\mathrm{d}x\,\mathrm{d}y + \int_0^1 y^2\,\mathrm{d}y && (13.0.15) \\
&= 1/3 - 2\cdot 1/2 \cdot 1/2 + 1/3. && (13.0.16)
\end{aligned}
$$

s.7.1.16.

$$
\begin{aligned}
a < b &\implies \mathsf{P}\{a < X < b\} = F(b) - F(a) = [F(b)-F(a)]^+ && (13.0.17) \\
a \geq b &\implies \mathsf{P}\{a < X < b\} = 0 = [F(b)-F(a)]^+, && (13.0.18)
\end{aligned}
$$

where the last equality follows from the fact that F is increasing.

s.7.1.18.

$$
\int_{-\infty}^\infty I_{0\leq x\leq 3}\,\mathrm{d}x = \int_0^3 \mathrm{d}x = 3.
$$

s.7.1.19.

$$\int x\, I_{0\le x\le 4}\, \mathrm{d}x = \int_0^4 x\, \mathrm{d}x = 16/2 = 8.$$

s.7.1.20.

$$\begin{aligned}\iint xy\, I_{0\le x\le 3}\, I_{0\le y\le 4}\, \mathrm{d}x\, \mathrm{d}y &= \int_0^3 x \int_0^4 y\, \mathrm{d}y\, \mathrm{d}x \\ &= \int_0^3 x \frac{y^2}{2}\bigg|_0^4 \mathrm{d}x \\ &= \int_0^3 x\cdot 8\, \mathrm{d}x = 8\cdot 9/2 = 4\cdot 9.\end{aligned}$$

s.7.1.21. Two solutions. First we integrate over y.

$$\begin{aligned}\iint I_{0\le x\le 3}\, I_{0\le y\le 4}\, I_{x\le y}\, \mathrm{d}x\, \mathrm{d}y &= \int I_{0\le x\le 3} \int I_{0\le y\le 4}\, I_{x\le y}\, \mathrm{d}y\, \mathrm{d}x && (13.0.19)\\ &= \int I_{0\le x\le 3} \int I_{\max\{x,0\}\le y\le 4}\, \mathrm{d}y\, \mathrm{d}x && (13.0.20)\\ &= \int_0^3 \int_{\max\{x,0\}}^4 \mathrm{d}y\, \mathrm{d}x && (13.0.21)\\ &= \int_0^3 y\Big|_{\max\{x,0\}}^4 \mathrm{d}x && (13.0.22)\\ &= \int_0^3 (4-\max\{x,0\})\, \mathrm{d}x && (13.0.23)\\ &= 12 - \int_0^3 \max\{x,0\}\, \mathrm{d}x && (13.0.24)\\ &= 12 - \int_0^3 x\, \mathrm{d}x && (13.0.25)\\ &= 12 - 9/2. && (13.0.26)\end{aligned}$$

Let's now instead first integrate over x.

$$\begin{aligned}\iint I_{0\le x\le 3}\, I_{0\le y\le 4}\, I_{x\le y}\, \mathrm{d}x\, \mathrm{d}y &= \int I_{0\le y\le 4} \int I_{0\le x\le 3}\, I_{x\le y}\, \mathrm{d}x\, \mathrm{d}y && (13.0.27)\\ &= \int_0^4 \int I_{0\le x\le \min\{3,y\}}\, \mathrm{d}x\, \mathrm{d}y && (13.0.28)\\ &= \int_0^4 \int_0^{\min\{3,y\}} \mathrm{d}x\, \mathrm{d}y && (13.0.29)\\ &= \int_0^4 \min\{3,y\}\, \mathrm{d}y && (13.0.30)\\ &= \int_0^3 \min\{3,y\}\, \mathrm{d}y + \int_3^4 \min\{3,y\}\, \mathrm{d}y && (13.0.31)\\ &= \int_0^3 y\, \mathrm{d}y + \int_3^4 3\, \mathrm{d}y && (13.0.32)\\ &= 9/2 + 3. && (13.0.33)\end{aligned}$$

s.7.1.22. Take c the normalization constant (why is $c = 1/4$), then using the previous exercise

$$\begin{aligned}
\mathsf{P}\{Y \le 2X\} &= \mathsf{E}[I_{Y\le 2X}] && (13.0.34)\\
&= c\int_1^3\int_2^4 I_{y\le 2x}\,\mathrm{d}y\,\mathrm{d}x && (13.0.35)\\
&= c\int_1^3\int I_{2\le y\le \min\{4,2x\}}\,\mathrm{d}y\,\mathrm{d}x && (13.0.36)\\
&= c\int_1^3 (\min\{4,2x\}-2)\,\mathrm{d}x && (13.0.37)
\end{aligned}$$

Now make a drawing of the function $(\min\{4,2x\}-2)$ on the interval $[1,3]$ to see that

$$\int_1^3 (min\{4,2x\}-2)\,\mathrm{d}x = \int_1^2 (2x-2)\,\mathrm{d}x + \int_2^3 (4-2)\,\mathrm{d}x. \tag{13.0.38}$$

I leave the rest of the computation to you.

s.7.1.23. The covariance might be a large number, which may suggest that the rvs X and Y are 'very' dependent. However, when $\mathsf{V}[X]$ and $\mathsf{V}[Y]$ are also large, the correlation can be small. Thus, correlation is a scaled type of covariance.

s.7.1.24. Answers: no and yes.
We have

$$C = \frac{\mathsf{V}[X]}{(\mathsf{E}[X])^2}, \tag{13.0.39}$$

which does not equal

$$\mathrm{Corr}(X,X) = \frac{\mathsf{Cov}[X,X]}{\sqrt{\mathsf{V}[X]\,\mathsf{V}[X]}} = 1 \tag{13.0.40}$$

in general (for instance, consider a degenerate random variable $X \equiv 1$). Next, consider a $N(1,100)$ random variable. Then,

$$C = 100/(1^2) = 100 > 1. \tag{13.0.41}$$

s.7.1.25.

1. We have

$$\mathsf{Cov}[X,X] = \mathsf{E}[XX] - \mathsf{E}[X]\,\mathsf{E}[X] = \mathsf{E}\left[X^2\right] - \mathsf{E}[X]^2 = \mathsf{V}[X]. \tag{13.0.42}$$

2. We have

$$\mathsf{Cov}[X,Y] = \mathsf{E}[XY] - \mathsf{E}[X]\,\mathsf{E}[Y] = \mathsf{E}[YX] - \mathsf{E}[Y]\,\mathsf{E}[X] = \mathsf{Cov}[Y,X]. \tag{13.0.43}$$

3. We have

$$\mathsf{Cov}[X,c] = \mathsf{E}[Xc] - \mathsf{E}[X]\,\mathsf{E}[c] = c\,\mathsf{E}[X] - c\,\mathsf{E}[X] = 0. \tag{13.0.44}$$

4. We have

$$\mathsf{Cov}[aX, Y] = \mathsf{E}[aXY] - \mathsf{E}[aX]\,\mathsf{E}[Y] = a\big(\mathsf{E}[XY] - \mathsf{E}[X]\,\mathsf{E}[Y]\big) = a\,\mathsf{Cov}[X, Y]. \tag{13.0.45}$$

5. We have

$$\begin{aligned}
\mathsf{Cov}[X+Y, Z] &= \mathsf{E}[(X+Y)Z] - \mathsf{E}[X+Y]\,\mathsf{E}[Z] && (13.0.46)\\
&= \mathsf{E}[XZ+YZ] - \big(\mathsf{E}[X] + \mathsf{E}[Y]\big)\mathsf{E}[Z] && (13.0.47)\\
&= \mathsf{E}[XZ] - \mathsf{E}[X]\,\mathsf{E}[Z] + \mathsf{E}[YZ] - \mathsf{E}[Y]\,\mathsf{E}[Z] && (13.0.48)\\
&= \mathsf{Cov}[X, Z] + \mathsf{Cov}[Y, Z]. && (13.0.49)
\end{aligned}$$

s.7.1.26. We have

$$\begin{aligned}
\mathsf{Cov}[X, Y] &= \mathsf{E}[(X - \mathsf{E}[X])(Y - \mathsf{E}[Y])] && (13.0.50)\\
&= \mathsf{E}[XY - X\,\mathsf{E}[Y] - Y\,\mathsf{E}[X] + \mathsf{E}[X]\,\mathsf{E}[Y]] && (13.0.51)\\
&= \mathsf{E}[XY] - \mathsf{E}[X]\,\mathsf{E}[Y] - \mathsf{E}[Y]\,\mathsf{E}[X] + \mathsf{E}[X]\,\mathsf{E}[Y] && (13.0.52)\\
&= \mathsf{E}[XY] - \mathsf{E}[X]\,\mathsf{E}[Y]. && (13.0.53)
\end{aligned}$$

When X and Y are independent, then $\mathsf{E}[XY] = \mathsf{E}[X]\,\mathsf{E}[Y]$, and then $\mathsf{Cov}[X, Y] = 0$.

s.7.1.27. By linearity of the covariance wea have

$$\begin{aligned}
\mathsf{Cov}[a(U+V), b(U-V)] &= a\Big(\mathsf{Cov}[U, b(U-V)] + \mathsf{Cov}[V, b(U-V)]\Big) && (13.0.54)\\
&= a\Big(b\big(\mathsf{Cov}[U,U] - \mathsf{Cov}[U,V]\big) + b\big(\mathsf{Cov}[V,U] - \mathsf{Cov}[V,V]\big)\Big) && (13.0.55)\\
&= a\Big(b\big(\mathsf{Cov}[U,U] - \mathsf{Cov}[U,V]\big) + b\big(\mathsf{Cov}[V,U] - \mathsf{Cov}[V,V]\big)\Big) && (13.0.56)\\
&= ab\Big(\mathsf{V}[U] - \mathsf{Cov}[U,V] + \mathsf{Cov}[V,U] - \mathsf{V}[V]\Big) && (13.0.57)\\
&= ab\Big(\mathsf{V}[U] - \mathsf{V}[V]\Big). && (13.0.58)
\end{aligned}$$

Alternatively one can also use the result from BH.7.1.26, according to which $\mathsf{Cov}[X, Y] = \mathsf{E}[XY] - \mathsf{E}[X]\,\mathsf{E}[Y]$.

s.7.1.28. With the hint: $\mathsf{E}[XY] = 1/\lambda^2$, when $X, Y \sim \text{Exp}(\lambda)$. Then, $L \sim \text{Exp}(2\lambda)$, since $f_L(x) = 2f_X(x)(1 - F_Y(x)) = 2\lambda e^{-2\lambda x}$. Therefore, $\mathsf{E}[L] = 1/2\lambda$. Also, by memoryless, $\mathsf{E}[M] = \mathsf{E}[L] + \mathsf{E}[X] = 3/2\lambda$. Hence, $\mathsf{E}[M]\,\mathsf{E}[L] = 3/4\lambda^2$. Hence, $\mathsf{E}[ML] - \mathsf{E}[M]\,\mathsf{E}[L] = 1/\lambda^{2-}3/4\lambda^2 = 1/4\lambda^2$.

s.7.1.29. We throw 10 fair dice. X_i denotes the number of dice that show the number i, $i = 1, \ldots, 6$.

s.7.1.30. No, this does not always hold, see BH.7.5.2. However, it does hold when X and Y are independent.

s.7.1.31. Since X,Y,Z are independent normally distributed variables, (X,Y,Z) is multivariate normally distributed. Hence, every linear combination of X,Y,Z is normally distributed. Note that every linear combination of the elements of W can be written as a linear combination of X,Y,Z. Hence, every linear combination of the elements of W is normally distributed. Hence, W is multivariate normally distributed.

s.7.2.1. Of course $X \in \{0,1,2,\ldots\}$.

s.7.2.2. For $X>0$, the first outcome should be a failure. Then, for j failures, we need to fail $j-1$ times and then once more. For $\mathsf{E}[X]$, if there is a success, we don't need another another experiment. However, in case of a fail, we need another experiment, and we start again. Thus, $\mathsf{E}[X] = q(1+\mathsf{E}[X]) \implies (1-q)\,\mathsf{E}[X] = q$.

s.7.2.3. With the regular method:

$$\begin{aligned}
\mathsf{P}\left\{X>j\right\} &= \sum_{i=j+1}^{\infty} \mathsf{P}\{X=i\} \\
&= p\sum_{i=j+1}^{\infty} q^i \\
&= p\sum_{i=0}^{\infty} q^{j+1+i} \\
&= pq^{j+1}\sum_{i=0}^{\infty} q^i \\
&= pq^{j+1}\frac{1}{1-q} = pq^{j+1}\frac{1}{p} = q^{j+1}.
\end{aligned}$$

s.7.2.4.

$$\begin{aligned}
\mathsf{E}[X] &= \sum_{i=0}^{\infty} i\,\mathsf{P}\{X=i\} \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} I_{j<i}\,\mathsf{P}\{X=i\} \\
&= \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} I_{j<i}\,\mathsf{P}\{X=i\} \\
&= \sum_{j=0}^{\infty}\sum_{i=j+1}^{\infty} \mathsf{P}\{X=i\} \\
&= \sum_{j=0}^{\infty} \mathsf{P}\left\{X>j\right\} \\
&= \sum_{j=0}^{\infty} q^{j+1} \\
&= q\sum_{j=0}^{\infty} q^j \\
&= q/(1-q) = q/p.
\end{aligned}$$

s.7.2.5.

$$\begin{aligned}
\mathsf{P}\{X \geq n+m \mid X \geq m\} &= \frac{\mathsf{P}\{X \geq n+m, X \geq m\}}{\mathsf{P}\{X \geq m\}} \\
&= \frac{\mathsf{P}\{X \geq n+m\}}{\mathsf{P}\{X \geq m\}} \\
&= \frac{q^{n+m}}{q^m} \\
&= q^n = \mathsf{P}\{X \geq n\}.
\end{aligned}$$

s.7.2.6.

$$\begin{aligned}
\mathsf{P}\{L \geq k\} &= \sum_i \sum_j I_{\min\{i,j\} \geq k} \mathsf{P}\left\{X = i, Y = j\right\} \\
&= \sum_{i \geq k} \sum_{j \geq k} \mathsf{P}\{X = i\} \mathsf{P}\left\{Y = j\right\} \\
&= \mathsf{P}\{X \geq k\} \mathsf{P}\{Y \geq k\} = q^k q^k = q^{2k}.
\end{aligned}$$

$\mathsf{P}\{L > i\}$ has the same form as $\mathsf{P}\{X > i\}$, but now with q^{2i} rather than q^i.

s.7.2.7. Immediate from the hint and [7.2.4].

s.7.2.8.

$$\begin{aligned}
\mathsf{E}[L] + \mathsf{E}[X] &= \frac{q^2}{1-q^2} + \frac{q}{1-q} \\
&= \frac{q}{1-q}\left(\frac{q}{1+q} + 1\right) \\
&= \frac{q}{1-q} \frac{1+2q}{1+q}
\end{aligned}$$

s.7.2.9.

$$\begin{aligned}
2\,\mathsf{E}[X] - \mathsf{E}[L] &= 2\frac{q}{1-q} - \frac{q^2}{1-q^2} \\
&= \frac{q}{1-q}\left(2 - \frac{q}{1+q}\right) \\
&= \frac{q}{1-q}\left(\frac{2+2q}{1+q} - \frac{q}{1+q}\right) \\
&= \frac{q}{1-q} \frac{2+q}{1+q}.
\end{aligned}$$

s.7.2.10.

$$\begin{aligned}
\mathsf{P}\{M=k\} &= \mathsf{P}\{\max\{X,Y\}=k\} \\
&= p^2\sum_{ij} I_{\max\{i,j\}=k}\, q^i q^j \\
&= 2p^2\sum_{ij} I_{i=k} I_{j<k}\, q^i q^j + p^2\sum_{ij} I_{i=j=k}\, q^i q^j \\
&= 2p^2 q^k \sum_{j<k} q^j + p^2 q^{2k} \\
&= 2p^2 q^k \frac{1-q^k}{1-q} + p^2 q^{2k}
\end{aligned}$$

s.7.2.11. It's just algebra

$$\begin{aligned}
\mathsf{P}\{M=k\} &= 2p^2 q^k \frac{1-q^k}{1-q} + p^2 q^{2k} \\
&= 2pq^k(1-q^k) + p^2 q^{2k} \\
&= 2pq^k + (p^2-2p)q^{2k} \\
&= 2\,\mathsf{P}\{X=k\} - \mathsf{P}\{L=k\},
\end{aligned}$$

where I use that $p^2-2p = p(p-2) = (1-q)(1-q-2) = -(1-q)(1+q) = -(1-q^2)$.

s.7.2.12.

$$\begin{aligned}
\mathsf{E}[M] &= \sum_k k\,\mathsf{P}\{M=k\} \\
&= \sum_k k(2\,\mathsf{P}\{X=k\} - \mathsf{P}\{L=k\}) \\
&= 2\,\mathsf{E}[X] - \mathsf{E}[L].
\end{aligned}$$

s.7.2.13.

$$\begin{aligned}
\mathsf{P}\{M\leq k\} &= \mathsf{P}\{X\leq k, Y\leq k\} \\
&= \mathsf{P}\{X\leq k\}\,\mathsf{P}\{Y\leq k\} \\
&= (1-\mathsf{P}\{X>k\})(1-\mathsf{P}\{Y>k\}) \\
&= (1-q^{k+1})^2.
\end{aligned}$$

s.7.2.14.

$$\begin{aligned}
\mathsf{P}\{M=k\} &= \mathsf{P}\{M\leq k\} - \mathsf{P}\{M\leq k-1\} \\
&= 1-2q^{k+1}+q^{2k+2} - (1-2q^k+q^{2k}) \\
&= 2q^k(1-q) + q^{2k}(q^2-1) \\
&= 2\,\mathsf{P}\{X=k\} - q^{2k}(1-q^2) \\
&= 2\,\mathsf{P}\{X=k\} - \mathsf{P}\{L=k\}.
\end{aligned}$$

s.7.2.15.

$$\begin{aligned}\mathsf{P}\{L=i, M=k\} &= 2\,\mathsf{P}\{X=i, Y=k\}\, I_{k>i} + \mathsf{P}\{X=Y=i\}\, I_{i=k} \\ &= 2p^2 q^{i+k}\, I_{k>i} + p^2 q^{2i}\, I_{i=k}.\end{aligned}$$

s.7.2.16.

$$\begin{aligned}\mathsf{P}\{M=k\} &= \sum_i \mathsf{P}\{L=i, M=k\} \\ &= \sum_i (2p^2 q^{i+k}\, I_{k>i} + p^2 q^{2i}\, I_{i=k}) \\ &= 2p^2 q^k \sum_{i=0}^{k-1} q^i + p^2 q^{2k} \\ &= 2pq^k(1-q^k) + p^2 q^{2k} \\ &= 2pq^k + (p^2 - 2p) q^{2k},\end{aligned}$$

s.7.2.17.

$$\begin{aligned}\mathsf{P}\{L=i\} &= \sum_k \mathsf{P}\{L=i, M=k\} \\ &= \sum_k (2p^2 q^{i+k}\, I_{k>i} + p^2 q^{2i}\, I_{i=k}) \\ &= 2p^2 q^i \sum_{k=i+1}^{\infty} q^k + p^2 q^{2i} \\ &= 2p^2 q^{2i+1} \sum_{k=0}^{\infty} q^k + p^2 q^{2i} \\ &= 2pq^{2i+1} + p^2 q^{2i} \\ &= pq^{2i}(2q+p) \\ &= (1-q) q^{2i} (q+1), \quad p+q=1, \\ &= (1-q^2) q^{2i}.\end{aligned}$$

s.7.2.18. With $t \geq s$,

$$\mathsf{P}\{X > t | X > s\} = \frac{\mathsf{P}\{X > t\}}{\mathsf{P}\{X > s\}} = e^{-\lambda t} e^{\lambda s} = e^{-\lambda(t-s)} = \mathsf{P}\{X > t - s\}.$$

s.7.2.19. It is essential that you know both methods to solve this integral.

$$\begin{aligned}\mathsf{E}[X] &= \int_0^\infty x f_X(x)\,\mathrm{d}x \\ &= \int_0^\infty x\lambda e^{-\lambda x}\,\mathrm{d}x \\ &= -xe^{-\lambda x}\Big|_0^\infty + \int_0^\infty e^{-\lambda x}\,\mathrm{d}x \\ &= -\frac{1}{\lambda} e^{-\lambda x}\Big|_0^\infty = \frac{1}{\lambda}.\end{aligned}$$

Substitution is also a very important technique to solve such integrals. Here we go again:

$$\begin{aligned}\mathsf{E}[X] &= \int_0^\infty x f_X(x)\,\mathrm{d}x \\ &= \int_0^\infty x\lambda e^{-\lambda x}\,\mathrm{d}x \\ &= \frac{1}{\lambda}\int_0^\infty u e^{-u}\,\mathrm{d}u,\end{aligned}$$

by the substitution $u = \lambda x \implies \mathrm{d}u = \mathrm{d}(\lambda x) \implies \mathrm{d}u = \lambda\,\mathrm{d}x \implies \mathrm{d}x = \mathrm{d}u/\lambda$. With partial integration (do it!), the integral evaluates to 1.

s.7.2.20. With the hint,

$$G_L(x) = \mathsf{P}\{L > x\} = \mathsf{P}\{X > x, Y > x\} = G_X(x)^2 = e^{-2\lambda x}.$$

The result follows since $F_L(x) = 1 - G_L(x)$.

s.7.2.21.

$$\begin{aligned}\mathsf{P}\{M \leq v\} &= \mathsf{E}[I_{M\leq v}] \\ &= \int_0^\infty\int_0^\infty I_{x\leq v, y\leq v} f_{XY}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^\infty\int_0^\infty I_{x\leq v, y\leq v} f_X(x) f_Y(y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^v f_X(x)\,\mathrm{d}x \int_0^v f_Y(y)\,\mathrm{d}y \\ &= F_X(v)F_Y(v) = (F_X(v))^2.\end{aligned}$$

s.7.2.22.

$$f_M(v) = F_M'(v) = 2F_X(v) f_X(v) = 2(1 - e^{-\lambda v})\lambda e^{-\lambda v}.$$

s.7.2.23.

$$\begin{aligned}\mathsf{E}[M] &= \int_0^\infty v f_M(v)\,\mathrm{d}v = \\ &= 2\lambda\int_0^\infty v(1 - e^{-\lambda v})e^{-\lambda v}\,\mathrm{d}v = \\ &= 2\lambda\int_0^\infty v e^{-\lambda v}\,\mathrm{d}v - 2\lambda\int_0^\infty v e^{-2\lambda v}\,\mathrm{d}v \\ &= 2\,\mathsf{E}[X] - \mathsf{E}[L],\end{aligned}$$

where the last equality follows from the previous exercises.

s.7.2.24. First the joint distribution. With $u \leq v$,

$$\begin{aligned}
F_{L,M}(u,v) &= \mathsf{P}\{L \leq u, M \leq v\} \\
&= 2 \iint I_{x\leq u, y \leq v, x \leq y} f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= 2 \int_0^u \int_x^v f_Y(y)\,\mathrm{d}y\, f_X(x)\,\mathrm{d}x && \text{independence} \\
&= 2 \int_0^u (F_Y(v) - F_Y(x)) f_X(x)\,\mathrm{d}x.
\end{aligned}$$

s.7.2.25. Taking partial derivatives,

$$\begin{aligned}
f_{L,M}(u,v) &= \partial_v \partial_u F_{L,M}(u,v) \\
&= 2\partial_v \partial_u \int_0^u (F_Y(v) - F_Y(x)) f_X(x)\,\mathrm{d}x \\
&= 2\partial_v \left\{ (F_Y(v) - F_Y(u)) f_X(u) \right\} \\
&= 2 f_X(u) \partial_v F_Y(v) \\
&= 2 f_X(u) f_Y(v).
\end{aligned}$$

s.7.2.26.

$$\begin{aligned}
f_M(v) &= \int_0^\infty f_{L,M}(u,v)\,\mathrm{d}u \\
&= 2 \int_0^v f_X(u) f_Y(v)\,\mathrm{d}u \\
&= 2 f_Y(v) \int_0^v f_X(u)\,\mathrm{d}u \\
&= 2 f_Y(v) F_X(v), \\
f_L(u) &= \int_0^\infty f_{L,M}(u,v)\,\mathrm{d}v \\
&= 2 f_X(u) \int_u^\infty f_Y(v)\,\mathrm{d}v \\
&= 2 f_X(u) G_Y(u).
\end{aligned}$$

s.7.2.27. First,

$$\begin{aligned}
\mathsf{P}\{X/n \approx x\} = \mathsf{P}\{X/n \in [i/n, (i+1)/n]\} = \mathsf{P}\{X \in [i, i+1]\} = pq^i \\
\approx pq^{nx} = \frac{\lambda}{n}\left(1 - \frac{\lambda}{n}\right)^{xn},
\end{aligned}$$

since $p = \lambda/n$, $q = 1 - p = 1 - \lambda/n$, and $i = nx$.

s.7.2.28.

$$M_{X/n}(s) = \mathsf{E}\left[e^{sX/n}\right] = \sum_i e^{si/n} pq^i = p \sum_i (qe^{s/n})^i = \frac{p}{1 - qe^{s/n}}.$$

With $p = \lambda/n$ this becomes

$$\begin{aligned}
M_{X/n}(s) &= \frac{\lambda/n}{1-(1-\lambda/n)(1+s/n+1/n^2\times(\cdots))} \\
&= \frac{\lambda/n}{\lambda/n - s/n + 1/n^2\times(\cdots)} \\
&= \frac{\lambda}{\lambda - s + 1/n\times(\cdots)} \\
&\to \frac{\lambda}{\lambda - s}, \quad \text{as } n\to\infty,
\end{aligned}$$

where we write $1/n^2\times(\cdots)$ for all terms that will disappear when we take the limit $n\to\infty$. This is just handy notation to hide details in which we are not interested.

s.7.2.29.

$$\begin{aligned}
\mathsf{E}[L/n] &= \frac{1}{n}\mathsf{E}[L] = \frac{1}{n}\frac{q^2}{1-q^2} \\
&= \frac{1}{n}\frac{(1-\lambda/n)^2}{1-(1-\lambda/n)^2} \\
&= \frac{1}{n}\frac{1-2\lambda/n+(\lambda/n)^2}{2\lambda/n+(\lambda/n)^2} \\
&= \frac{1-2\lambda/n+(\lambda/n)^2}{2\lambda+\lambda^2/n} \\
&\to \frac{1}{2\lambda}.
\end{aligned}$$

s.7.2.30. Take $p = \lambda/n$, $q = 1-\lambda/n$, and $k \approx xn$, hence $k/n \approx x$. Then,

$$\begin{aligned}
\mathsf{P}\{M/n = k/n\} &= 2pq^{k/n}(1-q^{k/n}) + p^2 q^{2k/n} \\
&= 2\frac{\lambda}{n}\left(1-\frac{\lambda}{n}\right)^{k/n}\left(1-\left(1-\frac{\lambda}{n}\right)^{k/n}\right) + \frac{\lambda^2}{n^2}\left(1-\frac{\lambda}{n}\right)^{2k/n} \\
&\to 2\lambda\,\mathrm{d}x e^{-\lambda x}\left(1-e^{-\lambda x}\right) + \lambda^2\,\mathrm{d}x^2 e^{-2\lambda}.
\end{aligned}$$

Now observe that the second term, proportional to $\mathrm{d}x^2$ can be neglected.

s.7.2.31.

$$\begin{aligned}
F_{L,M-L}(x,y) &= \mathsf{P}\{L \le x, M-L \le y\} \\
&= 2\mathsf{P}\{X \le x, Y-X \le y, X \le Y\} \\
&= 2\int_0^\infty \int_0^\infty I_{u\le x, v-u\le y, u\le v} f_{X,Y}(u,v)\,\mathrm{d}u\,\mathrm{d}v \\
&= 2\int_0^\infty \int_0^\infty I_{u\le x, v-u\le y, u\le v} \lambda^2 e^{-\lambda u} e^{-\lambda v}\,\mathrm{d}u\,\mathrm{d}v \\
&= 2\int_0^x \int_0^\infty I_{u\le v\le u+y} \lambda^2 e^{-\lambda u} e^{-\lambda v}\,\mathrm{d}v\,\mathrm{d}u \\
&= 2\int_0^x \lambda e^{-\lambda u} \int_u^{u+y} \lambda e^{-\lambda v}\,\mathrm{d}v\,\mathrm{d}u \\
&= 2\int_0^x \lambda e^{-\lambda u} (-e^{-\lambda v})\Big|_u^{u+y}\,\mathrm{d}u \\
&= 2\int_0^x \lambda e^{-\lambda u} (e^{-\lambda u} - e^{-\lambda(u+y)})\,\mathrm{d}u \\
&= 2\lambda \int_0^x e^{-2\lambda u}\,\mathrm{d}u - 2\lambda \int_0^x e^{-\lambda(2u+y)}\,\mathrm{d}u \\
&= 2\lambda \int_0^x e^{-2\lambda u}\,\mathrm{d}u - 2\lambda e^{-\lambda y} \int_0^x e^{-2\lambda u}\,\mathrm{d}u \\
&= (1-e^{-\lambda y}) 2\lambda \int_0^x e^{-2\lambda u}\,\mathrm{d}u \\
&= (1-e^{-\lambda y})(-e^{-2\lambda u})\Big|_0^x \\
&= (1-e^{-\lambda y})(1-e^{-2\lambda x}).
\end{aligned}$$

s.7.2.32. As $F_{L,M-L}(x,y) = F_Y(y)F_L(x)$. So the CDF factors as a function of x only and a function of y only. This implies that L and $M-L$ are independent, and moreover that $F_{M-L}(y) = F_Y(y)$, so $M-L \sim Y$. We can also see this from the joint PDF:

$$f_{L,M-L}(x,y) = \partial_x \partial_y (F_Y(y)F_L(x)) = f_Y(y) f_L(x),$$

so the joint PDF (of course) also factors. The independence now follows from BH 7.1.21. Because L and $M-L$ are independent, the conditional density equals the marginal density:

$$f_{M-L|L}(y|x) = \frac{f_{L,M-L}(x,y)}{f_L(x)} = \frac{f_Y(y) f_L(x)}{f_L(x)} = f_Y(y).$$

s.7.2.34. Suppose $j > 0$ (for $j = 0$ the maths is the same). Then,

$$\mathsf{P}\{M-L=j\} = 2\sum_{i=0}^\infty \mathsf{P}\{X=i, Y=i+j\} = \ = 2\sum_{i=0}^\infty pq^i pq^{i+j} = = 2p^2 q^j \sum_{i=0}^\infty q^{2i} = \ = 2p^2 q^j/(1-q^2).$$

s.7.2.35. Because either $M = L$ or $M > L$. Recall from earlier work that the factor 2 in the second equality follows from the fact that X, Y iid.

s.7.2.36.

$$\begin{aligned}
\mathsf{E}\left[(Y-X)\, I_{Y>X}\right] &= p^2 \sum_{ij} (j-i)\, I_{j>i} q^i q^j \\
&= p^2 \sum_i q^i \sum_{j=i+1}^{\infty} (j-i) q^j \\
&= p^2 \sum_i q^i q^i \sum_{k=1}^{\infty} k q^k, \quad k = j-i \\
&= p \sum_i q^{2i}\, \mathsf{E}[X] \\
&= \frac{p}{1-q^2}\, \mathsf{E}[X] \\
&= \frac{p}{1-q^2} \frac{q}{p} \\
&= \frac{q}{1-q^2}.
\end{aligned}$$

s.7.2.37.

$$\mathsf{E}[L] + 2\,\mathsf{E}\left[(Y-X)\, I_{Y>X}\right] = \frac{q^2}{1-q^2} + \frac{2q}{1-q^2} = \frac{q}{1-q}\frac{q+2}{1+q},$$

where I use that $1-q^2 = (1-q)(1+q)$.

s.7.2.38. To see why this might be true, I reason like this. After 'seeing' L, we want to restart. Let Z be the time from the restart to M. When $Z \sim \text{Geo}(p)$, it might happen that $Z = 0$ (with positive probability p). But if $Z = 0$, then $M = L$, and it that case, we should not restart. Hence, if $Z \sim \text{Geo}(p)$ we are 'double counting' when $Z = 0$. By including the condition $M > L$ and by taking $Z \sim \text{FS}(p)$ (so that $Z > 0$) I can prevent this.

s.7.2.39. With the hint:

$$\mathsf{E}\left[Z\, I_{M>L}\right] = \mathsf{E}[Z]\, \mathsf{E}\left[I_{M>L}\right] = \mathsf{E}[Z]\, \mathsf{P}\{M>L\} = \frac{1}{p}\frac{2pq}{1-q^2} = \frac{2q}{1-q^2},$$

We know that $\mathsf{E}[Z] = 1 + \mathsf{E}[X] = 1 + q/p = 1/p$, while

$$\begin{aligned}
\mathsf{P}\{M>L\} &= 1 - \mathsf{P}\{X=Y\} = 1 - \sum_{i=0}^{\infty} \mathsf{P}\{X=Y=i\} \\
&= 1 - \frac{p^2}{1-q^2} = \frac{1-q^2+p^2}{1-q^2} = \frac{2pq}{1-q^2}.
\end{aligned}$$

s.7.2.40. By independence,

$$\mathsf{E}\left[Z/n\, I_{M>L}\right] = \mathsf{E}[Z/n]\, \mathsf{P}\{M>L\}.$$

Then,

$$\mathsf{P}\{M>L\} = \frac{2pq}{1-q^2} = \frac{2\lambda/n(1-\lambda/n)}{1-(1-\lambda/n)^2} = \frac{2\lambda/n(1-\lambda/n)}{2\lambda/n - \lambda^2/n^2} = \frac{2(1-\lambda/n)}{2-\lambda/n} \to 1,$$

and

$$\mathsf{E}[Z/n] = \frac{1}{n}\mathsf{E}[Z] = \frac{1}{n}\frac{1}{p} = \frac{1}{n\lambda/n} = 1/\lambda.$$

s.7.3.1. a.

$$F(2,5) = P(X \le 2, Y \le 5) = P(X \le 2, Y \le 4) = F(2,4) = \frac{1}{4}$$

The second step is valid since the cumulative distribution function does not change by changing y from 5 to 4 thanks to property 3.

b. To obtain the joint pdf, use that $f_{X,Y}(x,y) = \frac{\partial^2}{\partial y \partial x} F(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} F(x,y)\right)$.
Since $\frac{\partial}{\partial x} F(x,y) = \frac{1}{4}(x-1)(y-2)$ for $1 < x < 3$, and $\frac{\partial}{\partial x} F(x,y) = 0$ for other values of x, we have that

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{4}(x-1), & \text{for } 1 < x < 3 \text{ and } 2 < y < 4, \\ 0, & \text{elsewhere.} \end{cases}$$

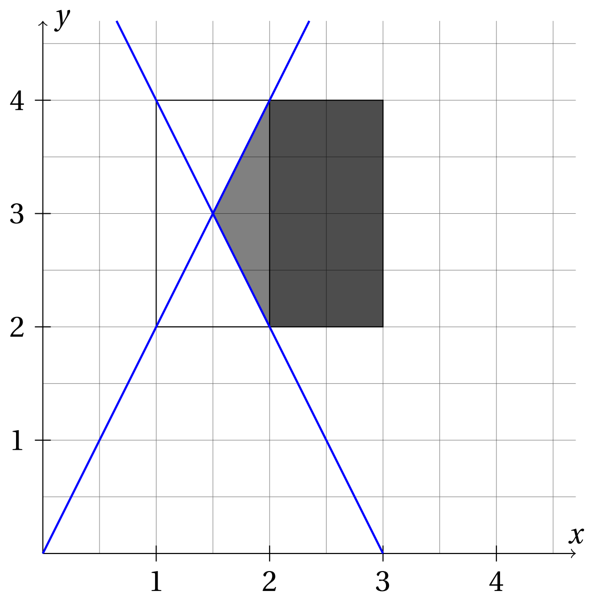
c. The simplest way of solving this question is by writing

$$\begin{aligned} P(2 < X < 3, 2 < Y < 4) &= F(3,4) - F(2,4) - F(3,2) + F(2,2) \\ &= 1 - \frac{1}{4} - 0 + 0 = \frac{3}{4}. \end{aligned}$$

Alternatively, one can integrate over the pdf from (b) to obtain the same result:

$$\begin{aligned} P(2 < X < 3, 2 < Y < 4) &= \int_2^3 \int_2^4 f(x,y) \mathrm{d}y \mathrm{d}x \\ &= \frac{1}{4} \int_2^3 (x-1) y\Big|_{y=2}^{y=4} \mathrm{d}x \\ &= \frac{1}{4}\left(x^2 - 2x\Big|_2^3\right) = \frac{1}{4}(9 - 6 - 4 + 4) = \frac{3}{4}. \end{aligned}$$

d. First, draw the integration area:



The domain on which the density is non-zero, is the complete shaded area. The downward-sloping line represents $2x+y=6$ and the upward-sloping line is $y=2x$.

We already know the integral over the dark shaded area from the previous subquestion. What remains is the lighter shaded triangular part on the left side.

First, we need to calculate the intersection of the two curves, which can be found by solving $6-2x=2x$, which gives $x=\frac{3}{2}$, and consequently $y=3$.

The integral of the triangular part of the dark shaded region is

$$\begin{aligned}
\int_{3/2}^{2}\int_{6-2x}^{2x}\frac{1}{4}(x-1)\,dy\,dx &= \int_{3/2}^{2}\frac{1}{4}(x-1)y\Big|_{6-2x}^{2x}dx \\
&= \frac{1}{4}\int_{3/2}^{2}(x-1)2x-(x-1)(6-2x)\,dx \\
&= \frac{1}{4}\int_{3/2}^{2}(x-1)(4x-6)\,dx \\
&= \frac{1}{4}[\frac{4}{3}x^3-5x^2+6x]_{3/2}^{2} \\
&= \frac{5}{48}\approx 0.1042
\end{aligned}$$

Finally, the joint probability asked for in the question is given by

$$P(Y<2X,2X+Y>6)=\frac{5}{48}+\frac{3}{4}=\frac{41}{48}\approx 0.8542$$

s.7.3.2. For $r\geq 0$, we have

$$\begin{aligned}
F_R(r) &= P(R\leq r) \\
&= P(X\leq Vr) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{vr} f_{X,V}(x,v)\,dx\,dv \\
&= \lambda\mu\int_0^{\infty}\int_0^{vr} e^{-\lambda x}e^{-\mu v}\,dx\,dv \\
&= \lambda\mu\int_0^{\infty} e^{-\mu v}\left[-\frac{1}{\lambda}e^{-\lambda x}\right]_0^{vr}dv \\
&= -\mu\int_0^{\infty} e^{-\mu v}(e^{-\lambda v r}-1)\,dv \\
&= -\mu\left[-\frac{1}{\mu+\lambda r}e^{-(\mu+\lambda r)v}+\frac{1}{\mu}e^{-\mu v}\right]_0^{\infty} \\
&= -\mu\left[\frac{1}{\mu+\lambda r}-\frac{1}{\mu}\right] \\
&= \frac{\lambda r}{\mu+\lambda r},
\end{aligned}$$

while $F_R(r) = 0$ when $r < 0$ since both X and V are nonnegative.

We see that (1) $F_R(-\infty) = 0$, (2) $F_R(\infty) = 1$, and (3) $F_R(r)$ is monotonically increasing in r, so $F_R(r)$ satisfies the conditions for being a valid CDF.

s.7.3.3. a. We integrate $f_{X,Y}(x,y)$ over its domain

$$\begin{aligned}
&\int_0^{1/2}\int_0^x cxydydx + \int_{1/2}^1\int_0^{1-x} cxydydx\\
=&c\left[\frac{1}{2}\int_0^{1/2} x^3dx + \frac{1}{2}\int_{1/2}^1 x(1-x)^2dx\right]\\
=&\frac{1}{2}c\left\{\left[\frac{1}{4}x^4\right]_0^{1/2} + \left[\frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{4}x^4\right]_{1/2}^1\right\}\\
&= \frac{1}{2}c\left\{\frac{1}{4}\frac{1}{16} + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} - \frac{1}{2}\frac{1}{4} + \frac{2}{3}\frac{1}{8} - \frac{1}{4}\frac{1}{16}\right\}\\
&= \frac{1}{2}c\frac{12-16+6-3+2}{24}\\
&= \frac{c}{48}
\end{aligned}$$

Since this integral should equal 1, $c = 48$.

Alternative Rewrite the probability density function to

$$f_{X,Y}(x,y) = \begin{cases} cxy & 0 \le y \le \frac{1}{2}, \quad y \le x \le 1-y \\ 0 & \text{otherwise} \end{cases}$$

Then,

$$\begin{aligned}
\int_0^{1/2}\int_y^{1-y} cxydxdy = c\int_0^{1/2} y\frac{1}{2}\left((1-y)^2 - y^2\right)\\
= \frac{1}{2}c\int_0^{1/2}\left(y - 2y^2\right)dy\\
= \frac{1}{2}c\left[\frac{1}{2}y^2 - \frac{2}{3}y^3\right]_0^{1/2}\\
= \frac{1}{2}c\left[\frac{1}{8} - \frac{2}{3}\frac{1}{8}\right] = \frac{c}{48}
\end{aligned}$$

Since this integral should equal 1, $c = 48$.

b. The conditional density function is given by

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

We first need the marginal density of Y.

$$\begin{aligned}
f_Y(y) &= 48\int_y^{1-y} xy\,dx \\
&= 48y\left[\frac{1}{2}(1-y)^2 - \frac{1}{2}y^2\right] \\
&= 24y\left[1-2y+y^2-y^2\right] \\
&= 24y(1-2y) \quad \text{for } 0 \le y \le \frac{1}{2}
\end{aligned}$$

and $f_Y(y) = 0$ otherwise.

Not required: We can check that this is a valid density function:

$$\begin{aligned}
\int_0^{1/2} 24y(1-2y)\,dy &= 24\left[\frac{1}{2}y^2 - \frac{2}{3}y^3\right]_0^{1/2} \\
&= 24\left[\frac{1}{2}\frac{1}{4} - \frac{2}{3}\frac{1}{8}\right]_0^{1/2} \\
&= 1
\end{aligned}$$

---

b. Now we can obtain the conditional density function

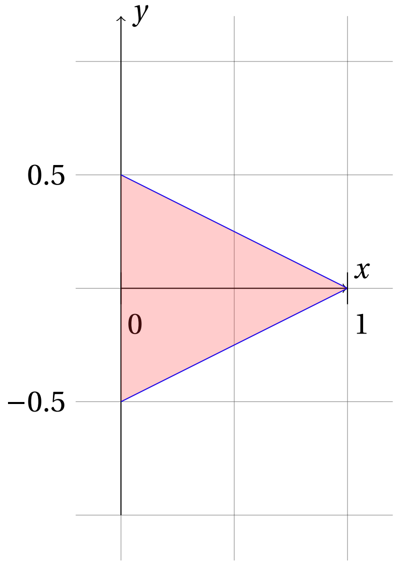
$$\begin{aligned}
f_{X|Y}(x|y) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\
&= \frac{48xy}{24y(1-2y)} = \frac{2x}{1-2y} \quad \text{for } y \le x \le 1-y, \text{ and } 0 \le y < \frac{1}{2}
\end{aligned}$$

and $f_{X|Y}(x|y) = 0$ otherwise.

This is a valid density function since $f(X|Y)(x|y) \ge 0$, and

$$\begin{aligned}
\int_y^{1-y} f_{X|Y}(x|y)\,dx &= \frac{2}{1-2y}\left[\frac{1}{2}x^2\right]_y^{1-y} \\
&= 1
\end{aligned}$$

s.7.3.4. a. The joint PDF is nonzero above the red-shaded area in the following graph. (draw, draw, draw!)



For the PDF for Y, using the graph above,

$$\begin{aligned}f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx \\ &= \int_0^{1-2|y|} 2\,dx \\ &= 2(1-2|y|) \quad \text{for } -1/2 < y < 1/2,\end{aligned}$$

and 0 otherwise. Since $-1/2 < y < 1/2$, we have $f_Y(y) \geq 0$. Also,

$$\begin{aligned}\int_{-\infty}^{\infty} f_Y(y)\,dy &= \int_{-1/2}^{1/2} 2(1-2|y|)\,dy \\ &= 2\int_0^{1/2}(1-2y)\,dy + 2\int_{-1/2}^{0}(1+2y)\,dy \\ &= 2\left(\frac{1}{2}-\frac{1}{4}+\frac{1}{2}-\frac{1}{4}\right) = 1.\end{aligned}$$

Probability density function for X.

$$\begin{aligned}f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}\,dy \\ &= \int_{-\frac{1}{2}(1-x)}^{\frac{1}{2}(1-x)} 2\,dy \\ &= 2(1-x) \quad \text{for } 0 \leq x < 1,\end{aligned}$$

and 0 otherwise. We have $f_X(x) \geq 0$, and also

$$\begin{aligned}\int_{-\infty}^{\infty} f_X(x)\,dx &= \int_0^1 2(1-x)\,dx \\ &= 2\left[x - \frac{1}{2}x^2\right]_0^1 \\ &= 2\left(1-\frac{1}{2}\right) = 1.\end{aligned}$$

Since $-1/2 < y < 1/2$, we have $f_Y(y) \geq 0$. Also,

$$\begin{aligned}\int_{-\infty}^{\infty} f_Y(y)dy &= \int_{-1/2}^{1/2} 2(1-2|y|)dy \\ &= 2\int_0^{1/2}(1-2y)dy + 2\int_{-1/2}^{0}(1+2y)dy \\ &= 2\left(\frac{1}{2}-\frac{1}{4}+\frac{1}{2}-\frac{1}{4}\right) = 1.\end{aligned}$$

Probability density function for X.

$$\begin{aligned}f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}\,dy \\ &= \int_{-\frac{1}{2}(1+x)}^{\frac{1}{2}(1+x)} 2dy \\ &= 2(1+x) \quad \text{for } -1 < x \leq 0,\end{aligned}$$

and 0 otherwise. We have $f_X(x) \geq 0$, and also

$$\begin{aligned}\int_{-\infty}^{\infty} f_X(x)dx &= \int_{-1}^{0} 2(1+x)dx \\ &= 2\left[x+\frac{1}{2}x^2\right]_{-1}^{0} \\ &= 2\left(1-\frac{1}{2}\right) = 1.\end{aligned}$$

b. Using the definition of a conditional probability density function, we have

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{1}{1-2|y|} \quad \text{for } -\frac{1}{2} < y < \frac{1}{2}, 0 < x < 1-2|y|$$

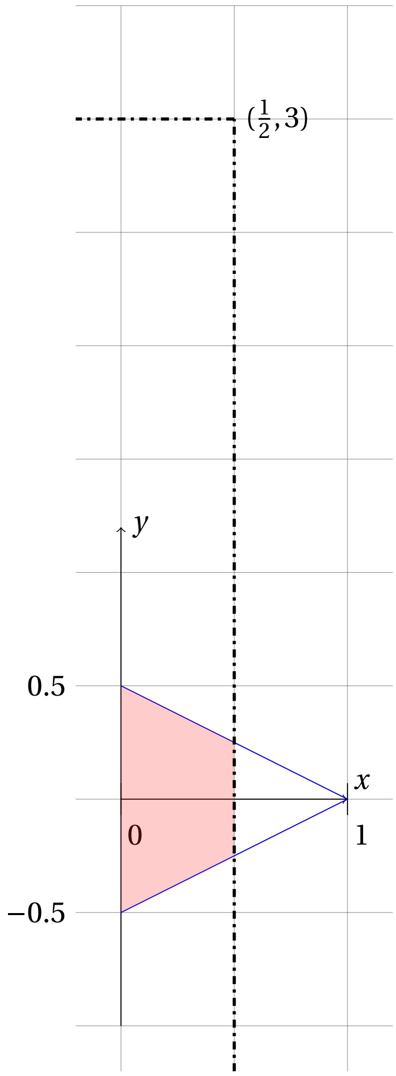
and 0 otherwise.

The required expected value is (bounds are crucial!)

$$\mathrm{E}[X|Y=y] = \int_0^{1-2|y|} \frac{x}{1-2|y|}dx = \frac{1}{2}(1-2|y|).$$

If $y = -1/2$ or $y = 1/2$, we find $\mathrm{E}[X|Y=y] = 0$, which makes sense based on the figure above. Similarly, if $y = 0$, we find $\mathrm{E}[X|Y=y] = 1/2$ since the density of x is uniform over $[0,1]$.

The point $(\frac{1}{2}, 3)$ is located as in the picture below. We need to integrate $f_{X,Y}(x,y)$ over the entire area on the lower left side of this point. It is crucial to realize that the joint PDF is only nonzero in the red shaded area. Moreover, it's very helpful that the PDF has a constant value of 2 above this area. So we just need to calculate the the area of the red shape in the following graph and multiply this by 2.



The dash dotted line intersects the line $y = \frac{1}{2}(1-x)$ at $x = \frac{1}{2}$, so $y = \frac{1}{4}$. The line intersects $y = -\frac{1}{2}(1-x)$ at $x = \frac{1}{2}$, so $y = -\frac{1}{4}$. The whole area of the triangle is $\frac{1}{2}$. The area *not* in red is $\frac{1}{2} \cdot (\frac{1}{4} - (-\frac{1}{4})) \cdot (1 - \frac{1}{2}) = \frac{1}{8}$. So

$$F_{X,Y}\left(\frac{1}{2}, 3\right) = 2\left(\frac{1}{2} - \frac{1}{8}\right) = \frac{3}{4}.$$

s.7.3.5. a. $f_{X,Y}(x, y)$ is a joint probability density function if

1. If $f_{X,Y}(x, y)$ satisfies

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x, y)dxdy = 1$$

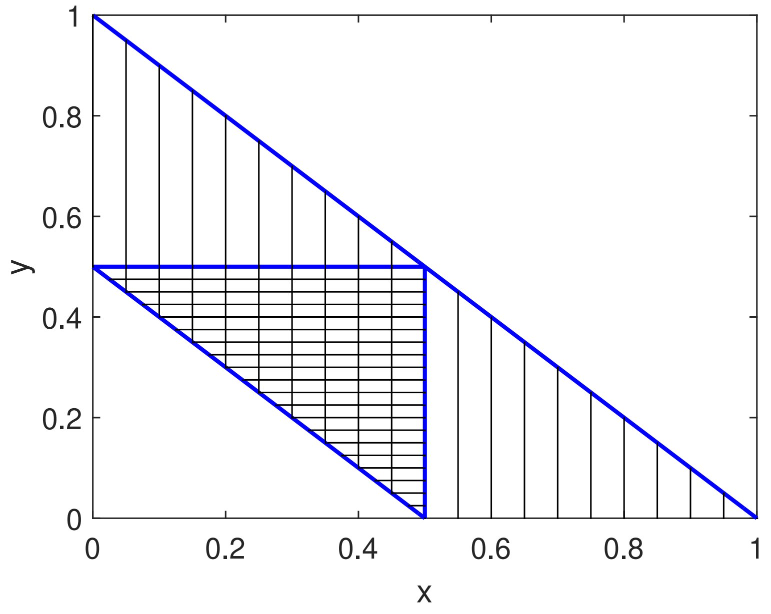
2. $f_{X,Y}(x, y) \geq 0$ for all x and y.

We have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy\,dx = \int_0^1\int_0^{1-x}\frac{c}{1-x}\,dy\,dx$$
$$= c\int_0^1 \frac{1-x}{1-x}\,dx = 1$$

So to satisfy condition (1), we need to set $c = 1$.
Check Condition (2): $f_{X,Y}(x,y) \geq 0$ for all x, y.
For $0 < x < 1$, $f_{X,Y}(x,y) > 0$. Outside of this interval $f_{X,Y}(x,y) = 0$. So, we have that $f_{X,Y}(x,y) \geq 0$ for all x, y.

b. The following graph is used for questions b and c.



To obtain $P(X+Y>1/2)$, we need to integrate $f_{X,Y}$ over the vertically hatched area. We do this by integrating over the larger triangle defined by $x+y<1$, and then subtract the white triangle in the lower left corner defined by $x+y<\frac{1}{2}$.

$$\begin{aligned}
P\left(X+Y>\frac{1}{2}\right) &= \int_0^1\int_0^{1-x}\frac{1}{1-x}dydx - \int_0^{1/2}\int_0^{1/2-x}\frac{1}{1-x}dydx \\
&= 1-\int_0^{1/2}\frac{1/2-x}{1-x}dx \\
&= 1-\int_0^{1/2}\left(1-\frac{1}{2}\frac{1}{1-x}\right)dx \\
&= 1-\frac{1}{2}+\frac{1}{2}\int_0^{1/2}\frac{1}{1-x}dx \\
&= \frac{1}{2}+\frac{1}{2}\left[-\ln(1-x)\right]_0^{1/2} \\
&= \frac{1}{2}-\frac{1}{2}\ln\left(\frac{1}{2}\right) = 0.8466
\end{aligned}$$

c. Using the definition of conditional probability

$$P\left(X<\frac{1}{2}, Y<\frac{1}{2}\,\middle|\, X+Y>\frac{1}{2}\right) = \frac{P\left(X<\frac{1}{2}, Y<\frac{1}{2}, X+Y>\frac{1}{2}\right)}{P\left(X+Y>\frac{1}{2}\right)}$$

The integral in the numerator is the integral over the horizontally hatched triangle. Easiest is to first integrate over the square $[0,1/2]\times[0,1/2]$ and then subtract the lower white triangle, which we have already calculated in the previous question to be $\frac{1}{2}+\frac{1}{2}\ln\left(\frac{1}{2}\right)$.

$$\begin{aligned}
\int_0^{1/2}\int_0^{1/2}\frac{1}{1-x}dydx &= \frac{1}{2}\int_0^{1/2}\frac{1}{1-x}dx \\
&= \frac{1}{2}\left[-\ln(1-x)\right]_0^{1/2} \\
&= -\frac{1}{2}\ln\left(\frac{1}{2}\right)
\end{aligned}$$

Subtracting the lower triangle from the square, we see that the integral over the horizontally hatched triangle equals

$$-\frac{1}{2}\ln\left(\frac{1}{2}\right)-\frac{1}{2}-\frac{1}{2}\ln\left(\frac{1}{2}\right) = -\frac{1}{2}-\ln\left(\frac{1}{2}\right)$$

We can then calculate

$$P\left(X<\frac{1}{2}, Y<\frac{1}{2}|X+Y>\frac{1}{2}\right) = \frac{-\frac{1}{2}-\ln\left(\frac{1}{2}\right)}{\frac{1}{2}-\frac{1}{2}\ln\left(\frac{1}{2}\right)} \approx 0.23$$

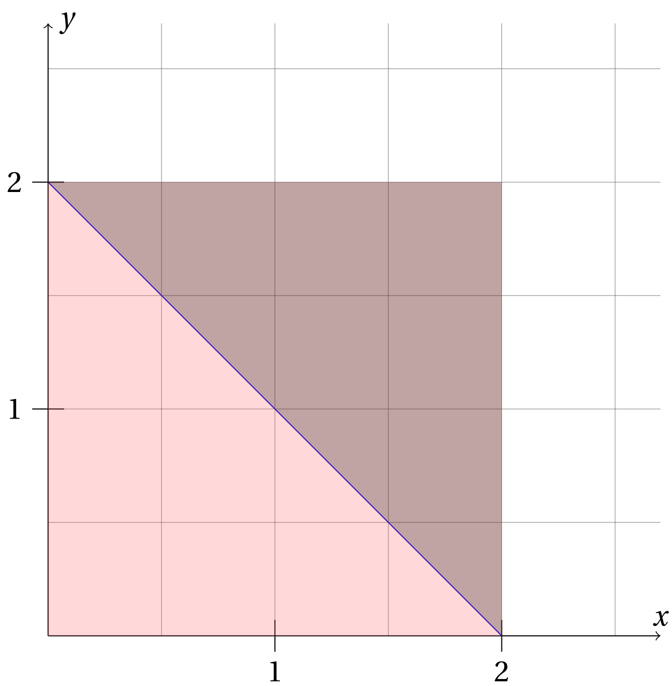
s.7.3.6. a. Since $f_{XY}(x,y)$ is a joint probability density function, we should have 1. $f_{X,Y}(x,y)\geq 0$. This is satisfied since $x,y\geq 0$. 2.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\mathrm{d}x\mathrm{d}y = 1,$$

. We can calculate the integral as follows.

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\mathrm{d}x\mathrm{d}y = 1 &\iff \frac{3}{16}\int_0^c\int_0^c xy^2\mathrm{d}x\mathrm{d}y = 1\\
&\iff \frac{3}{32}\int_0^c \left(x^2y^2\Big|_0^c\right)\mathrm{d}y = 1\\
&\iff \frac{3c^2}{32}\int_0^c y^2\mathrm{d}y = 1\\
&\iff \frac{3c^2}{96}y^3\Big|_0^c = 1\\
&\iff 3c^5 = 96 \iff c = 2
\end{aligned}$$

b. First, draw the area over which the integral is taken.



We want to integrate over the darkest area. Hence, for every value of y, x varies between $2-y$ and 2. Hence, the required probability can be calculated as follows:

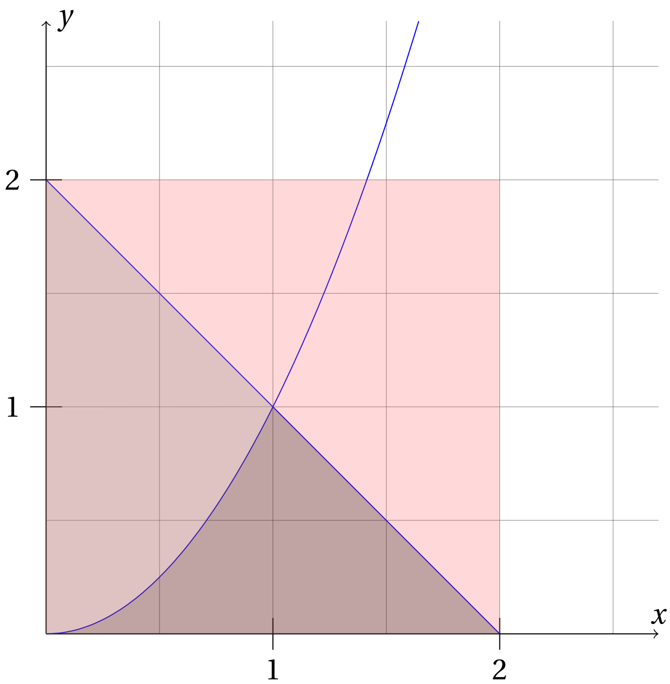
Solution 1

$$
\begin{aligned}
&P(X+Y>2) = \\
&= \int_0^2 \int_{2-y}^2 f_{X,Y}(x,y)\mathrm{d}x\mathrm{d}y \\
&= \frac{3}{16}\int_0^2 \int_{2-y}^2 xy^2\mathrm{d}x\mathrm{d}y \\
&= \frac{3}{32}\int_0^2 \left(x^2y^2\Big|_{2-y}^2\right)\mathrm{d}y \\
&= \frac{3}{32}\int_0^2 (4y^2-(2-y)^2y^2)\mathrm{d}y \\
&= \frac{3}{32}\int_0^2 (4y^3-y^4)\mathrm{d}y \\
&= \frac{3}{32}\left(y^4-\frac{1}{5}y^5\Big|_0^2\right) \\
&= \frac{3}{32}\left(16-\frac{32}{5}\right) \\
&= \frac{3}{2}-\frac{3}{5}=\frac{9}{10}
\end{aligned}
$$

Solution 2

$$
\begin{aligned}
&P(X+Y>2) = \\
&= \int_0^2 \int_{2-x}^2 f_{X,Y}(x,y)\mathrm{d}y\mathrm{d}x \\
&= \frac{3}{16}\int_0^2 \int_{2-x}^2 xy^2\mathrm{d}y\mathrm{d}x \\
&= \frac{3}{16}\int_0^2 \left(\frac{1}{3}xy^3\Big|_{2-x}^2\right)\mathrm{d}x \\
&= \frac{1}{16}\int_0^2 (8x-x(2-x)^3)\mathrm{d}x \\
&= \frac{1}{16}\int_0^2 (x^4+12x^2-6x^3)\mathrm{d}x \\
&= \frac{1}{16}\left(\frac{1}{5}x^5+4x^3-\frac{3}{2}x^4\Big|_0^2\right) \\
&= \frac{1}{16}\left(\frac{32}{5}+32-24\right) \\
&= \frac{2}{5}+\frac{8}{16}=\frac{9}{10}
\end{aligned}
$$

c. First, draw the area over which the integral is taken.



The conditional probability is given by

$$P(Y<X^2|X+Y<2) = \frac{P(X+Y<2 \cap Y<X^2)}{P(X+Y<2)}.$$

From part (b) we have $P(X+Y<2) = 1-P(X+Y>2) = \frac{1}{10}$, which means the probability of falling into one of the two darkest areas equals $\frac{1}{10}$. The probability $P(X+Y<2\cap Y<X^2)$ is given by the integral over the darkest area in the plot.

Solution 1

$$\begin{aligned}
P(X+Y<2\cap Y<X^2) &= \int_0^1\int_0^{x^2} f_{X,Y}(x,y)\mathrm{d}y\mathrm{d}x + \int_1^2\int_0^{2-x} f_{X,Y}(x,y)\mathrm{d}y\mathrm{d}x\\
&= \frac{3}{16}\left[\int_0^1\int_0^{x^2} xy^2\mathrm{d}y\mathrm{d}x + \int_1^2\int_0^{2-x} xy^2\mathrm{d}y\mathrm{d}x\right]\\
&= \frac{1}{16}\left[\int_0^1\left(xy^3\Big|_0^{x^2}\right)\mathrm{d}x + \int_1^2\left(xy^3\Big|_0^{2-x}\right)\mathrm{d}x\right]\\
&= \frac{1}{16}\left[\int_0^1 x^7\mathrm{d}x + \int_1^2 x(2-x)^3\mathrm{d}x\right]\\
&= \frac{1}{16}\left[\int_0^1 x^7\mathrm{d}x + \int_1^2 (-x^4+6x^3-12x^2+8x)\mathrm{d}x\right]\\
&= \frac{1}{128}\left[x^8\Big|_0^1\right] + \frac{1}{16}\left(-\frac{1}{5}x^5+\frac{3}{2}x^4-4x^3+4x^2\right)\Bigg|_1^2\\
&= \frac{1}{128} + \frac{1}{16}\left(-\frac{32}{5}+24-32+16+\frac{1}{5}-\frac{3}{2}+4-4\right) = \frac{17}{640}
\end{aligned}$$

Solution 2 (easier)

$$\begin{aligned}
P(X+Y<2\cap Y<X^2) &= \int_0^1\int_{\sqrt{y}}^{2-y} f_{X,Y}(x,y)\mathrm{d}x\mathrm{d}y\\
&= \frac{3}{16}\left[\int_0^1\int_{\sqrt{y}}^{2-y} xy^2\mathrm{d}x\mathrm{d}y\right]\\
&= \frac{3}{32}\left[\int_0^1\left(x^2y^2\Big|_{\sqrt{y}}^{2-y}\right)\mathrm{d}y\right]\\
&= \frac{3}{32}\left[\int_0^1 (2-y)^2y^2-y^3\mathrm{d}y\right]\\
&= \frac{3}{32}\left[\int_0^1 4y^2-5y^3+y^4\mathrm{d}y\right]\\
&= \frac{3}{32}\left[\frac{4}{3}y^3-\frac{5}{4}y^4+\frac{1}{5}y^5\Big|_0^1\right]\\
&= \frac{3}{32}\left(\frac{4}{3}-\frac{5}{4}+\frac{1}{5}\right) = \frac{17}{640}
\end{aligned}$$

Using either Solution 1 or Solution 2, we get the final answer:

$$P(Y<X^2|X+Y<2) = \frac{P(X+Y<2\cap Y<X^2)}{P(X+Y<2)} = \frac{\frac{17}{640}}{\frac{1}{10}} = \frac{17}{64}.$$

s.7.4.1.

s.7.4.2. Us the hint. For $k = 0, 1, \ldots, n$,

$$\mathsf{P}\{X = k, N = n\} = \mathsf{P}\{X = k, Y = n-k\} = pq^k pq^{n-k} = p^2 q^n.$$

(Have we used independence somewhere?) Now observe that the right hand side does not depend on k. This implies that $\mathsf{P}\{X = k|N = n\}$ also does not depend on k. (Why?) But, since $\mathsf{P}\{X = k|N = n\}$ is a true PMF, is must be that $\sum_{k=0}^{n} \mathsf{P}\{X = k|N = n\}$ adds up to 1. These two ideas put together imply that $\mathsf{P}\{X = k|N = n\} = 1/(n+1)$.

With Bayes' expression

$$\mathsf{P}\{X = k|N = n\} = \frac{\mathsf{P}\{X = k, N = n\}}{\mathsf{P}\{N = n\}},$$

it follows that

$$\mathsf{P}\{N = n\} = \frac{\mathsf{P}\{X = k, N = n\}}{\mathsf{P}\{X = k|N = n\,\mathsf{P}\{N = n\}\}} = (n+1)p^2 q^n.$$

s.7.4.3. Just reasoning as if there is no problem, i.e., applying Bayes' rule in a naive way,

$$\begin{aligned}
F_T(t|x) &= \mathsf{P}\{T \leq t|X = x\} = \mathsf{P}\{X + Y \leq t|X = x\} \\
&= \mathsf{P}\{Y \leq t - x, X = x\} / \mathsf{P}\{X = x\} = \mathsf{P}\{Y \leq t - x\}\, \mathsf{P}\{X = x\} / \mathsf{P}\{X = x\} \\
&= \mathsf{P}\{Y \leq t - x\}, 0 \leq x \leq t.
\end{aligned}$$

where I use that Y and X are independent to split the probability.

The problem with this derivation is that we multiply and divide by 0 ($= \mathsf{P}\{X = x\}$) just as if all is ok. But hopefully, you know that when we multiply and divide by zero, we can get any answer we like. A better way is as follows. Note beforehand that I do not expect that you could have come up with such an answer, but you should definitely study it.

The first step is to realize that PDF $f_{T|X}(t|x) = f_{TX}(t,x)/f_X(x)$ *is* well defined; we don't divide by zero because $f_X(x) > 0$ on $x \geq 0$. By the proof of BH.8.2.1 we see that $f_{TX}(t,x) = f_X(x) f_Y(t-x)\, I_{0 \leq x \leq t}$, where I include the indicator to ensure that we don't run out of the support of X and T. Thus,

$$f_{T|X}(t|x) = \frac{f_{TX}(t,x}{f_X(x)} = f_X(x) f_Y(t-x)\, I_{0 \leq x \leq t} / f_X(x) = f_Y(t-x)\, I_{0 \leq x \leq t}.$$

Now we know that a conditional PDF is a full-fledged PDF. So we can use idea that to *define* the conditional CDF as follows:

$$F_{T|X}(t|x) := \int_0^t f_{T|X}(v|x)\, I_{0 \leq x \leq v}\, \mathrm{d}v = \int_x^t \lambda e^{-\lambda(v-x)}\, \mathrm{d}v = \int_0^{t-x} \lambda e^{-\lambda v}\, \mathrm{d}v = 1 - e^{-\lambda(t-x)}.$$

Isn't it a bit strange that we get the same answer? How to get out of this situation in a technically correct way is one of the hard parts of (mathematical) probability, and certainly not something we can deal with in this course. All books on elementary[1] probability, and lecturers similarly, struggle with this problem; this course is not an exception, nor am I.

[1] When in mathematics someething is elementary, it doesn't necessarily mean that that thing is simple. In fact, it can be very difficult. Elementary means that we just dont use very advanced mathematical concepts.)

b. See part a.

c. By the above,

$$
\begin{aligned}
&f_{X|T}(x|t) = f_{TX}(t,x)/f_T(t), \\
&f_{TX}(t,x) = f_X(x) f_Y(t-x)\, I_{0\le x\le t} \lambda^2 e^{-\lambda x} e^{-\lambda(t-x)}\, I_{0\le x\le t} = \lambda^2 e^{-\lambda t}\, I_{0\le x\le t}, \\
&\qquad \implies f_{X|T}(x|t) \propto \lambda^2 e^{-\lambda t}\, I_{0\le x\le t},
\end{aligned}
$$

where the last follows because $f_T(t)$ is just a normalization constant. Now we use some real nice, but subtle, reasoning to avoid computing f_T by means of marginalizing out x from $f_{T,X}(t,x)$. Observe that $f_{TX}(t,x)$ is constant *as a function of* x on $0 \le x \le t$ (in other words, the RHS does not depend on x on this interval). But $f_{X|T}(x|t)$ is also a real PDF. This implies that the constant $f_T(t)$ (since it does not depend on x) must be such that $f_{X|T}(x|t)$ integrates to 1 on $0 \le x \le t$. The only possibility is that $f_{X|T}(x|t) = t^{-1}\, I_{0\le x\le t}$.

This reasoning gives some offspin. We can conclude that

$$f_T(t) = f_{TX}(t,x)/f_{T|X}(t,x) = \lambda^2 t e^{-\lambda t}.$$

This is more than a nice trick. Recall it, as it is not only used more often in the book, but also in more advanced courses on data science and machine learning.

s.7.4.4. c. Here is the answer. The ideas are important, you'll need them during nearly any course in statistics, given the importance of the normal distribution.

$$f_{Y,Z}(y,z) = \int \frac{1}{2\pi} e^{-(y-x)^2/2} e^{-(z-x)^2/2} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, \mathrm{d}x.$$

It remains to simplify $(y-x)^2 + (z-x)^2 + x^2$. With a bit of work, it follows that this can be written as

$$3(x-(y+z)/3)^2 - (y+z)^2/3 + y^2 + z^2.$$

When plugging this in the integral, the last two terms appear in front of the integral. The term $(y+z)/2$ is just a shift, hence can be neglected in the integration over x. The 3 has to be absorbed in the standard deviation $\sigma = 1/\sqrt{3}$. And therefore,

$$f_{Y,Z}(y,z) = \frac{1}{2\pi}\frac{1}{\sqrt{3}} e^{-y^2/2-z^2/2+(y+z)^2/6}.$$

s.7.4.5.

s.7.4.6. Using the hint: $F(x,y)$ is the area of an (infinite) square lying south west of the point (x,y). Add and subtract such (infinite) squares until the square $[a_1,a_2]\times[b_1,b_2]$ is covered exactly once. Realize that in the process, the square $(-\infty,a_1]\times(-\infty,b_1]$ is subtracted twice.

s.7.4.7. From the hint,

$$\begin{aligned}\mathsf{P}\{Y_1 < cY_2\} &= \int\int I_{x<cy}\lambda_1 e^{-\lambda_1 x}\lambda_2 e^{-\lambda_2 y}\,\mathrm{d}x\,\mathrm{d}y = \lambda_1\lambda_2\int_0^\infty e^{-\lambda_1 x}\int_{x/c}^\infty e^{-\lambda_2 y}\,\mathrm{d}y\,\mathrm{d}x\\ &= \lambda_1\int_0^\infty e^{-\lambda_1 x}e^{-\lambda_2 x/c}\,\mathrm{d}x = \frac{\lambda_1}{\lambda_1+\lambda_2/c}.\end{aligned}$$

Check the result for $c = 0$ and $c = \infty$.

I prefer to use conditioning, like this:

$$\begin{aligned}\mathsf{P}\{Y_1 < cY_2\} &= \int \mathsf{P}\{Y_1 < cY_2 | Y_1 = x\}\,\lambda_1 e^{-\lambda_1 x}\,\mathrm{d}x = \int \mathsf{P}\{Y_2 > x/c | Y_1 = x\}\,\lambda_1 e^{-\lambda_1 x}\,\mathrm{d}x\\ &= \int e^{-\lambda x/c}\lambda_1 e^{-\lambda_1 x}\,\mathrm{d}x,\end{aligned}$$

and the rest goes as before. Actually, I tend to use conditioning as it helps to make the reasoning easier. In this case, suppose that I know that $Y_1 = x$, what can I say about $\mathsf{P}\{Y_2 > cx\}$?

BTW, conditioning does not always make things simpler. When rvs are dependent, then you have to watch out.

s.7.4.8.

s.7.4.9. First check [7.1.28].

In general, I am always very careful with such 'shortcuts' such as $\max\{X, Y\} + \min\{X, Y\} = X + Y$. As a matter of fact, I try to avoid such arguments because it is easy to go wrong. Seemingly plausible arguments are often wrong due to overlooked dependency or non-linearity (effects of higher moments).

It is useful to write $\max\{x, y\} = x\,I_{x\geq y} + y\,I_{y>x}$, and something similar for the minimum. In the present case, $\mathsf{Cov}[X,Y] = \mathsf{E}[XY] - \mathsf{E}[X]\,\mathsf{E}[Y]$, and, similarly, $\mathsf{Cov}[M,L] = \mathsf{E}[ML] - \mathsf{E}[M]\,\mathsf{E}[L]$, where M is max, and L is min. With the above indicators, it is simple to show that $\mathsf{E}[ML] = \mathsf{E}[XY]$:

$$\begin{aligned}ML &= (X\,I_{X\geq Y} + Y\,I_{Y\geq X})((X\,I_{X<Y} + Y\,I_{Y<X})\\ &= XY\,I_{X\geq Y} + XY\,I_{Y<X} = XY\end{aligned}$$

since $I_{X\geq Y}\,I_{X<Y} = 0$.

However, take $X, Y \sim \text{Exp}(\lambda)$. Then, $\mathsf{E}[M] = 3/(2\lambda)$ and $\mathsf{E}[L] = 1/(2\lambda)$, but $\mathsf{E}[X] = \mathsf{E}[Y] = 1/\lambda$.

s.7.4.10. a. In my notation, $X_i = 0 \implies Y_i \neq 0$ and $X_i \neq 0 \implies Y_i = 0$. The reason is that in step i, the drunkard makes a step left or right OR up or down. However, s/he cannot move to the right and up at the same time.

Here is an argument based on recursion. (By now I hope you see that I like this method in particular).

$$\mathsf{E}\left[R_n^2\right] = \mathsf{E}\left[(R_{n-1} + X_n + Y_n)^2\right],$$

but R_{n-1} and $X_n + Y_n$ are independent, and $\mathsf{E}\left[(X_n + Y_n)^2\right] = 1$. Using the recursion, $\mathsf{E}\left[R_n^2\right] = n$.

s.7.4.11. b. Use the hint. Then, if we choose two points at random from the sample, then $(x_i - x_j)(y_i - y_j)$ is the area spanned by these two points. More generally, I have n choices for my first point, and also n choices for the second point (if both points are the same, the area of the rectangle is 0, so we don't have to exclude such choices). Hence, the expected area of the rectangle spanned by the two random points (X, Y) and $(\tilde{X}, \tilde{Y})$ is

$$\frac{1}{n^2}\sum_{i,j}(x_i - x_j)(y_i - y_j).$$

Simplify this to show that

$$2\frac{1}{n}\sum_i x_i y_i - 2\bar{x}\bar{y} = 2r$$

Hence, by part a., the expected area is twice the covariance.

Why is $\mathsf{Cov}[X, a] = 0$ for a a constant? Because the 'area' of rectangles, all with the same y-coordinate, is zero, i.e., they lie on a line.

c. This is the part of the exercise that explains what the above is all about. Since there is a direct relation between covariance and area, we can use geometric arguments to derive (and memorize!) all properties of covariance! Write property i. of covariance as $\mathsf{Cov}[X, Y] = \mathsf{Cov}[Y, X]$. Suppose I flip the x and y-axis, does the area of a rectangle change? For property ii., what happens to the area of rectangle if you stretch the sides? For property iii., realize that this is just a shift of a rectangle that leaves its area invariant. For property iv., what happens to the area if you put an extra rectangle on top or to the right?

BTW, property iii. follows directly from property iv. In iv., take W_3 equal to a constant a_2, in other words $\mathsf{P}\{W_3 = a_2\} = 1$. We know that $\mathsf{Cov}[X, a] = 0$ for a constant a.

Here are some final remarks.

Let's put all the above in a very general frame. The covariance has a number of interesting properties:

1. It is bilinear, that is, the covariance is linear in both arguments. The linearity in the first argument means that $\mathsf{Cov}[X+Y, Z] = \mathsf{Cov}[X, Z] + \mathsf{Cov}[Y, Z]$ and $\mathsf{Cov}[aX, Z] = a\,\mathsf{Cov}[X, Z]$ for $a \in \mathbb{R}$. The linearity in the second argument means that $\mathsf{Cov}[X, Y+Z] = \mathsf{Cov}[X, Y] + \mathsf{Cov}[X, Z]$ and $\mathsf{Cov}[X, aZ] = a\,\mathsf{Cov}[X, Z]$ for $a \in \mathbb{R}$.
2. It is symmetric: $\mathsf{Cov}[X, Y] = \mathsf{Cov}[Y, X]$, from which we define $\mathsf{V}[X] = \mathsf{Cov}[X, X]$.
3. $\mathsf{Cov}[X, a] = 0$ for all $a \in R$.

If you memorize the first two properties of covariance, all the rest follows.

Now we do some geometry. Take three vectors $x, y, z \in \mathbb{R}^2$ (it's easy to generalize to $\mathbb{R}^n$). Then we know that the area $D(x, y)$ of the parallelogram spanned by vectors x and y satisfies the following properties.

1. Area is bilinear. The linearity in the first argument means that $D(x+y, z) = D(x, z) + D(y, z)$ and $D(ax, z) = aD(x, z)$ for $a \in \mathbb{R}$. (Just make a drawing to convince you about this.) The linearity in the second argument means that $D(x, y+z) = D(x, y) + D(x, z)$ and $D(x, az) = aD(x, z)$ for $a \in \mathbb{R}$.

2. $D(x,x) = 0$; there is no area between x and x.

3. $D((1,0),(0,1)) = 1$; the area of the square with side 1 is 1.

In fact, thex first property means that stretching vectors and stacking parallelograms result in stretching and adding areas. The second says that the area of a parallelogram spanned by two parallel vectors is zero. The third specifies that the area of the unit square is 1.

Now it can be proven that there exists just one function D that satisfies these properties. In fact, this is the determinant of the matrix with as columns the vectors that span the parallellogram. Moreover, it can be shown that the second property can be replaced by the skew-symmetric property: $D(x,y) = -D(x,y)$. (Note that $D(x,x) = -D(x,x) \implies 2D(x,x) = 0 \implies D(x,x) = 0$.)

Let us use the properties to compute the area of a parallelogram spanned by the vectors $x = (a,b)$ and $y = (c,d)$ in 2D. Then

$$\begin{aligned} D(x,y) &= D((a,b),(c,d)) = D(a(1,0)) + b(0,1), c(1,0) + d(0,1)) \\ &= adD((1,0),(0,1)) + bcD((0,1),(1,0)) = ad - bc, \end{aligned}$$

where we use bilinearity in the first step, and skew-symmetry in the second and third. And this is indeed the determinant of the matrix with x and y as columns.

So, all in all, this is what I remembered throughout the years: the covariance and the determinant are bi-linear forms, the first is symmetric, the second skew- (or anti-)symmetric.

Finally, I don't see why the areas of the rectangles have to have a sign in this problem. Interestingly, for the determinant, the areas of the parallelograms do have to have a sign to make the concept useful for physics.

s.7.4.12. a. Follows directly from the hint.

Check the hint!

c. If X_i and Y_j are iid, it must be that $w_1 = n/(n+m)$.

b. Can we make some further progress, just by keeping a clear mind? Well, in fact we can by using our insights of part c. If we have $n+m$ iid measurements of which we call n measurements of type X_i, and m of type Y_j, then

$$\mathsf{V}\left[\hat{\theta}_1\right] = \mathsf{E}\left[\left(\frac{1}{n}\sum_i X_i - \theta\right)^2\right] = n^{-2}\,\mathsf{E}\left[\left(\sum_i (X_i - \theta)\right)^2\right] = n^{-2}\,\mathsf{V}\left[\sum_i X_i\right] = \mathsf{V}\left[X_1\right]/n = \sigma^2/n.$$

So, $n = \sigma^2/\mathsf{V}\left[\hat{\theta}_1\right]$, and likewise $m = \sigma^2/\mathsf{V}\left[\hat{\theta}_2\right]$. Finally, plug this into our earlier expression for w_1 to get

$$w_1 = \frac{n}{n+m} = \frac{\sigma^2/\mathsf{V}\left[\hat{\theta}_1\right]}{\sigma^2/\mathsf{V}\left[\hat{\theta}_1\right] + \sigma^2/\mathsf{V}\left[\theta_2\right]} = \frac{\mathsf{V}\left[\hat{\theta}_2\right]}{\mathsf{V}\left[\hat{\theta}_1\right] + \mathsf{V}\left[\theta_2\right]}.$$

If we check our earlier insight, then we see that if $\mathsf{V}\left[Y_j\right] = 0$, then $\mathsf{V}\left[\theta_2\right] = 0$, hence $w_1 = 0$ in that case. This is precisely what we wanted.

Let us finally use the hint of BH to check that the above expression for w_1 is correct.

$$\mathsf{E}\left[(\hat{\theta}-\theta)^2\right] = \mathsf{E}\left[(w_1(\hat{\theta}_1-\theta)+w_2(\hat{\theta}_2-\theta))^2\right] = \mathsf{V}\left[w_1\hat{\theta}_1\right] + \mathsf{V}\left[w_2\hat{\theta}_2\right],$$

by independence. Take the w's out of the variances, then write $w_2 = 1 - w_1$, take ∂_{w_1} of the expression, set the result to 0, and solve for w_1. You'll get the above expression.

s.7.4.13. a. Multinomial.

b. With the hint we end up at $X_1 + X_2 \sim \text{Bin}(n, p^2 + 2p(1-p))$.

c. Here is a short intermezzo on finding a recursion for the sum of a number of Bernouilli rvs. Let S_n be the number of successes in the binomial, and write $g_n(i) = \mathsf{P}\{S_n = i\}$ for this case. Then,

$$\begin{aligned} g_n(i) &= g_{n-1}(i-1)p + g_{n-1}(i)q \\ &= (g_{n-2}(i-2)p + g_{n-2}(i-1)q)p + (g_{n-2}(i-1)p + g_{n-2}(i)q)q \\ &= g_{n-2}(i-2)p^2 + g_{n-2}(i-1)2pq + g_{n-2}(i)q^2. \end{aligned}$$

I also know that $g_n(i) = \binom{n}{i}p^i q^{n-i}$. End of intermezzo.

Now compare the recursion with $f_n(i)$ for the genes tox the expression for the binomial. They are nearly the same, except that in the genes case, the 'n' seems to run twice as fast. I then tried the guess $f_n(i) = \binom{2n}{i}p^i q^{2n-i}$. For you, plug it in, and show that it works.

So, what was my overall approach? I used recursion, but got stuck. Then I used recursion for a simpler case whose solution I know by heart. I compared the recursions for both cases to see whether I could recognize a pattern. This lead me to a guess, which I verified by plugging it in. Using recursion is not guaranteed to work, of course, but often it's worth a try.

Now, looking back, I realize that it is as if individual n adds the outcome of two coin flips (with values in AA, Aa or aa) to the sum S_n of A's. For you to solve: what is the distribution of two coin flips? Next, S_n is just the sum of n individual 'double coin flips'. Hence, what must the distribution of S_n be?

d. It is easiest to work with $f(p) = \log \mathsf{P}\{X_1 = k, X_2 = l, X_3 = m\}$. With part a. this can be written as

$$f(p) = C + (2k + l)\log p + (l + 2m)\log(1-p),$$

where C is a constant (the log of the normalization constant). (BTW, with this you can check your answer for part a.) Compute $\mathrm{d}f(p)/\mathrm{d}p = 0$, because at this p, $\log f$, hence f itself, is maximal. Observe that C drops out of the computation, because when differentiating, it disappears.

e. Now we like to know what p maximizes $\mathsf{P}\{X_3 = n - i\}$. Take $g(q) = \log \mathsf{P}\{X_3 = n - i\}$, then

$$g(q) = C + i\log(1-q^2) + 2(n-i)\log q.$$

(With this, check your answer of part b.) Again, take the derivative (with respect to q), and solve for q.

s.7.4.14. a. It is given that $\mathsf{P}\{T \le t \mid D = 1\} = G(t)$ and $\mathsf{P}\{T \le t \mid D = 0\} = H(t)$. From Theorem 5.3.1.i, we have that we can associate a rv. to a CDF F. Sometimes we say that the CDF F /induces/ a rv. X. So let us use this here to say that G induces the rv. T_1 and H induces T_0. So the /sensitivity/ is $\mathsf{P}\{T_1 > t_0\} = 1 - G(t_0)$ and the /specificity/ is $\mathsf{P}\{T_1 < t_0\} = H(t_0)$.

To make the ROC plot, I first made two plots, one of the sensitivity and the other for 1 minus the specificity, i.e., $1 - H(t_0)$. Then, in the ROC plot, we put a specificity of s on the x-axis, then we search for a t such that $1 - H(t) = s$, and then we plug this t into $1 - G(t)$ to get the sensitivity. To help you understand this better, check that $s = 0 \implies t = b \implies 1 - G(t) = 0$. Moreover, check that $s = 1 \implies t = a \implies 1 - G(t) = 1$. Hence, the ROC curve starts in the origin and stops at the point $(1, 1)$.

With this insight, the area under the ROC curve can be written as

$$\int_0^1 (1 - G(H^{-1}(1-s)))\,\mathrm{d}s = 1 - \int_0^1 G(H^{-1}(1-s))\,\mathrm{d}s = 1 - \int_a^b G(t)h(t)\,\mathrm{d}t,$$

where, in the last step, we use the 1D change of variable $H(t) = 1 - s \implies h(t)\,\mathrm{d}t = -\mathrm{d}s$. It remains to interpret the integral, so let's plug in the definitions:

$$\int_a^b G(t)h(t)\,\mathrm{d}t = \int_a^b \mathsf{P}\{T_1 \le t\} f_{T_0}(t)\,\mathrm{d}t = \int_a^b \mathsf{P}\{T_1 \le T_0 \mid T_0 = t\} f_{T_0}(t)\,\mathrm{d}t = \mathsf{P}\left\{T_{1 \le T_0}\right\}.$$

s.8.1.1. Recall that $\mathsf{V}[X] = \mathsf{E}\left[X^2\right] - (\mathsf{E}[X])^2$; so we have to deal with the function $g(x) = x^2$ because $\mathsf{E}\left[X^2\right] = \mathsf{E}\left[g(X)\right]$. Note that even to properly define the variance, we have to deal with a function that is not one-to-one everywhere on $\mathbb{R}$.

s.8.1.2.

$$X \in \{0, \ldots, 5\} \implies Z \in \{0, 3, 6, 9, 12, 15\}, \quad \text{and not in } \{0, 1, 2, \ldots, 14, 15\}, \tag{13.0.59}$$

$$z = g(x) = 3x, \tag{13.0.60}$$

$$p_Z(z) = \sum_{x : g(x) = z} p_X(x) = \frac{1}{6} I_{z \in \{0,3,6,9,12,15\}}, \tag{13.0.61}$$

$$F_Z(z) = \frac{1}{6} \sum_{x=0}^{z} I_{x \in \{0,3,6,9,12,15\}}. \tag{13.0.62}$$

s.8.1.3. To get the derivative of g^{-1}, consider the equality $g(g^{-1}(y)) = y$. Then, taking derivatives with respect to y at both sides, and applying the chain rule,

$$g(g^{-1}(y)) = y \implies \frac{\mathrm{d}}{\mathrm{d}y} g(g^{-1}(y)) = 1 \iff g'(g^{-1}(y)) \frac{\mathrm{d}}{\mathrm{d}y} g^{-1}(y) = 1 \implies \frac{\mathrm{d}}{\mathrm{d}y} g^{-1}(y) = 1/g'(x),$$

where we use that $g^{-1}(y) = x$. Notice why g is assumed increasing: now we know that $g'(x) \neq 0$.

s.8.1.4. When working the CDFs, we need to solve the problem $\{x : g(x) \le y\}$. If we take $g(x) = \sin x + x/100$ then this is really messy. In fact, to solve this, we first solve for the set $x : g(x) = y$, which might still be hard, but requires less work than check each and every interval.

With PDFs we only have to require *locally* that g is one-to-one, and we don't have to work with inequalities, but can direcly focus on the set $\{x : g(x) = y\}$.

In 2D, functions can have saddle points, i.e., points in which the function increases in one direction and decreases in another. Then finding the set of points x such that[2] $g(x, y) \leq (u, v)$ (which we need if we want to express $\mathsf{P}\left\{g(X, Y) \leq (u, v)\right\}$ in terms of the distribution $F_{X,Y}$) is not a particularly attractive task, to say the least.

See also the inverse function theorem, which will be covered in more detail next block.

s.8.1.5. Note that $g(x) = x^2$ is not monotone increasing, moreover, $g^{-1}(y)$ does not exist (in $\mathbb{R}$) for $y < 0$. We split the line into disjoint intervals in which g is either strictly increasing or decreasing, and then we apply the above rule in each of the intervals. Since $g'(x) = 2x$ and $x = \pm\sqrt{y}$,

$$f_Y(y) = f_X(\sqrt{y})\frac{1}{2\sqrt{y}} + f_X(-\sqrt{y})\frac{1}{2\sqrt{y}}.$$

s.8.1.6. Take $y = g(x) = \lambda x$. Then,

$$f_Y(y) = f_X(x)\frac{\mathrm{d}x}{\mathrm{d}y} = f_X(x)\frac{1}{g'(x)} = e^{-y/\lambda}\frac{1}{\lambda}.$$

With this,

$$\mathsf{E}[Y] = \int_0^\infty y f_Y(y)\,\mathrm{d}y = \int_0^\infty y e^{-y/\lambda}\frac{1}{\lambda}\,\mathrm{d}y.$$

To solve this integral, I recognize y/λ in the exponent, and I want to get rid of the $1/\lambda$ factor. Hence, I write $u = y/\lambda$, and use this to see that

$$u = y/\lambda \implies \mathrm{d}u = \mathrm{d}y/\lambda \implies \mathrm{d}y = \lambda\,\mathrm{d}u.$$

Then, including a and b for the boundaries to show explicitly what is going on when changing the variables

$$\int_a^b y/\lambda e^{-y/\lambda}\,\mathrm{d}y = \int_{a/\lambda}^{b/\lambda} u e^{-u}\lambda\,\mathrm{d}u = \lambda\int_{a/\lambda}^{b/\lambda} u e^{-u}\,\mathrm{d}u.$$

Applying this to our case so that $a = 0/\lambda = 0$ and $b = \infty/\lambda = \infty$,

$$\mathsf{E}[Y] = \lambda\int_0^\infty u e^{-u}\,\mathrm{d}u = \lambda\,\mathsf{E}[X].$$

s.8.1.7. When we have the density f_Y and the function g, then the substitution rule says that,

$$\int_a^b f_Y(g(x))g'(x)\,\mathrm{d}x = \int_{g(a)}^{g(b)} f_Y(y)\,\mathrm{d}y.$$

2

We also want that the transformation from X to Y does not affect the probability of the set (event) $A = [a, b]$, hence,

$$\int_{g(a)}^{g(b)} f_Y(y)\,\mathrm{d}y = \int_a^b f_X(x)\,\mathrm{d}x.$$

Combining the above two equations gives that

$$\int_a^b f_Y(g(x))g'(x)\,\mathrm{d}x = \int_a^b f_X(x)\,\mathrm{d}x.$$

Since this holds for any a and b, it follows that

$$f_Y(g(x))g'(x) = f_X(x).$$

s.8.1.8.

s.8.1.9.

s.8.1.10.

s.8.1.11.

$$X \in [0,5] \implies Z \in [0,15], \tag{13.0.63}$$
$$z = 3x = g(x) \implies x = z/3, \tag{13.0.64}$$
$$f_Z(z) = f_X(x)\frac{\mathrm{d}x}{\mathrm{d}z}, \tag{13.0.65}$$
$$\frac{\mathrm{d}z}{\mathrm{d}x} = 3, \tag{13.0.66}$$
$$f_Z(z) = f_X(z/3)\frac{1}{3}. \tag{13.0.67}$$

$F_Z(u) = 1$ for $u \geq 15$ and $F_Z(u) = 0$ for $u \leq 0$. When $0 \leq u \leq 15$,

$$F_Z(u) = \int_0^u f_X(z/3)\frac{1}{3}\,\mathrm{d}z = \frac{1}{5}\int_0^u I_{0\leq z/3\leq 5}\frac{1}{3}\,\mathrm{d}z \tag{13.0.68}$$
$$= \frac{1}{5}\int_0^u I_{0\leq z\leq 15}\frac{1}{3}\,\mathrm{d}z = \frac{u}{15}. \tag{13.0.69}$$

s.8.1.12.

$$X \in [0,5] \implies Z \in [0,125], \tag{13.0.70}$$
$$z = x^3 = g(x) \implies x = z^{1/3}, \tag{13.0.71}$$
$$f_Z(z) = f_X(x)\frac{\mathrm{d}x}{\mathrm{d}z}, \tag{13.0.72}$$
$$\frac{\mathrm{d}z}{\mathrm{d}x} = 3x^2 = 3z^{2/3}, \tag{13.0.73}$$
$$f_Z(z) = f_X(z^{1/3})\frac{1}{3z^{2/3}}. \tag{13.0.74}$$

When $F_Z(u) = 1$ for $u \ge 125$ and $F_Z(u) = 0$ for $u \le 0$. When $0 \le u \le 125$,

$$\begin{aligned} F_Z(u) &= \int_0^u f_X(z^{1/3}) \frac{1}{3z^{2/3}} \, \mathrm{d}z = \frac{1}{5} \int_0^u I_{0 \le z^{1/3} \le 5} \frac{1}{3z^{2/3}} \, \mathrm{d}z && (13.0.75) \\ &= \frac{1}{5} \int_0^u I_{0 \le z \le 125} \frac{1}{3z^{2/3}} \, \mathrm{d}z && (13.0.76) \\ &= \frac{1}{5} \int_0^u \frac{1}{3z^{2/3}} \, \mathrm{d}z = \frac{1}{5} z^{1/3} \Big|_0^u = u^{1/3}/5. && (13.0.77) \end{aligned}$$

s.8.1.13.

$$\begin{aligned} z = g(x) = (x - \mu)/\sigma, &\implies x = \sigma z + \mu && (13.0.78) \\ f_Z(z) &= f_X(x) \frac{\mathrm{d}x}{\mathrm{d}z}, && (13.0.79) \\ \frac{\mathrm{d}z}{\mathrm{d}x} &= \frac{1}{\sigma}, && (13.0.80) \\ f_Z(z) &= f_X(x)\sigma = \sigma f_X(\sigma z + \mu) && (13.0.81) \end{aligned}$$

and now using the density of $X \sim \mathrm{Norm}(\mu, \sigma)$,

$$f_Z(z) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(\sigma z + \mu - \mu)^2/2\sigma^2} \sigma = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}. \qquad (13.0.82)$$

s.8.1.14.

$$\begin{aligned} z = g(x) = e^{-x} &\implies x = -\log z, && (13.0.83) \\ x \in (0, \infty) &\implies z \in (0, 1), && (13.0.84) \\ f_Z(z) &= f_X(x) \frac{\mathrm{d}x}{\mathrm{d}z}, && (13.0.85) \\ \frac{\mathrm{d}z}{\mathrm{d}x} &= -e^{-x}, \quad \text{Don't forget to take the abs value next,} && (13.0.86) \\ f_Z(z) &= f_X(x) e^x = e^{-x} e^x = 1\, I_{0<z<1}, && (13.0.87) \end{aligned}$$

where we include the domain of Z in the last equality.

s.8.1.15.

$$\begin{aligned} (u, v) = (x + y, x - y) = g(x, y) &\implies (x, y) = ((u + v)/2, (u - v)/2), && (13.0.88) \\ \frac{\partial(u, v)}{\partial(x, y)} &= \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \implies |-2| = 2, && (13.0.89) \\ f_{U,V}(u, v) &= f_{X,Y}(x, y) \frac{\partial(x, y)}{\partial(u, v)} = f_{X,Y}((u + v)/2, (u - v)/2)/2 && (13.0.90) \\ &= \frac{1}{4\pi} e^{-((u+v)/2)^2/2} e^{-((u-v)/2)^2/2} && (13.0.91) \\ &= \frac{1}{4\pi} e^{-u^2/4 - v^2/4}, && (13.0.92) \end{aligned}$$

where we work out the squares and simplify. Hence, U and V are independent and normally distributed with mean 0 and $\sigma = \sqrt{2}$. This is in line with our earlier definition of a multi-variate normal distribution.

s.8.1.16. 1. $Z = Y^4 \in [0, \infty)$ for $Y \in (-\infty, \infty)$;

2. $Y = X^3 + a \in (a, a+1)$ for $X \in (0,1)$;
3. $U = |V| + b \in [b, \infty)$ for $V \in (-\infty, \infty)$;
4. $Y = e^{X^3} \in (0, \infty)$ for $X \in (-\infty, \infty)$;
5. $V = U\, I_{U \le c} \in (-\infty, c]$ for $U \in (-\infty, \infty)$;
6. $Y = \sin(X) \in [-1, 1]$ for $X \in (-\infty, \infty)$;
7. $Y = \frac{X_1}{X_1 + X_2} \in (0,1)$ for $X_1 \in (0, \infty)$ and $X_2 \in (0, \infty)$;
8. $Z = \log(UV) \in (-\infty, \infty)$ for $U \in (0, \infty)$ and $V \in (0, \infty)$.

s.8.1.17. When the variables become dependent, the Jacobian becomes zero. For instance, in the latter case,

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1/y & -x/y^2 \\ -y/x^2 & 1/x \end{vmatrix} = \frac{1}{xy} - \frac{x}{y^2}\frac{y}{x^2} = 0. \tag{13.0.93}$$

Moreover, the function g is not locally one-to-one.

s.8.1.18. If we would not add this extra variable, we cannot use the change of variables theorem. We also need a function to deal with the scaling. In the change of variables theorem, this is the Jacobian.

There is also another problem. Consider the function $g(x, y)$ that maps $\mathbb{R}^2$ to $\mathbb{R}$. The inverse set $\{(x, y) : g(x, y) = z\}$ can be quite complicated, while the set $\{y : g(x, y) = z\}$ for a fixed x is hopefully just one point. Hence, the mapping $(x, y) \to (x, g(x, y))$ is, at least locally, one-to-one.

It is possible to deal with the more general problem, but this requires much more theory than we need for this course.

s.8.1.19. From BH.8.1.4: Z chi-square $\implies X = \sqrt{Z} \sim \text{Norm}(0,1)$. Then, from BH.8.1.9,

$$X^2 + Y^2 = (\sqrt{2T}\cos U)^2 + (\sqrt{2T}\sin U)^2 = 2T\left(\cos^2 U + \sin^2 U\right) = 2T \sim \text{Exp}(1/2), \tag{13.0.94}$$

when $X, Y \sim \text{Norm}(0,1)$.

s.8.1.20. Take $g(x, y) = (x, w) = (x, (x + y)/2)$. Then, $y = 2w - x$.

$$\frac{\partial(x,w)}{\partial(x,y)} = \begin{vmatrix} 1 & 0 \\ 1/2 & 1/2 \end{vmatrix} = 1/2, \tag{13.0.95}$$

$$f_{X,W}(x,w) = f_{X,Y}(x,y)\frac{\partial(x,y)}{\partial(x,w)} = \frac{1}{\pi(1+x^2)}\frac{1}{\pi(1+(2w-x)^2)}2, \tag{13.0.96}$$

$$f_W(w) = \int_{-\infty}^{\infty} f_{X,W}(x,w)\,\mathrm{d}x = \frac{2}{\pi^2}\int_{-\infty}^{\infty}\frac{1}{1+x^2}\frac{1}{1+(2w-x)^2}\,\mathrm{d}x. \tag{13.0.97}$$

The expectation of a Cauchy distributed r.v. X is not well-defined because $\mathsf{E}[|X|] = \infty$. As a consequence, taking the average of some outcomes (i.e. a sample) will also not give a sensible answer.

s.8.1.21.

s.8.1.22.

s.8.1.23.

s.8.1.24.

s.8.1.25.

s.8.1.26.

s.8.1.27.

s.8.1.28.

s.8.1.29. Incorrect: The support of T is $(0,2)$ whereas the support of any beta distribution is $(0,1)$. Hence, T does not have a beta distribution for some a, b.
Also see page 378 of the book for the distribution of the sum of two uniform distributions. This might help your intuition for this solution.

s.8.1.30. We use that the PDF integrates to 1:

$$1 = \int_0^1 \frac{1}{\beta(1,b)}(1-x)^{b-1}\,\mathrm{d}x = \frac{1}{\beta(1,b)}\left[-\frac{1}{b}(1-x)^b\right]_0^1 = \frac{1}{\beta(1,b)b}.$$

Hence, $\beta(1,b) = \frac{1}{b}$.

s.8.1.31. The scaling factor $\beta(a,b)$ is a positive constant, so we may as well leave it out and maximize $x^{a-1}(1-x)^{b-1}$. Note that its derivative (to x) is given by

$$\begin{aligned}\frac{d}{dx}x^{a-1}(1-x)^{b-1} &= ((a-1)(1-x)-(b-1)x)x^{a-2}(1-x)^{b-2}\\ &= ((a-1)-(a+b-2)x)x^{a-2}(1-x)^{b-2}.\end{aligned}$$

Setting this to zero yields $x = \frac{a-1}{a+b-2}$ as the only candidate for an interior optimum. Since $a, b > 1$, we have $0 < x < 1$. If $a, b > 1$, then the PDF converges to 0 as $x \to 0$ or $x \to 1$, so then we conclude that $x = \frac{a-1}{a+b-2}$ indeed yields a maximum. (Think about this last sentence; most students do not use the information that $a, b > 1$ correctly.)

s.8.1.32. A prior is a distribution reflecting one's information or belief about a parameter before updating it with information.

It is harder than you might think, hardly any student gives a completely satisfactory answer here. Compare your solution to the definition above. If they are different, try to understand how exactly your solution was different and determine which definition is better.

A conjugate prior is a prior distribution such that the posterior distribution is in the same family of distributions.

s.8.1.33. Dirichlet distribution. The Beta distribution is a special case of the Dirichlet distribution, because binomial is a special case of multinomial. Of course, this can also be shown directly using the formula.

s.8.1.34. The prior is $p \sim \text{Beta}(1,1)$. The posterior is $p|X=k \sim \text{Beta}(1+k, 1+n-k)$.

s.8.1.35. Let X denote the number of heads.

1. Your posterior is $p|X=900 \sim \text{Beta}(910,110)$.
2. Your friend's posterior is $p|X=900 \sim \text{Beta}(901,101)$.
3. The mean of your posterior is $\frac{910}{910+110} = \frac{91}{102} \approx 0.892$; the mean of your friend's posterior is $\frac{901}{901+101} = \frac{901}{1002} \approx 0.899$. The difference is small, so the effect of the prior distribution is small if you have a lot of data. This effect is known as *washing out the prior*.

s.8.1.36.

s.8.1.37.

s.8.1.38. This states that the PMF of the Beta-Binomial distribution,

$$P(X=k) = \binom{n}{k} \frac{\beta(a+k, b+n-k)}{\beta(a,b)},$$

sums to 1. To see this, we have to rewrite the beta functions in terms of binomial coefficients:

$$\begin{aligned}\frac{1}{\beta(a,b)} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} = \frac{(a+b-1)!}{(a-1)!(b-1)!} &= (a+b-1)\binom{a+b-2}{a-1},\\ \frac{1}{\beta(a+k,b+n-k)} &= (a+b+n-1)\binom{a+b+n-2}{a+k-1}.\end{aligned}$$

Plugging this in gives the result.

s.8.1.39. $\mathsf{V}[X] = n/\lambda^2$, $\mathsf{E}[X] = n/\lambda$, $\text{SCV} = 1/n$.

s.8.1.40.

s.8.1.41. We fill in $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ to find

$$f_Y(y) = \varphi\left(\sqrt{y}\right) y^{-1/2} = \frac{1}{\sqrt{2\pi}} y^{-1/2} e^{-(\sqrt{y})^2/2} = \frac{1}{\sqrt{2\pi}} y^{-1/2} e^{-y/2},$$

so $a = \frac{1}{2}$ and $\lambda = \frac{1}{2}$.

s.8.1.42. Incorrect: the scale parameters λ need to be the same *and* both random variables need to be independent.

s.8.1.43. The base case is $n = 1$. We have $\Gamma(1) = \int_0^\infty e^{-x}\,\mathrm{d}x = 1 = 0!$, so the statement holds for $n = 1$. Now let $k \in \mathbb{N}$ be arbitrary and assume that the statement holds for $n = k$, i.e. that $\Gamma(k) = (k-1)!$. Then

$$\Gamma(k+1) = k\Gamma(k) = k(k-1)! = k! = ((k+1)-1)!, \tag{13.0.98}$$

so the statement also holds for $n = k+1$. By mathematical induction, we conclude that $\Gamma(n) = (n-1)!$ for all positive integers n.

s.8.1.44. Incorrect: It is the other way around, the Gamma distribution is the conjugate prior of the Poisson distribution. This statement doesn't make much sense, for example one would need to say for which parameter of the Gamma distribution it is the prior. In addition, the parameters of the Gamma distribution can be any positive real number, so the conjugate prior of (either parameter) of the Gamma distribution is a continuous distribution, so in particular not the Poisson distribution.

s.8.1.45. $X + Y \sim \text{Gamm}(11, 2)$ and $\frac{X}{X+Y} \sim \text{Beta}(4, 7)$.

s.8.1.46.

1. Minimum
2. Maximum
3. Median

s.8.2.1. From the hint, we first focus on a set $\{V \le 0\} = \{1/T \le 0\}$. Now, $1/T \le 0 \iff T \le 0$. And therefore $\mathsf{P}\{V \le 0\} = \mathsf{P}\{T \le 0\} = F_T(0)$.

If $v < 0$, then $1/T \le v \le 0 \iff 1/v \le T \le 0$. Therefore $F_V(v) = F_T(0) - F_T(1/v)$.

If $v > 0$, then $1/T \le v$ when $T < 0$ or $T \ge 1/v$. Hence, $F_V(v) = F_T(0) + 1 - F_T(1/v)$.

s.8.2.2. a. I remember this: $f_{X,Y}(x, y)\,\mathrm{d}x\,\mathrm{d}y = f_{R,\Theta}(r, \theta)\,\mathrm{d}r\,\mathrm{d}\theta$. From this,

$$f_{R,\Theta}(r,\theta) = f_{X,Y}(x, y)\left|\frac{\partial(x, y)}{\partial(r,\theta)}\right|.$$

Now, since $x = r\cos\theta$ and $y = r\sin\theta$,

$$\frac{\partial(x, y)}{\partial(r,\theta)} = \begin{pmatrix}\cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta\end{pmatrix},$$

which has determinant equal to r. It is given that $f_{X,Y}(x, y) = g(x^2 + y^2) = g(r^2)$. Hence,

$$f_{R,\Theta}(r,\theta) = f_{X,Y}(x, y)r = g(r^2)r,$$

with $r \ge 0, \theta \in [0, 2\pi$. The RHS does not depend on θ. Hence, $f_\Theta(\theta)$ must be a constant.

b. Use the hint. Since g is a constant, $f_{R,\Theta}(r,\theta) \propto r$. Thus,

$$\int_0^1 \int_0^{2\pi} r\,\mathrm{d}r\,\mathrm{d}\theta = 2\pi(1/2)r^2|_0^1 = \pi.$$

So, $1/\pi$ is the normalization constant.

c. $f_{X,Y}(x, y) = \exp - x^2/\sqrt{2\pi}\exp - y^2/\sqrt{2\pi} = \exp -(x^2 + y^2)/2\pi = \exp - r^2/2\pi$. Indeed, $f_{X,Y}(x, y)$ has the form $g(x^2 + y^2)$. The rest is as in part b.

s.8.2.3. I always start with this line: $f_{T,U}(t,u)\,\mathrm{d}t\,\mathrm{d}u = f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y$. Then,

$$\frac{\partial(t,u)}{\partial(x,y)} = \begin{pmatrix} 1/y & -x/y^2 \\ 1 & 0 \end{pmatrix} = x/y^2.$$

We don't need to take absolute signs in the last expression because X, Y are positive rvs. Next, $x = u$, $y = u/t$. With this,

$$f_{T,U}(t,u) = f_{X,Y}(x,y)\left(\frac{\partial(t,u)}{\partial(x,y)}\right)^{-1} = f_{X,Y}(u,u/t)y^2/x = f_X(u)f_Y(u/t)u/t^2.$$

b. Use part a.

$$f_T = \frac{1}{t^2}\int_0^\infty x f_X(x) f_Y(x/t)\,\mathrm{d}x.$$

Since f_X and f_Y are not given explicitly, we cannot make further progress.

All and all, division of rvs is not so simple.

s.8.2.4. a.

$$\begin{aligned} f_{X,T}(x,t) &= f_{X,Y}(x,y)|\frac{\partial(x,y)}{\partial(x,t)}|, \\ \frac{\partial(x,t)}{\partial(x,y)} &= \begin{pmatrix} 1 & 0 \\ y & x \end{pmatrix} = x. \\ &\implies \\ f_{X,T}(x,t) &= f_{X,Y}(x,y)/x = f_{X,Y}(x,t/x)/x, \end{aligned}$$

since $y = t/x$. Finally, for f_T, marginalize x out by integration.

b. Just do the algebra. With part a. you have the answer, so you can check.

s.8.2.6. a. I did not attempt any smart tricks. Take as transform $u = s/t$ and $v = s+t$, where I associate s to T_1 and t to T_2. Then,

$$\frac{\partial(u,v)}{\partial(s,t)} = \begin{pmatrix} 1/t & -s/t^2 \\ 1 & 1 \end{pmatrix} = \frac{1}{t} + \frac{s}{t^2} = \frac{t+s}{t^2} = \frac{v}{t^2}$$

With a bit of algebra: $s = uv/(u+1)$ and $t = v/(u+1)$. Therefore the Jacobian becomes equal to $(u+1)^2/v$. Next,

$$\begin{aligned} f_{U,V}(u,v) &= f_{T_1,T_2}(s,t)\frac{\partial(s,t)}{\partial(u,v)} = f_{T_1}(uv/(u+1)f_{T_2}(v/(u+1))\frac{(u+1)^2}{v} \\ &= \lambda^2 \exp(-\lambda uv/(u+1) - \lambda v/(u+1))\frac{(u+1)^2}{v} \\ &= \lambda^2 \exp(-\lambda v)\frac{(u+1)^2}{v}. \end{aligned}$$

This factors into one term with only v and another with only u. Hence, U and V are independent.

b. With the hint,

$$\begin{aligned}\mathsf{P}\{T_1 < T_2\} &= \int_0^\infty \mathsf{P}\{T_1 < T_2 \,|\, T_1 = s\} f_{T_1}(s)\,\mathrm{d}s \\ &= \int_0^\infty \mathsf{P}\{s < T_2 \,|\, T_1 = s\} \lambda_1 e^{-\lambda_1 s}\,\mathrm{d}s \\ &= \lambda_1 \int_0^\infty e^{-\lambda_2 s} e^{-\lambda_1 s}\,\mathrm{d}s = \frac{\lambda_1}{\lambda_1+\lambda_2}.\end{aligned}$$

c. See the hint. Alice first has to wait for the first server to become free. The expected time in queue is $1/(\lambda_1+\lambda_2)$. If server 1 is the first, then Alice spends a time $1/\lambda_1$ in service. Thus, the total time is

$$\frac{1}{\lambda_1+\lambda_2} + \frac{\lambda_1}{\lambda_1+\lambda_2}\frac{1}{\lambda_1} + \frac{\lambda_2}{\lambda_1+\lambda_2}\frac{1}{\lambda_2} = \frac{3}{\lambda_1+\lambda_2}.$$

s.8.2.7. Let us solve the question from first principles. At the end, I'll give the short solution based on Beta-Binomial conjugacy.

Let $f(p)$ be our prior density (In the exercise it is taken to be uniform). Then

$$\mathsf{P}\left\{p > r\right\} = \int I_{p>r} f(p)\,\mathrm{d}p = \int_r^1 f(p)\,\mathrm{d}p$$

is our belief that $p > r$. For this exercise, we are interested in the relation $\mathsf{P}\left\{p > r\right\} \geq c$. For instance, suppose we take $c = 0.95$, then we like to know which value for r achieves that $\mathsf{P}\left\{p > r\right\} \geq c$?

We can start with one trial, i.e., $n = 1$. Then we analyze the case for $n = 2$, and so on, and hope to see a pattern. Here are the standard steps of Bayesian reasoning.

1. I want to know the density $f_1(p|N = 1)$, i.e, the density of p after having seen one successful test. (Note here that I am careful about notation. We do $n = 1$ trials, and then the number of successes is given by the random variable N.)

2. Now I use Bayes' rule:

$$f_1(p|N = 1) = \frac{f_1(p, N = 1)}{\mathsf{P}\{N = 1\}} = \frac{f_1(N = 1|p)}{\mathsf{P}\{N = 1\}} f(p).$$

   Here $f(p)$ acts as the prior density on p.

3. It is clear that $f_1(N = 1|p) = p$, because we know that an item passes a test with probability p, when p is given.

4. Perhaps I don't need $\mathsf{P}\{N = 1\}$ if I can guess it (though see below), but here it is just for completeness' sake.

$$\mathsf{P}\{N = 1\} = \int f(N = 1|p) f(p)\,\mathrm{d}p = \int_0^1 p\,\mathrm{d}p = 1/2,$$

   because the prior $f(p) = I_{p\in[0,1]}$, i.e., uniform on $[0,1]$, i.e, it is Beta(1, 1).

5. With this, $f_1(p|N=1) = \frac{p}{1/2} I_{p\in[0,1]} = 2p\, I_{p\in[0,1]}$.

6. Thus, $\mathsf{P}\{p > r \,|\, N = 1\} = \int_0^1 I_{p>r} f_1(p|N=1)\,\mathrm{d}p = \int_r^1 2p\,\mathrm{d}p = 1 - r^2$.

Sometimes we are lucky and we don't have to compute the denominator in Bayes' formula. We did this earlier, but let's show again how this works.

$$f_1(p|N=1) = \frac{f_1(p, N=1)}{\mathsf{P}\{N=1\}} \propto f_1(N=1|p) f(p) = p\, I_{0\leq p\leq 1}.$$

Now $f_1(p|N=1)$ is a PDF, hence must integrate to 1. Thus, $\int_0^1 p\,\mathrm{d}p = 1/2$, must be the normalization constant by which we have to divide to turn f_1 into a real PDf. In this case we don't save any work, but sometimes this really helps, in particular when dealing with integrals with Beta distributed rvs.

Now generalize to larger n, compute $f_2(p|N=2)$, then for $n=3$, and so on, until you see the pattern.

We can also directly use the ideas of the book. Starting with a prior Beta(1,1), after n 'wins', the distribution becomes Beta$(1+n, 1)$. Then,

$$\mathsf{P}\{p > r\} = \frac{\Gamma(n+2)}{\Gamma(n+1)\Gamma(1)} \int_r^1 p^n\,\mathrm{d}p = 1 - r^{n+1}. = (n+1)p^{n+1}|_r^1 = 1 - r^{n+1}.$$

s.8.2.8. a. By the hint and the fact that U_j is uniform on $[0,1]$, so that $1-U_j$ is also uniform, the last equality of the hint implies that $\mathsf{P}\{1-U_j \leq 1-e^{-c}\} = \mathsf{P}\{U_j \leq 1-e^{-c}\} = 1-e^{-c}$. But then, $X_j \sim \text{Exp}(1)$.

b. The sum of n iid exponentials is Gamma(n,λ). And so, if $S_n = \sum_i^n X_i$, then $\mathsf{P}\{S_n \leq x\} = \int_0^x f(y)\,\mathrm{d}y$, with $f(y)$ the gamma density with n and $\lambda = 1$.

Just to test my skills, I used MGFs, because I know that the MGF of a sum of iid rvs is the product of the MGF of one them. Since $e^{\log u} = u$,

$$\mathsf{E}\left[e^{-s\log U}\right] = \int_0^1 e^{-s\log u}\,\mathrm{d}u = \int_0^1 u^{-s}\,\mathrm{d}u.$$

If $s \geq 1$ this does not converge (convince yourself that you understand this). With $s < 1$,

$$\mathsf{E}\left[e^{-s\log U}\right] = \frac{1}{-s+1} u^{-s+1}|_0^1 = \frac{1}{1-s}.$$

Therefore,

$$\begin{aligned}\mathsf{E}\left[e^{-sS_n}\right] &= \mathsf{E}\left[e^{-s\log U_1 - s\log U_2 - \cdots - s\log U_n}\right] = \left(\mathsf{E}\left[e^{-s\log U}\right]\right)^n \\ &= \left(\frac{1}{1-s}\right)^n,\end{aligned}$$

and this is the MGF of a Gamma$(n, \lambda = 1)$ rvs.

s.8.2.9.

$$M_Y(s) = \mathsf{E}\left[\exp sY\right] = p\sum_{i=0}^{\infty} e^{spi} q^i = p/(1-qe^{sp}).$$

Now, use that $e^{sp)} \approx 1 + sp$ for $p \ll 1$. (This is easier than using l' Hopital's rule as BH do in their solution). Hence, the denominator becomes $\approx 1-(1-p)(1+sp) = p(1-s) - sp^2 \approx p(1-s)$ when $p \ll 1$ Hence,

$$M_Y(s) \approx p/(p(1-s) = 1/(1-s).$$

In the limit $p \to 0$ the LHS converges to the RHS, which is the MGF of an exponential rv. For the rest, follow the solution of BH.

Here is another line of attack. Let us first use probability theory to find out what is $\sum_{i=0}^{\infty} q^1$ for some $|q| < 1$. Take $X \sim \mathrm{Geo}(p)$, so that X corresponds to the number of failures (tails say) until we see a success (heads say). So, X corresponds to the number of tails until we see a heads. Now if we keep on throwing, then we know that eventually a heads will appear. Therefore $p + pq + pq^2 + \cdots = 1$, that, is $p\sum_i^{\infty} q^i = 1$. But this implies that $\sum_{i=0}^{\infty} q^i = 1/p = 1/(1-q)$.

By similar reasoning, if we keep on throwing the coin until we see r heads then we know that $p^r \sum_{i=0}^{\infty} \binom{r+i-1}{r} q^i = 1$. Therefore,

$$\sum_{i=0}^{\infty} \binom{r+i-1}{r} q^i = \frac{1}{p^2} = \frac{1}{(1-q)^r}.$$

With this insight, for $X \sim \mathrm{NBin}(p, n)$

$$\begin{aligned} M_X(s) &= p^r \sum_{i=0}^{\infty} \binom{r+i-1}{r} q^i e^{si} = p^r \sum_{i=0}^{\infty} \binom{r+i-1}{r} (e^s q)^i \\ &= \frac{p^r}{(1-qe^s)^r} \approx \left(\frac{p}{p(1-s)}\right)^r, \end{aligned}$$

where we use again Taylor's expansion for $p \ll 1$.x

s.9.1.2.

$$\begin{aligned} \mathsf{E}[X] &= \int_0^1 (v-b)\, I_{b\geq \alpha v}\, \mathrm{d}v = \int_0^{b/\alpha} v\, \mathrm{d}v - b\int_0^{b/\alpha} \mathrm{d}v \\ &= b^2/2\alpha^2 - b^2/\alpha = \frac{b^2}{\alpha^2}\frac{1-2\alpha}{2}. \end{aligned}$$

Clearly, $\alpha > 1/2$ to ensure that $\mathsf{E}[X] > 0$.

s.9.1.3.

s.9.1.4. $\mathsf{E}[X] = 1 + q\,\mathsf{E}[X]$, because we have to throw at least once, and with probability q, we start again. Hence, $\mathsf{E}[X] = 1/(1-q) = 1/p$.

s.9.1.5. Suppose the first throw is a success, then we need $r-1$ more successes, if the first throw is a failure, we are back at 'hole one'. Thus, $N_r = pN_{r-1} + q(1+N_r)$. Simplifying (and using that $p/(1-q) = 1$) gives $N_r = N_{r-1} + q/p$, which implies $N_r = rq/p$.

s.9.1.6.

$$N_r = N_{r-1} + p\cdot 1 + q(1+N_r) \implies N_r = N_{r-1}/p + 1/p \implies N_r = \sum_{i=1}^{r} 1/p^i. \tag{13.0.99}$$

s.9.1.7. Let X be the outcome of the die throw (note that X is a random variable) and let A be the event that the outcome is even. Then

$$\mathsf{E}[X\,|\,A] = 2\,\mathsf{P}\{X=2|A\} + 4\,\mathsf{P}\{X=4|A\} + 6\,\mathsf{P}\{X=6|A\} = \tfrac{1}{3}\cdot(2+4+6) = 4.$$

We conclude that $\mathsf{E}[X\,|\,A] = 4$.

s.9.1.8. For 4: take $i = b-1$. Then solve for α in $\alpha(b-1) = \alpha(b-2)/2 + 1/2$, because $p_b = 1$. This gives $\alpha = 1/b$.

s.9.1.10.

s.9.1.12. We have that $\mathsf{E}[Y - \mathsf{E}[Y|X]] = 0$. Hence, $\mathsf{E}[Y - \mathsf{E}[Y|X]]\,\mathsf{E}[X] = 0$. Then define $h(X) = \mathsf{E}[Y|X]$ and apply BH.9.3.9 to see that $\mathsf{E}[(Y - \mathsf{E}[Y|X])h(X)] = 0$. From the definition of the covariance, $\mathsf{Cov}[W,Z] = \mathsf{E}[WZ] - \mathsf{E}[W]\,\mathsf{E}[Z]$, we have shown that both terms are zero.

s.9.1.13. 1. Since Adam keeps $b/2$ and does the gamble with $a = b/2$, we have

$$\mathsf{E}[X] = b/2 + \tfrac{1}{5}\cdot 4(b/2) + \tfrac{4}{5}\cdot 0 = 0.9b.$$

2. The computation is the same as in part 1., but with X instead of b:

$$\mathsf{E}[Y|X] = X/2 + \tfrac{1}{5}\cdot 4(X/2) + \tfrac{4}{5}\cdot 0 = 0.9X.$$

Note that the result is a random variable.
3. Using Adam's law (and linearity of expectation), we conclude that:

$$\mathsf{E}[Y] = \mathsf{E}[\mathsf{E}[Y|X]] = \mathsf{E}[0.9X] = 0.9\,\mathsf{E}[X] = 0.81b.$$

In general, if Adam would do this n times, the expected amount of money he has after n such gambles would be $0.9^n b$. This would be very difficult to show without Adam's law!

s.9.1.14. We have $\mathsf{E}[X|N] = Np$, so using Adam's law (and linearity of expectation), we conclude that $\mathsf{E}[X] = \mathsf{E}[\mathsf{E}[X|N]] = \mathsf{E}\left[Np\right] = \mathsf{E}[N]\,p = \lambda p$.

This is in accordance with $X \sim \text{Pois}(\lambda p)$, which was shown in the chicken-egg story.

Some students reported answers like $\lambda^2 p$. This is wrong, and can be immediately seen by checking units: the unit of λ being 1 per time.

Others wrote $\mathsf{E}[X\,|\,N=n]\,np$, hence $\mathsf{E}[X] = \mathsf{E}[\mathsf{E}[X\,|\,N]] = \mathsf{E}\left[np\right] = np$.

Apparently, such students are not aware of the idea that $\mathsf{E}[X\,|\,N]$ is a random variable. When this happens during the exam, you will score 0 points for that particular part of a question.

s.9.1.15. Incorrect: $\mathsf{E}[X \mid A]$ is a number since A is an event, whereas $\mathsf{E}[X \mid I_A]$ is a random variable since I_A is a random variable. A correct statement is $\mathsf{E}[X \mid A] = \mathsf{E}[X \mid I_A = 1]$.

s.9.1.16. Correct, if X and Y are independent, then $\mathsf{E}[Y \mid X] = \mathsf{E}[Y]$ which is a constant (formally, a degenerate random variable). Since the variance of a constant is 0, we conclude that $\mathsf{V}[\mathsf{E}[Y \mid X]] = 0$.

s.9.1.17. 1. We compute $\mathsf{E}[X \mid X \geq a]$ as follows:

$$\begin{aligned}
\mathsf{E}[X \mid X \geq a] &= \int_0^\infty y f(y|A)\,\mathrm{d}y \\
&= \int_0^\infty y \frac{\lambda e^{-\lambda y} I_{y \geq a}}{e^{-\lambda a}}\,\mathrm{d}y \\
&= \lambda \int_a^\infty y e^{-\lambda(y-a)}\,\mathrm{d}y \\
&= -y e^{-\lambda(y-a)}\Big|_a^\infty + \int_a^\infty e^{-\lambda(y-a)}\,\mathrm{d}y \\
&= a - \frac{1}{\lambda} e^{-\lambda(y-a)}\Big|_a^\infty = a + \frac{1}{\lambda}.
\end{aligned}$$

2. The result also follows from the memoryless property, which states that conditional on the event that $X \geq a$, we have that $X - a | X \geq a \sim \mathrm{Exp}(\lambda)$.

s.9.1.18. 1. Note that $X \mid A \sim \mathrm{Bin}(10, 0.5)$, so $\mathsf{E}[X \mid A] = 10 \cdot 0.5 = 5$.
2. Note that $X \mid A^c \sim \mathrm{Bin}(10, 0.8)$, so $\mathsf{E}[X \mid A^c] = 10 \cdot 0.8 = 8$.
3. By LOTE we have $\mathsf{E}[X] = \mathsf{P}\{A\}\,\mathsf{E}[X \mid A] + \mathsf{P}\{A^c\}\,\mathsf{E}[X \mid A^c] = 0.9 \cdot 5 + 0.1 \cdot 8 = 5.3$.
4. Note that $\mathsf{P}\{B \mid A\} = 0.5^4$ and $\mathsf{P}\{B \mid A^c\} = 0.8^4$. By LOTP we have

$$\mathsf{P}\{B\} = \mathsf{P}\{A\}\,\mathsf{P}\{B \mid A\} + \mathsf{P}\{A^c\}\,\mathsf{P}\{B \mid A^c\} = 0.9 \cdot 5 + 0.1 \cdot 8 = 0.09721.$$

5. By Bayes' rule $\mathsf{P}\{A \mid B\} = \frac{\mathsf{P}\{B \mid A\}\mathsf{P}\{A\}}{\mathsf{P}\{B\}} \approx 0.57864$.
6. Note that $\mathsf{E}[X \mid A, B] = 4 + 6 \cdot 0.5 = 7$ and $\mathsf{E}[X \mid A^c, B] = 4 + 6 \cdot 0.8 = 8.8$. By LOTP with extra conditioning we have

$$\mathsf{P}\{X \mid B\} = \mathsf{P}\{A \mid B\}\,\mathsf{E}[X \mid A, B] + \mathsf{P}\{A^c \mid B\}\,\mathsf{E}[X \mid A^c, B] \approx 7.75844.$$

7. By LOTE we have $\mathsf{P}\{B\}\,\mathsf{E}[X \mid B] + \mathsf{P}\{B^c\}\,\mathsf{E}[X \mid B^c] = \mathsf{E}[X] = 5.3$. We know $\mathsf{P}\{B\}$ and $\mathsf{E}[X \mid B]$, so solving this for $\mathsf{E}[X \mid B^c]$ yields $\mathsf{E}[X \mid B^c] \approx 5.035$.

One or more students wrote the LOTE as $\mathsf{E}[X] = \sum_Y \mathsf{E}[X \mid Y]\,\mathsf{P}\{Y\}$. This is wrong, as you cannot sum over a rv. This is correct: $\mathsf{E}[X] = \sum_y \mathsf{E}[X \mid Y = y]\,\mathsf{P}\{Y = y\}$, so sum over the *outcomes* of a rv.

s.9.1.19. The marginal density of X is given by $f_X(x) = 2(1-x)$.

So the conditional density is given by $f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{I_{x \leq y}}{1-x}$. Hence,

$$\mathsf{E}[Y \mid X = x] = \int_0^1 y \frac{I_{x \leq y}}{1-x}\,\mathrm{d}y = \frac{1}{1-x}\int_x^1 y\,\mathrm{d}y = \frac{1}{1-x}\left[\tfrac{1}{2}y^2\right]_x^1 = \frac{\frac{1}{2}\left(1-x^2\right)}{1-x} = \tfrac{1}{2}(1+x).$$

We conclude that $\mathsf{E}[Y \mid X] = \frac{1}{2}(1+X)$.

The marginal density of Y is given by $f_Y(y) = 2y$.

So the conditional density is given by $f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{I_{x\leq y}}{y}$. So

$$\mathsf{E}\left[X \,\middle|\, Y=y\right] = \int_0^1 x\frac{I_{x\leq y}}{y}\,\mathrm{d}x = \frac{1}{y}\int_0^y x\,\mathrm{d}x = \tfrac{1}{2}y.$$

We conclude that $\mathsf{E}[X \mid Y] = \frac{1}{2}Y$.

Some students wrote for instance $\mathsf{E}[X \mid Y] = y/2$. Apparently, such students are not aware of the idea that $\mathsf{E}[X \mid N]$ is a random variable. When this happens during the exam, you will score 0 points for that particular part of a question.

s.9.1.20. Note that $\mathsf{E}[X \mid X \geq a] \geq a > \mathsf{E}[X \mid X < a]$. By LOTE:

$$\begin{aligned}\mathsf{E}[X] &= \mathsf{P}\{X \geq a\}\,\mathsf{E}[X \mid X \geq a] + \mathsf{P}\{X < a\}\,\mathsf{E}[X \mid X < a] \\ &< \mathsf{P}\{X \geq a\}\,\mathsf{E}[X \mid X \geq a] + \mathsf{P}\{X < a\}\,\mathsf{E}[X \mid X \geq a] \\ &= \mathsf{E}[X \mid X \geq a],\end{aligned}$$

where the inequality is strict since $\mathsf{P}\{X < a\} > 0$.

s.9.1.21. With the hint,

$$\mathsf{E}[N \mid X] = \mathsf{E}[N - X \mid X] + \mathsf{E}[X \mid X] = \mathsf{E}[N - X] + X = \lambda(1-p) + X.$$

As a check, $\mathsf{E}[\mathsf{E}[N \mid X]] = \mathsf{E}\left[\lambda(1-p) + X\right] = \lambda(1-p) + \lambda p = \lambda = \mathsf{E}[N]$.

Here is straightforward computation. You should check each and every step as they are based on pattern recognition.

$$\begin{aligned}\mathsf{E}[N \mid X = k] &= \sum_{n=k}^{\infty} n\,\mathsf{P}\{N = n \mid X = k\} && (13.0.100)\\ &= \frac{1}{\mathsf{P}\{X=k\}}\sum_{n=k}^{\infty} ne^{-\lambda}\frac{\lambda^n}{n!}\binom{n}{k}p^k(1-p)^{n-k} && (13.0.101)\\ &= \frac{1}{\mathsf{P}\{X=k\}}\sum_{n=k}^{\infty} ne^{-\lambda}\frac{1}{n!}\frac{n!}{k!(n-k)!}(\lambda p)^k(\lambda(1-p))^{n-k} && (13.0.102)\\ &= \frac{e^{-\lambda p}(\lambda p)^k/k!}{\mathsf{P}\{X=k\}}\sum_{n=k}^{\infty} ne^{-\lambda(1-p)}\frac{1}{(n-k)!}(\lambda(1-p))^{n-k} && (13.0.103)\\ &= \sum_{n=k}^{\infty} ne^{-\lambda(1-p)}\frac{1}{(n-k)!}(\lambda(1-p))^{n-k} && (13.0.104)\\ &= \sum_{n=0}^{\infty} (n+k)e^{-\lambda(1-p)}\frac{1}{n!}(\lambda(1-p))^{n} && (13.0.105)\\ &= k + \sum_{n=0}^{\infty} ne^{-\lambda(1-p)}\frac{1}{n!}(\lambda(1-p))^{n} && (13.0.106)\\ &= k + \lambda(1-p). && (13.0.107)\end{aligned}$$

Hence, $\mathsf{E}[N \mid X] = \lambda(1-p) + X$. Since $\mathsf{E}[X] = \lambda p$, we get $\mathsf{E}[N] = \lambda$ with Adam's law, as above.

s.9.1.22.

s.9.1.23. By Eve's law,

$$\mathsf{V}[Y] = \mathsf{E}[\mathsf{V}[Y \mid X]] + \mathsf{V}[\mathsf{E}[Y \mid X]] \geq \mathsf{V}[\mathsf{E}[Y \mid X]], \tag{13.0.108}$$

since $\mathsf{V}[Y \mid X] \geq 0$ for all X, which implies that $\mathsf{E}[\mathsf{V}[Y \mid X]] \geq 0$.

s.9.1.24. Conditional on Z, Y is a constant, and the variance of a constant is 0. Hence, $\mathsf{V}[Y|Z] = 0$.

s.9.1.25. Incorrect. Counterexample: Let $Y \sim \text{Bern}(1/2)$ and A be the event $Y = 0$. then $\text{Var}(Y|A)$ and $\text{Var}(Y|A^c)$ are both 0, but $\text{Var}(Y)$=1/4.

s.9.1.26. $\mathsf{V}[Y|X]$ is a random variable, but $\mathsf{V}[Y|X = x]$ is a constant.

s.9.1.27. Define $g(X) = \mathsf{E}[Y|X]$. Then,

$$\begin{aligned}
\mathsf{E}\left[(Y - \mathsf{E}[Y|X])^2|X\right] &= \mathsf{E}\left[(Y - g(X))^2|X\right] && (13.0.109)\\
&= \mathsf{E}\left[Y^2 - 2Yg(X) + g(X)^2|X\right] && (13.0.110)\\
&= \mathsf{E}\left[Y^2|X\right] - 2\,\mathsf{E}\left[Yg(X)|X\right] + \mathsf{E}\left[g(X)^2|X\right] && (13.0.111)\\
&= \mathsf{E}\left[Y^2|X\right] - 2g(X)\,\mathsf{E}[Y|X] + g(X)^2 && (13.0.112)\\
&= \mathsf{E}\left[Y^2|X\right] - 2g(X)^2 + g(X)^2 && (13.0.113)\\
&= \mathsf{E}\left[Y^2|X\right] - (\mathsf{E}[Y|X])^2 && (13.0.114)
\end{aligned}$$

s.9.1.28. Using Eve's Law we have

$$\mathsf{V}[W] = \mathsf{V}[\mathsf{E}[W|X]] + \mathsf{E}[\mathsf{V}[W|X]] = \mathsf{V}[0] + \mathsf{E}\left[X^2\right] = 0 + \mu^2 + \sigma^2 = \mu^2 + \sigma^2. \tag{13.0.115}$$

s.9.2.1. Using the hint, for a. $\mathsf{E}[T] = \mathsf{E}\left[\sum_j \mathsf{E}\left[R_j\right] I_{R=j}\right] = \sum_j \mathsf{E}\left[R_j\right] \mathsf{E}\left[I_{R=j}\right] = (\mu_1 + \mu_2 + \mu_3)/3$.

For b., $\mathsf{E}\left[T^2|R\right]$, realize that $\mathsf{E}\left[R_j^2\right] = \mu_j^2 + \sigma_j^2$, because $\mathsf{V}\left[R_j\right] = \mathsf{E}\left[R_j^2\right] - (\mathsf{E}\left[R_j\right])^2$. Finally, with these ideas,

$$\begin{aligned}
\mathsf{V}[\mathsf{E}[T \mid R]] &= \mathsf{E}\left[(\mathsf{E}[T \mid R])^2\right] - (\mathsf{E}[\mathsf{E}[T \mid R]])^2\\
&= (\mu_1^2 + \mu_2^2 + \mu_3^2)/3 + (\mu_1 + \mu_2 + \mu_3)^2/9.
\end{aligned}$$

s.9.2.2.

$$\begin{aligned}
\mathsf{E}[X_{n+1}|X_n = 100] &= 100 + pf100 - (1-p)f100 = 100(1 - f + 2pf)\\
\mathsf{E}[X_{n+1}|X_n] &= X_n(1 - f + 2pf)\\
\mathsf{E}[X_{n+1}] &= (1 - f + 2pf)\,\mathsf{E}[X_n]\\
\mathsf{E}[X_{n+1}] &= (1 - f + 2pf)^2\,\mathsf{E}[X_{n-1}] = (1 - f + 2pf)^{n+1} X_0.
\end{aligned}$$

s.9.2.4. Use that $P^2 = P$ (indicator funtion), Adam and Eve, and that $N \sim \text{Pois}(8\lambda)$,

$$
\begin{aligned}
\mathsf{E}[Y|P] &= \mathsf{E}[PX|P] = \mathsf{E}[X]\,\mathsf{E}[P|P] = \mu P, & \mathsf{V}[Y|P] &= \mathsf{V}[XP|P] = P^2\,\mathsf{V}[X|P] = P\sigma^2 \\
\mathsf{E}[Y] &= \mu p, & \mathsf{V}[Y] &= \mathsf{E}[\mathsf{V}[Y|P]] + \mathsf{V}[\mathsf{E}[Y|P]] = \sigma^2 p + \mu^2 p(1-p), \\
\mathsf{E}[S|N] &= N\,\mathsf{E}[Y], & \mathsf{V}[S|N] &= N\,\mathsf{V}[Y] \\
\mathsf{E}[N] &= 8\lambda, & \mathsf{V}[N] &= 8\lambda.
\end{aligned}
$$

Now use BH.9.6.1. It's just a matter of filling in how.

s.9.2.5. a. Here you should assume that the X_i are not yet known. Thus, the expectation over X_i is taken with respect to the CDF F_X. Using the independence of X_j and S_j, $I_{S_j=i}\, I_{S_j=k} = 0$ if $i \neq k$, and that $\mathsf{E}\left[I_{S_j=k}\right] = 1/n$,

$$
\begin{aligned}
\mathsf{E}[Y_j] &= \sum_i \mathsf{E}[X_i]\,\mathsf{E}\left[I_{S_j=i}\right] = \mu, \\
\mathsf{E}\left[Y_j^2\right] &= \mathsf{E}\left[\sum_k \sum_l X_k X_l\, I_{S_j=k}\, I_{S_j=l}\right] = \mathsf{E}\left[\sum_k X_k^2\, I_{S_j=k}\right] = \sum_k \mathsf{E}\left[X^2\right] n^{-1} = \mathsf{E}\left[X^2\right], \\
\mathsf{V}[Y_j] &= \mathsf{E}\left[Y_j^2\right] - (\mathsf{E}[Y_j])^2 = \sigma^2.
\end{aligned}
$$

b. Now we are given the outcomes (samples) $X_i = x_i$ of n experiments. I prefer to write $D = X_1, \ldots, X_n$ as it is shorter. Noting that S_j and D are independent, and that $\mathsf{E}[X_k|D] = X_k$,

$$
\mathsf{E}[Y_j|D] = \sum_k X_k\,\mathsf{E}\left[I_{S_j=k}|D\right] = \frac{1}{n}\sum_k X_k := \bar{X}
$$

Observe that this average need not be the same as μ!

The conditional variance. Since S_j and S_k are independent when $j \neq k$, it must be that $Y_j|D$ and $Y_k|D$ are also conditionally independent. Moreover, $\{Y_j|D\}$ are conditionally iid. Therefore,

$$
\begin{aligned}
\mathsf{E}\left[Y_j^2|D\right] &= \mathsf{E}\left[\sum_k \sum_l X_k X_l\, I_{S_j=k}\, I_{S_j=l}|D\right] \\
&= \mathsf{E}\left[\sum_k X_k^2\, I_{S_j=k}|D\right] = \sum_k X_k^2\,\mathsf{E}\left[I_{S_j=k}|D\right] \\
&= \frac{1}{n}\sum_k X_k^2, \\
\mathsf{V}[Y_j|D] &= \frac{1}{n}\sum_k X_k^2 - (\bar{X})^2 = \frac{1}{n}\sum_k (X_k - \bar{X})^2 = \frac{n-1}{n}\sigma^2, \\
\mathsf{V}[\bar{Y}|D] &= \mathsf{V}\left[\frac{1}{n}\sum_j Y_j|D\right] = \frac{1}{n^2}\sum_j \mathsf{V}[Y_j|D] = \frac{1}{n}\mathsf{V}[Y_1|D].
\end{aligned}
$$

c. For $\mathsf{E}[\bar{Y}]$ use linearity and Adam's law:

$$
\mathsf{E}[\bar{Y}] = \mathsf{E}[\mathsf{E}[\bar{Y}|D]] = \frac{1}{n}\sum_k \mathsf{E}[X_k] = \mathsf{E}[X] = \mu.
$$

Here are the details for $\mathsf{V}\left[\bar{Y}\right]$. Using BH.6.3.3 and BH.6.3.4,

$$\begin{aligned}
\mathsf{E}\left[\mathsf{V}\left[\bar{Y}|D\right]\right] &= \frac{1}{n}\mathsf{E}\left[\mathsf{V}\left[Y_1|D\right]\right] = \frac{1}{n^2}\mathsf{E}\left[\sum_{i=1}^{n}(X_i-\bar{X})^2\right] \\
&= \frac{n-1}{n^2}\mathsf{E}\left[\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2\right] = \frac{n-1}{n^2}\mathsf{E}\left[S_n^2\right] = \frac{(n-1)\sigma^2}{n^2} \\
\mathsf{V}\left[\mathsf{E}\left[\bar{Y}|D\right]\right] &= \mathsf{V}\left[\bar{X}\right] = \frac{1}{n^2}\sum_i \mathsf{V}\left[X_i\right] = \frac{1}{n}\sigma^2.
\end{aligned}$$

Now use Eve's law to add both terms to get $V\bar{Y}$.

d. We add randomness twice, first we draw samples to get D, and then we draw randomly from D.

The extra exercise: immediate from Example 1.4.22. We are not interested in the sequence of the bootstrap sample. BTW, the story that goes for me with this example is the 'balls and bars story'. I have n balls to distribute over k boxes. Hence, there are $k-1$ bars to separate the boxes. For the bootstrap sample, I have to distribute n bootstrap samples (the X_i^*) over n boxes (the initial sample X_i.)

If n is small, say $n=4$. Does it make sense to take more than 1000 bootstrap samples?

s.9.2.7. a. Using the hint gives us $\mathsf{E}\left[N\,|\,\lambda\right]=\lambda$ and $\mathsf{V}\left[N\,|\,\lambda\right]=\lambda$.

Now use Adam and Eve.

b. Just copy the formulas of BH.9.6.1

c. With the hint, observe that $\text{Exp}(()1) = \Gamma 1, 1$. In the relevant formula of BH.8.4.5 ($\mathsf{P}\left\{Y=y\right\}$), take $t=r_0=b_0=1$ and conclude that $\mathsf{P}\left\{N=n\right\}=2^{-n-1}$. Hence, $N \sim \text{Geo}(1/2)$.

d. Same story. The relevant formula is $f_1(\lambda|y)$.

s.9.2.10. a. From the hint,

$$\begin{aligned}
\mathsf{E}\left[T\right] = \mathsf{E}\left[\mathsf{E}\left[T|p\right]\right] &= \frac{1}{\beta(a,b)}\int_0^1 \frac{1}{p}p^{a-1}(1-p)^{b-1}\,\mathrm{d}p \\
&= \frac{1}{\beta(a,b)}\int_0^1 p^{a-2}(1-p)^{b-1}\,\mathrm{d}p = \frac{\beta(a-1,b)}{\beta(a,b)} \\
&= \frac{a+b-1}{a-1} = 1+\frac{b}{a-1}.
\end{aligned}$$

To get the last equation, use the definition of $\beta(a,b)$ in terms of factorials (see the Bayes' billiards story) to simplify. This is easy, many terms cancel.

b. Take $Y=1+G$, then Y has the first success distribution since G is geometric. Hence, $\mathsf{E}\left[Y\right]=(a+b)/a=1+b/a$. Clearly, this is smaller than $1+b/(a-1)=\mathsf{E}\left[T\right]$.

But why is this so?

I must miss something here. The prior is Beta(a,b). Then Beta-Binomial conjugacy story, we assume that Vishy won $a-1$ games, and lost $b-1$ games. My guess for Vishy winning the next game would be $(a-1)/(a+b-2)$, not $a/(a+b)$. But I make an error here. Check the BH problem 9.57. You'll see that we should indeed use $a/(a+b)$! Tricky!

c. Immediate from BH.8.3.3: $p|X=7 \sim \text{Beta}(a+7,b+3)$.

s.9.2.11. a. By the hint,

$$f(p|X_1 = x_1) \propto f(p, X_1 = x_1) \propto p^{a-1} q^{b-1} p^{x_1} q^{1-x_1} \propto p^{a+x_1-1} q^{b+(1-x_1)-1}.$$

Hence, $p|X_1 = x_1 \sim \text{Beta}(a + x_1, b + (1 - x_1))$. We can now use this as prior to see that $p|X_1 = x_a, X_2 = x_2 \sim \text{Beta}(1 + x_1 + x_2, 1 + (1 - x_1) + (1 - x_2))$, and so on. Hence, $p|X_1, \dots X_n \sim \text{Beta}(1 + S_k, 1 + n - S_k)$.

b. With the hint, $\mathsf{P}\{X_{n+1} = 1|p\} = p$ and $\mathsf{P}\{S_n = k|p\} = \binom{n}{k} p^k q^{n-k} \propto p^k q^{n-k}$. Also $X_{n+1}|p$ and $S_n|p$ are conditionally independent. Therefore,

$$\mathsf{P}\{X_{n+1} = 1, S_n = k|p\} \propto p p^k q^{n-k} = p^{k+1} q^{n-k},$$

which in turn implies that

$$\mathsf{P}\{X_{n+1} = 1|S_n = k, p\} \propto = p^{k+1} q^{n-k}.$$

Hence, $X_{n+1}|S_n = k, p \sim \text{Beta}(k+2, n-k+1)$. Now, $X_{n+1} \in \{0, 1\}$, so that $\mathsf{P}\{X_{n+1} = 1|S_n = k, p\} = \mathsf{E}[X_{n+1}|S_n = k, p] = (k+2)/(n+3)$, since $X_{n+1}|S_n = k, p \sim \text{Beta}(k+2, n-k+1)$.

The last step is to realize that $\mathsf{E}[X_{n+1}|S_n = k] = \mathsf{E}[\mathsf{E}[X_{n+1}|S_n = k, p] | S_{n=k}]$.

Here is another way to get the same result.

$$\begin{aligned}
\mathsf{P}\{S_n = k\} &= \frac{1}{n+1}, \text{ by Bayes' billiard,} \\
\mathsf{P}\{X_{n+1} = 1, S_n = k\} &= \int_0^1 \mathsf{P}\{X_{n+1} = 1, S_n = k \,|\, p\} f(p)\,\mathrm{d}p = \int_0^1 p \binom{n}{k} p^k (1-p)^{n-k} f(p)\,\mathrm{d}p \\
&= \frac{k+1}{n+1} \int_0^1 \binom{n+1}{k+1} p^{k+1} (1-p)^{n-k} f(p)\,\mathrm{d}p \\
&= \frac{k+1}{n+1} \frac{1}{n+2}, \text{ again with Bayes' billiard,} \\
\mathsf{P}\{X_{n+1} = 1 \,|\, S_n = k\} &= \mathsf{P}\{X_{n+1} = 1, S_n = k\} / \mathsf{P}\{S_n = k\}.
\end{aligned}$$

Now simplify.

s.9.2.12.

$$\begin{aligned}
\mathsf{E}[S_n|p] &= np \\
\mathsf{E}[p] &= \frac{1}{\beta(a,b)} \int_0^1 p p^{a-1} q^{b-1}\,\mathrm{d}p = \frac{\beta(a+1,b)}{\beta(a,b)} = \frac{a}{a+b} = 1/2 \\
\mathsf{E}[\mathsf{E}[S_n|p]] &= n\,\mathsf{E}[p] = n/2. \\
\mathsf{V}[S_n|p] &= npq \\
\mathsf{E}[\mathsf{V}[S_n|p]] &= n\,\mathsf{E}[pq] = n\,\mathsf{E}[p] - n\,\mathsf{E}[p^2] = n/2 - n\,\mathsf{E}[p^2] \\
\mathsf{E}[p^2] &= \frac{1}{\beta(a,b)} \int_0^1 p^2 p^{a-1} q^{b-1}\,\mathrm{d}p = \frac{\beta(a+2,b)}{\beta(a,b)} = \frac{a(a+1)}{(a+b)(a+b+1)} = \frac{2}{2 \cdot 3} = 1/3 \\
\mathsf{V}[\mathsf{E}[S_n|p]] &= \mathsf{V}[np] = n^2\,\mathsf{V}[p] = n^2/12.
\end{aligned}$$

The rest of Eve's law is now trivial.

b. We start with a Beta$(1,1)$ prior on p. After the first win, the prior gets updated to Beta$(1+1,1)$, after a loss to Beta$(1,1+1)$. Reasoning like this, after a wins and $j-a$ losses, the distribution for a win becomes Beta$(1+a, a+j-1)$. Therefore, by using the hint in the book, $\mathsf{E}\left[p|S_j = a\right] = (a+1)/(j+2)$.

c. When somebody doesn't give me any information about what team can win, then any outcome must be equally likely. (What else can it be?) This is also my way to understand the expression in BH.8.3.2. Hence, $\mathsf{P}\{X=k\} = 1/(n+1)$. Observe that we use the prior $p \sim$ Beta$(1,1)$.

When the prior is Beta$(a, j-a)$, we should get the negative hypergeometric distribution, see the remark in BH.8.3.3.

d. Shanille scores the first and missed the second. Hence, there are 98 shots left, out which she has to score 49. Thus, we ask for $\mathsf{P}\left\{S_{98} = 49|p\right\}$, where $p \sim$ Beta$(a=1, b=1)$ is the prior since she hit $a=1$ out of $a+b=2$ shots. This places us in the situation of part c above, with $n=98$. Hence, $\mathsf{P}\left\{S_{98} = 49|p\right\} = 1/99$.

s.10.1.1. f is the expectation of something non-negative. Then, work out the square and apply linearity of the expectation. As $f \geq 0$, it can have most one root, hence $D \leq 0$. But $D = 4\,\mathsf{E}[XY] - 4\,\mathsf{E}\left[X^2\right]\mathsf{E}\left[Y^2\right]$.

s.10.1.3.

s.10.1.4.

s.10.1.5.

s.10.1.6.

s.10.1.8. Equality holds for functions that are both convex and concave. The only functions that are both convex and concave are affine functions, i.e., functions of the type $g(x) = ax+b$. Assuming that g is twice differentiable, we can show this as follows. Convexity is equivalent to $g''(x) \geq 0$ and concavity is equivalent to $g''(x) \leq 0$. This means $g''(x) = 0$ and the only functions for which this holds are affine functions.

If you like maths, consider generalizing the condition. Is it necessary to assume that g is twice differentiable? For instance, it is not hard to prove that a convex function is continuous. Consider now a point at which g is convex and concave at the same time, does it follow that g is twice differentiable at such a point?

s.10.1.10. In the equation of the hint, take expectations at both sides. Realize that $\mathsf{E}\left[I_{X\geq a}\right] = \mathsf{P}\{X \geq a\}$. Next, for any rv, $|X| \geq 0$. Hence, we can apply the simple form of Markov's inequality to get the result of the book.

s.10.1.11. By definition, equation (1) is Chebyshev's inequality. Letting $a = c\sigma_X$ we get (4). Equation (3) follows from multiplying (1) by -1, adding 1 and using the complement rule. Equation (2) is not equivalent to any of the others, as this is not how reversing inequalities works.

s.10.1.12. This will result in the trivial bound $\mathsf{P}\{|X-\mu| \geq a\} \leq B$, for some $B \geq 1$. But we already know that every probability is at most one. So the bound does not tell us anything interesting.

s.10.1.13. In this (pathological) example we get from Markov's inequality that $P(X \geq 2) \leq \frac{E(X)}{2} = \frac{1}{4}$. This means the Markov bound is tight, as it is equal to the probability that X exceeds 2. From Chernoff's bound we get

$$P(X \geq 2) \leq \frac{E(e^{tX})}{e^{2t}} = \frac{3+e^{2t}}{4e^{2t}} = \frac{1}{4}\left(1+\frac{3}{e^{2t}}\right) > \frac{1}{4} \quad \forall t > 0.$$

Hence here the Markov bound is tighter. We use the facts from probability theory that $E(X) = \frac{1}{2}$ and that $E(e^{tX}) = \frac{3}{4} + e^{2t}\frac{1}{4}$ in this example.

s.10.1.14.

s.10.1.15.

s.10.1.16.

s.10.1.17. Divide by the std corresponds to the standard transformation $(X-\mu)/\sigma$. Like this, I don't have to remember anything new. Algebra gives the formula of the book.

s.10.1.18.

s.10.1.19.

s.10.1.20.

s.10.1.21. Here is the reason. $\bar{Z}_n$ is the sum of n normal rvs Z_j, hence normal itself. As each of these Z_j is standard normal, $\mathsf{E}\left[\bar{Z}_n\right] = 0$, and $\mathsf{V}\left[\bar{Z}_n\right] = n^{-2}\sum_j \mathsf{V}\left[Z_j\right] = 1/n$, by independence. Therefore, $\sqrt{n}\bar{Z}_n \sim N(0,1) \implies (\sqrt{n}\bar{Z}_n)^2 \sim \chi_1^2$, where we use Definition 10.4.1 and Theorem 10.4.2 in the last step.

s.10.1.22.

s.10.1.23. Let X be the r.v. corresponding to the number of heads. Then $X \sim \text{Binomial}\left(100, \frac{1}{2}\right)$, which has moments $\mathsf{E}[X] = 100 \cdot \frac{1}{2} = 50$ and $\mathsf{V}[X] = 100 \cdot \frac{1}{2} \cdot \frac{1}{2} = 25$. By symmetry of the $\text{Binomial}\left(100, \frac{1}{2}\right)$ distribution,

$$\mathsf{P}\{X \leq 40\} = \mathsf{P}\{X \geq 60\}. \tag{13.0.116}$$

Hence, using Chebyshev's inequality,

$$\begin{aligned}
\mathsf{P}\{X \leq 40\} &= \frac{1}{2}\mathsf{P}\{|X-50| \geq 10\} && (13.0.117)\\
&= \frac{1}{2}\mathsf{P}\{|X-50| \geq 10\} && (13.0.118)\\
&\leq \frac{1}{2}\frac{\mathsf{V}[X]}{10^2} && (13.0.119)\\
&= \frac{1}{2}\frac{25}{100} = \frac{1}{8}. && (13.0.120)
\end{aligned}$$

Hence,

$$\mathsf{P}\{X \le 40\} \le \frac{1}{8}. \tag{13.0.121}$$

s.10.1.25. We assemble m observations of Y_j (hence, we throw the coin nm times). Suppose we see M times that $|Y_j - \mu| > \epsilon$. Then we expect that $M/m < \sigma^2/n\epsilon$.

Thus, Chebyshev's inequality makes a statement about sample means of size n, say.

s.10.1.26. First fix some $\epsilon > 0$. Now take some n and determine the fraction of outliers, that is, count how many of the sample means $Y_1 = \sum_{i=1}^n X_i/n$, $Y_2 = \sum_{i=n+1}^{2n} X_i/n, \ldots$ lie outside the interval $[\mu - \epsilon, \mu + \epsilon]$ and divide by the number of samples taken. The WLLN says this: If the sample averages Y_1, Y_2 are taken over larger sets of the X_j, i.e., n is larger so that we put more throws in a batch, then the fraction of outliers become smaller.

s.10.1.27. The SLLN says nothing about individual sample paths, i.e., strings of outcomes like $H, T, H, T, \ldots$. In fact, the probability of obtaining any particular sample path has zero probability. Instead, the SLLN makes a statement about sets of sample paths. For the coin it says that it is virtually impossible to pick a path from the set of paths whose long-run fraction of heads is not equal to $1/2$.

s.10.2.3. Take W as in the hint and $Z = 1$. By the inequality of Cauchy-Schwarz, $(\mathsf{E}[W])^2 \ge \mathsf{E}\left[W^2\right]$. The LHS is σ^4, the RHS is $\mathsf{E}\left[(X-\mu)^4\right]$. The rest follows right away from the definition of kurtosis.

s.10.2.4. a. $\le$ Immediate from the hint.

b. =: immediate from the hint

c.

$$\mathsf{P}\{X > Y - 3\} = \mathsf{P}\{X > Y + 3\} + \mathsf{P}\{Y - 3 \le X \le Y + 3\}.$$

Both terms on the RHS are non-negative.

d. Use the hint. $(\mathsf{E}[XY])^2 \le \mathsf{E}\left[X^2\right]\mathsf{E}\left[Y^2\right] = (\mathsf{E}\left[X^2\right])^2 \le \mathsf{E}\left[X^4\right]$, where we use that X and Y are iid, so that $\mathsf{E}\left[X^2\right]$ and $\mathsf{E}\left[Y^2\right]$ are equal.

e. =: since X and Y are independent, $\mathsf{V}[Y|X] = \mathsf{V}[Y]$.

f. From the hint, $\mathsf{P}\{|X+Y| > 3\} \le \mathsf{E}[|X+Y|]/3 \le \mathsf{E}[|X|]/3 + \mathsf{E}[|Y|]/3 = 2\mathsf{E}[|X|]/3 \le \mathsf{E}[|X|]$. Why is there not an $<$ in the last step?

s.10.2.6. a. I did things a bit differently than in the book. Take $S_n = \sum_{i=1}^n X_i$ with $X_i \sim \text{Bern}(p)$. Then I know this:

$$\mathsf{P}\{S_n = k\} = \binom{n}{k} p^k (1-p)^{n-k} \to e^{-\lambda}\lambda^k/k! = \mathsf{P}\{N = k\}, \quad \text{if } N \sim \text{Pois}(\lambda),$$

for $n \to \infty$, $p \to 0$ but such that $pn = \lambda$. I also know from the CTL that $S_n \sim N(np, np(1-p))$ if n becomes large. But, $N(np, np(1-p)) \to N(\lambda, \lambda)$ in the above limit. Now take $\lambda = n$ to see that $\text{Pois}(\lambda) \sim N(n, n)$.

b. Check the solution manual. Then, with $\mu = \sigma = \lambda = n$, and $n \gg 1$,

$$\begin{aligned}\Phi(n+1/2) - \Phi(n-1/2) &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{n-1/2}^{n+1/2} e^{-(x-\mu)/2\sigma^2} \,\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi n}} \int_{-1/2}^{1/2} e^{-x^2/2n} \,\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi n}} \int_{-1/2}^{1/2} (1 - x^2/2n) \,\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi n}} (1 - 1/(24n)).\end{aligned}$$

So, we found another term to approximate $n!$ yet better.

s.10.2.7. Since $X_n \sim \text{Pois}(n)$, $\mathsf{E}[X_n] = n$, $\mathsf{V}[X_n] = n$. Using the hints, with Y_n the standardized version of X_n:

$$\begin{aligned}M_{Y_n}(s) &= \sum_{i=0}^{\infty} e^{-n} n^i / i! \cdot e^{s(i-n)/\sqrt{n}} = e^{-n} e^{s\sqrt{n}} \sum_{i=0}^{\infty} (n e^{s/\sqrt{n}})^i / i! \\ &= \exp\{-n + s\sqrt{n} + n e^{-s/\sqrt{n}}\}.\end{aligned}$$

With Taylor's expansion for e^x to second order,

$$-n + s\sqrt{n} + n e^{-s/\sqrt{n}} \approx -n + s\sqrt{n} + n\left(1 - s/\sqrt{n} + s^2/2n\right) = s^2/2.$$

Now follow the proof of the CTL, BH.10.3.1.

s.10.2.8. a. Define I_n as the success indicator: it is 1 if I win, and 0 if I loose. For round 1, suppose I win, then $Y_1 = Y_0/2 + 1.7Y_0/2 = 1.35Y_0$. If I lose, $Y_1 = Y_0/2 + 0.5Y_0/2 = 0.75Y_0$. Therefore,

$$Y_n = Y_{n-1}(1.35)^{I_n}(0.75)^{1-I_n}.$$

With this expression, the rest is simple, just follow BH.10.3.7. It turns out that $Y_n \to \infty$ as $n \to \infty$.

b. Use the hint.

$$\begin{aligned}Y_n &= Y_{n-1}(1+0.7\alpha)^{I_n}(1-0.5\alpha)^{1-I_n} \implies \\ \log Y_n &= \log Y_{n-1} + I_n \log(1+0.7\alpha) + (1-I_n)\log(1-0.5\alpha) \\ &= \log Y_0 + \log(1+0.7\alpha) \sum_{i=1}^{n} I_i + \log(1-0.5\alpha). \sum_{i=1}^{n} (1-I_i)\end{aligned}$$

By the strong law, $\sum I_i/n \to 1/2$ and $\sum(1-I_i)/n \to 1/2$. Therefore

$$n^{-1} \log Y_n \to 0.5\log(1+0.7\alpha) + 0.5\log(1-0.5\alpha) = 0.5\log((1+0.7\alpha)(1-0.5\alpha)) = g(\alpha)$$

For the maximum, take the derivative with respect to α. This gives $\alpha = 2/7$.

s.10.2.10. a. $\mathsf{P}\{N=n\}=\mathsf{P}\{X_1<1, X_2<1, \cdots X_{n-1}<1, X_n>1\}$. But, then N must have the first success distribution, and $N-1$ be geometric.

b. Let X_i be the inter-arrival time between jobs $i-1$ and i. Then $S_n=\sum_i^n X_i$ is the arrival time of job n. We want that $S_{M-1}<10\leq S_M$. Since the X_i are $\sim \text{Exp}(\lambda)$, $S_n \sim \text{Pois}(\lambda t)$.

c. The sum of n iid Exp(1) rvs is Gamma$(n,1)$. Since $\bar{X}_n$ has mean 1, $X_n \sim \text{Gamma}(n,n)$. Then $\mathsf{V}[X_n]=1/n$ (I just looked it up in the back of the book). By the CLT, $\bar{X}_n$ is approximated well by a Norm(μ,σ^2) rv with $\mu=1, \sigma^2=1/n$.

s.11.0.1. I *have remembered* that $\mathsf{V}[X]=\mathsf{E}[X]$ when $X\sim\text{Exp}(\lambda)$. Since $\mathsf{V}[X]=\mathsf{E}\left[X^2\right]-(\mathsf{E}[X])^2$, $\mathsf{E}\left[X^2\right]=2/\lambda^2$. Applying this to L, we see that $\mathsf{E}\left[L^2\right]=2/(2\lambda)^2=1/2\lambda^2$. Moreover, $\mathsf{V}[L]=1/4\lambda^2$.

Next, $f_M(x)=2f_x(x)F_Y(x)$. Hence,

$$\mathsf{E}\left[M^2\right]=\int x^2 2\lambda e^{-\lambda x}(1-e^{-\lambda x}\,\mathrm{d}x=2\int x^2\lambda e^{-\lambda x}\,\mathrm{d}x-\int x^2 2\lambda e^{-2\lambda x}\,\mathrm{d}x.$$

The first integral is just 2 times $\mathsf{E}\left[X^2\right]$, the second is $\mathsf{E}\left[L^2\right]$. Hence, $\mathsf{E}\left[M^2\right]=4/\lambda^2-1/2\lambda^2=7/2\lambda^2$. Finally, $\mathsf{V}[M]=7/2\lambda^2-9/4\lambda^2=5/4\lambda^2$.

s.11.0.2.

1. Since (X,Y) are bivariate normally distributed, every linear combination of X and Y is normally distributed. Note that every linear combination of $(X+Y)$ and $(X-Y)$ can be written as a linear combination of X and Y. Hence, every linear combination of $(X+Y)$ and $(X-Y)$ is normally distributed. Hence, $(X+Y, X-Y)$ is bivariate normally distributed.

2. By the story above, both X and Y are normally distributed. We have

$$\mathsf{E}[X+Y]=\mathsf{E}[X]+\mathsf{E}[Y]=\mu+\mu=2\mu, \tag{13.0.122}$$

and

$$\mathsf{E}[X-Y]=\mathsf{E}[X]-\mathsf{E}[Y]=\mu-\mu=0. \tag{13.0.123}$$

Moreover,

$$\mathsf{V}[X+Y]=\mathsf{V}[X]+\mathsf{V}[Y]+2\,\mathsf{Cov}[X,Y]=2\sigma^2+2\rho\sigma^2=2(1+\rho)\sigma^2. \tag{13.0.124}$$

Simlarly,

$$\mathsf{V}[X-Y]=\mathsf{V}[X]+\mathsf{V}[-Y]+2\,\mathsf{Cov}[X,-Y]=\mathsf{V}[X]+\mathsf{V}[Y]-2\,\mathsf{Cov}[X,Y] \tag{13.0.125}$$
$$=2\sigma^2-2\rho\sigma^2=2(1-\rho)\sigma^2. \tag{13.0.126}$$

So we have found that $X+Y\sim N(2\mu, 2(1+\rho)\sigma^2)$ and $X-Y\sim N(0,2(1-\rho)\sigma^2$.

3. We have

$$\mathsf{Cov}[X+Y,X-Y]=\mathsf{Cov}[X,X]-\mathsf{Cov}[X,Y]+\mathsf{Cov}[Y,X]-\mathsf{Cov}[Y,Y] \tag{13.0.127}$$
$$=\mathsf{V}[X]-\mathsf{V}[Y]=\sigma^2-\sigma^2=0. \tag{13.0.128}$$

Write $U = X+Y$, $V = X-Y$. Plugging all the parameters into the formula for the joint pdf of a bivariate normal distribution (see https://en.wikipedia.org/wiki/Multivariate_normal_distribution#Bivariate_case), we obtain

$$f_{U,V}(u,v) = \frac{1}{2\pi\sqrt{2(1+\rho)\sigma^2 2(1-\rho)\sigma^2}} \exp\left(-\frac{1}{2}\left[\frac{(u-2\mu)^2}{2(1+\rho)\sigma^2} + \frac{v^2}{2(1-\rho)\sigma^2}\right]\right). \tag{13.0.129}$$

s.11.0.3. This question tests your modeling skills too.

In hindsight, the questions have to reorganized a bit. The capital at the end of the ith week is $I_i = I_{i-1} + X_i - C_i$.

Suppose claims arrive at the beginning of the week, and contributions arrive at the end of the week (people prefer to send in their claims early, but they prefer to pay their contribution as late as possible). If we don't have sufficient money in cash, then we cannot pay a claim. Thus, $\max\{I_0 - C_1\}$ is our capital just before the contribution arrives. Hence, I_1' is our capital at the end of week 1 under the assumption that we never pay out more than we have in cash. Likewise for I_2'

$\bar{I}_n$ is the lowest capital we have seen for the first n weeks.

In the supermarket setting, I_i is our inventory is we can be temporarily out of stock, but as soon as new deliveries—so called replenishments—arrive then we serve the waiting customers immediately. The model with I' corresponds to a setting is which we consider unmet demand as lost.

$$\mathsf{P}\{I_0 <= 0\} = \mathsf{P}\{2 + X_1 - C_1 < 0\} = \frac{1}{10}\sum_{i=1}^{10} \mathsf{P}\{C_1 > 2+i\} = \frac{1}{10}\sum_{i=1}^{5} \mathsf{P}\{C_1 > 2+i\} \tag{13.0.130}$$

$$= \frac{1}{10}\sum_{i=1}^{5} \frac{6-i}{9}. \tag{13.0.131}$$

When grading, I realized that questions 8 was not quite reasonable to ask as an exam question. We graded this leniently. As I find it too boring to compute these probabilities by hand, here is the python code. The ideas in the code are highly interesting and useful. The main data structure here is a dictionary, one of the most used data structure in python. I don't have the R code yet, so if you take the (unwise) decision to stick to only R, you have to wait a bit until somebody sends me the R code for this problem.

Python Code

```python
C = {}
for i in range(0, 9):
    C[i] = 1 / 9

X = {}
for i in range(1, 11):
    X[i] = 1 / 10


```

```
I0 = 2

I1 = {}
for k, p in X.items():
    for l, q in C.items():
        i = I0 + k - l
        I1[i] = I1.get(i, 0) + p * q

print("I1, ", sum(I1.values()))  # check


# compute P(I1<0):
P = sum(r for i, r in I1.items() if i < 0)
print(P)


I2 = {}
for i, r in I1.items():
    for k, p in X.items():
        for l, q in C.items():
            j = i + k - l
            I2[j] = I2.get(j, 0) + r * p * q

print("I2 ", sum(I2.values()))  # just a check

# compute P(I2<0):
P = sum(r for i, r in I2.items() if i < 0)
```

Interestingly, $I_i' \geq 1$. (This is so simple to see that I first did it wrong.)

Mistake: note that X_i and C_i are discrete rvs, not continuous. The sum of two uniform random variables is not uniform. For example, think of the sum of two die throws. Is getting 2 just as likely as getting 7?

s.11.0.4. We have

$$\mathsf{Cov}\left[X, Y\right] = \mathsf{Cov}\left[X, X^2\right] = \mathsf{E}\left[X X^2\right] - \mathsf{E}\left[X\right]\mathsf{E}\left[X^2\right] = 0 - 0 \cdot 2.5 = 0. \tag{13.0.132}$$

Hence, $\mathrm{Corr}(X, Y) = 0$.
Yes, for instance, take $X \sim \mathrm{Unif}(\{0, 1\})$. Then,

$$\mathsf{Cov}\left[X, Y\right] = \mathsf{E}\left[X X^2\right] - \mathsf{E}\left[X\right]\mathsf{E}\left[X^2\right] = 0.5 - 0.5 \cdot 0.5 = 0.25. \tag{13.0.133}$$

s.11.0.5. 1. The interpretation is: the time until the first component fails. That is, the time until the machine stops working.

2. Let $\lambda = 10$. We have

$$\begin{aligned}
\mathsf{P}\{\text{machine not failed at time } T\} &= \mathsf{P}\{\min\{X_1, X_2\} > T\} && (13.0.134)\\
&= \mathsf{P}\{X_1 > T, X_2 > T\} && (13.0.135)\\
&= \mathsf{P}\{X_1 > T\}\,\mathsf{P}\{X_2 > T\} && (13.0.136)\\
&= e^{-\lambda T} \cdot e^{-\lambda T} && (13.0.137)\\
&= e^{-(2\lambda)T} && (13.0.138)\\
&= e^{-20T} && (13.0.139)\\
& && (13.0.140)
\end{aligned}$$

3. Note that

$$\mathsf{P}\{\min\{X_1, X_2\} \leq T\} = 1 - \mathsf{P}\{\min\{X_1, X_2\} > T\} = 1 - e^{-20T}. \qquad (13.0.141)$$

   Note that this is the cdf of an exponential distribution with parameter 20. Hence, $\min\{X_1, X_2\} \sim \exp(20)$.

4. The expected time until the machine fails is

$$\mathsf{E}[\min\{X_1, X_2\}] = 1/20, \qquad (13.0.142)$$

   i.e., 3 minutes. Apparently, the machine is not very robust.

s.11.0.6. 1. We have

$$\begin{aligned}
\mathsf{P}\{X + Y > 1\} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I_{X+Y>1} f_{X,Y}(x, y)\,\mathrm{d}y\,\mathrm{d}x && (13.0.143)\\
&= \int_0^1 \int_{1-x}^{1} \frac{6}{7}(x+y)^2\,\mathrm{d}y\,\mathrm{d}x && (13.0.144)\\
&= \frac{6}{7}\int_0^1 \left[\frac{1}{3}(x+y)^3\right]_{y=1-x}^{1} \mathrm{d}x && (13.0.145)\\
&= \frac{2}{7}\int_0^1 \Big((x+1)^3 - (x+1-x)^3\Big)\,\mathrm{d}x && (13.0.146)\\
&= \frac{2}{7}\int_0^1 \Big((x+1)^3 - 1\Big)\,\mathrm{d}x && (13.0.147)\\
&= \frac{2}{7}\left[\frac{1}{4}(x+1)^4 - x\right]_{x=0}^{1} && (13.0.148)\\
&= \frac{1}{14}\left[(x+1)^4 - 4x\right]_{x=0}^{1} && (13.0.149)\\
&= \frac{1}{14}\Big(\big((1+1)^4 - 4\big) - \big((0+1)^4 - 0\big)\Big) && (13.0.150)\\
&= \frac{1}{14}\Big(16 - 4 - 1\Big) && (13.0.151)\\
&= \frac{11}{14}. && (13.0.152)
\end{aligned}$$

2. We have

$$\mathsf{Cov}[U,V] = \mathsf{E}[UV] - \mathsf{E}[U]\,\mathsf{E}[V]. \tag{13.0.153}$$

First, we compute

$$\mathsf{E}[UV] = \int_0^1 \int_0^{1-u} 2uv \,\mathrm{d}v \,\mathrm{d}u \tag{13.0.154}$$
$$= \int_0^1 [uv^2]_{v=0}^{1-u} \,\mathrm{d}u \tag{13.0.155}$$
$$= \int_0^1 \big(u(1-u)^2 - 0\big) \,\mathrm{d}u \tag{13.0.156}$$
$$= \int_0^1 u(1-2u+u^2) \,\mathrm{d}u \tag{13.0.157}$$
$$= \int_0^1 (u-2u^2+u^3) \,\mathrm{d}u \tag{13.0.158}$$
$$= \Big[\frac{1}{2}u^2 - \frac{2}{3}u^3 + \frac{1}{4}u^4\Big]_{u=0}^1 \tag{13.0.159}$$
$$= \frac{1}{2} - \frac{2}{3} + \frac{1}{4} \tag{13.0.160}$$
$$= \frac{1}{12}. \tag{13.0.161}$$

Next,

$$\mathsf{E}[U] = \int_0^1 \int_0^{1-u} 2u \,\mathrm{d}v \,\mathrm{d}u \tag{13.0.162}$$
$$= \int_0^1 2u \int_0^{1-u} 1 \,\mathrm{d}v \,\mathrm{d}u \tag{13.0.163}$$
$$= \int_0^1 2u(1-u) \,\mathrm{d}u \tag{13.0.164}$$
$$= 2\int_0^1 (u-u^2) \,\mathrm{d}u \tag{13.0.165}$$
$$= 2\Big[\frac{1}{2}u^2 - \frac{1}{3}u^3\Big]_{u=0}^1 \tag{13.0.166}$$
$$= 2\Big(\frac{1}{2} - \frac{1}{3}\Big) \tag{13.0.167}$$
$$= \frac{1}{3} \tag{13.0.168}$$

By symmetry, $\mathsf{E}[V] = \frac{1}{3}$. Hence,

$$\mathsf{Cov}[U,V] = \mathsf{E}[UV] - \mathsf{E}[U]\,\mathsf{E}[V] \tag{13.0.169}$$
$$= \frac{1}{12} - \frac{1}{3}\frac{1}{3} \tag{13.0.170}$$
$$= \frac{1}{12} - \frac{1}{9} \tag{13.0.171}$$
$$= -\frac{1}{36}. \tag{13.0.172}$$

s.11.0.7. Since $(u,v) = g(x,y) = (x+y, x-y)$,

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = |-2| = 2.$$

Moreover, $x = (u+v)/2$, $y = (u-v)/2$, so that

$$f_{U,V}(u,v) = f_{X,Y}(x,y)\frac{\partial(x,y)}{\partial(u,v)} = f_{X,Y}(x,y)/2 = f_X(x)f_Y(y)/2 = \frac{1}{2}.$$

The difficulty is in the domain, however. Note that x and y satisfy $0 \leq x \leq 1$, $0 \leq y \leq 1$. So $0 \leq (u+v)/2 \leq 1$ and $0 \leq (u-v)/2 \leq 1$, which simplifies to $-v \leq u \leq 2-v$ and $v \leq u \leq 2+v$, which can also be written as $|v| \leq u \leq 2-|v|$.

s.11.0.8. Note that $u(x,y) = \min\{x,y\} = x\,I_{x\leq y} + y\,I_{x>y}$. With a similar expression for v we find for the Jacobian:

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{bmatrix} I_{x\leq y} & I_{y<x} \\ I_{y<x} & I_{x\leq y} \end{bmatrix} = |\,I_{x\leq y} - I_{x>y}| = 1.$$

If $(U,V) = g(X,Y)$, then $g^{-1}(u,v) = \{(u,v),(v,u)\}$, i.e., a set of two points.

If X, Y iid with PDF, then $f_{X,Y}(x,y) = f(x)f(y)$.

s.11.0.9. a.

$$F \geq 0 \implies 2 < y \tag{13.0.173}$$

$$F \leq 1 \implies F(3,y) \leq 1 \implies F(3,4) = 1 \tag{13.0.174}$$

b. $F(3,7) = 1$.

c. $f(x,y) = \partial_x\partial_y F(x,y) = (x-1)/4$ for $x \in (1,3)$, $y \in (2,4)$ and 0 elsewhere.

d.

$$\mathsf{P}\{2 < X < 3\} = F_X(3) - F_X(2) \tag{13.0.175}$$

$$= F_{X,Y}(3,4) - F_{X,Y}(2,4) = 1 - 1\cdot 2/8 = 3/4. \tag{13.0.176}$$

e. Make a drawing of the rectangle $[2,3]\times[2,4]$. Then check what parts of this are covered by $F_{X,Y}$.

$$\mathsf{P}\{2 < X < 3, 2 < Y < 3\} = F_{X,Y}(3,3) - F_{X,Y}(2,3) - F_{X,Y}(3,2) + F_{X,Y}(2,2). \tag{13.0.177}$$

The rest is just number plugging.

f. Use the fundamental bridge and c.

$$\begin{aligned}
\mathsf{P}\{Y<2X\} &= \mathsf{E}[I_{Y<2X}] && (13.0.178)\\
&= \iint I_{y<2x} f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y && (13.0.179)\\
&= \frac{1}{4}\iint I_{y<2x}\, I_{2<y<4}\, I_{1<x<3}(x-1)\,\mathrm{d}x\,\mathrm{d}y && (13.0.180)\\
&= \frac{1}{4}\int_1^3 (x-1)\int I_{2<y<\min\{2x,4\}}\,\mathrm{d}y\,\mathrm{d}x && (13.0.181)\\
&= \frac{1}{4}\int_1^3 (x-1)(\min\{2x,4\}-2)\,\mathrm{d}x && (13.0.182)\\
&= \frac{1}{4}\int_1^2 (x-1)(2x-2)\,\mathrm{d}x + \frac{1}{4}\int_2^3 (x-1)(4-2)\,\mathrm{d}x. && (13.0.183)
\end{aligned}$$

Finishing the computation must be easy for you now (and if not, practice real hard).

g. As X, Y continuous, the answer is equal to that of f.

h. Similar to f. but a bit more involved.

$$\begin{aligned}
\mathsf{P}\{Y<2X, Y+2X>6\} &= \mathsf{E}\left[I_{Y<2X,Y>6-2X}\right] && (13.0.184)\\
&= \iint I_{y<2x,y>6-2x} f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y && (13.0.185)\\
&= \frac{1}{4}\iint I_{y<2x,y>6-2x}\, I_{2<y<4}\, I_{1<x<3}(x-1)\,\mathrm{d}x\,\mathrm{d}y && (13.0.186)\\
&= \frac{1}{4}\int_1^3 (x-1)\int I_{\max\{2,6-2x\}<y<\min\{2x,4\}}\,\mathrm{d}y\,\mathrm{d}x && (13.0.187)\\
&= \frac{1}{4}\int_1^3 (x-1)[\min\{2x,4\}-\max\{2,6-2x\}]^+\,\mathrm{d}x, && (13.0.188)
\end{aligned}$$

where we need the $[\cdot]^+$ to ensure the positivity of $\min\{2x,4\}-\max\{2,6-2x\}$. To see this, make a graph of the function $\min\{2x,4\}-\max\{2,6-2x\}$. Also, from this graph,

$$= \frac{1}{4}\int_{3/2}^2 (x-1)(2x-6+2x)\,\mathrm{d}x + \frac{1}{4}\int_2^3 (x-1)(4-2)\,\mathrm{d}x. \qquad (13.0.189)$$

The rest is for you.

s.11.0.10. The function $g(x) = x^4$ is not one-to-one on $\mathbb{R}$. It is, however, locally, one-to-one, around the roots of U. (In this course we don't deal with complex numbers, for your interest, it can be proven that the equation $x^4 - y$ has, in general, four roots in the complex plane.)

We need to be bit careful with applying the change of variables formula, but we are OK if we apply it locally around the roots $U^{1/4}$ and $-U^{1/4}$. However, mind that we also should take care of the domain of V, so it might be that these roots don't lie in the domain of V.

With all this, let's first tackle the Jacobian, and then get the domain right with indicators.

$$u = g(v) = v^4 \implies v = \pm u^{1/4}, \tag{13.0.190}$$

$$f_U(u)\,\mathrm{d}u = f_V(v)\,\mathrm{d}v \implies f_U(u) = f_V(v)\frac{\mathrm{d}v}{\mathrm{d}u}, \tag{13.0.191}$$

$$\frac{\mathrm{d}u}{\mathrm{d}v} = 4v^3 = 4u^{3/4}\, I_{v\geq 0} - 4u^{3/4}\, I_{v<0}, \tag{13.0.192}$$

$$f_U(u) = \frac{f_V(-u^{1/4})}{4(-u)^{3/4}}\, I_{-u^{1/4}\in(-3,0)} + \frac{f_V(u^{1/4})}{4(u)^{3/4}}\, I_{u^{1/4}\in[0,2)} \tag{13.0.193}$$

$$= \frac{f_V(-u^{1/4})}{4(-u)^{3/4}}\, I_{u\in(0,81)} + \frac{f_V(u^{1/4})}{4(u)^{3/4}}\, I_{u\in[0,16)}. \tag{13.0.194}$$

If V has the uniform distribution, then $f_V(v) = \frac{1}{5}$ for $v \in (-3,2)$, so

$$f_U(u) = \frac{1}{20(-u)^{3/4}}\, I_{u\in(0,81)} + \frac{1}{20(u)^{3/4}}\, I_{u\in[0,16)}. \tag{13.0.195}$$

s.11.0.11. Here is a direct approach.

$$x = \tan u = g(u) \implies u = \arctan x \tag{13.0.196}$$

$$\frac{\mathrm{d}x}{\mathrm{d}u} = \frac{1}{\cos^2 u} = \frac{\sin^2 u + \cos^2 u}{\cos^2 u} = \tan^2 u + 1 = x^2 + 1, \tag{13.0.197}$$

$$f_X(x) = f_U(u)\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{\pi}\, I_{u\in(0,\pi)}\frac{1}{1+x^2} \tag{13.0.198}$$

$$= \frac{1}{\pi(1+x^2)}\, I_{\arctan x\in(0,\pi)} = \frac{1}{\pi(1+x^2)}. \tag{13.0.199}$$

In the last equation we just shifted the tan from $(-\pi/2, \pi/2]$ to the interval $(0,\pi)$. The tan has also a proper inverse in $(0,\pi)$ (make a drawing of tan to see this), hence all is well-defined.

s.11.0.12. No. The only relevant information is the amount of legs won by each player.

s.11.0.13. Our current information can be represented as: $A_{10} = 6$.

s.11.0.14. We have $A_n \sim \mathrm{Bin}(n,p)$.

s.11.0.15. Let f_0 denote the prior distribution of p. Then for the posterior pdf we find by Bayes' theorem:

$$f_1(p|A_n = k) = \frac{\mathsf{P}\{A_n = k\,|p\}\, f_0(p)}{\mathsf{P}\{A_n = k\}} \tag{13.0.200}$$

$$= \frac{\binom{n}{k} p^k (1-p)^{n-k}\cdot 1}{\mathsf{P}\{A_n = k\}} \tag{13.0.201}$$

$$\propto p^k(1-p)^{n-k}, \tag{13.0.202}$$

in which we recognize the pdf of a $\mathrm{Beta}(k+1, n-k+1)$ distribution (up to a normalizing constant). Hence, $p|A_n = k \sim \mathrm{Beta}(k+1, n-k+1)$.

s.11.0.16. Important: we have already observed 10 legs with an outcome, with which we have updated our belief. Hence, we should use the *posterior* distribution given $A_{10} = 6$ in this exercise! (It's easy to make a mistake here.) Think about it. Suppose instead we had observed 1000 legs and Amy had won 990 of them (i.e., $A_{1000} = 990$). Wouldn't we use this information if someone offered us a bet?

Note that Bob should wins the match if and only if he wins the next three legs. Let W_k be short-hand notation for the event "Bob wins the kth leg". Then, observing that W_{11}, W_{12}, W_{13} are independent, and using the LOTP in the fourth step, we obtain

$$\begin{aligned}
\mathsf{P}\{\text{Bob wins the match} \mid A_{10} = 6\} &= \mathsf{P}\{W_{11} \cap W_{12} \cap W_{13} \mid A_{10} = 6\} && (13.0.203)\\
&= \mathsf{P}\{W_{11} \mid A_{10} = 6\}\, \mathsf{P}\{W_{12} \mid A_{10} = 6\}\, \mathsf{P}\{W_{13} \mid A_{10} = 6\} && (13.0.204)\\
&= \mathsf{P}\{W_{11} \mid A_{10} = 6\}^3 && (13.0.205)\\
&= \int_0^1 \mathsf{P}\left\{I_{11} \mid p, A_{10} = 6\right\}^3 f_1(p|A_{10} = 6)\,dp && (13.0.206)\\
&= \int_0^1 (1-p)^3 \cdot \frac{p^6(1-p)^4}{\beta(7,5)}\,dp && (13.0.207)\\
&= \frac{\beta(7,8)}{\beta(7,5)} \int_0^1 \frac{p^6(1-p)^7}{\beta(7,8)}\,dp && (13.0.208)\\
&= \frac{\beta(7,8)}{\beta(7,5)}. && (13.0.209)
\end{aligned}$$

(Note that we very explicitly do all the steps here. It might be more intuitively clear if you skip the first few steps and write

$$\mathsf{P}\{\text{Bob wins the match} \mid A_{10} = 6\} = \int_0^1 (1-p)^3 f_1(p|A_{10} = 6)\,dp, \qquad (13.0.210)$$

and work from there).

s.11.0.17. We have

$$\begin{aligned}
\mathsf{P}\{\text{Bob wins the match} \mid A_{10} = 6\} &= \frac{\beta(7,8)}{\beta(7,5)} && (13.0.211)\\
&= \left(\frac{6!7!}{14!}\right) / \left(\frac{6!4!}{11!}\right) && (13.0.212)\\
&= \frac{7!/4!}{14!/11!} && (13.0.213)\\
&= \frac{7 \cdot 6 \cdot 5}{14 \cdot 13 \cdot 12} && (13.0.214)\\
&= 5/52 && (13.0.215)\\
&= 0.0962. && (13.0.216)
\end{aligned}$$

s.11.0.18. Our expected profit when taking the bet is

$$300 \cdot \mathsf{P}\{\text{Bob wins the match} \mid A_{10} = 6\} - 10 \cdot \mathsf{P}\left\{\text{Amy wins the match} \mid A_{10} = 6\right\} \tag{13.0.217}$$
$$= 300 \cdot \frac{5}{52} - 10 \cdot (1 - \frac{5}{52}) \tag{13.0.218}$$
$$= 19.808. \tag{13.0.219}$$

So we expect to make a profit of €19.81. Hence, you should take the bet.

s.11.0.19.

s.11.0.20. By the linearity of expectation and BH Theorem 9.3.9:

$$\begin{aligned}\mathsf{E}\left[(Y - \mathsf{E}[Y|X] - h(X))^2\right] &= \mathsf{E}\left[(Y - \mathsf{E}[Y|X])^2 - 2(Y - \mathsf{E}[Y|X])h(X) + (h(X))^2\right] \\ &= \mathsf{E}\left[(Y - \mathsf{E}[Y|X])^2\right] - \mathsf{E}[2(Y - \mathsf{E}[Y|X])h(X)] + \mathsf{E}\left[(h(X))^2\right] \\ &= \mathsf{E}\left[(Y - \mathsf{E}[Y|X])^2\right] + \mathsf{E}\left[(h(X))^2\right].\end{aligned}$$

Since $\mathsf{E}\left[(h(X))^2\right] \geq 0$, we conclude that $\mathsf{E}\left[(Y - \mathsf{E}[Y|X] - h(X))^2\right] \geq \mathsf{E}\left[(Y - \mathsf{E}[Y|X])^2\right]$ for any function h, so $\mathsf{E}[Y|X]$ is the predictor of Y based on X with the lowest mean squared error, i.e. the best predictor of Y based on X.

s.11.0.21.

$$\mathsf{E}\left[\tilde{X} \,\middle|\, Y\right] = \mathsf{E}\left[X - \hat{X} \,\middle|\, Y\right] = \mathsf{E}[X \mid Y] - \mathsf{E}[\mathsf{E}[X \mid Y] \mid Y] \tag{13.0.220}$$
$$= \mathsf{E}[X \mid Y] - \mathsf{E}[X \mid Y]\,\mathsf{E}[1 \mid Y] \tag{13.0.221}$$
$$= \mathsf{E}[X \mid Y] - \mathsf{E}[X \mid Y]\,1 = 0 \tag{13.0.222}$$

s.11.0.22.

$$\mathsf{E}\left[\tilde{X}\right] = \mathsf{E}\left[\mathsf{E}\left[\tilde{X} \,\middle|\, Y\right]\right] = \mathsf{E}[0 \mid Y] = 0. \tag{13.0.223}$$

This means that the estimation error $\tilde{X}$ does not have bias.

s.11.0.23.

$$\mathsf{E}\left[\tilde{X}\hat{X}\right] = \mathsf{E}\left[\mathsf{E}\left[\tilde{X}\hat{X} \,\middle|\, Y\right]\right] \tag{13.0.224}$$
$$= \mathsf{E}\left[\mathsf{E}\left[\tilde{X}\,\mathsf{E}[X \mid Y] \,\middle|\, Y\right]\right] \tag{13.0.225}$$
$$= \mathsf{E}\left[\mathsf{E}[X \mid Y]\,\mathsf{E}\left[\tilde{X} \,\middle|\, Y\right]\right] \tag{13.0.226}$$
$$= \mathsf{E}[\mathsf{E}[X \mid Y]\,0 \mid Y] = 0 \tag{13.0.227}$$

Here, in the rest of the exercises about this topic, we have seen the most terrible mistakes during grading. Hence, study the reasonging applied very carefully, and ensure you know the motivation behind each and every step. There will be questions in the exam about this, and you have to be able to use the arguments. If not, you fail the exam; simple as that. So, you are warned!

s.11.0.24. Using the previous exercises,

$$\mathsf{Cov}\left[\hat{X},\tilde{X}\right] = \mathsf{E}\left[\hat{X}\tilde{X}\right] - \mathsf{E}\left[\hat{X}\right]\mathsf{E}\left[\tilde{X}\right] = 0 - \mathsf{E}\left[\hat{X}\right]0 = 0. \tag{13.0.228}$$

Next, from the definition of $\tilde{X} = X - \hat{X} \implies X = \tilde{X} + \hat{X}$. The variance of the sum is the sum of the variances since $\hat{X}$ and $\tilde{X}$ are uncorrelated.

s.11.0.25. Since $\mathsf{E}\left[\tilde{X}\right] = 0$,

$$\begin{aligned}
\mathsf{V}\left[\tilde{X}\right] &= \mathsf{E}\left[\tilde{X}^2\right] && (13.0.229)\\
&= \mathsf{E}\left[\mathsf{E}\left[\tilde{X}^2 \,\middle|\, Y\right]\right] && (13.0.230)\\
&= \mathsf{E}\left[\mathsf{E}\left[(X-\hat{X})^2 \,\middle|\, Y\right]\right] && (13.0.231)\\
&= \mathsf{E}\left[\mathsf{E}\left[(X-\mathsf{E}[X \mid Y])^2 \,\middle|\, Y\right]\right] && (13.0.232)\\
&= \mathsf{E}\left[\mathsf{V}[X \mid Y]\right], && (13.0.233)
\end{aligned}$$

where the last equation follow from the definition of $\mathsf{V}[X \mid Y]$.

For Eve's law, use the above and the previous exercise to see that

$$\mathsf{V}[X] = \mathsf{V}\left[\hat{X}\right] + \mathsf{V}\left[\tilde{X}\right] = \mathsf{V}\left[\mathsf{E}[X \mid Y]\right] + \mathsf{E}\left[\mathsf{V}[X \mid Y]\right]. \tag{13.0.234}$$

s.11.0.26. From Probability Theory we know $E\left(\frac{Y_n}{n}\right) = \frac{1}{2}$ and $V\left(\frac{Y_n}{n}\right) = \frac{1}{4n}$. Then by Chebyshev's inequality,

$$\lim_{n\to\infty} P\left(\left|\frac{Y_n}{n} - \frac{1}{2}\right| \geq \varepsilon\right) \leq \lim_{n\to\infty} \frac{V\left(\frac{Y_n}{n}\right)}{\varepsilon^2} = \lim_{n\to\infty} \frac{1}{4n\varepsilon^2} = 0 \quad \forall \varepsilon > 0.$$

s.11.0.27. The Cauchy distribution has no mean to converge to.

s.11.0.28. 1. This follows from the inequality $\mathsf{P}\{X \geq a\} \leq \mathsf{E}\left[f(X)\right]/f(a)$ with $f(x) = e^{tX^2}$.

2. The chi-square distribution with 1 degree of freedom is the Gamma distribution with $n = 1/2$ and $\lambda = 1/2$, so $M_{Z^2}(t) = (1-2t)^{-1/2}$.

3. We find $\mathsf{P}\{|Z| > 3\} \leq e^{-9t}(1-2t)^{-1/2}$. So we want to minimize $e^{-9t}(1-2t)^{-1/2}$. It is easier if we take the logarithm first and minimize $-9t - 1/2\log(1-2t)$. Its derivative to t is $-9 + \frac{1}{1-2t}$, so setting the derivative to 0 yields $t = 4/9$. This gives us the bound $\mathsf{P}\{|Z| > 3\} \leq e^{-4}(1-2t)^{-1/2} \approx 0.055$.

4. We have $\mathsf{P}\{|Z| > 3\} = \mathsf{P}\left\{Z^4 > 81\right\}$ since $|Z| > 3$ if and only if $Z^4 > 81$. The inequality now directly follows from Markov's inequality.

5. By Markov's inequality,

$$\mathsf{P}\{|Z| > 3\} = \mathsf{P}\left\{Z^{2n} > 9^n\right\} \leq \frac{\mathsf{E}\left[Z^{2n}\right]}{9^n} = \frac{1}{9^n}\frac{(2n)!}{2^n n!}.$$

From the formula for $\mathsf{E}\left[Z^{2n}\right]$ we see that $\mathsf{E}\left[Z^{2(n+1)}\right] = (2n+1)\,\mathsf{E}\left[Z^{2n}\right]$. We now consider what happens when incrementing n. If $n < 4$ then $2n+1 < 9$, so then incrementing n improves the bound, for $n = 4$ incrementing n doesn't change the bound and for $n > 4$ the bound becomes weaker. So we get the best possible bound for $n = 4$ and $n = 5$, which yields the bound

$$\mathsf{P}\{|Z| > 3\} \le \frac{1\cdot 3\cdot 5\cdot 7}{9^4} \approx 0.016,$$

which is the best bound obtained so far.

s.11.0.29. We have

$$\begin{aligned}
\mathsf{P}\{X \le 1\} &= \int_0^1 \frac{3}{4}x(2-x)\,dx && (13.0.235)\\
&= \frac{3}{4}\left[x^2 - \frac{1}{3}x^3\right]_0^1 dx && (13.0.236)\\
&= \frac{3}{4}\left(1 - \frac{1}{3}\right) && (13.0.237)\\
&= 1/2. && (13.0.238)
\end{aligned}$$

An argument based on the fact that f_X is a parabola centered around 1 is also fine.

Correct initial integral: 0.5 point.
Correct solution: 0.5 point.

s.11.0.30. Let Y denote the payout in thousands of euros. Then, $Y = \min\{X, 1\}$. We find

$$\begin{aligned}
\mathsf{E}[Y|X \le 1] &= \mathsf{E}[X|X \le 1] && (13.0.239)\\
&= \int_0^1 x f_X(x|X \le 1)\,dx && (13.0.240)\\
&= \int_0^1 x \frac{f_X(x)}{\mathsf{P}\{X \le 1\}}\,dx && (13.0.241)\\
&= 2\int_0^1 x\frac{3}{4}x(2-x)\,dx && (13.0.242)\\
&= \frac{3}{2}\int_0^1 (2x^2 - x^3)\,dx && (13.0.243)\\
&= \frac{3}{2}\left[\frac{2}{3}x^3 - \frac{1}{4}x^4\right]_0^1 && (13.0.244)\\
&= \frac{3}{2}\left(\frac{2}{3} - \frac{1}{4}\right) && (13.0.245)\\
&= \frac{5}{8}. && (13.0.246)
\end{aligned}$$

Writing down $\mathsf{E}[X|X \le 1]$: 0.5 point.
Correct derivation and solution: 1 point.

s.11.0.31. By the law of total expectation,

$$\mathsf{E}[Y] = \mathsf{P}\{X \le 1\}\,\mathsf{E}[Y|X \le 1] + \mathsf{P}\{X > 1\}\,\mathsf{E}[Y|X > 1] \tag{13.0.247}$$
$$= \frac{1}{2}\frac{5}{8} + \frac{1}{2}\cdot 1 \tag{13.0.248}$$
$$= \frac{13}{16}. \tag{13.0.249}$$

Mentioning law of total expectation: 0.5 point.
Correctly using law of total expectation: 0.5 point.
Correct solution: 0.5 point.

s.11.0.32. By Adam's law,

$$\mathsf{E}[T] = \mathsf{E}[\mathsf{E}[T|Y]] \tag{13.0.250}$$
$$= \mathsf{E}\left[\frac{3}{2}Y\right] \tag{13.0.251}$$
$$= \frac{3}{2}\mathsf{E}[Y] \tag{13.0.252}$$
$$= \frac{3}{2}\cdot\frac{13}{16} = \frac{39}{32}. \tag{13.0.253}$$

Mentioning and using Adam's law: 0.5 point.
Correct solution: 0.5 point.

Note: if in your interpretation, $Y = y$ is actually given (I realized that the question is ambiguous), then $\frac{3}{2}y$ (lowercase!) will also be regarded as correct.

s.11.0.33. Let I_i be the indicator r.v. for the ith poem taking more time to read than each of poem 1, 2 and 3. Then:

$$\begin{aligned}\mathsf{P}\{I_i = 1\} &= \mathsf{P}\{X_i > X_1, X_i > X_2, X_i > X_3\}\\ &= \mathsf{P}\{X_i = \max\{X_1, X_2, X_3, X_i\}\}\\ &= \frac{1}{4},\end{aligned}$$

by symmetry. Then $\mathsf{E}\left[\Sigma_{i=4}^{5} I_i\right] = \frac{1}{4}\cdot 2 = \frac{1}{2}$.

s.11.0.34. In order to answer this question, we want to know $P(X_i - X_1 > 1)$, for $i = 4,5$. We first consider $i = 4$. Since X_4 and X_1 are jointly normal distributed, $X_4 - X_1$ is also normally distributed, with $\mathsf{E}[X_4 - X_1] = \mathsf{E}[X_4] - \mathsf{E}[X_1] = 0$ and

$$\begin{aligned}\mathsf{V}[X_4 - X_1] &= \mathsf{V}[X_4] + \mathsf{V}[-X_1] + 2\,\mathsf{Cov}[X_4, -X_1]\\ &= 1 + 1 - 2\cdot\frac{1}{2}\cdot 1\cdot 1\\ &= 1\end{aligned}$$

Then we know $X_4 - X_1$ follows a standard normal distribution and $P(X_4 - X_1 > 1) = 0.16$ Similarly, $P(X_5 - X_1 > 1) = 0.16$. So the average number of the remaining poems that take 1 minute more to read than the first poem is $0.16\cdot 2 = 0.32$.

s.11.0.35. $\mathsf{Cov}[X_1 - cX_4, X_4] = \mathsf{Cov}[X_1 - cX_4, X_4] = \mathsf{Cov}[X_1, X_4] - c\mathsf{V}[X_4] = \frac{1}{2} - c$, so for $c = \frac{1}{2}$, we have that $X_1 - cX_4$ and X_4 are uncorrelated. Since $(X_1, ..., X_5)$ has the multivariate normal distribution, it follows that $X_1 - cX_4$ and X_4 are independent.

s.11.0.36. Since X and Y are independent, their joint PDF is the product of their marginal PDFs. This gives:

$$f(x, y) = (1e^{-1x})(2e^{-2y}) = 2e^{-x-2y}$$

One mistake, zero points

s.11.0.37. Calculating this integral gives:

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\,\mathrm{d}x\,\mathrm{d}y =& \int_0^{\infty}\int_0^{\infty} 2e^{-x-2y}\,\mathrm{d}x\,\mathrm{d}y \\ =& \int_0^{\infty} 2e^{-2y)}[-e^{-x}]_0^{\infty}\,\mathrm{d}y \\ =& \int_0^{\infty} 2e^{-2y}\,\mathrm{d}y \\ =& \int_0^{\infty} h(y)\,\mathrm{d}y \\ =& 1\end{aligned}$$

Where $h(y)$ is the PDF of Y.
One mistake, zero points

s.11.0.38. Similar to example 7.2.2., we get that by 2D LOTUS:

$$\begin{aligned}\mathsf{E}[|X - Y|] =& \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x-y| f(x, y)\,\mathrm{d}x\,\mathrm{d}y \\ =& \int_0^{\infty}\int_y^{\infty} (x-y)(2e^{-x-2y})\,\mathrm{d}x\,\mathrm{d}y + \int_0^{\infty}\int_0^{y} (y-x)(2e^{-x-2y})\,\mathrm{d}x\,\mathrm{d}y \\ =& \int_0^{\infty}\left([(x-y)\cdot -2e^{-x-2y}]_y^{\infty} - \int_y^{\infty} -2e^{-x-2y}\,\mathrm{d}x\right)\mathrm{d}y \\ &+ \int_0^{\infty}\left([(y-x)\cdot -2e^{-x-2y}]_0^{y} - \int_0^{y} 2e^{-x-2y}\,\mathrm{d}x\right)\mathrm{d}y \\ =& \int_0^{\infty} [-2e^{-x-2y}]_y^{\infty}\,\mathrm{d}y + \int_0^{\infty} 2ye^{-2y} - [-2e^{-x-2y}]_0^{y}\,\mathrm{d}y \\ =& \int_0^{\infty} 2e^{-3y}\,\mathrm{d}y + \int_0^{\infty} 2ye^{-2y} + 2e^{-3y} - 2e^{-2y}\,\mathrm{d}y \\ =& \left[-\tfrac{2}{3}e^{-3y}\right]_0^{\infty} + \left[-ye^{-2y} + \tfrac{1}{2}e^{-2y} - \tfrac{2}{3}e^{-3y} + e^{-2y}\right]_0^{\infty} \\ =& \tfrac{2}{3} + \tfrac{1}{2} + \tfrac{2}{3} - 1 = \tfrac{5}{6}.\end{aligned}$$

One point for writing down the integral correctly using LOTUS and splitting it up correctly. Two points for the computations.

s.11.0.39. It loads the required packages and creates one sample with 500 observations from a $\mathcal{N}(50, 200)$-distribution. Then for all observations it standardizes and takes the square. The empirical CDF of the standardized values is plotted against the PDF of a sample from a chi-square distribution with 1 degree of freedom. It can be seen they look very much alike. This is expected as for $Z \sim \mathcal{N}(0,1)$ we have $Z^2 \sim \chi_1^2$.
0.5 points for mentioning data is standardized, 0.5 points for mentioning a squared standard normal r.v. is chi-square.

s.11.0.40. For $y \in (0,1)$:

$$F_{X_1}(y) = \mathsf{P}\left\{X_1 \le y\right\} = \mathsf{P}\left\{(U_1)^{1/a} \le y\right\} = \mathsf{P}\left\{U_1 \le y^a\right\} = F_{U_1}(y^a).$$

We know that $F_{U_1}(y) = y$ for $y \in (0,1)$. Hence $F_{X_1}(y) = F_{U_1}(y^a) = y^a$. Then

$$f_{X_1}(y) = \frac{\partial y^a}{\partial y} = a y^{a-1} \quad \forall y \in (0,1).$$

Now we can say $f_{X_2}(y) = b y^{b-1}$ for all $y \in (0,1)$. Both PDFs are 0 outside of this region.

Grading scheme:

- Correct application of transformation theorem or CDF technique 0.5pt.
- Most of: the correct bounds, the verification the theorem is applicable, no mistakes in calculation 0.5pt.

s.11.0.41. $X_1 \sim \text{Beta}(a, 1)$ and $X_2 \sim \text{Beta}(b, 1)$.

Grading scheme:

- Correct 0.5pt.

s.11.0.42. For $y \in (0,1)$ we have that

$$F_{1-B}(y) = \mathsf{P}\left\{1 - B \le y\right\} = \mathsf{P}\left\{B \ge 1 - y\right\} = 1 - \mathsf{P}\left\{B \le 1 - y\right\} = 1 - F_B(1-y).$$

We can write this as

$$\begin{aligned}
1 - F_B(1-y) &= 1 - \int_0^{1-y} \frac{x^{p-1}(1-x)^{q-1}}{\beta(p,q)}\,\mathrm{d}x \\
&= 1 - \int_1^{y} \frac{(1-x)^{p-1}(x)^{q-1}}{\beta(p,q)}\,\mathrm{d}(1-x) \\
&= 1 - \int_y^{1} \frac{(1-x)^{p-1}(x)^{q-1}}{\beta(p,q)}\,\mathrm{d}x \\
&= 1 - \int_y^{1} \frac{(1-x)^{p-1}(x)^{q-1}}{\beta(q,p)}\,\mathrm{d}x \\
&= \int_0^{y} \frac{(1-x)^{p-1}(x)^{q-1}}{\beta(q,p)}\,\mathrm{d}x \\
&= F_D(y)
\end{aligned}$$

For $D \sim \text{Beta}(q,p)$. Since both r.v.s have support $(0,1)$ and have the same CDF on this support we conclude $1-B \sim \text{Beta}(q,p)$. *Remark.* This can also be shown by looking at the PDF, using a similar derivation.

Grading scheme:

- Noting the Beta function is symmetric 0.5pt.
- Calculating the correct inverse transformation 0.5pt.
- Correct application transformation theorem 0.5pt.
- Most of: the correct bounds, the verification the theorem is applicable, no mistakes in calculation 0.5pt.
- **OR:** The derivation as above correct 2pt.
- **OR:** A reasonable attempt at a story proof 1pt.

s.11.0.43. For a PDF to be valid it needs to be non-negative and integrate to 1. Clearly f_Z is non-negative, so let's check the other condition. We know from parts (b) and (c) that $1-X_2 \sim \text{Beta}(1,b)$. Then,

$$\begin{aligned}\int_0^1 f_Z(y)\,\mathrm{d}y &= \int_0^{\frac{1}{2}} f_{X_1}(y)\,\mathrm{d}y + \int_{\frac{1}{2}}^1 f_{1-X_2}(y)\,\mathrm{d}y\\ &= \int_0^{\frac{1}{2}} a y^{a-1}\,\mathrm{d}y + \int_{\frac{1}{2}}^1 b(1-y)^{b-1}\,\mathrm{d}y\\ &= \left(\frac{1}{2}\right)^a + \left(\frac{1}{2}\right)^b.\end{aligned}$$

This should equal 1. The easy solution is $a=b=1$. The above can also be solved to obtain $b = -\frac{\ln(1-2^{-a})}{\ln 2}$, hence there are infinitely many solutions for $a,b>0$. For $a=b$, $a=b=1$ is the only solution and Z then follows the Beta(1,1) distribution.

Grading scheme:

- Correct integration 0.5pt.
- Found at least one other combination or showed such a combination must exist 0.5pt.
- Noticing that for $a=b=1$ there is a Beta distribution 0.5pt.

s.11.0.44. Since X and Y are independent, their joint PDF is the product of their marginal PDFs. This gives:

$$\begin{aligned}f(x,y) &= \left(\frac{1}{3-1}\right)(2e^{-2y})\\ &= \left(\frac{1}{2}\right)(2e^{-2y})\\ &= e^{-2y}, \quad \text{for } y>0 \text{ and } x \in [1,3]\end{aligned}$$

0.5 points for the correct expression, 0.5 points for the boundary

s.11.0.45. We have

$$\begin{aligned}\mathsf{P}\{Y \le X\} &= \int_1^3 \int_0^x e^{-2y} dy dx \\ &= \int_1^3 [-\frac{1}{2}e^{-2y}]_0^x dx \\ &= \int_1^3 (-\frac{1}{2}e^{-2x} + \frac{1}{2}) dx \\ &= -\frac{1}{2}\int_1^3 e^{-2x} dx + 1 \\ &= -\frac{1}{2}[-\frac{1}{2}e^{-2x}]_1^3 + 1 \\ &= \frac{1}{4}(e^{-6} - e^{-2}) + 1 \\ &= 1 - \frac{e^{-2} - e^{-6}}{4}.\end{aligned}$$

So $\mathsf{P}\{X \le Y\} = 1 - P(X \le Y) = \frac{e^{-2}-e^{-6}}{4}$.
One point for writing down an integral with the correct bounds. One point for the computations.

s.11.0.46. The code computes the sample mean for the minimum of $X \sim \text{Exp}(1)$ and $Y \sim \text{Exp}(\frac{1}{2})$. Since X and Y are independent, we have $\min(X, Y) \sim \text{Exp}(1+\frac{1}{2})$. So then $\mathsf{E}[\min(X, Y)] = \frac{1}{1\frac{1}{2}} = \frac{2}{3}$. By the law of large numbers, the sample mean converges to the population mean for large enough samples. Hence, we expect the output to be approximately $2/3$. *0.5 points for explaining what the code does. 1 points for computing the population mean (with correct argumentation). 0.5 points for mentioning the (strong/weak/any) law of large numbers.*

s.11.0.47.

$$\begin{aligned}&\mathsf{Cov}[2N_1 + N_2, 2N_1 - N_2] \\ &= \mathsf{Cov}[2N_1, 2N_1] - \mathsf{Cov}[2N_1, N_2] + \mathsf{Cov}[N_2, 2N_1] - \mathsf{Cov}[N_2, N_2] \cdots \text{(0.5 point)} \\ &= 4 \cdot Var(N_1) - Var(N_2) \\ &= 4\lambda_1 - \lambda_2 \neq 0. \cdots \text{(0.5 point)}\end{aligned}$$

s.11.0.48. Since N_1 and N_2 are independent, $N \sim Pois(\lambda_1 + \lambda_2)$.(0.5 point)
$X|N \sim Bin(N, p)$ and $X \sim Pois((\lambda_1 + \lambda_2)p)$ by the Chicken-egg theory. (0.5 point)

s.11.0.49. Let $Y = N - X$ be the number of customers that do not apply for a credit card. Then we know $Y \sim Pois((\lambda_1 + \lambda_2)q)$ with $q = 1 - p$, and X and Y are independent. (0.5 point)

$$\begin{aligned}\mathsf{Cov}[N, X] &= \mathsf{Cov}[X + Y, X] \\ &= \mathsf{Cov}[X, X] + \mathsf{Cov}[Y, X] \\ &= Var(X) \cdots \text{(0.5 point)} \\ &= (\lambda_1 + \lambda_2)p \cdots \text{(0.5 point)}\end{aligned}$$

Then it follows that

$$\begin{aligned}\rho_{X,N} &= \frac{\mathrm{Cov}[N,X]}{sd(N)\cdot sd(X)}\\ &= \frac{(\lambda_1+\lambda_2)p}{\sqrt{\lambda_1+\lambda_2}\cdot\sqrt{(\lambda_1+\lambda_2)p}}\\ &= \sqrt{p}.\cdots(0.5\text{ point})\end{aligned}$$

s.11.0.50. Line 1: Load the package "mvtnorm" so that we can use the function *rmvnorm.*
Line 2: Set a random seed to reproduce the same results.
Line 3: Generate a 3×3 identity matrix.
Line 4: Generate a vector of 0's.
Line 5: Generate 100 multivariate normally distributed variables $\mathbf{X} = x_1, x_2, x_3$ with mean equal to $(0,0,0)$ and variance equal to an identity matrix.
Line 6: Calculate the covariance matrix of the first 2 columns of matrix X.
(0.5 points for mentioning at least 3 of the above.)

s.11.0.51. We know by independence of X and Y that $X-Y \sim \mathcal{N}(0,2)$. By the fact that $cZ \sim \mathcal{N}(0,c^2)$ for all $c \in \mathbb{R}$, using $c=\sqrt{2}$ we get the same distribution.
One point for the fact $X-Y \sim \mathcal{N}(0,2)$, one point for a correct conclusion that this equals the density of Z.

s.11.0.52. Using 2D LOTUS, substitution, and the integral equation above, we obtain

$$\begin{aligned}E|X-Y| &= \int_{\infty}^{\infty} |\sqrt{2}z|\phi(z)dz\\ &= \int_{-\infty}^{\infty} |\sqrt{2}z|\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz\\ &= \sqrt{2}\int_{-\infty}^{\infty} |z|\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz\\ &= \sqrt{2}\int_{0}^{\infty} z\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz + \sqrt{2}\int_{-\infty}^{0}(-z)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz\\ &= \sqrt{2}\int_{0}^{\infty} z\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz - \sqrt{2}\int_{\infty}^{0} u\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2}du\\ &= \sqrt{2}\int_{0}^{\infty} z\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz + \sqrt{2}\int_{0}^{\infty} u\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2}du\\ &= 2\sqrt{2}\int_{0}^{\infty} z\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz\end{aligned}$$

0.5 points for the first integral, 1 point for splitting up correctly, 0.5 points for simplifying correctly.

s.11.0.53. Integration by substitution yields:

$$
\begin{aligned}
E(|X-Y|) =& 2\sqrt{2}\int_0^{\infty} z\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz \\
=& 2\sqrt{2}\int_{0^2}^{\infty^2} \frac{1}{\sqrt{2\pi}}e^{-u}du \\
=& 2\sqrt{2}\int_{0}^{\infty^2} \frac{1}{\sqrt{2\pi}}e^{-u}du \\
=& \frac{2}{\sqrt{\pi}}\left(1-e^{-\infty^2}\right) = \frac{2}{\sqrt{\pi}}
\end{aligned}
$$

0.5 points for the right substitution, 0.5 points for the rest of the computations.

s.11.0.54. It loads the required packages and creates one sample with 10000 observations from a r.v. $Y \sim \mathcal{N}(1,2)$. Then for all observations y_i it calculates e^{y_i} and stores it into a vector. Finally, it estimates the mean of a log-normal r.v. $X = e^Y$.
0.5 points for mentioning that for observations of a normal r.v. the exponent is taken. 0.5 points for stating a mean of X is estimated.

s.11.0.55. The random vector (X, Y) is uniformly distributed on $(-\pi,\pi)^2$. Hence, the joint pdf is given by

$$
\begin{aligned}
f(x,y) =& \left(\frac{1}{\pi-(-\pi))}\right)\left(\frac{1}{(\pi-(-\pi))}\right) \\
=& \frac{1}{4\pi^2}
\end{aligned}
$$

for $-\pi < x < \pi$ and $-\pi < x < \pi$. *0.5 points for the correct solution, 0.5 points for the boundaries.*

s.11.0.56. Note that $N^2 = c^2(X^2+Y^2)$. Using LOTUS:

$$\begin{aligned}
\mathsf{E}\left[N^2\right] =& \mathsf{E}\left[c^2(X^2+Y^2)\right] \\
=& \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c^2\left(x^2+y^2\right) f(x,y)\,dx\,dy \\
=& c^2\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left(x^2+y^2\right)\left(\frac{1}{4\pi^2}\right)dx\,dy \\
=& c^2\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{x^2+y^2}{4\pi^2}\,dx\,dy \\
=& \frac{c^2}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} x^2+y^2\,dx\,dy \\
=& \frac{c^2}{4\pi^2}\int_{-\pi}^{\pi}\left[\frac{x^3}{3}+y^2x\right]_{-\pi}^{\pi}dy \\
=& \frac{c^2}{4\pi^2}\int_{-\pi}^{\pi}\left(\left(\frac{\pi^3}{3}+y^2\pi\right)-\left(\frac{(-\pi)^3}{3}-y^2\pi\right)\right)dy \\
=& \frac{c^2}{4\pi^2}\int_{-\pi}^{\pi}\left(\frac{2\pi^3}{3}+2y^2\pi\right)dy \\
=& \frac{c^2}{4\pi^2}\int_{-\pi}^{\pi}\frac{2}{3}\pi^3+2\pi y^2\,dy \\
=& \frac{c^2}{4\pi^2}\left[\frac{2}{3}\pi^3 y+\frac{2}{3}\pi y^3\right]_{-\pi}^{\pi} \\
=& \frac{c^2}{4\pi^2}\left(\frac{2}{3}\pi^4+\frac{2}{3}\pi^4+\frac{2}{3}\pi^4+\frac{2}{3}\pi^4\right) \\
=& \frac{c^2}{4\pi^2}\frac{8}{3}\pi^4 = c^2\frac{2\pi^2}{3}
\end{aligned}$$

So then since $c^2\frac{2\pi^2}{3}=1 \implies c^2=\frac{3}{2\pi^2}$. We have that $c=\sqrt{\frac{3}{2\pi^2}}=\frac{\sqrt{\frac{3}{2}}}{\pi}=\frac{\sqrt{1\frac{1}{2}}}{\pi}>0$. Where you should use $c>0$.

1 point for $N = c^2(X^2+Y^2)$ and writing down the integral correctly using LOTUS. 1 point for the calculations to find the expectation. 1 point for finding the correct value of c.

s.11.0.57. The code computes the integral over the entire domain of a Cauchy random variable. Hence, it returns a value of one.
0.5 points for explaining what the code does. 0.5 points for mentioning the correct output.

s.11.0.58. We first find the distribution of Y_1.

$$F_{Y_1}(y)=\mathsf{P}\left\{Y_1\le y\right\}=\mathsf{P}\left\{2\lambda X_1\le y\right\}=\mathsf{P}\left\{X_1\le\frac{y}{2\lambda}\right\}=F_{X_1}\left(\frac{y}{2\lambda}\right).$$

We can easily find that

$$F_{X_1}(y)=\int_0^y \lambda e^{-\lambda x}\,\mathrm{d}x = 1-e^{-\lambda y}$$

for $y > 0$, and 0 elsewhere. Then $F_{Y_1} = 1 - e^{-y/2}$, and we conclude $Y_1 \sim \text{Expo}(\frac{1}{2})$. By symmetry, $Y_2 \sim \text{Expo}(\frac{1}{2})$. Note Y_1 and Y_2 are independent. By the convolution theorem, we know that

$$\begin{aligned} f_{Y_1+Y_2}(t) &= \int_0^t f_{Y_1}(t-s) f_{Y_2}(s)\, \mathrm{d}s \\ &= \int_0^t \frac{1}{2} e^{-\frac{t-s}{2}} \frac{1}{2} e^{-\frac{s}{2}}\, \mathrm{d}s \\ &= \frac{1}{4} \int_0^t e^{-\frac{t}{2}}\, \mathrm{d}s \\ &= \frac{1}{4} t e^{-\frac{t}{2}} \end{aligned}$$

for $t > 0$, and 0 elsewhere.

Grading scheme:

- Derived the correct distribution of Y_i 0.5pt.
- Noticed that Y_i are independent to apply convolution theorem 0.5pt.
- Convolution theorem correctly applied 0.5pt.
- Most of: the correct bounds, no mistakes in calculation 0.5pt.

s.11.0.59. B.H. show that $\sum_{i=1}^{n} X_i \sim \text{Gamma}(n, \lambda)$, whereas $nX_i \sim \text{Expo}\left(\frac{\lambda}{n}\right)$. The distributions are different since the first one is a sum of independent random variables, whereas the latter is one random variable that is scaled.

Grading scheme:

- Correct distributions given 0.5pt.
- Reason why 0.5pt. (very very lenient here)

s.11.0.60. We know the PDF of Z is given by

$$f_Z(x) = \frac{1}{2^n \Gamma(n)} x^{n-1} e^{-\frac{x}{2}}$$

for $x > 0$, and 0 elsewhere. In (a) we have shown the PDF of S. Then, by the convolution formula we get

$$\begin{aligned} f_W(w) &= \int_0^w \frac{1}{4}(w-x)e^{-\frac{(w-x)}{2}} \frac{1}{2^n\Gamma(n)} x^{n-1} e^{-\frac{x}{2}} \, \mathrm{d}x \\ &= \frac{e^{-\frac{w}{2}}}{2^{n+2}\Gamma(n)} \left(\int_0^w w x^{n-1} \, \mathrm{d}x - \int_0^w x^n \, \mathrm{d}x \right) \\ &= \frac{e^{-\frac{w}{2}}}{2^{n+2}\Gamma(n)} \left(\frac{w^{n+1}}{n} - \frac{w^{n+1}}{n+1} \right) \\ &= \frac{w^{n+1} e^{-\frac{w}{2}}}{2^{n+2}\Gamma(n)} \left(\frac{1}{n(n+1)} \right) \\ &= \frac{w^{n+1} e^{-\frac{w}{2}}}{2^{n+2}\Gamma(n+2)} \end{aligned}$$

for $t > 0$ and 0 elsewhere. This is the Gamma$\left(n+2, \frac{1}{2}\right)$ distribution.

Grading scheme:

- Noting Z follows a Gamma distribution with correct parameters 0.5pt. (to be lenient)
- Applying the convolution theorem 0.5pt.
- Recognition of final distribution 0.5pt.
- Most of: the correct bounds, no mistakes in calculation 0.5pt.
- No recognition that a sum of Gamma distributions with different rate parameters does not work as you want it to -0.5pt (if applicable).

s.11.0.61. Notice that the function $|x|$ is not one-to-one on $(-1,1)$, hence we cannot use the transformation theorem here. We see that since $U \in (-1,1)$, $B \in [0,1)$. Then we see that

$$F_B(y) = \mathsf{P}\left\{B \le y\right\} = \mathsf{P}\left\{|X| \le y\right\} = \mathsf{P}\left\{-y \le X \le y\right\} = F_X(y) - F_X(-y)$$

For $y \in [0,1]$. We know $F_X(y) = \frac{y+1}{2}$, so then $F_X(y) - F_X(-y) = \frac{y+1}{2} - \frac{-y+1}{2} = y$, and we conclude that $B \sim \text{Unif}(0,1)$. Then we know $\mathsf{E}[B] = \frac{1}{2}$.

Grading scheme:

- Correct derivation with CDF 0.5pt
- No mistakes in the above 0.5pt.
- Expectation 0.5pt.

s.11.0.62. Using the given relation of M_X, we see that

$$M_X(t) = e^t M_X(-t) = e^t \mathsf{E}\left[e^{-tX}\right] = \mathsf{E}\left[e^{t(1-X)}\right] = M_{1-X}(t)$$

and since the MGF determines the distribution we can immediately say that $X \sim 1-X$. Then it must be that $\mathsf{E}[X] = \mathsf{E}[1-X]$ and then by linearity we have that $\mathsf{E}[X] = \frac{1}{2}$. This is not enough information to conclude what distribution X has, we can only see that it must be symmetric around $\frac{1}{2}$.

Grading scheme:

- Note $X \sim 1-X$ 0.5pt.
- Correct expectation 0.5pt.
- Cannot conclude the same distribution 0.5pt.
- **OR:** Correctly solved the differential equation/took the derivative and concluded the result 1pt.
- Cannot conclude the same distribution 0.5pt.

s.11.0.63. As usual, we start with the CDF of B, this is known to be $F_B(y) = y$ for $y \in [0,1]$ (and 1 for $y > 1$, 0 for $y < 0$). Then we have that

$$\begin{aligned}
F_X(y) &= \mathsf{P}\{X \leq y\} \\
&= \mathsf{P}\left\{\kappa + \lambda \ln\left(\frac{B}{1-B}\right) \leq y\right\} \\
&= \mathsf{P}\left\{\ln\left(\frac{B}{1-B}\right) \leq \frac{y-\kappa}{\lambda}\right\} \\
&= \mathsf{P}\left\{B \leq \frac{\exp\frac{y-\kappa}{\lambda}}{1+\exp\frac{y-\kappa}{\lambda}}\right\} \\
&= F_B\left(\frac{\exp\frac{y-\kappa}{\lambda}}{1+\exp\frac{y-\kappa}{\lambda}}\right) \\
&= \frac{\exp\frac{y-\kappa}{\lambda}}{1+\exp\frac{y-\kappa}{\lambda}}
\end{aligned}$$

for $y \in \mathbf{R}$.

Grading scheme:

- CDF technique derivation 0.5pt.
- No mistakes 0.5pt.

s.11.0.64. Notice that F_X maps the real line onto the interval $(0,1)$. Then, Q_X must map the interval $(0,1)$ onto **R**. We find the inverse of the CDF of X as follows:

$$\begin{aligned}
&z = F_X(y) \implies \\
&z = \frac{e^y}{1+e^y} \implies \\
&z = \frac{1}{1+e^{-y}} \implies \\
&e^{-y}z = 1-z \implies \\
&e^y = \frac{z}{1-z} \implies \\
&y = \ln\frac{z}{1-z} = Q_X(z)
\end{aligned}$$

for $z \in (0,1)$.

Grading scheme:

- Mention the idea to invert the CDF 0.5pt. (of course this includes the people who did so)
- Correct inversion 0.5pt.
- Correct bounds for the quantile function 0.5pt bonus.

s.11.0.65. The length of Y is $n = 600$; Each element of Y is a mean of $k = N = 250$ i.i.d. Exp(4) r.v.s. The expectation is $\frac{1}{\lambda} = \frac{1}{4}$ and the variance is $\frac{1}{k\lambda^2} = \frac{1}{4000} = 0.00025$.

Grading scheme:

- 0.5 for getting both the length $n = 600$ and $k = 250$ correct (no partial credit);
- 0.5 for $\lambda = 4$, expectation $\frac{1}{4}$ and the factor $\frac{1}{16}$ in the variance (no partial credit);
- 0.5 for the factor $\frac{1}{k}$ in the variance.

s.11.0.66. The sum of k i.i.d. Exp(λ) r.v.s. has the Gamma(k,λ) distribution. Hence, the mean of k i.i.d. Exp(λ) r.v.s. has the Gamma($k, k\lambda$) distribution. In this exercise, $k = 250$ and $k\lambda = 1000$.
Grading scheme:

- 0.5 for Gamma with first parameter k
- 0.5 for the second parameter

s.11.0.67. By the CLT, $Z_1 \mathrel{\dot\sim} \text{Norm}(0,1)$. Hence, $Y_1 = \mu + \sigma/\sqrt{n}Z_1 \mathrel{\dot\sim} \text{Norm}(0.25, 0.0625/n)$.
Grading scheme:

- 0.5 for mentioning CLT and the distribution of Z_1;

- 0.5 for the approximate distribution of Y_1.

s.11.0.68. In the limit $k \to \infty$, each Z_i has the standard normal distribution by CLT, but the sum of ℓ powers of normal distributions has a non-trivial CDF.

- 0.5 for explaining that S does not converge to a constant; no points for just CLT (unless previous question was not answered).

s.11.0.69. By LLN, S does converge to a constant as $\ell \to \infty$, however, it converges to $\mathsf{E}\left[Z_1^{53}\right]$ for that fixed value of k. By symmetry, we have $\mathsf{E}\left[Z_1^{53}\right] = 0$. However, the gamma distribution is right-skewed, which implies $\mathsf{E}\left[T^{53}\right] > 0$. Hence, it does not converge to $\mathsf{E}\left[T^{53}\right]$.
Grading scheme:

- 0.5 for concluding that S converges to a constant using LLN.
- 0.5 for explaining why the constant is not equal to $\mathsf{E}\left[T^{53}\right]$.

s.11.0.70. Standard consequence of exponential rvs.
Grading:

- The time to the next arrival is not λ. -0.5.

s.11.0.71. Use the memoryless property of the exp distribution. When a job arrives first, we can model this as if we start from $n+1$ until we hit 0. Likewise, when a job leaves first, we start from $n-1$. The last term is the expected time until an event happens.
Grading:

- Some people write $P(S = X)$ and give this a positive probability. That is a grave mistake: -0.5.

s.11.0.72. Just fill in the expression in the previous exercise and check that the RHS and LHS match.

s.11.0.73. Use conditioning on L.

$$\mathsf{E}[T] = \mathsf{E}[\mathsf{E}[T|L(0)]] = \mathsf{E}\left[L(0)/(\mu-\lambda)\right] = \frac{\rho}{1-\rho}\frac{1}{\mu-\lambda} = \frac{\lambda}{\mu^2(1-\rho)^2}.$$

s.11.0.74. If $\lambda > \mu$, jobs arrive faster than they can be served. In such cases the queueing process drifts to infinity, in expectation.

The case $\lambda = \mu$ is difficult, and I don't expect you to discuss this.

s.11.0.75. We have

$$\mathsf{P}\{X \le 1\} = \int_0^1 \frac{3}{4} x(2-x)\,dx \tag{13.0.254}$$
$$= \frac{3}{4}\left[x^2 - \frac{1}{3}x^3\right]_0^1 dx \tag{13.0.255}$$
$$= \frac{3}{4}\left(1 - \frac{1}{3}\right) \tag{13.0.256}$$
$$= 1/2. \tag{13.0.257}$$

An argument based on the fact that f_X is a parabola centered around 1 is also fine.

Correct initial integral: 0.5 point.

Correct solution: 0.5 point.

s.11.0.76. Note that $Y = \min\{X, 1\}$. We find

$$\mathsf{E}[Y|X \le 1] = \mathsf{E}[X|X \le 1] \tag{13.0.258}$$
$$= \int_0^1 x f_X(x|X \le 1)\,dx \tag{13.0.259}$$
$$= \int_0^1 x \frac{f_X(x)}{\mathsf{P}\{X \le 1\}}\,dx \tag{13.0.260}$$
$$= 2\int_0^1 x \frac{3}{4} x(2-x)\,dx \tag{13.0.261}$$
$$= \frac{3}{2}\int_0^1 (2x^2 - x^3)\,dx \tag{13.0.262}$$
$$= \frac{3}{2}\left[\frac{2}{3}x^3 - \frac{1}{4}x^4\right]_0^1 \tag{13.0.263}$$
$$= \frac{3}{2}\left(\frac{2}{3} - \frac{1}{4}\right) \tag{13.0.264}$$
$$= \frac{5}{8}. \tag{13.0.265}$$

Writing down $\mathsf{E}[X|X \le 1]$: 0.5 point.

Correct derivation and solution: 1 point.

s.11.0.77. By the law of total expectation,

$$\mathsf{E}[Y] = \mathsf{P}\{X \le 1\}\,\mathsf{E}[Y|X \le 1] + \mathsf{P}\{X > 1\}\,\mathsf{E}[Y|X > 1] \tag{13.0.266}$$
$$= \frac{1}{2}\frac{5}{8} + \frac{1}{2} \cdot 1 \tag{13.0.267}$$
$$= \frac{13}{16}. \tag{13.0.268}$$

Mentioning law of total expectation: 0.5 point.

Correctly using law of total expectation: 0.5 point.

Correct solution: 0.5 point.

s.11.0.78. By Adam's law,

$$\begin{aligned} \mathsf{E}[Z] &= \mathsf{E}[\mathsf{E}[Z|Y]] && (13.0.269) \\ &= \mathsf{E}\left[\frac{3}{2}Y\right] && (13.0.270) \\ &= \frac{3}{2}\mathsf{E}[Y] && (13.0.271) \\ &= \frac{3}{2}\cdot\frac{13}{16} = \frac{39}{32}. && (13.0.272) \end{aligned}$$

Mentioning and using Adam's law: 0.5 point.
Correct solution: 0.5 point.

Note: if in your interpretation, $Y = y$ is actually given (I realized that the question is ambiguous), then $\frac{3}{2}y$ (lowercase!) will also be regarded as correct.

s.11.0.79. The length of Y is $N = 500$; Each element of Y is a mean of $k = n = 200$ i.i.d. Exp(0.5) r.v.s. The expectation is $\frac{1}{\lambda} = 2$ and the variance is $\frac{1}{k\lambda^2} = \frac{1}{200\cdot 1/4} = 0.02$.

Grading scheme:

- 0.5 for getting both the length $N = 500$ and $k = 200$ correct (no partial credit);
- 0.5 for $\lambda = 0.5$, expectation 2 and the factor $\frac{1}{\lambda^2} = 4$ in the variance (no partial credit);
- 0.5 for the factor $\frac{1}{k}$ in the variance.

s.11.0.80. The sum of k i.i.d. Exp(λ) r.v.s. has the Gamma(k,λ) distribution. Hence, the mean of k i.i.d. Exp(λ) r.v.s. has the Gamma$(k, k\lambda)$ distribution. In this exercise, $k = 200$ and $k\lambda = 100$.
Grading scheme:

- 0.5 for Gamma with first parameter k
- 0.5 for the second parameter

s.11.0.81. By the CLT, $Z_1 \dot\sim \text{Norm}(0,1)$. Hence, $Y_1 = \mu + \sigma/\sqrt{n}Z_1 \dot\sim \text{Norm}(2, 4/n)$.
Grading scheme:

- 0.5 for mentioning CLT and the distribution of Z_1;
- 0.5 for the approximate distribution of Y_1.

s.11.0.82. In the limit $k \to \infty$, each Z_i has the standard normal distribution by CLT, but the sum of ℓ powers of normal distributions has a non-trivial CDF.

- 0.5 for explaining that S does not converge to a constant; no points for just CLT (unless previous question was not answered).

s.11.0.83. By LLN, S does converge to a constant as $\ell \to \infty$, however, it converges to $\mathsf{E}\left[Z_1^{29}\right]$ for that fixed value of k. By symmetry, we have $\mathsf{E}\left[Z_1^{29}\right] = 0$. However, the gamma distribution is right-skewed, which implies $\mathsf{E}\left[T^{29}\right] > 0$. Hence, it does not converge to $\mathsf{E}\left[T^{29}\right]$.
Grading scheme:

- 0.5 for concluding that S converges to a constant using LLN.
- 0.5 for explaining why the constant is not equal to $\mathsf{E}\left[T^{29}\right]$.

s.11.0.84.

$$\begin{aligned}
&\mathsf{Cov}\left[N_1+N_2, N_2-2N_3\right] \\
=&\mathsf{Cov}\left[N_1, N_2\right] - \mathsf{Cov}\left[N_1, 2N_3\right] + \mathsf{Cov}\left[N_2, N_2\right] - \mathsf{Cov}\left[N_2, 2N_3\right] \cdots \text{(0.5 point)} \\
=&Var(N_2) \\
=&\lambda_2 > 0. \cdots \text{(0.5 point)}
\end{aligned}$$

s.11.0.85. Since N_1, N_2 and N_3 are independent, $N \sim Pois(\lambda_1+\lambda_2+\lambda_3)$.(0.5 point)
$X|N \sim Bin(N, p)$, and $X \sim Pois((\lambda_1+\lambda_2+\lambda_3)p)$ by the Chicken-egg theory. (0.5 point)

s.11.0.86. Let $Y = N - X$ be the number of customers that do not buy cheese. Then we know $Y \sim Pois((\lambda_1+\lambda_2+\lambda_3)q)$ with $q = 1-p$, and X and Y are independent. (0.5 point)

$$\begin{aligned}
\mathsf{Cov}\left[N, X\right] &= \mathsf{Cov}\left[X+Y, X\right] \\
&= \mathsf{Cov}\left[X, X\right] + \mathsf{Cov}\left[Y, X\right] \\
&= Var(X) \cdots \text{(0.5 point)} \\
&= (\lambda_1+\lambda_2+\lambda_3)p \cdots \text{(0.5 point)}
\end{aligned}$$

Then it follows that

$$\begin{aligned}
\rho_{X,N} &= \frac{\mathsf{Cov}\left[N, X\right]}{sd(N)\cdot sd(X)} \\
&= \frac{(\lambda_1+\lambda_2+\lambda_3)p}{\sqrt{\lambda_1+\lambda_2+\lambda_3}\cdot\sqrt{(\lambda_1+\lambda_2+\lambda_3)p}} \\
&= \sqrt{p}. \cdots \text{(0.5 point)}
\end{aligned}$$

s.11.0.87. Line 1: Load the package "mvtnorm" so that we can use the function *rmvnorm*.
Line 2: Set a random seed to reproduce the same results.
Line 3: Generate a 2×2 identity matrix A=$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$.
Line 4: Generate a vector $B = (1, 2)$.
Line 5: Generate 50 multivariate normally distributed variables $\mathbf{X} = x_1, x_2$ with mean equal to B and variance equal to A.
Line 6: Calculate the correlation of the 3rd and 41th rows of matrix X.
(0.5 points for mentioning at least 3 of the above.)

s.11.0.88. a. substitute the definition of S

b. Linearity of expectation

c. On the set $\{T = t\}$, $T = t$. Hence we can replace T by t.

d. On the set $\{T = t\}$, $T = t$. Hence we can replace T by t. And the (conditional) expectation of a constant is that constant.
Each property missed, e.g, linearity of expectation, minus 0.5.

s.11.0.89. We can use the result of part 1. Since $\mathsf{P}\{T = t\} = 1$, $\mathsf{E}[R] = 1/\mu$, $\mathsf{E}[N(t)] = \lambda t$, and indepence of R_i and N, and R_i iid,

$$\begin{aligned}
\mathsf{E}[S] &= t + \mathsf{E}\left[\sum_{i=1}^{N(t)} R_i\right] \\
&= t + \mathsf{E}\left[\mathsf{E}\left[\sum_{i=1}^{N(t)} R_i \,\middle|\, N(t)\right]\right] \\
&= t + \mathsf{E}[N(t)\,\mathsf{E}[R]] \\
&= t + \mathsf{E}[N(t)]\,\mathsf{E}[R] = t + \lambda t/\mu.
\end{aligned}$$

Typically, 0.5 point for each correct line, but I subtracted points If you say that $\mathsf{E}[N(T)\,\mathsf{E}[R]] = \mathsf{E}[R]$, or write $n\,\mathsf{E}[R]$ as final answer (apparently you did not get the idea that N is an rv.)

s.11.0.90. Here is an EXPLANATION:

1. T is a simulated service time of a job without interruptions.
2. N is is the simulated number of failures that occur during the service of the job
3. R is then a vector of simulated durations of each of the interruptions
4. Hence, S is the total time the job spends at the server
5. We sample the total service time a number of times, and count how often (mean or variance) this exceeds a threshold.

The points earned depends on how clear you explanation is.

Quite a few of you don't seem to understand what it means to *explain* something. Here are some answers that I saw that certainly don't explain what the code does:

1. 'The code does what's stated in the exercise.'. What's the explanation here? The question is also not: do you understood what the code does?
2. 'T is uniform rv, N is a Poisson rv, R is a also a uniform rv. We repeat this a few times.' Like this you just read the code, but I know you can read, so this type of answer is quite useless.

3. 'We collect a subset of the samples and print that.' This answer is just a repition of the hint.

So, next time, when you are asked to explain something, explain it. In your professional carreer, if you read out loud to your customer what s/he can read for him/herself, you'll be fired pretty soon.

s.11.0.91. (i). Since g is increasing, we have $Z \geq a$ if and only if $g(Z) \geq g(a)$, so $\{Z \geq a\}$ and $\{g(Z) \geq g(a)\}$ are the same event. Hence, $\mathsf{P}\{Z \geq a\} = \mathsf{P}\left\{g(Z) \geq g(a)\right\}$.
(ii). Since g is positive, we have $|g(Z)| = g(Z)$ and $g(a) > 0$. Hence, the inequality follows directly from Markov's inequality with r.v. $g(Z)$ and constant $g(a) > 0$.

Remarks and grading scheme:

- In part (ii), it is not sufficient to just say that it holds by Markov's inequality, also explain how you apply Markov's inequality.
- Since it was asked to indicate **clearly** where you use that g is positive and increasing, don't just say "since g is positive and increasing" twice. For part (i) the positivity is not needed, it is only needed that g is increasing. For part (ii), it is not needed that g is increasing, only the positivity is needed. Moreover, some explanation should be given how it is used.
- g is not necessarily a linear transformation, as some of you claimed.
- Not every positive and increasing function is convex or concave.
And using Jensen's inequality certainly doesn't work in (i): you are asked to prove an equality of probabilities, not an inequality of expectations.
- In part (i), the **if and only if** is really important. If you just mention that $g(Z) \geq g(a)$ if $Z \geq a$, then you are only proving that $\mathsf{P}\{Z \geq a\} \leq \mathsf{P}\left\{g(X) \geq g(a)\right\}$, because if you just say "$g(Z) \geq g(a)$ if $Z \geq a$", $g(Z) \geq g(a)$ could still be true in cases where $Z \geq a$ is not, and hence $g(Z) \geq g(a)$ can still have a larger probability. (I've not been strict on this when grading.)
- Grading: 0.5 for a sufficient explanation for (i), 0.5 for a sufficient explanation for (ii) and 0.5 for clearly indicating where it is used that g is positive and increasing and writing a clear answer overall.

s.11.0.92. Note that $g(x) = e^{tx^2}$ is positive and increasing on $(0, \infty)$ for $t > 0$. By applying the inequality of the first question with $a = 3$ we find

$$\mathsf{P}\{|Y| > 3\} \leq e^{-9t}\,\mathsf{E}\left[e^{t|Y|^2}\right] = e^{-9t}\,\mathsf{E}\left[e^{tY^2}\right].$$

Remarks and grading scheme:

- Most of you did not apply the inequality from the previous part to solve this question, but instead used Chernoff's inequality as follows:

$$\mathsf{P}\{|Z|>3\} = \mathsf{P}\left\{Z^2>9\right\} \le e^{-9t}\,\mathsf{E}\left[e^{t|Z|^2}\right] = e^{-9t}\,\mathsf{E}\left[e^{tZ^2}\right],$$

  where the first equality holds since $|Z|>3$ if and only if $\mathsf{P}\left\{Z^2>9\right\}$. This is also correct, but takes a bit more time.

- Don't write nonsense like $e^{-3t}\,\mathsf{E}\left[e^{t|Z|}\right] = e^{-9t}\,\mathsf{E}\left[e^{tZ^2}\right]$, just to make it look like you solved the exercise although you didn't.

- Grading scheme: 1 point for solving the exercise. Partial credit (0.5) for a correct and relevant application of Chernoff's inequality.

s.11.0.93. Since $Y^2 \sim \chi^2_1$, we have $\mathsf{E}\left[e^{tY^2}\right] = \mathsf{E}\left[e^{tX}\right] = M_X(t) = (1-2t)^{-1/2}$.

So we minimize $e^{-9t}\,\mathsf{E}\left[e^{tY^2}\right] = e^{-9t}(1-2t)^{-1/2}$. It is easier if we take the logarithm first and minimize $-9t - \frac{1}{2}\log(1-2t)$. Its derivative to t is $-9+\frac{1}{1-2t}$, so setting the derivative to 0 yields $t=4/9$. The second derivative to t is $\frac{2}{(1-2t)^2}>0$ (the value at $t=4/9$ is 162), so the second order condition holds.

This yields $\mathsf{P}\{|Y|>3\} \le e^{-4}(1-8/9)^{-1/2} \approx 0.0549$.

Remarks and grading scheme:

- Unless explicitly noted, you have to give an analytical answer and derive everything using just pen and paper.

- It is essential that you include some derivations for computing the (second) derivative.

- It is not necessary to take the logarithm first, but it makes the derivation easier, and hence it is less likely that you make errors.

- Grading scheme: 0.5 for arguing that $\mathsf{E}\left[e^{tZ^2}\right] = (1-2t)^{-1/2}$ with sufficient explanation; 1 for taking the derivative and calculating that it is 0 at $t=4/9$; (0.5 if small mistake is made but resulting t satisfies $0<t<1/2$, or if it is remarked that this should be the case); 0.5 for calculating the second derivative and checking the second order condition;
  0.5 for filling in $t=4/9$ to provide the upper bound (if an incorrect value of t is found, this point can be given only if the resulting bound is between 0.0001 and 1, or if it is explicitly noted that the answer does not make sense).

s.11.0.94. By Bayes' rule we have

$$\begin{aligned}
f_1(\lambda|X_1=x_1) &= \frac{f_{X_1|\lambda}(x_1|\lambda)f_0(\lambda)}{f_{X_1}(x_1)} && (13.0.273)\\
&\propto f_{X_1|\lambda}(x_1|\lambda)f_0(\lambda) && (13.0.274)\\
&= \frac{b^a}{\Gamma(a)}\lambda^{a-1}e^{-b\lambda}\lambda e^{-\lambda x_1} && (13.0.275)\\
&\propto \lambda^a e^{-(b+x_1)\lambda}, && (13.0.276)
\end{aligned}$$

in which we recognize the pdf of a Gamma$(a+1, b+x_1)$ distribution (up to a scaling constant). Hence, the posterior distribution λ given $X_1 = x_1$ is Gamma$(a+1, b+x_1)$.

Applying Bayes' rule: 1 point.
Finding the expression for the posterior: 1 point.
recognizing a $Gamma(a+1, b+x_1)$ dist: 0.5 point.

s.11.0.95. Yes. The posterior distribution is in the same family of distributions (Gamma) as the prior. Hence, John has a conjugate prior.

Correct answer and motivation: 1 point.

s.11.0.96. The posterior after observing $X_1 = x_1$ becomes our new prior. Hence, our new prior is a Gamma$(a+1, b+x_1)$ distribution. From question 1 it follows that the prior after observing $X_2 = x_2$ then is a Gamma$(a+2, b+x_1+x_2)$ distribution. Hence, iterating this process, we find that the posterior distribution of λ after observing $X_1 = x_1, \dots, X_n = x_n$ is a Gamma$(a+n, b+\sum_{i=1}^n x_i)$ distribution.

Noting that the posterior becomes the new prior: 0.5 point.
Correct derivation and answer: 1 point.

s.11.0.97. After selecting tunnel B, which takes 3 minutes to travel, the mouse is back in the pit again, and the process starts over again.

s.11.0.98.

$$\mathsf{E}[T] = \mathsf{E}[T|X=A]/3 + \mathsf{E}[T|X=B]/3 + \mathsf{E}[T|X=C]/3. \tag{13.0.277}$$

$$\mathsf{E}[T|X=A] = 2 \tag{13.0.278}$$

$$\mathsf{E}[T|X=B] = 3 + \mathsf{E}[T] \tag{13.0.279}$$

$$\mathsf{E}[T|X=C] = 4 + \mathsf{E}[T]. \tag{13.0.280}$$

Solving gives $\mathsf{E}[T] = 9$.

Grading

- Not using the result of subquestion 1: no points.

s.11.0.99.

$$\mathsf{V}[T|X=A] = 0 \tag{13.0.281}$$

$$\mathsf{V}[T|X=B] = \mathsf{V}[T] \tag{13.0.282}$$

$$\mathsf{V}[T|X=C] = \mathsf{V}[T] \tag{13.0.283}$$

$$\mathsf{E}[\mathsf{V}[T|X]] = \mathsf{V}[T]\,2/3. \tag{13.0.284}$$

$$\mathsf{E}[T|X] = 2\,I_{X=A} + (3+\mathsf{E}[T])\,I_{X=B} + (4+\mathsf{E}[T])\,I_{X=C} \tag{13.0.285}$$

$$= 2\,I_{X=A} + 12\,I_{X=B} + 13\,I_{X=C} \tag{13.0.286}$$

$$\mathsf{V}[\mathsf{E}[T|X]] = 4\cdot 2/9 + 144\cdot 2/9 + 169\cdot 2/9 =: \alpha \tag{13.0.287}$$

$$\mathsf{V}[T] = \mathsf{V}[T]\,2/3 + \alpha \quad \text{EVE} \tag{13.0.288}$$

$$\mathsf{V}[T] = 3\alpha. \tag{13.0.289}$$

Here we use that $I_{X=A}$ etc are independent and Bernoulli distributed with success probability p, hence $\mathsf{V}[I_{X=A}] = pq = 1/3 \cdot 2/3$.

Grading

- Not using EVE: no points.
- I saw this: $\mathsf{E}[T^2] = \ldots + (3 + \mathsf{E}[])^2 1/3 + \ldots$. This is not correct of course.

s.11.0.100. By the strong law of large numbers, any sequence of tunnel selections that excludes tunnel A has probability zero.

Grading:

- Mention the LLN somehow. If not: 0 points.

s.11.0.101. a. On the set $\{N(t) = n\}$ $N(t) = n$. Hence we can replate $N(t)$ by n. Then use linearity of the expectation.

b. Since $\mathsf{E}[R]$ is a constant, we can take it out from the variance as a square.

c. Variance of Poisson rv is λt

Each property missed, e.g, linearity of expectation, minus 0.5.

s.11.0.102. Since $T = t$ a.s.,

$$\mathsf{E}[S] = \mathsf{E}[T] + \mathsf{E}[N(T)]\,\mathsf{E}[R] = t + r\,\mathsf{E}[N(t)] = t + r\lambda t.$$

The answer should also be simplied to show that you use all information that is available. Stopping at, e.g., $\mathsf{E}[N(t)\,\mathsf{E}[R]]$ is not completely sufficient. Here are some wrong answers. It's interesting to try to understand why.

$$\mathsf{E}[NR\,|\,N] \neq \mathsf{E}[NR] = \mathsf{E}[N]\,\mathsf{E}[R] \tag{13.0.290}$$

$$\mathsf{E}[S] \neq \mathsf{E}[T] + N(t)\,\mathsf{E}[R]. \tag{13.0.291}$$

Typically, 0.5 point for each correct line, but I subtracted points If you say that $\mathsf{E}[N(T)\,\mathsf{E}[R]] = \mathsf{E}[R]$, or write $n\,\mathsf{E}[R]$ as final answer (apparently you did not get the idea that N is an rv.)

s.11.0.103. Here is an EXPLANATION:

1. T is a simulated service time of a job without interruptions.
2. N is is the simulated number of failures that occur during the service of the job
3. R is then a vector of simulated durations of each of the interruptions
4. Hence, S is the total time the job spends at the server
5. We sample the total service time a number of times, and count how often (mean or variance) this exceeds a threshold.

The points earned depends on how clear you explanation is.

Quite a few of you don't seem to understand what it means to *explain* something. Here are some answers that I saw that certainly don't explain what the code does:

1. 'The code does what's stated in the exercise.'. What's the explanation here? The question is also not: do you understood what the code does?

2. 'T is uniform rv, N is a Poisson rv, R is a also a uniform rv. We repeat this a few times.' Like this you just read the code, but I know you can read, so this type of answer is quite useless.

3. 'We collect a subset of the samples and print that.' This answer is just a repition of the hint.

So, next time, when you are asked to explain something, explain it. In your professional carreer, if you read out loud to your customer what s/he can read for him/herself, you'll be fired pretty soon.

s.11.0.104. We start with finding the CDF of Y.

$$F_Y(y) = \mathsf{P}\left\{Y \leq y\right\} = \mathsf{P}\left\{e^X \leq y\right\} = \mathsf{P}\left\{X \leq \ln y\right\} = F_X(\ln y).$$

Then we differentiate this integral, and we obtain our PDF. Using the FTC, we get

$$\begin{aligned}
F_X(\ln y) &= \int_{-\infty}^{\ln y} f_X(x)\,\mathrm{d}x \implies \\
\frac{\mathrm{d}}{\mathrm{d}y}F_X(\ln y) &= \frac{\mathrm{d}}{\mathrm{d}y}\int_{-\infty}^{\ln y} f_X(x)\,\mathrm{d}x \\
&= f_X(\ln y)\frac{\mathrm{d}\ln y}{\mathrm{d}y} \\
&= f_X(\ln y)\frac{1}{y} \\
&= \frac{1}{\sqrt{2\pi}\mu y}\exp\left(-\frac{1}{2\mu^2}(\ln y - \mu)^2\right)
\end{aligned}$$

for $y > 0$. Here $f_X(x)$ is the PDF of the normal random variable X.

Grading scheme:

- No deduction if second parameter is assumed to be std.dev instead of variance, even though the parametrization should be very clear in this course and other courses.

- Noticing a suitable transformation 0.5pt.

- Correctly applying the transformation theorem 0.5pt.

- Most of: the correct bounds, no mistakes in calculation 0.5pt.

s.11.0.105. They are independent. The book proves that $X_1 + X_2$ and $X_1 - X_2$ are independent. Then it must be that $e^{X_1+X_2} = Y_1 Y_2$ and $e^{X_1-X_2} = \frac{Y_1}{Y_2}$ are also independent.

Grading scheme:

- Noticing the independence of the sum and difference of the X_i's 0.5pt.
- Transformations of independent random variables preserve independence 0.5pt. (lenient)

s.11.0.106. Since $U = Y_1 Y_2$ and $V = \frac{Y_1}{Y_2}$, we can write the inverse functions $Y_1 = \sqrt{UV}$ and $Y_2 = \sqrt{\frac{U}{V}}$. These functions are one-to-one and C^1, so we can write the Jacobian matrix

$$J = \begin{pmatrix} \frac{\sqrt{V}}{2\sqrt{U}} & \frac{\sqrt{U}}{2\sqrt{V}} \\ \frac{1}{2\sqrt{UV}} & -\frac{1}{2V\sqrt{UV}} \end{pmatrix},$$

which has absolute determinant $\frac{1}{2V}$. Since X_1 and X_2 are independent, it must be that Y_1 and Y_2 are independent. Then

$$f_{Y_1,Y_2}(y_1, y_2) = \frac{1}{2\pi\sigma^2 y_1 y_2} \exp\left(-\frac{1}{2\sigma^2}\left((\ln y_1 - \mu)^2 + (\ln y_2 - \mu)^2\right)\right)$$

for $y_1, y_2 \in \mathbf{R}_+$. Then, by the transformation theorem, we have that

$$\begin{aligned} f_{U,V}(u,v) &= f_{Y_1,Y_2}\left(\sqrt{uv}, \sqrt{\frac{u}{v}}\right)\frac{1}{2v} \\ &= \frac{1}{4\pi\sigma^2 uv} \exp\left(-\frac{1}{2\sigma^2}\left((\ln\sqrt{uv} - \mu)^2 + (\ln\sqrt{\frac{u}{v}} - \mu)^2\right)\right) \end{aligned}$$

For $u, v \in \mathbf{R}_+$.

Grading scheme:

- Correct inverses 0.5pt.
- Correct absolute determinant of the Jacobian matrix 0.5pt.
- Noticing independence of Y_1, Y_2 0.5pt.
- Correct application of transformation theorem 0.5pt.
- Most of: the correct bounds, no mistakes in calculation, noticing the theorem is applicable, the inverses are 1-1 and C^1 0.5pt.

s.11.0.107. Since we are dealing with discrete uniform, we see that $\mathsf{P}\{B = k\} = \mathsf{P}\{U = k\} + \mathsf{P}\{U = -k\}$ for $k = 1, 2, \ldots, n$, and $\mathsf{P}\{B = 0\} = \mathsf{P}\{U = 0\}$. Hence, we can immediately say that

$$f_B(b) = \begin{cases} \frac{2}{2n+1} & 0 < b \le n \\ \frac{1}{2n+1} & b = 0 \end{cases} .$$

Then,

$$\mathsf{E}[B] = 0\frac{1}{2n+1} + \sum_{i=1}^{n} \frac{2i}{2n+1} = \frac{n(n+1)}{2}\frac{2}{2n+1} = \frac{n^2+n}{2n+1}$$

Grading scheme:

- Correct PMF 0.5pt.
- Correct expectation 0.5pt.

s.11.0.108. First, note that $Z = a + (b-a)X$. Then, we know that

$$F_Z(y) = \mathsf{P}\left\{X(b-a) + a \le y\right\} = \mathsf{P}\left\{X \le \frac{y-a}{b-a}\right\} = \int_0^{\frac{y-a}{b-a}} f_X(s)\,\mathrm{d}s.$$

Then, by the FTC, we get

$$f_Z(y) = f_X(\frac{y-a}{b-a})\frac{1}{b-a} = \frac{(y-a)^{p-1}(b-y)^{q-1}}{\beta(p,q)(b-a)^{p+q-1}}$$

for $a < y < b$.

Grading scheme:

- Correct rewriting of Z 0.5pt.
- Correct CDF 0.5pt.
- Correct PDF 0.5pt.
- No mistakes, correct bounds etc. 0.5pt.

s.11.0.109. We start as usual by considering the CDF of $Q = |Z|$. This shows us that

$$F_Q(y) = \mathsf{P}\left\{Q \le y\right\} = \mathsf{P}\left\{|Z| \le y\right\} = \mathsf{P}\left\{-y \le Z \le y\right\} = F_Z(y) - F_Z(-y).$$

We see that

$$F_Z(y) = \int_{-b}^{y} \frac{3}{4b^3}(b^2 - s^2)\,\mathrm{d}s = \frac{3}{4b}y - \frac{1}{4b^3}y^3 + \frac{1}{2}$$

after filling all values given and integrating. This holds for $0 < y < b$, the CDF is 1 for $y \geq b$, and 0 for $y \leq 0$. Then it must be that

$$f_Q(y) = \frac{\mathrm{d}}{\mathrm{d}y}(F_Z(y) - F_Z(-y)) = \frac{3}{2b} - \frac{3}{2b^3}y^2$$

for $0 < y < b$.

Grading scheme:

- Difference of CDF 0.5pt.
- Difference of CDF of Z correct, and derivative 1pt.
- Most of: no mistakes, correct bounds etc. 0.5pt.

s.11.0.110. The length of Y is $n = 750$; Each element of Y is a mean of $k = N = 300$ i.i.d. Exp(1/3) r.v.s. The expectation is $\frac{1}{\lambda} = 3$ and the variance is $\frac{1}{k\lambda^2} = \frac{1}{300 \cdot 1/9} = 0.03$.

Grading scheme:

- 0.5 for getting both the length $n = 750$ and $k = 300$ correct (no partial credit);
- 0.5 for $\lambda = 1/3$, expectation 3 and the factor $\frac{1}{\lambda^2} = \frac{1}{1/9}$ in the variance (no partial credit);
- 0.5 for the factor $\frac{1}{k}$ in the variance.

s.11.0.111. The sum of k i.i.d. Exp(λ) r.v.s. has the Gamma(k, λ) distribution. Hence, the mean of k i.i.d. Exp(λ) r.v.s. has the Gamma($k, k\lambda$) distribution. In this exercise, $k = 300$ and $k\lambda = 100$.
Grading scheme:

- 0.5 for Gamma with first parameter k
- 0.5 for the second parameter

s.11.0.112. By the CLT, $Z_1 \dot{\sim} \text{Norm}(0,1)$. Hence, $Y_1 = \mu + \sigma/\sqrt{n}Z_1 \dot{\sim} \text{Norm}(3, 9/n)$.
Grading scheme:

- 0.5 for mentioning CLT and the distribution of Z_1;
- 0.5 for the approximate distribution of Y_1.

s.11.0.113. In the limit $k \to \infty$, each Z_i has the standard normal distribution by CLT, but the sum of ℓ powers of normal distributions has a non-trivial CDF.

- 0.5 for explaining that S does not converge to a constant; no points for just CLT (unless previous question was not answered).

s.11.0.114. By LLN, S does converge to a constant as $\ell \to \infty$, however, it converges to $\mathsf{E}\left[Z_1^{71}\right]$ for that fixed value of k. By symmetry, we have $\mathsf{E}\left[Z_1^{71}\right] = 0$. However, the gamma distribution is right-skewed, which implies $\mathsf{E}\left[T^{71}\right] > 0$. Hence, it does not converge to $\mathsf{E}\left[T^{71}\right]$.
Grading scheme:

- 0.5 for concluding that S converges to a constant using LLN.
- 0.5 for explaining why the constant is not equal to $\mathsf{E}\left[T^{71}\right]$.

s.11.0.115. Using Bayes' rule we have

$$\begin{aligned} f_1(p|Y_1 = y_1) &= \frac{\mathsf{P}\{Y_1 = y_1|p\} f_0(p)}{\mathsf{P}\{Y_1 = y_1\}} && (13.0.292)\\ &\propto \mathsf{P}\{Y_1 = y_1|p\} f_0(p) && (13.0.293)\\ &= \binom{y_1 + r - 1}{y_1}(1-p)^r p^{y_1} \frac{1}{B(a,b)} p^{a-1}(1-p)^{b-1} && (13.0.294)\\ &\propto (1-p)^{r+b-1} p^{a+y_1-1}, && (13.0.295) \end{aligned}$$

in which we recognize the pdf of a Beta$(a + y_1, b + r)$ distribution (up to a constant factor). Hence, the posterior distribution of p given $Y_1 = y_1$ is a Beta$(a + y_1, b + r)$ distribution.

s.11.0.116. Yes. The posterior distribution is in the same family of distributions (Beta) as the prior. Hence, Amy has a conjugate prior.

s.11.0.117. The posterior after observing $Y_1 = y_1$ becomes our new prior. Hence, our new prior is a Beta$(a + y_1, b + r)$ distribution. From question 1 it follows that the prior after observing $Y_2 = y_2$ then is a Beta$(a + y_1 + y_2, b + 2r)$ distribution. Hence, iterating this process, we find that the posterior distribution of p after observing $Y_1 = y_1, \dots, Y_n = y_n$ is a Beta$(a + \sum_{i=1}^n y_i, b + rn)$ distribution.

s.11.0.118. Let $V = Y^2$, then $V \sim \chi_1^2$, so $V \sim$ Gamma$(\frac{1}{2}, \frac{1}{2})$. We have

$$\begin{aligned} \mathsf{E}\left[Y^{2n+2}\right] = \mathsf{E}\left[V^{n+1}\right] &= \frac{\Gamma(n+3/2)}{(1/2)^{n+1}\Gamma(1/2)} = \frac{2^{n+1}\Gamma(n+3/2)}{\Gamma(1/2)}\\ \mathsf{E}\left[Y^{2n}\right] = \mathsf{E}\left[V^{n}\right] &= \frac{\Gamma(n+1/2)}{(1/2)^{n}\Gamma(1/2)} = \frac{2^{n}\Gamma(n+1/2)}{\Gamma(1/2)}. \end{aligned}$$

Since $\Gamma(n+3/2) = (n+1/2)\Gamma(n+1/2)$ we conclude that

$$\mathsf{E}\left[Y^{2n+2}\right] = \frac{2^{n+1}\Gamma(n+3/2)}{\Gamma(1/2)} = 2(n+1/2)\frac{2^{n}\Gamma(n+1/2)}{\Gamma(1/2)} = (2n+1)\,\mathsf{E}\left[Y^{2n}\right].$$

Remarks and grading scheme:

- It is NOT true that $\mathsf{E}\left[Y^{2n+2}\right] = \mathsf{E}\left[Y^2\right]\mathsf{E}\left[Y^{2n}\right]$. This would only be true if Y^2 and Y^{2n} would be uncorrelated. But clearly, they are positively correlated: if Y^2 is large, then so is Y^{2n}.

- While induction is a good strategy to try when you have to prove a statement for all $n \in \mathbb{N}$, it is certainly not the only option.

- Grading: 0.5 for introducing $V = Y^2$ and arguing $V \sim \text{Gamma}(\frac{1}{2}, \frac{1}{2})$ with some explanation,
  0.5 for using the given expression to calculate $\mathsf{E}\left[Y^{2n+2}\right]$ and $\mathsf{E}\left[Y^{2n}\right]$,
  0.5 for using a property of the Gamma function to finish the answer.

s.11.0.119. By applying the previous exercise with $n = 1$, we obtain that $\mathsf{E}\left[Y^4\right] = 3\,\mathsf{E}\left[Y^2\right] = 3$. By applying the previous exercise with $n = 3$ and $n = 2$, we obtain that $\mathsf{E}\left[Y^8\right] = 7\,\mathsf{E}\left[Y^6\right] = 7 \cdot 5\,\mathsf{E}\left[Y^4\right] = 7 \cdot 5 \cdot 3 = 105$.

Remarks and grading scheme:

- If the exercise explicitly asks to use the previous exercise, don't do it in a different way.

- You should really know that $\mathsf{E}\left[Y^2\right] = 1$ for $Y \sim \text{Norm}(0, 1)$.

- Grading: 0.5 for a correct solution for $\mathsf{E}\left[Y^4\right]$ and 0.5 for a correct solution for $\mathsf{E}\left[Y^8\right]$.

s.11.0.120. We have $\mathsf{P}\{|Y| > 4\} = \mathsf{P}\left\{Y^4 > 256\right\}$ since $|Y| > 4$ if and only if $Y^4 > 256$. The inequality now directly follows from Markov's inequality.

Remarks and grading scheme:

- This exercise was made quite well.

- Grading scheme: 0.5 for (i) and 0.5 for (ii).

s.11.0.121. By Markov's inequality,

$$\mathsf{P}\{|Y| > 4\} = \mathsf{P}\left\{Y^{2n} > 16^n\right\} \leq \frac{\mathsf{E}\left[Y^{2n}\right]}{16^n}.$$

From the formula for $\mathsf{E}\left[Y^{2n}\right]$ we see that $\mathsf{E}\left[Y^{2(n+1)}\right] = (2n+1)\,\mathsf{E}\left[Y^{2n}\right]$. We now consider what happens when incrementing n. If $n < 8$ then $2n + 1 < 16$, so then incrementing n improves the bound, but for $n \geq 8$ the bound becomes weaker. So we get the best possible bound for $n = 8$.
Remarks and grading scheme:

- Many people tried to use induction in this exercise, but that doesn't work. While it is good to think of induction when you have to prove a statement for all $n \in \mathbb{N}$, it is certainly not the only option.

- In fact, following a similar strategy as in the previous exercise would have helped here.

- Grading scheme: 0.5 for $\mathsf{P}\{|Y| > 4\} \leq \frac{\mathsf{E}[Y^{2n}]}{16^n}$, 1 for showing that the best bound is obtained for $n = 8$.

s.11.0.122. Since job interarrival and departure times are exponentially distributed, we can use that $B(h) \sim \text{Pois}(\lambda h)$ and $D(h) = 0 \implies S > h$, hence $\mathsf{P}\{S > h | L(0) = n\} = e^{-\mu n h}$.

Mentioning that both are Poisson is also fine, but see the next question.

s.11.0.123.

$$\mathsf{P}\{B(h) = 0, D(h) = 1 | L(0) = n\} = e^{-\lambda h} \mu n h e^{-\mu n h} + o(h() \tag{13.0.296}$$

$$= (1 - \lambda h)\mu n h(1 - \mu n h) + o(h) = \mu n h + o(h). \tag{13.0.297}$$

Note that $X > h \implies B(h) = 0$. We also know that for $h << 1$, the rv $D(h)$ is nearly Poisson distributed with mean $\mu n h$. The first $o(h)$ is necessary because during the time h also two departures can occur and then the departure rates are not the same. Before the departure, people leave at rate μn, but after the first departure they leave at rate $\mu(n-1)$. However, since two or more departures have very small, in fact have $o(h)$ probability, we can capture all such details in the $o(h)$ terms.

I don't require the explanation about this subtle point.

s.11.0.124. Use conditional expectation and the above results to see that

$$\mathsf{E}[L(t+h) | L(t) = n] = n\mathsf{P}\{B(h) = 0, D(h) = 0\} + (n+1)\mathsf{P}\{B(h) = 1, D(h) = 0\} \tag{13.0.298}$$

$$+ (n-1)\mathsf{P}\{B(h) = 0, D(h) = 1\} + o(h) \tag{13.0.299}$$

$$= n e^{-\lambda h} e^{-\mu n h} + (n+1)\lambda h + (n-1)\mu n h + o(h) \tag{13.0.300}$$

$$= n(1 - \lambda h)(1 - \mu n h) + (n+1)\lambda h + (n-1)\mu n h + o(h) \tag{13.0.301}$$

$$= n - n(\lambda + \mu n)h + (n+1)\lambda h + (n-1)\mu n h + o(h) \tag{13.0.302}$$

$$= n + (\lambda - \mu n)h + o(h). \tag{13.0.303}$$

s.11.0.125. Replace n by $L(t)$ in $\mathsf{E}[L(t+h)|L(t)]$ to see that

$$\mathsf{E}[L(t+h)|L(t)] = L(t) + (\lambda - \mu L(t))h + o(h). \tag{13.0.304}$$

Take expectations left and right and use Adam's law.

s.11.0.126. The PDF of Y is given by

$$= \frac{1}{\sqrt{2\pi}\sigma y} \exp\left(-\frac{1}{2\sigma^2}(\ln y - \mu)^2\right)$$

for $y > 0$.

Grading scheme:

- Correct 0.5pt.

s.11.0.127. Notice that, since Y is a normal rv, the log of Y is log-normal. Then, taking the ln on both sides, we get that

$$\ln W_k = \ln k - \ln 5 - 2\ln Y_k.$$

From the book, we know that if $\ln Y \sim \mathcal{N}(\mu, \sigma^2)$, then it must be that

$$-2\ln Y \sim \mathcal{N}(-2\mu, 4\sigma^2),$$

and that

$$\ln W_k = \ln k - \ln 5 - 2\ln Y \sim \mathcal{N}(\ln k - \ln 5 - 2\mu, 4\sigma^2).$$

Thus, $W_k \sim \mathscr{L}\mathcal{N}(\ln k - \ln 5 - 2\mu, 4\sigma^2)$

Grading scheme:

- The idea to take logs 0.5pt.
- The rest correct 1pt.

s.11.0.128. Clearly, $W_k = \frac{k}{l} W_l$, and thus we can see that

$$W_k W_l = \frac{k}{l} W_k^2,$$
$$\frac{W_k}{W_l} = \frac{k}{l}.$$

These are independent, since one is a constant.

Grading scheme:

- Correct probability 0.5pt.
- Correct conclusion 0.5pt.

s.11.0.129. Since $U = X_1 + X_2$ and $V = X_1 - X_2$, we can write the inverse functions $X_1 = \frac{1}{2}(U+V)$ and $X_2 = \frac{1}{2}(U-V)$. These functions are one-to-one and C^1, so we can write the Jacobian matrix

$$J = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix},$$

which has absolute determinant $\frac{1}{2}$. Since X_1 and X_2 are independent,

$$f_{X_1,X_2}(x_1, x_2) = \frac{1}{2\pi\sqrt{x_1 x_2}} \exp\left(-\frac{x_1 + x_2}{2}\right)$$

for $x_1, x_2 \in \mathbf{R}_+$. Then, by the transformation theorem, we have that

$$\begin{aligned} f_{U,V}(u,v) &= f_{X_1,X_2}\left(\frac{1}{2}(u+v), \frac{1}{2}(u-v)\right)\frac{1}{2} \\ &= \frac{1}{2\pi\sqrt{u^2-v^2}}\exp\left(-\frac{u}{2}\right) \end{aligned}$$

For $-\infty < v < u < \infty$.

Grading scheme:

- Correct inverses 0.5pt.
- Correct absolute determinant of the Jacobian matrix 0.5pt.
- Correct application of transformation theorem 0.5pt.
- Most of: the correct bounds, no mistakes in calculation, noticing the theorem is applicable, the inverses are 1-1 and C^1, independence of X_1 and X_2 etc. 0.5pt.

s.11.0.130. So we can see that both X and Y are uniformly distributed on $(-\sqrt{\pi}, \sqrt{\pi})$. Then their joint PDF is simply:

$$\begin{aligned} f(x,y) =& \left(\frac{1}{\sqrt{\pi}-(-\sqrt{\pi}))}\right)\left(\frac{1}{(\sqrt{\pi}-(-\sqrt{\pi}))}\right) \\ =& \frac{1}{4\pi} \end{aligned}$$

One mistake, zero points.

s.11.0.131.

$$\begin{aligned} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy =& \int_{-\sqrt{\pi}}^{\sqrt{\pi}}\int_{-\sqrt{\pi}}^{\sqrt{\pi}} \frac{1}{4\pi}\,dx\,dy \\ =& 1 \end{aligned}$$

One mistake, zero points

s.11.0.132. Note that $R = \sqrt{X^2 + X^2}$, so then $R^2 = X^2 + Y^2$. Using LOTUS:

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(x^2+y^2\right)f(x,y)\,dx\,dy &= \int_{-\sqrt{\pi}}^{\sqrt{\pi}}\int_{-\sqrt{\pi}}^{\sqrt{\pi}}\left(x^2+y^2\right)\left(\frac{1}{4\pi}\right)dx\,dy\\
&= \int_{-\sqrt{\pi}}^{\sqrt{\pi}}\int_{-\sqrt{\pi}}^{\sqrt{\pi}}\frac{x^2+y^2}{4\pi}\,dx\,dy\\
&= \frac{1}{4\pi}\int_{-\sqrt{\pi}}^{\sqrt{\pi}}\int_{-\sqrt{\pi}}^{\sqrt{\pi}}x^2+y^2\,dx\,dy\\
&= \frac{1}{4\pi}\int_{-\sqrt{\pi}}^{\sqrt{\pi}}\left[\frac{x^3}{3}+y^2x\right]_{-\sqrt{\pi}}^{\sqrt{\pi}}dy\\
&= \frac{1}{4\pi}\int_{-\sqrt{\pi}}^{\sqrt{\pi}}\left(\frac{\pi^{\frac{3}{2}}}{3}+\sqrt{\pi}y^2-\frac{(-\pi)^{\frac{3}{2}}}{3}+\sqrt{\pi}y^2\right)dy\\
&= \frac{1}{4\pi}\int_{-\sqrt{\pi}}^{\sqrt{\pi}}\left(\frac{\pi^{\frac{3}{2}}}{3}+\sqrt{\pi}y^2+\frac{\pi^{\frac{3}{2}}}{3}+\sqrt{\pi}y^2\right)dy\\
&= \frac{1}{4\pi}\int_{-\sqrt{\pi}}^{\sqrt{\pi}}\frac{2\pi^{\frac{3}{2}}}{3}+2\sqrt{\pi}y^2\,dy\\
&= \frac{1}{4\pi}\left[\frac{2\pi^{\frac{3}{2}}}{3}y+\frac{2\sqrt{\pi}y^3}{3}\right]_{-\sqrt{\pi}}^{\sqrt{\pi}}\\
&= \frac{1}{4\pi}\left(\frac{2\pi^2}{3}+\frac{2\pi^2}{3}+\frac{2(-\pi)^2}{3}+\frac{2(-\pi)^2}{3}\right)\\
&= \frac{1}{4\pi}\frac{8\pi^2}{3}=\frac{2\pi}{3}
\end{aligned}$$

One point for finding that $R^2 = X^2 + Y^2$ and writing down the integral correctly using LOTUS. 2 points for the remaining calculations.

s.11.0.133. It loads the required packages and creates two samples with 100000 observations from respectively a $\mathcal{N}(50,200)$- and $\mathcal{N}(20,100)$-distribution. Then for all paired observations it computes the product and takes the mean to estimate $E(XY)$.
0.5 points if it is mentioned a product is taken and an average is computed/estimated.

s.11.0.134. As the samples are generated independently we would expect $E(XY) = E(x)E(Y) = 50 * 20 \approx 1000$. This is indeed shown by the code.
0.5 points if independence is mentioned. Which would then result in $E(XY) = E(X)E(Y)$

s.11.0.135. The initial distribution is $\mathsf{P}\{X = x, Y = y\} = 1/9,\ x, y = 1,2,3$. Catherine wins iff $X > Y$, i.e., iff $(X,Y) \in \{(2,1),(3,1),(3,2)\}$. So we have

$$\mathsf{P}\{X = x, Y = y | X > Y\} = \frac{\mathsf{P}\{X = x, Y = y, X > Y\}}{\mathsf{P}\{X > Y\}} \tag{13.0.305}$$

$$= \frac{1/9}{1/3} = 1/3, \tag{13.0.306}$$

for $(x,y) \in \{(2,1),(3,1),(3,2)\}$.

s.11.0.136. We have

$$C = \begin{cases} 0 & X < Y \\ 1/2 & X = Y \\ X - Y & X > Y. \end{cases} \tag{13.0.307}$$

Hence,

$$\mathsf{E}[C|X > Y] = \mathsf{E}[X - Y|X > Y] \tag{13.0.308}$$
$$= \frac{1}{3}(2-1) + \frac{1}{3}(3-1) + \frac{1}{3}(3-2) \tag{13.0.309}$$
$$= 4/3. \tag{13.0.310}$$

s.11.0.137. We have, by the law of total expectation,

$$\mathsf{E}[C] = \mathsf{P}\{X < Y\}\,\mathsf{E}[C|X < Y] + \mathsf{P}\{X = Y\}\,\mathsf{E}[C|X = Y] + \mathsf{P}\{X > Y\}\,\mathsf{E}[C|X > Y] \tag{13.0.311}$$
$$= 0 + \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3}\frac{4}{3} \tag{13.0.312}$$
$$= 11/18. \tag{13.0.313}$$

s.11.0.138. By Adam's law,

$$\mathsf{E}[S] = \mathsf{E}[\mathsf{E}[S|C]] \tag{13.0.314}$$
$$= \mathsf{E}\left[\frac{3}{2}C\right] \tag{13.0.315}$$
$$= \frac{3}{2}\mathsf{E}[C] \tag{13.0.316}$$
$$= \frac{3}{2}\frac{11}{18} \tag{13.0.317}$$
$$= \frac{33}{36}. \tag{13.0.318}$$

s.11.0.139. Since (X, Y) is uniformly distributed on $(-\sqrt{e}, \sqrt{e})^2$, their joint PDF is simply:

$$f(x, y) = \left(\frac{1}{\sqrt{e} - (-\sqrt{e}))}\right)\left(\frac{1}{(\sqrt{e} - (-\sqrt{e}))}\right)$$
$$= \frac{1}{4e},$$

for $x \in (-\sqrt{e}, \sqrt{e})$ and $y \in (-\sqrt{e}, \sqrt{e})$. *0.5 points for the solution, 0.5 points for the correct bounds.*

s.11.0.140. Note that $S = X^2 + Y^2$. Using LOTUS and symmetry in x and y:

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(x^2+y^2\right)f(x,y)\,dx\,dy =& \int_{-\sqrt{e}}^{\sqrt{e}}\int_{-\sqrt{e}}^{\sqrt{e}}\left(x^2+y^2\right)\left(\frac{1}{4e}\right)dx\,dy\\
=&\frac{1}{4e}\int_{-\sqrt{e}}^{\sqrt{e}}\int_{-\sqrt{e}}^{\sqrt{e}}(x^2+y^2)\,dx\,dy\\
=&\frac{1}{4e}\int_{-\sqrt{e}}^{\sqrt{e}}\int_{-\sqrt{e}}^{\sqrt{e}}x^2\,dx\,dy+\frac{1}{4e}\int_{-\sqrt{e}}^{\sqrt{e}}\int_{-\sqrt{e}}^{\sqrt{e}}y^2\,dx\,dy+\\
=&\frac{2}{4e}\int_{-\sqrt{e}}^{\sqrt{e}}\int_{-\sqrt{e}}^{\sqrt{e}}x^2\,dx\,dy\\
=&\frac{1}{2e}\int_{-\sqrt{e}}^{\sqrt{e}}[\frac{1}{3}x^3]_{\sqrt{e}}^{\sqrt{e}}\,dy\\
=&\frac{1}{6e}\int_{-\sqrt{e}}^{\sqrt{e}}(e^{3/2}+e^{3/2})\,dy\\
=&\frac{2}{6e}2\sqrt{e}e^{3/2}\\
=&\frac{4e^2}{6e}=\frac{2}{3}e.
\end{aligned}$$

One point for finding $S^2 = X^2 + Y^2$ and writing down the integral correctly using LOTUS. Two points for the remaining calculations.

s.11.0.141. The output of -1 means that the correlation between X and Y, is -1. This makes sense since $Y = 10 - X$, so obviously the correlation is -1 as Y completely determines the value of X. If Y goes up by 1, X always goes down by exactly 1. This can also be seen in the graph where X and Y always sum op to 10 and there is are linear negative relationship between them. No points are deviated from the line, X and Y will always move together.

s.11.0.142. First notice that $X_A|N \sim \text{Bin}(N,p)$ and $X_B|N \sim \text{Bin}(N,1-p)$.By the chicken-egg theory, $X_A \sim \text{Pois}(\lambda p)$ and $X_B \sim \text{Pois}(\lambda(1-p))$ are independent. (0.5 points)
Then it follows

$$Var(X_A - X_B) = Var(X_A) + Var(X_B) = \lambda p + \lambda(1-p) = \lambda \cdots \text{ (0.5 points)}$$

And

$$\begin{aligned}
&\text{Cov}[X_B, N]\\
=&\text{Cov}[X_B, X_A + X_B]\\
=&\text{Cov}[X_B, X_A] + \text{Cov}[X_B, X_B]\\
=&Var(X_B)\\
=&\lambda(1-p)\cdots \text{ (0.5 points)}
\end{aligned}$$

Then

$$\rho_{X_B,N} = \frac{\mathsf{Cov}\left[X_B, N\right]}{sd(X_B)sd(N)} = \frac{\lambda(1-p)}{\sqrt{\lambda(1-p)}\sqrt{\lambda}} = \sqrt{1-p} \cdots \text{ (0.5 points)}$$

s.11.0.143. First notice that $X_j \sim \text{Bin}(500, \frac{1}{4}), j = A, B, C, D.$(0.5 points)
Then we know $\boldsymbol{X} = (X_A, X_B, X_C, X_D) \sim Mult_4(500, \frac{1}{4})$.(0.5 points)
Using the property of a Multinomial distribution,

$$\mathsf{Cov}\left[X_B, X_C\right] = -\frac{500}{4^2}, \cdots \text{ (0.5 points)}$$

$$\rho_{X_B,X_C} = \frac{\mathsf{Cov}\left[X_B, X_C\right]}{sd(X_B)sd(X_C)} = -\frac{500/4^2}{500(1/4)(3/4)} = -\frac{1}{3}. \cdots \text{ (0.5 points)}$$

s.11.0.144. Line 1: Load the package "mvtnorm" so that we can use the function *rmvnorm*.
Line 2: Set a random seed to reproduce the same results.
Line 5: Generate a vector $C = (3, 5)$.
Line 6: Generate a 2×2 matrix of D=$\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$.
Line 7: Generate 200 multivariate normally distributed variables $\mathbf{X} = x_1, x_2$ with mean equal to $(3, 5)$ and variance equal to matrix D.
Line 8: Calculate the average value for each column of matrix X.
(0.5 points for mentioning at least 3 of the above.)

s.11.0.145. Since all of the first t days are equally likely to have the lowest return, by symmetry, $\mathsf{P}\{A_t\} = \frac{1}{t}$.

s.11.0.146. To solve this exercise we use permutations. First notice that in $t+1$ days, there are in total $(t+1)!$ possible combination of the daily returns. Since only the lowest 2 daily returns should be on day $t+1$ and day t, the order of the remaining $t-1$ daily returns does not matter. Thus, $\mathsf{P}\{A_t \cap A_{t+1}\} = \frac{(t-1)!}{(t+1)!} = \frac{1}{t(t+1)}$.

s.11.0.147. To solve this exercise we use permutations. First notice that in t days, there are in total $t!$ possible combination of the daily returns. Since the lowest daily return should be on day t, we need to sort the rest $t-1$ daily returns. Further notice that between day $s+1$ and $t-1$ the daily returns only have to be higher than X_{t+1}, no other restrictions. So we need $t-(s+1)$ our of the remaining $t-1$ daily returns to fill the days between day $s+1$ and $t-1$, in total $\begin{pmatrix} t-1 \\ t-s-1 \end{pmatrix}(t-s-1)!$ possible combinations. Finally, the lowest out of the remaining s

daily returns need to be on day s, and the order of the remaining does not matter. Thus we have:

$$\begin{aligned}
\mathsf{P}\{A_s \cap A_t\} &= \frac{\binom{t-1}{t-s-1}(t-s-1)!(s-1)!}{t!} \\
&= \frac{(s-1)!(t-1)!}{s!t!} \\
&= \frac{1}{st} \\
&= \mathsf{P}\{A_s\}\,\mathsf{P}\{A_t\}.
\end{aligned}$$

s.11.0.148. First notice that $\mathsf{Cov}[N, I_t] = \mathsf{E}[NI_t] - \mathsf{E}[N]\,\mathsf{E}[I_t]$. Since $\mathsf{E}[I_t] = \mathsf{P}\{A_t\} = \frac{1}{t}$, we need to find out $\mathsf{E}[NI_t]$ and $\mathsf{E}[N]$. Since $N = \sum_{k=1}^{t} I_k$,

$$\begin{aligned}
\mathsf{E}[N] &= \mathsf{E}\left[\sum_{k=1}^{t} I_k\right] \\
&= \sum_{k=1}^{t} \mathsf{E}[I_k] \\
&= \sum_{k=1}^{t} \frac{1}{k}
\end{aligned}$$

$$\begin{aligned}
\mathsf{E}[NI_t] &= \mathsf{E}[\mathsf{E}[NI_t \mid I_t]] \\
&= \mathsf{E}\left[\sum_{k=1}^{t} I_k I_t \,\middle|\, I_t = 1\right] \mathsf{P}\{I_t = 1\} \\
&= \mathsf{E}\left[\sum_{k=1}^{t-1} I_k + 1\right] \mathsf{P}\{I_t = 1\} \\
&= \frac{1}{t}\sum_{k=1}^{t-1} \frac{1}{k} + \frac{1}{t}
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathsf{Cov}[N, I_t] &= \mathsf{E}[NI_t] - \mathsf{E}[N]\,\mathsf{E}[I_t] \\
&= \frac{1}{t}\sum_{k=1}^{t-1} \frac{1}{k} + \frac{1}{t} - \frac{1}{t}\sum_{k=1}^{t} \frac{1}{k} \\
&= \frac{1}{t} - \frac{1}{t^2} = \frac{t-1}{t^2}.
\end{aligned}$$

Alternatively, note that $\mathsf{Cov}\left[I_i, I_j\right] = 0$ if $i \neq j$ since I_i and I_j are independent if $i \neq j$. Hence,

$$\begin{aligned}\mathsf{Cov}[N, I_t] &= \mathsf{Cov}\left[\sum_{k=1}^{t} I_k, I_t\right] \\ &= \sum_{k=1}^{t} \mathsf{Cov}[I_k, I_t] \\ &= \mathsf{Cov}[I_t, I_t] = \mathsf{V}[I_t] = \mathsf{E}\left[I_t^2\right] - \mathsf{E}[I_t]^2 \\ &= \frac{1}{t} - \frac{1}{t^2} = \frac{t-1}{t^2}.\end{aligned}$$

s.11.0.149. First notice that $X_A|X \sim \text{Bin}(X, p)$ and $X_B|X \sim \text{Bin}(X, 1-p)$. By the chicken-egg theory, $X_A \sim \text{Pois}(\lambda p)$ and $X_B \sim \text{Pois}(\lambda(1-p))$ are independent. (0.5 points)
Then it follows

$$Var(X_A - X_B) = Var(X_A) + Var(X_B) = \lambda p + \lambda(1-p) = \lambda \cdots \text{ (0.5 points)}$$

And

$$\begin{aligned}&\mathsf{Cov}[X_A, X] \\ =&\mathsf{Cov}[X_A, X_A + X_B] \\ =&\mathsf{Cov}[X_A, X_A] + \mathsf{Cov}[X_A, X_B] \\ =&Var(X_A) \\ =&\lambda p \cdots \text{ (0.5 points)}\end{aligned}$$

Then

$$\rho_{X_A, X} = \frac{\mathsf{Cov}[X_A, X]}{sd(X_A)sd(X)} = \frac{\lambda p}{\sqrt{\lambda p}\sqrt{\lambda}} = \sqrt{p} \cdots \text{ (0.5 points)}$$

s.11.0.150. First notice that $X_j \sim \text{Bin}(1000, \frac{1}{3}), j = A, B, C$. (0.5 points)
Then we know $\boldsymbol{X} = (X_A, X_B, X_C) \sim Mult_3(1000, \frac{1}{3})$. (0.5 points)
Using the property of a Multinomial distribution,

$$\mathsf{Cov}[X_A, X_C] = -\frac{1000}{3^2}, \cdots \text{ (0.5 points)}$$

$$\rho_{X_A, X_C} = \frac{\mathsf{Cov}[X_A, X_C]}{sd(X_A)sd(X_C)} = -\frac{1000/3^2}{1000(1/3)(2/3)} = -\frac{1}{2}. \cdots \text{ (0.5 points)}$$

s.11.0.151. Line 1: Load the package "mvtnorm" so that we can use the function *rmvnorm*.
Line 2: Set a random seed to reproduce the same results.
Line 6: Combine vector A, B, C to generate a matrix D=$\begin{pmatrix}1 & 2 & 1\\ 2 & 3 & 1\\ 1 & 1 & 8\end{pmatrix}$

Line 7: Generate 200 multivariate normally distributed variables $\mathbf{X} = x_1, x_2, x_3$ with mean equal to $(1, 2, 1)$ and variance equal to matrix D.
Line 8: Calculate the correlation of the sum of column 1-2 of matrix X and the 3rd column of matrix X.
(0.5 points for correct answer of at least 3 of the above.)

s.11.0.152. So we can see that both X and Y are uniformly distributed on $(-\sqrt{7}, \sqrt{7})$. Then their joint PDF is simply:

$$
\begin{aligned}
f(x, y) =& \Big(\frac{1}{\sqrt{7} - (-\sqrt{7}))}\Big)\Big(\frac{1}{(\sqrt{7} - (-\sqrt{7}))}\Big) \\
=& \frac{1}{28}
\end{aligned}
$$

One mistake, zero points.

s.11.0.153.

$$
\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\, dx\, dy =& \int_{-\sqrt{7}}^{\sqrt{7}}\int_{-\sqrt{7}}^{\sqrt{7}} \frac{1}{28}\, dx\, dy \\
=& 1
\end{aligned}
$$

One mistake, zero points.

s.11.0.154. Note that $S = X^2 + Y^2$. Using LOTUS:

$$
\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(x^2+y^2\right)f(x,y)\,dx\,dy &= \int_{-\sqrt{7}}^{\sqrt{7}}\int_{-\sqrt{7}}^{\sqrt{7}}\left(x^2+y^2\right)\left(\frac{1}{28}\right)dx\,dy\\
&= \int_{-\sqrt{7}}^{\sqrt{7}}\int_{-\sqrt{7}}^{\sqrt{7}}\frac{x^2+y^2}{28}\,dx\,dy\\
&= \frac{1}{28}\int_{-\sqrt{7}}^{\sqrt{7}}\int_{-\sqrt{7}}^{\sqrt{7}}x^2+y^2\,dx\,dy\\
&= \frac{1}{28}\int_{-\sqrt{7}}^{\sqrt{7}}\left[\frac{x^3}{3}+y^2x\right]_{-\sqrt{7}}^{\sqrt{7}}dy\\
&= \frac{1}{28}\int_{-\sqrt{7}}^{\sqrt{7}}\left(\frac{7^{\frac{3}{2}}}{3}+\sqrt{7}y^2-\frac{(-7)^{\frac{3}{2}}}{3}+\sqrt{7}y^2\right)dy\\
&= \frac{1}{28}\int_{-\sqrt{7}}^{\sqrt{7}}\left(\frac{7^{\frac{3}{2}}}{3}+\sqrt{7}y^2+\frac{7^{\frac{3}{2}}}{3}+\sqrt{7}y^2\right)dy\\
&= \frac{1}{28}\int_{-\sqrt{7}}^{\sqrt{7}}\frac{2}{3}7^{\frac{3}{2}}+2\sqrt{7}y^2\,dy\\
&= \frac{1}{28}\left[\frac{2}{3}7^{\frac{3}{2}}y+\frac{2}{3}\sqrt{7}y^3\right]_{-\sqrt{7}}^{\sqrt{7}}\\
&= \frac{1}{28}\left(\frac{2}{3}7^2+\frac{2}{3}7^2+\frac{2}{3}(-7)^2+\frac{2}{3}(-7)^2\right)\\
&= \frac{1}{28}\frac{8}{3}7^2=\frac{14}{3}=4\frac{2}{3}
\end{aligned}
$$

One point for finding $S^2 = X^2 + Y^2$ and writing down the integral correctly using LOTUS. Two points for the remaining calculations.

s.11.0.155. It loads the required packages and defines a vector of zeros of length `N`. Then in the for-loop it creates a random point in the unit circle. It computes the distance from this point to the origin and stores it in `distances`. Finally, it takes the mean to estimate the mean distance from a randomly chosen point in the unit circle to the origin.
0.5 points for mentioning an average distance is between a random point and the origin is estimated. This point has an x-coordinate and a y-coordinate.

s.11.0.156. The mean distance from a randomly selected point in the unit circle to the origin is $\approx \frac{2}{3}$.
Trivial

s.11.0.157. Since the current month and all previous 11 months are equally likely to have the lowest return, by symmetry, $\mathsf{P}\{A_t\} = \frac{1}{12}$.

s.11.0.158. Since $N = \sum_{k=12}^{t} I_k$ and $\mathsf{E}[I_t] = \mathsf{P}\{A_t\} = \frac{1}{12}$,

$$\mathsf{E}[N] = \mathsf{E}\left[\sum_{k=12}^{t} I_k\right] = \sum_{k=12}^{t}\mathsf{E}[I_k] = \frac{t-11}{12}.$$

Then notice that $\mathrm{Cov}[N, I_t] = \mathrm{E}[NI_t] - \mathrm{E}[N]\,\mathrm{E}[I_t]$. Since $\mathrm{E}[I_t] = \mathrm{P}\{A_t\} = \frac{1}{12}$ and $\mathrm{E}[N] = \frac{t-11}{12}$ we need to find $\mathrm{E}[NI_t]$.

$$\begin{aligned}
\mathrm{E}[NI_t] &= \mathrm{E}[\mathrm{E}[NI_t \mid I_t]] \\
&= \mathrm{E}\left[\sum_{k=12}^{t} I_k I_t \,\middle|\, I_t = 1\right] \mathrm{P}\{I_t = 1\} \\
&= \mathrm{E}\left[\sum_{k=12}^{t-1} I_k + 1\right] \mathrm{P}\{I_t = 1\} \\
&= \left(\frac{t-12}{12} + 1\right)\frac{1}{12} = \frac{t}{144}
\end{aligned}$$

Thus:

$$\begin{aligned}
\mathrm{Cov}[N, I_t] &= \mathrm{E}[NI_t] - \mathrm{E}[N]\,\mathrm{E}[I_t] \\
&= \frac{t}{144} - \frac{t-11}{12}\frac{1}{12} = \frac{11}{144}.
\end{aligned}$$

s.11.0.159. To solve this exercise we use permutations. First notice that since we involve 2 months, there are in total 12+1=13 possible combination of the monthly returns. Since only the lowest 2 monthly returns should be on month $t+1$ and month t, the order of the remaining 13-2=11 monthly returns does not matter. Thus, $\mathrm{P}\{A_t \cap A_{t+1}\} = \frac{(13-2)!}{13!} = \frac{1}{12*13} = \frac{1}{156}$.
Since $\mathrm{P}\{A_t\} = \mathrm{P}\{A_{t+1}\} = \frac{1}{12}$, $\mathrm{P}\{A_t \cap A_{t+1}\} \neq \mathrm{P}\{A_t\}\,\mathrm{P}\{A_{t+1}\}$, and A_t and A_{t+1} are not independent.

s.11.0.160. To solve this exercise we use permutations. First notice that in $t+1$ months, there are in total $(t+1)!$ possible combination of the monthly returns. Since only the lowest 2 monthly returns should be on month $t+1$ and month t, the order of the remaining $t-1$ monthly returns does not matter. Thus, $\mathrm{P}\{B_t \cap B_{t+1}\} = \frac{(t-1)!}{(t+1)!} = \frac{1}{t(t+1)}$.

s.11.0.161. Let I_i be the indicator r.v. for the ith book having more page than each of book 1 and book 2. Then:

$$\begin{aligned}
\mathrm{P}\{I_i = 1\} &= \mathrm{P}\{X_i > X_1, X_i > X_2\} \\
&= \mathrm{P}\{X_i = \max\{X_1, X_2, X_i\}\} \\
&= \frac{1}{3},
\end{aligned}$$

by symmetry. Then $\mathrm{E}\left[\Sigma_{i=3}^{6} I_i\right] = \frac{1}{3} \cdot 4 = \frac{4}{3}$.

s.11.0.162. In order to answer this question, we want to know $P(X_i - X_1 > 1)$, for $i = 3, \ldots, 6$. We first consider $i = 3$. Since X_3 and X_1 are jointly normal distributed, $X_3 - X_1$ is also normally distributed, with $\mathrm{E}[X_3 - X_1] = \mathrm{E}[X_3] - \mathrm{E}[X_1] = 0$ and

$$\begin{aligned}
\mathrm{V}[X_3 - X_1] &= \mathrm{V}[X_3] + \mathrm{V}[-X_1] + 2\,\mathrm{Cov}[X_3, -X_1] \\
&= 1 + 1 - 2 \cdot \frac{1}{2} \cdot 1 \cdot 1 \\
&= 1
\end{aligned}$$

Then we know $X_3 - X_1$ follows a standard normal distribution and $P(X_3 - X_1 > 1) = 0.16$ Similarly, $P(X_4 - X_1 > 1) = P(X_5 - X_1 > 1) = P(X_6 - X_1 > 1) = 0.16$. So the average number of the remaining books that has 100 pages more than the first book is $0.16 \cdot 4 = 0.64$.

s.11.0.163. $\mathsf{Cov}[X_1 - cX_3, X_3] = \mathsf{Cov}[X_1 - cX_3, X_3] = \mathsf{Cov}[X_1, X_3] - c\mathsf{V}[X_3] = \frac{1}{2} - c$, so for $c = \frac{1}{2}$, we have that $X_1 - cX_3$ and X_3 are uncorrelated. Since $(X_1, ..., X_6)$ has the multivariate normal distribution, it follows that $X_1 - cX_3$ and X_3 are independent.

s.11.0.164. a. On the set $\{N(t) = n\}$ $N(t) = n$. Hence we can replate $N(t)$ by n.

b. Variance of sum of iid rvs is sum of variance of rv.

c. Use a and b, substitute $N(t)$ for n and use the definition of conditional expectation.

d. Since $\mathsf{V}[R]$ is a constant, we can take out of the expectation. Expectation of Poisson rv is λt Each property missed, e.g, linearity of expectation, minus 0.5.

s.11.0.165. Since $T = t$ a.s.,

$$\mathsf{E}[S] = \mathsf{E}[T] + \mathsf{E}[N(T)]\,\mathsf{E}[R] = a/2 + r\lambda\,\mathsf{E}[T] = a/2 + r\lambda a/2.$$

We know that $\mathsf{E}[T] = a/2$, Hence writing $\mathsf{E}[N(t)] = \lambda t$, is not ok. Some other strange things that I saw:

$$R \neq rT \tag{13.0.319}$$
$$\mathsf{E}[R] \neq ra/2 \tag{13.0.320}$$
$$\tag{13.0.321}$$

Typically, 0.5 point for each correct line, but I subtracted points If you say that $\mathsf{E}[N(T)\,\mathsf{E}[R]] = \mathsf{E}[R]$, or write $n\,\mathsf{E}[R]$ as final answer (apparently you did not get the idea that N is an rv.)

s.11.0.166. Here is an EXPLANATION:

1. T is a simulated service time of a job without interruptions.
2. N is is the simulated number of failures that occur during the service of the job
3. R is then a vector of simulated durations of each of the interruptions
4. Hence, S is the total time the job spends at the server
5. We sample the total service time a number of times, and count how often (mean or variance) this exceeds a threshold.

The points earned depends on how clear you explanation is.

Quite a few of you don't seem to understand what it means to *explain* something. Here are some answers that I saw that certainly don't explain what the code does:

1. 'The code does what's stated in the exercise.'. What's the explanation here? The question is also not: do you understood what the code does?

2. 'T is uniform rv, N is a Poisson rv, R is a also a uniform rv. We repeat this a few times.' Like this you just read the code, but I know you can read, so this type of answer is quite useless.

3. 'We collect a subset of the samples and print that.' This answer is just a repition of the hint.

So, next time, when you are asked to explain something, explain it. In your professional carreer, if you read out loud to your customer what s/he can read for him/herself, you'll be fired pretty soon.

s.11.0.167. The length of Y is $N = 1000$; Each element of Y is a mean of $k = n = 100$ i.i.d. Exp(2) r.v.s. The expectation is $\frac{1}{\lambda} = \frac{1}{2}$ and the variance is $\frac{1}{k\lambda^2} = \frac{1}{400} = 0.0025$.

Grading scheme:

- 0.5 for getting both the length $N = 1000$ and $k = 100$ correct (no partial credit);
- 0.5 for $\lambda = 2$, expectation $\frac{1}{2}$ and the factor $\frac{1}{4}$ in the variance (no partial credit);
- 0.5 for the factor $\frac{1}{k}$ in the variance.

s.11.0.168. The sum of k i.i.d. Exp(λ) r.v.s. has the Gamma(k, λ) distribution. Hence, the mean of k i.i.d. Exp(λ) r.v.s. has the Gamma($k, k\lambda$) distribution. In this exercise, $k = 100$ and $k\lambda = 200$.
Grading scheme:

- 0.5 for Gamma with first parameter k
- 0.5 for the second parameter

s.11.0.169. By the CLT, $Z_1 \dot\sim \text{Norm}(0,1)$. Hence, $Y_1 = \mu + \sigma/\sqrt{n} Z_1 \dot\sim \text{Norm}(0.5, 0.25/n)$.
Grading scheme:

- 0.5 for mentioning CLT and the distribution of Z_1;
- 0.5 for the approximate distribution of Y_1.

s.11.0.170. In the limit $k \to \infty$, each Z_i has the standard normal distribution by CLT, but the sum of ℓ powers of normal distributions has a non-trivial CDF.

- 0.5 for explaining that S does not converge to a constant; no points for just CLT (unless previous question was not answered).

s.11.0.171. By LLN, S does converge to a constant as $\ell \to \infty$, however, it converges to $\mathsf{E}\left[Z_1^{37}\right]$ for that fixed value of k. By symmetry, we have $\mathsf{E}\left[Z_1^{37}\right] = 0$. However, the gamma distribution is right-skewed, which implies $\mathsf{E}\left[T^{37}\right] > 0$. Hence, it does not converge to $\mathsf{E}\left[T^{37}\right]$.
Grading scheme:

- 0.5 for concluding that S converges to a constant using LLN.
- 0.5 for explaining why the constant is not equal to $\mathsf{E}\left[T^{37}\right]$.

s.11.0.172. (i). Since f is increasing, we have $X \geq a$ if and only if $f(X) \geq f(a)$, so $\{X \geq a\}$ and $\{f(X) \geq f(a)\}$ are the same event. Hence, $\mathsf{P}\{X \geq a\} = \mathsf{P}\left\{f(X) \geq f(a)\right\}$.
(ii). Since f is positive, we have $|f(X)| = f(X)$ and $f(a) > 0$. Hence, the inequality follows directly from Markov's inequality with r.v. $f(X)$ and constant $f(a) > 0$.

Remarks and grading scheme:

- In part (ii), it is not sufficient to just say that it holds by Markov's inequality, also explain how you apply Markov's inequality.
- Since it was asked to indicate **clearly** where you use that f is positive and increasing, don't just say "since f is positive and increasing" twice. For part (i) the positivity is not needed, it is only needed that f is increasing. For part (ii), it is not needed that f is increasing, only the positivity is needed. Moreover, some explanation should be given how it is used.
- f is not necessarily a linear transformation, as some of you claimed.
- Not every positive and increasing function is convex or concave.
  And using Jensen's inequality certainly doesn't work in (i): you are asked to prove an equality of probabilities, not an inequality of expectations.
- In part (i), the **if and only if** is really important. If you just mention that $f(X) \geq f(a)$ if $X \geq a$, then you are only proving that $\mathsf{P}\{X \geq a\} \leq \mathsf{P}\left\{f(X) \geq f(a)\right\}$, because if you just say "$f(X) \geq f(a)$ if $X \geq a$", $f(X) \geq f(a)$ could still be true in cases where $X \geq a$ is not, and hence $f(X) \geq f(a)$ can still have a larger probability. (I've not been strict on this when grading.)
- Grading: 0.5 for a sufficient explanation for (i), 0.5 for a sufficient explanation for (ii) and 0.5 for clearly indicating where it is used that f is positive and increasing and writing a clear answer overall.

s.11.0.173. Note that $f(x) = e^{tx^2}$ is positive and increasing on $(0, \infty)$ for $t > 0$. By applying the inequality of the first question with $a = 2$ we find

$$\mathsf{P}\{|Z| > 2\} \leq e^{-4t}\,\mathsf{E}\left[e^{t|Z|^2}\right] = e^{-4t}\,\mathsf{E}\left[e^{tZ^2}\right].$$

Remarks and grading scheme:

- Most of you did not apply the inequality from the previous part to solve this question, but instead used Chernoff's inequality as follows:

$$\mathsf{P}\{|Z| > 2\} = \mathsf{P}\left\{Z^2 > 4\right\} \le e^{-4t}\,\mathsf{E}\left[e^{t|Z|^2}\right] = e^{-4t}\,\mathsf{E}\left[e^{tZ^2}\right],$$

  where the first equality holds since $|Z| > 2$ if and only if $\mathsf{P}\left\{Z^2 > 4\right\}$. This is also correct, but takes a bit more time.

- Don't write nonsense like $e^{-2t}\,\mathsf{E}\left[e^{t|Z|}\right] = e^{-4t}\,\mathsf{E}\left[e^{tZ^2}\right]$, just to make it look like you solved the exercise although you didn't.

- Grading scheme: 1 point for solving the exercise. Partial credit (0.5) for a correct and relevant application of Chernoff's inequality.

s.11.0.174. Since $Z^2 \sim \chi^2_1$, we have $\mathsf{E}\left[e^{tZ^2}\right] = \mathsf{E}\left[e^{tY}\right] = M_Y(t) = (1-2t)^{-1/2}$.

So we minimize $e^{-4t}\,\mathsf{E}\left[e^{tZ^2}\right] = e^{-4t}(1-2t)^{-1/2}$. It is easier if we take the logarithm first and minimize $-4t - \frac{1}{2}\log(1-2t)$. Its derivative to t is $-4 + \frac{1}{1-2t}$, so setting the derivative to 0 yields $t = 3/8$. The second derivative to t is $\frac{2}{(1-2t)^2} > 0$ (the value at $t = 3/8$ is 32), so the second order condition holds.

This yields $\mathsf{P}\{|Z| > 2\} \le e^{-3/2}(1-3/4)^{-1/2} \approx 0.446$.

Remarks and grading scheme:

- Unless explicitly noted, you have to give an analytical answer and derive everything using just pen and paper.

- It is essential that you include some derivations for computing the (second) derivative.

- It is not necessary to take the logarithm first, but it makes the derivation easier, and hence it is less likely that you make errors.

- Grading scheme: 0.5 for arguing that $\mathsf{E}\left[e^{tZ^2}\right] = (1-2t)^{-1/2}$ with sufficient explanation; 1 for taking the derivative and calculating that it is 0 at $t = 3/8$; (0.5 if small mistake is made but resulting t satisfies $0 < t < 1/2$, or if it is remarked that this should be the case); 0.5 for calculating the second derivative and checking the second order condition;
  0.5 for filling in $t = 3/8$ to provide the upper bound (if an incorrect value of t is found, this point can be given only if the resulting bound is between 0.01 and 1, or if it is explicitly noted that the answer does not make sense).

s.11.0.175. The initial distribution is $\mathsf{P}\left\{X = x, Y = y\right\} = 1/9$, $x, y = 1,2,3$. Amy wins iff $X > Y$, i.e., iff $(X,Y) \in \{(2,1),(3,1),(3,2)\}$. So we have

$$\mathsf{P}\left\{X = x, Y = y | X > Y\right\} = \frac{\mathsf{P}\left\{X = x, Y = y, X > Y\right\}}{\mathsf{P}\{X > Y\}} \tag{13.0.322}$$

$$= \frac{1/9}{1/3} = 1/3, \tag{13.0.323}$$

for $(x,y) \in \{(2,1),(3,1),(3,2)\}$.

s.11.0.176. We have

$$A = \begin{cases} 0 & X < Y \\ 1/2 & X = Y \\ X - Y & X > Y. \end{cases} \tag{13.0.324}$$

Hence,

$$\mathsf{E}[A|X > Y] = \mathsf{E}[X - Y|X > Y] \tag{13.0.325}$$
$$= \frac{1}{3}(2-1) + \frac{1}{3}(3-1) + \frac{1}{3}(3-2) \tag{13.0.326}$$
$$= 4/3. \tag{13.0.327}$$

s.11.0.177. We have, by the law of total expectation,

$$\mathsf{E}[A] = \mathsf{P}\{X < Y\}\,\mathsf{E}[A|X < Y] + \mathsf{P}\{X = Y\}\,\mathsf{E}[A|X = Y] + \mathsf{P}\{X > Y\}\,\mathsf{E}[A|X > Y] \tag{13.0.328}$$
$$= 0 + \frac{1}{3}\cdot\frac{1}{2} + \frac{1}{3}\frac{4}{3} \tag{13.0.329}$$
$$= 11/18. \tag{13.0.330}$$

s.11.0.178. By Adam's law,

$$\mathsf{E}[T] = \mathsf{E}[\mathsf{E}[T|A]] \tag{13.0.331}$$
$$= \mathsf{E}\left[\frac{3}{2}A\right] \tag{13.0.332}$$
$$= \frac{3}{2}\mathsf{E}[A] \tag{13.0.333}$$
$$= \frac{3}{2}\frac{11}{18} \tag{13.0.334}$$
$$= \frac{33}{36}. \tag{13.0.335}$$

s.11.0.179. Let $Y = Z^2$, then $Y \sim \chi^2_1$, so $Y \sim \text{Gamma}(\frac{1}{2}, \frac{1}{2})$. We have

$$\mathsf{E}\left[Z^{2n+2}\right] = \mathsf{E}\left[Y^{n+1}\right] = \frac{\Gamma(n+3/2)}{(1/2)^{n+1}\Gamma(1/2)} = \frac{2^{n+1}\Gamma(n+3/2)}{\Gamma(1/2)}$$
$$\mathsf{E}\left[Z^{2n}\right] = \mathsf{E}\left[Y^{n}\right] = \frac{\Gamma(n+1/2)}{(1/2)^{n}\Gamma(1/2)} = \frac{2^{n}\Gamma(n+1/2)}{\Gamma(1/2)}.$$

Since $\Gamma(n+3/2) = (n+1/2)\Gamma(n+1/2)$ we conclude

$$\mathsf{E}\left[Z^{2n+2}\right] = \frac{2^{n+1}\Gamma(n+3/2)}{\Gamma(1/2)} = 2(n+1/2)\frac{2^{n}\Gamma(n+1/2)}{\Gamma(1/2)} = (2n+1)\,\mathsf{E}\left[Z^{2n}\right].$$

Remarks and grading scheme:

- It is NOT true that $\mathsf{E}\left[Z^{2n+2}\right] = \mathsf{E}\left[Z^2\right]\mathsf{E}\left[Z^{2n}\right]$. This would only be true if Z^2 and Z^{2n} would be uncorrelated. But clearly, they are positively correlated: if Z^2 is large, then so is Z^{2n}.

- While induction is a good strategy to try when you have to prove a statement for all $n \in \mathbb{N}$, it is certainly not the only option.

- Grading: 0.5 for introducing $Y = Z^2$ and arguing $Y \sim \text{Gamma}(\frac{1}{2}, \frac{1}{2})$ with some explanation, 0.5 for using the given expression to calculate $\mathsf{E}\left[Z^{2n+2}\right]$ and $\mathsf{E}\left[Z^{2n}\right]$ and 0.5 for using a property of the Gamma function to finish the answer.

s.11.0.180. By using the previous exercise with $n = 1$, we get $\mathsf{E}\left[Z^4\right] = 3\,\mathsf{E}\left[Z^2\right] = 3$.

s.11.0.181. We have $\mathsf{P}\{|Z| > 3\} = \mathsf{P}\left\{Z^4 > 81\right\}$ since $|Z| > 3$ if and only if $Z^4 > 81$. The inequality now directly follows from Markov's inequality.

Remarks and grading scheme:

- This exercise was made quite well.

- Grading scheme: 0.5 for (i) and 0.5 for (ii).

s.11.0.182. By Markov's inequality,

$$\mathsf{P}\{|Z| > 3\} = \mathsf{P}\left\{Z^{2n} > 9^n\right\} \leq \frac{\mathsf{E}\left[Z^{2n}\right]}{9^n}.$$

From the formula for $\mathsf{E}\left[Z^{2n}\right]$ we see that $\mathsf{E}\left[Z^{2(n+1)}\right] = (2n+1)\,\mathsf{E}\left[Z^{2n}\right]$. We now consider what happens when incrementing n. If $n < 4$ then $2n+1 < 9$, so then incrementing n improves the bound, for $n = 4$ incrementing n doesn't change the bound and for $n > 4$ the bound becomes weaker. So we get the best possible bound for $n = 4$ and $n = 5$. We have $\mathsf{E}\left[Z^8\right] = 7\,\mathsf{E}\left[Z^6\right] = 7 \cdot 5\,\mathsf{E}\left[Z^4\right] = 105$. Hence,

$$\mathsf{P}\{|Z| > 3\} \leq \frac{105}{9^4} \approx 0.016.$$

Remarks and grading scheme:

- Many people tried to use induction in this exercise, but that doesn't work. While it is good to think of induction when you have to prove a statement for all $n \in \mathbb{N}$, it is certainly not the only option.

- In fact, following a similar strategy as in the previous exercise would have helped here.

- Grading scheme: 0.5 for $\mathsf{P}\{|Z| > 3\} \leq \frac{\mathsf{E}[Z^{2n}]}{9^n}$, 1 for showing that the best bound is obtained for $n = 4$ and $n = 5$, and 0.5 for calculating the resulting bound.

s.11.0.183. We can use the result of part 1.

a. On the set $\{N(t) = n\}$ $N(t) = n$. Hence we can replate $N(t)$ by n.

b. Linearity of expectation

c. definition of conditional expectation.

Each property missed, e.g, linearity of expectation, minus 0.5.

s.11.0.184.

$$\mathsf{E}[S] = \mathsf{E}[T] + \mathsf{E}[N(T)]\,\mathsf{E}[R] = 1/\mu + r\lambda\,\mathsf{E}[T] = 1/\mu + r\lambda/\mu.$$

Typically, 0.5 point for each correct line, but I subtracted points If you say that $\mathsf{E}[N(T)\,\mathsf{E}[R]] = \mathsf{E}[R]$, or write $n\,\mathsf{E}[R]$ as final answer (apparently you did not get the idea that N is an rv.)

s.11.0.185. Here is an EXPLANATION:

1. T is a simulated service time of a job without interruptions.
2. N is is the simulated number of failures that occur during the service of the job
3. R is then a vector of simulated durations of each of the interruptions
4. Hence, S is the total time the job spends at the server
5. We sample the total service time a number of times, and count how often (mean or variance) this exceeds a threshold.

The points earned depends on how clear you explanation is.

Quite a few of you don't seem to understand what it means to *explain* something. Here are some answers that I saw that certainly don't explain what the code does:

1. 'The code does what's stated in the exercise.'. What's the explanation here? The question is also not: do you understood what the code does?
2. 'T is uniform rv, N is a Poisson rv, R is a also a uniform rv. We repeat this a few times.' Like this you just read the code, but I know you can read, so this type of answer is quite useless.
3. 'We collect a subset of the samples and print that.' This answer is just a repition of the hint.

So, next time, when you are asked to explain something, explain it. In your professional carreer, if you read out loud to your customer what s/he can read for him/herself, you'll be fired pretty soon.

s.11.0.186. Since births and deaths are exponentially distributed, we can use that $B(h) \sim \text{Pois}((\lambda X(t)+\theta)h)$ and $D(h) \sim \text{Pois}(\mu X(0)h)$ when $X(0) = n$.

The subtlety is due to the fact that during the time h also multiple arrivals and departures can occur, but since these rates depend on the number people in the system, these rates need not be constant during the time interval h. However, since such events have very small, in fact have $o(h)$ probability, we can capture all such details in the $o(h)$ terms.

Grading: mention the use of exponential and Poisson distribution: +1/2.

s.11.0.187.

$$\mathsf{P}\{B(h)=0, D(h)=1|X(0)=n\} = e^{-(\lambda n+\theta)h}\mu n h e^{-\mu n h} = (1-(\lambda n+\theta)h)\mu n h(1-\mu n h)+o(h) = \mu n h+o(h). \tag{13.0.336}$$

Grading:

- Skipping the algebra: -1/2.

s.11.0.188.

$$\begin{aligned}
\mathsf{E}[X(t+h)|X(t)=n] &= n\,\mathsf{P}\{B(h)=0, D(h)=0\} + (n+1)\,\mathsf{P}\{B(h)=1, D(h)=0\} && (13.0.337)\\
&\quad + (n-1)\,\mathsf{P}\{B(h)=0, D(h)=1\} + o(h) && (13.0.338)\\
&= ne^{-(\lambda n+\theta)h}e^{-\mu n h} + (n+1)(\lambda n+\theta)h + (n-1)\mu n h + o(h) && (13.0.339)\\
&= n(1-(\lambda n+\theta)h)(1-\mu n h) + (n+1)(\lambda n+\theta)h + (n-1)\mu n h + o(h) && (13.0.340)\\
&= n + (\lambda n+\theta-\mu n)h + o(h). && (13.0.341)
\end{aligned}$$

Grading:

- Show also how to simplify the results of the first question. If not, -1/2.

s.11.0.189. Replace n by $X(t)$ in $\mathsf{E}[X(t+h)|X(t)]$ to see that

$$\mathsf{E}[X(t+h)|X(t)] = X(t) + (\lambda-\mu)X(t)h + \theta h + o(h). \tag{13.0.342}$$

Take expectations left and right and use Adam's law.

Grading:

- No points for not mentioning Adam's law, or showing in some way that you used it.
- Using Adam's law in the wrong way, i.e, not replacing the n by $X(t)$ at most 1/2.

s.11.0.190. Since X and Y are independent, their joint PDF is the product of their marginal PDFs. This gives:

$$f(x,y) = \left(\frac{1}{(3-1)}\right)\left(\frac{1}{(3-1)}\right) = \frac{1}{4}$$

For $1 < x < 3$ and $1 < y < 3$ and 0 otherwise.
One mistake, zero points.

s.11.0.191. Calculating this integral gives:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_{1}^{3}\int_{1}^{3}\frac{1}{4}\,dx\,dy$$
$$=1$$

One mistake, zero points.

s.11.0.192. Step 1. Find the expectation $E(|X-Y|)$. Using LOTUS.

$$\begin{aligned} E(|X-Y|) &= \int_1^3\int_1^3 |x-y|\left(\frac{1}{4}\right)dx\,dy \\ &= \int_1^3\int_y^3 (x-y)\left(\frac{1}{4}\right)dx\,dy + \int_1^3\int_1^y (y-x)\left(\frac{1}{4}\right)dx\,dy \\ &= \frac{1}{3}+\frac{1}{3} \\ &= \frac{2}{3} \end{aligned}$$

Step 2. Find the squared expectation $|X-Y|^2$. Using LOTUS.

$$\begin{aligned} E(|X-Y|^2) &= \int_1^3\int_1^3 |x-y|^2\left(\frac{1}{4}\right)dx\,dy \\ &= \int_1^3\int_1^3 (x-y)^2\left(\frac{1}{4}\right)dx\,dy \\ &= \int_1^3\int_1^3 (x^2-2xy+y^2)\left(\frac{1}{4}\right)dx\,dy \\ &= \frac{2}{3} \end{aligned}$$

Step 3. Find the variance of $|X-Y|$.

$$\begin{aligned} \mathsf{V}[|X-Y|] &= E(|X-Y|^2) - E(|X-Y|)^2 \\ &= \frac{2}{3}-\left(\frac{2}{3}\right)^2 \\ &= \frac{2}{9} \end{aligned}$$

Step 4. Find the standard deviation of $|X-Y|$.

$$\begin{aligned} SD(|X-Y|) &= \sqrt{\mathsf{V}[|X-Y|]} \\ &= \sqrt{\frac{2}{9}} \\ &= 0.4714 \end{aligned}$$

One point for writing down the integral for $E|X-Y|$ and splitting it up correctly. One point for $E|X-Y|^2$. One point for finding $SD(|X-Y|)$ in the correct way.

s.11.0.193. It loads the required packages and creates one sample with 100000 observations from a $\mathcal{N}(50, 200)$-distribution. Then for all observations it subtracts its mean and tests if the new value is within 2 standard deviations of the mean.
0.5 points for mentioning the mean is subtracted and it is checked if the value found is smaller than 2 times the s.d.

s.11.0.194. By Theorem 5.4.5 we get that $P(|X-\mu| < 2\sigma) \approx 0.95$, this is also shown in the code.
0.5 points for making a comparison between the theorem and the answer in the code. Conclusion should be that they give similar results.

s.11.0.195.

s.11.0.196.

s.11.0.197.

s.11.0.198.

s.11.0.199.

s.11.0.200.

s.11.0.201. By the book, we know that $\min\{X_1, X_2, \dots, X_n\} \sim \text{Expo}(n\lambda)$.

Grading scheme:

- Correct 0.5pt.

s.11.0.202. We start with the definition:

$$\begin{aligned}
M_W(t) &= \mathsf{E}\left[e^{Wt}\right] \\
&= \int_{-\infty}^{\infty} e^{wt} \frac{\lambda}{2} e^{-\lambda|w|} \, \mathrm{d}w \\
&= \frac{\lambda}{2} \int_{-\infty}^{0} e^{(t+\lambda)w} \, \mathrm{d}w + \frac{\lambda}{2} \int_{0}^{\infty} e^{(t-\lambda)w} \, \mathrm{d}w \\
&= \frac{\lambda}{2} \frac{1}{\lambda + t} + \frac{\lambda}{2} \frac{1}{\lambda - t} \\
&= \frac{\lambda^2}{\lambda^2 - t^2},
\end{aligned}$$

where we used the assumptions $t > -\lambda$ and $t < \lambda$ to make the first and second integral converge respectively. Hence, this MGF is defined only for $|t| < \lambda$.

Grading scheme:

- Definition 0.5pt.

- Split the integral 0.5pt.
- Correct integral calculation 0.5pt.
- Correct bounds 0.5pt.

s.11.0.203. We know that $M_{X-Y}(t) = M_X(t)M_Y(-t)$, and that $M_X(t) = M_Y(t) = \frac{\lambda}{\lambda - t}$ for $t < \lambda$. Then, we show that

$$\frac{\lambda}{\lambda - t}\frac{\lambda}{\lambda + t} = \frac{\lambda^2}{\lambda^2 - t^2},$$

for $t < \lambda$ *and* $-t < \lambda$, or $|t| < \lambda$. Since the MGF uniquely determines the distribution, we know that $X - Y \sim W$.

Grading scheme:

- Correct MGF and bounds 1pt.

s.11.0.204. Let $Q = |X - Y| \sim |W|$. We start with the definition:

$$\begin{aligned}
M_Q(t) &= \mathsf{E}\left[e^{Qt}\right] \\
&= \mathsf{E}\left[e^{|W|t}\right] \\
&= \int_{-\infty}^{\infty} e^{|w|t}\frac{\lambda}{2}e^{-\lambda|w|}\,\mathrm{d}w \\
&= \frac{\lambda}{2}\int_{-\infty}^{0} e^{(\lambda-t)w}\,\mathrm{d}w + \frac{\lambda}{2}\int_{0}^{\infty} e^{(t-\lambda)w}\,\mathrm{d}w \\
&= \frac{\lambda}{2}\left(\frac{1}{\lambda - t} - \frac{1}{t-\lambda}\right) \\
&= \frac{\lambda}{\lambda - t},
\end{aligned}$$

where we need $t < \lambda$ to make both integrals converge. This is again an exponential MGF!

Grading scheme:

- Correct integration 1pt.
- Correct bounds 0.5pt.

s.11.0.205. Using Bayes' rule we have

$$f_1(p|X_1 = x_1) = \frac{\mathsf{P}\left\{X_1 = x_1|p\right\}f_0(p)}{\mathsf{P}\left\{X_1 = x_1\right\}} \tag{13.0.343}$$

$$\propto \mathsf{P}\left\{X_1 = x_1|p\right\}f_0(p) \tag{13.0.344}$$

$$= \binom{x_1 + r - 1}{x_1}(1-p)^r p^{x_1}\frac{1}{B(a,b)}p^{a-1}(1-p)^{b-1} \tag{13.0.345}$$

$$\propto (1-p)^{r+b-1}p^{a+x_1-1}, \tag{13.0.346}$$

in which we recognize the pdf of a Beta$(a+x_1, b+r)$ distribution (up to a constant factor). Hence, the posterior distribution of p given $X_1 = x_1$ is a Beta$(a+x_1, b+r)$ distribution.

s.11.0.206. Yes. The posterior distribution is in the same family of distributions (Beta) as the prior. Hence, Amy has a conjugate prior.

s.11.0.207. The posterior after observing $X_1 = x_1$ becomes our new prior. Hence, our new prior is a Beta$(a+x_1, b+r)$ distribution. From question 1 it follows that the prior after observing $X_2 = x_2$ then is a Beta$(a+x_1+x_2, b+2r)$ distribution. Hence, iterating this process, we find that the posterior distribution of p after observing $X_1 = x_1, \ldots, X_n = x_n$ is a Beta$(a+\sum_{i=1}^n x_i, b+rn)$ distribution.

s.11.0.208. We start by trying to find a formula for $F_Y(y)$. After drawing the function $y = \frac{1}{x}$ in the x, y-plane, it becomes obvious that

$$F_Y(y) = \mathsf{P}\left\{Y \le y\right\} = \begin{cases} \mathsf{P}\{X \le 0\} + \mathsf{P}\left\{X \ge \frac{1}{y}\right\} & \text{if } y > 0 \\ \mathsf{P}\left\{\frac{1}{y} \le X \le 0\right\} & \text{if } y < 0 \end{cases}.$$

Draw it if this is not clear!
To calculate less, we notice that $\mathsf{P}\{X \le 0\} = \frac{1}{2}$, by symmetry. Then,

$$F_Y(y) = \begin{cases} \frac{1}{2} + \mathsf{P}\left\{X \ge \frac{1}{y}\right\} & \text{if } y > 0 \\ \frac{1}{2} & \text{if } y = 0 \\ \frac{1}{2} - \mathsf{P}\left\{X \le \frac{1}{y}\right\} & \text{if } y < 0 \end{cases}.$$

See also exercise 8.9.11.

Grading scheme:

- Correct cases 0.5pt.
- No mistakes etc. 0.5pt.

s.11.0.209.

$$f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y} F_Y(y) = \begin{cases} \frac{\mathrm{d}}{\mathrm{d}y}(\frac{1}{2} + \mathsf{P}\left\{X \ge \frac{1}{y}\right\}) & \text{if } y > 0 \\ \frac{\mathrm{d}}{\mathrm{d}y}(\frac{1}{2} - \mathsf{P}\left\{X \le \frac{1}{y}\right\}) & \text{if } y < 0 \end{cases}$$
$$= \begin{cases} -\frac{\mathrm{d}}{\mathrm{d}y} \mathsf{P}\left\{X \le \frac{1}{y}\right\} & \text{if } y > 0 \\ -\frac{\mathrm{d}}{\mathrm{d}y} \mathsf{P}\left\{X \le \frac{1}{y}\right\} & \text{if } y < 0 \end{cases}.$$

We can write, by definition:

$$\mathsf{P}\left\{X \le \frac{1}{y}\right\} = \int_{-\infty}^{\frac{1}{y}} f_X(s)\,\mathrm{d}s,$$

such that by the FTC,

$$-\frac{\mathrm{d}}{\mathrm{d}y}\mathsf{P}\left\{X \le \frac{1}{y}\right\} = \frac{1}{y^2} f_X(\frac{1}{y})$$

for $y \neq 0$.

Grading scheme:

- Correct calculations 0.5pt.
- No mistakes etc. 0.5pt.
- Alternatively, use the transformation theorem to show this, if you didn't use the result from part (a). Be careful to correctly apply it.

s.11.0.210. When $\nu = 1$, X follows a Cauchy distribution. Then, Y must also be Cauchy.

Grading scheme:

- Correct 1pt.

s.11.0.211. It converges to the standard normal distribution.

Grading scheme:

- Correct 0.5pt.

s.11.0.212. For $\nu > 1$, we have that

$$f_Y(y) \propto \left(y + \frac{1}{\nu y}\right)^{\frac{\nu}{2}+\frac{1}{2}}.$$

The FOC tells us that the mode is at

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}y}\left(y + \frac{1}{\nu y}\right)^{\frac{\nu}{2}+\frac{1}{2}} = \left(y + \frac{1}{\nu y}\right)^{\frac{\nu}{2}-\frac{1}{2}}\left(1 - \frac{1}{\nu y^2}\right) = 0 \implies \\
&1 - \frac{1}{\nu y^2} = 0 \implies \\
&y = \pm\frac{\sqrt{\nu}}{\nu}.
\end{aligned}$$

Clearly, these must be the maximum values f_Y takes on; if they were minima the PDF would not integrate to unity, and they cannot be saddle points (the only other option as the PDF is symmetric) since then there would be a different maximum, which the FOC would show. Alternatively, you could look at the second derivative.

Grading scheme:

- Correct FOC 1pt.
- Something about it being a maximum (lenient) 0.5pt.

s.11.0.213. By Bayes' rule we have

$$f_1(\lambda|Y_1 = y_1) = \frac{f_{Y_1|\lambda}(y_1|\lambda) f_0(\lambda)}{f_{Y_1}(y_1)} \tag{13.0.347}$$

$$\propto f_{Y_1|\lambda}(y_1|\lambda) f_0(\lambda) \tag{13.0.348}$$

$$= \frac{b^a}{\Gamma(a)} \lambda^{a-1} e^{-b\lambda} \lambda e^{-\lambda x_1} \tag{13.0.349}$$

$$\propto \lambda^a e^{-(b+x_1)\lambda}, \tag{13.0.350}$$

in which we recognize the pdf of a Gamma$(a+1, b+y_1)$ distribution (up to a scaling constant). Hence, the posterior distribution λ given $Y_1 = y_1$ is Gamma$(a+1, b+y_1)$.

Applying Bayes' rule: 1 point.
Finding the expression for the posterior: 1 point.
recognizing a $Gamma(a+1, b+x_1)$ dist: 0.5 point.

s.11.0.214. The posterior after observing $Y_1 = y_1$ becomes our new prior. Hence, our new prior is a Gamma$(a+1, b+y_1)$ distribution. From question 1 it follows that the prior after observing $Y_2 = y_2$ then is a Gamma$(a+2, b+y_1+y_2)$ distribution. Iterating this process (i.e., by mathematical induction), we find that the posterior distribution of λ after observing $Y_1 = y_1, \ldots, Y_n = y_n$ is a Gamma$(a+n, b+\sum_{i=1}^n y_i)$ distribution.

Noting that the posterior becomes the new prior: 0.5 point.
Correct derivation and answer: 1 point.

s.11.0.215. Yes. The posterior distribution is in the same family of distributions (Gamma) as the prior. Hence, John has a conjugate prior.

Correct answer and motivation: 1 point.