Comparison of Models Describing the Domino Effect and Literature Review

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Abstract

The domino effect has been a subject of discussion in physics and applied mathematics for many decades and dominoes have been a children's toy for much longer than that. This paper provides an overview of the current literature, analyzing the varying experimental designs and a comparison of theoretical models and their merits. It is clear that this is a topic of considerable complexity and the current models still have many constraints. Additionally the pedagogical merits of studying this system are considered.

Introduction

The domino effect refers to the toppling of a chain of equally distributed rectangular prisms (of equal height, thickness and width), with a uniform mass distribution, standing on their smallest face. This effect has been explored both experimentally and theoretically. The main question that most authors tackle is, what is the relationship between the spacing between dominoes and the final/asymptotic velocity of the domino collisions.

Many models have been developed but due to the complexity, currently, there is no definitive model that describes the behaviour of the system. Additionally, there has been a lack of comparison between different models, with most authors opting to only compare their model with experimental data.

This paper will discuss and compare the varying assumptions made by different authors, give an overarching view of how the problem can be modeled and compare models numerically to analyze how well they agree.

Motivation and Interest

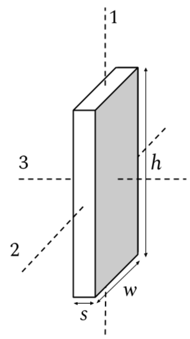

Figure 1: A single domino with width w, height h and thickness s. The three axis of rotation are labeled 1, 2 and 3.

The most fascinating behaviours of the domino system arise when one considers problem in its full complexity. Since standard dominoes are not very massive, (8.5 ± 0.1 g from manufacturer [1]), experimentally, it is difficult to find a surface on which the domino will not slip at all. Standard practice for experiments is to use sandpaper in order to increase the friction but even so it is difficult to eliminate slipping as the domino falls and especially when one domino strikes the next domino.

The possibility of slipping allows there to be rotation about the first axis of the domino. In three dimensions, dominoes have three principle moments of inertia, one about each axis in as seen in Figure 1. The axis about which dominoes rotate in general is the second or the intermediate axis. This axis has a moment of inertia that is in between the first and third axis and the theorem states that the rotation will be unstable if the axis of rotation is not exactly aligned with the second principal axis [2].

The instability generates rotation about the first axis of the domino. During the collision, the domino may hit the next one with one corner rather than the full width of the top edge, causing more rotation about the first axis. This effect will slow down the dominoes as less energy is transferred from one domino to the next. Experimentally, it was seen that (in general with spacing between dominoes that is not too large) the rotation will self correct and eventually one domino will make full contact with the next, causing the domino to rotate with a faster angular velocity in the direction of propagation [3].

This series of events was observed to cause the velocity of dominoes to continuously increase then decrease [3]. So it is possible that in the three dimensional case, there is no final asymptotic velocity for falling dominoes.

Dominoes Slipping

Another factor that contributes to the varying velocity of dominoes falling in succession becomes more obvious when dominoes are spaced further apart. The faster a domino is rotating, the more likely it will be able to overcome the static friction and slip on the surface rather than simply tilting. If this occurs, the domino will strike the next domino lower down. Since the torque proportional to how far away from the axis of rotation the force is applied, the next domino falls slower. But if it rotates slower than it is less likely to slip and may only tip, so then it is likely to hit the next domino higher giving it more torque.

The pattern of slipping and tipping was observed experimentally and was seen to also cause fluctuations in the speed of the domino wave [3]. Since this phenomenon is not dependent on the width of the domino, we can isolate this it from the previous effect by considering a two-dimensional domino.

This is the first simplification that can be made to the domino system. Unfortunately it still leaves a lot of complexity. If the dominoes are allowed to slip, it doesn't simply slide across the surface because it is rotating about its center of mass while falling. The domino may "skip" across the surface as it tilts, overcomes friction, rotates while not in contact with the surface then falls enough to make contact again.

The interactions between the surface the dominoes rest on and the domino adds much complexity on top of the considerations of interactions between dominoes.

Simplifying Assumptions and Hypothesis

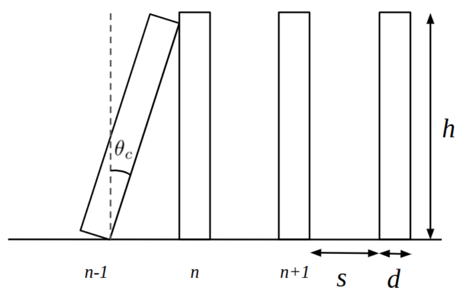

Figure 2: A possible arrangement of dominoes. With h as the domino height, d as the thickness, and s as the distance between dominoes. θ_c is the angle that the leading edge of the domino makes with the vertical when it collides with the next domino. Domino n is the leading domino in the chain. As soon as it collides with the next domino, it becomes domino $n-1$.

Fortunately, there are many more simplifications that can be made to make the problem more approachable. Most notably, all models the simplifying assumption that there is sufficient friction at the tipping point such that dominoes do not slip during the collision. The consequence of this assumption is that the domino which is a 3D object can be reduced to 2D as the width of the domino is not an important parameter. So the only relevant lengths of the domino are its height and its thickness. This was proven to be true through dimensional analysis by Sun in 2020 [4].

Of all the models, Efthimiou and Johnson [5] made the most limiting assumptions. The most unrealistic was considering a domino as a point mass atop a massless rod. It was shown by Larham [6] that this model does not agree with experimental data. The model vastly overestimates the speed of the domino wave especially at separation ratios (ratio of separation to height of domino) of less than 0.6. Since no other models make this assumption and the model has already been proven to not reflect reality, it will not be considered in the subsequent model comparisons.

While the most important assumption that must be made by any model is the nature of the collision between dominoes, there are a few other differences that should be discussed first that more subtly affect the models.

Thickness of Dominoes

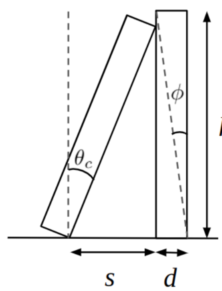

Figure 3: $\arcsin\left(\frac{s}{h}\right) = \theta_c$ and $\arctan\left(\frac{d}{h}\right) = \phi$ where ϕ is the angle measured between the diagonal and the vertical of the domino.

For a domino to fall, it must be tilted enough that its center of mass moves past the front supporting edge. This becomes relevant when considering the energy of the system, specifically how much energy is necessary for the wave to continue propagating.

For some arbitrary height of domino and separation, and assuming that the system begins with no angular velocity. What is the limit on the thickness of the domino such that the first domino would be able to cause the next domino to fall? The kinetic energy of the first domino needs to be sufficient to push the second domino from the rest position to when the center of mass of the domino is above or just past the tilting point.

Using conservation of energy, the kinetic energy that the first domino would have is simply the difference in potential energy between when its center of mass is at its highest and when it collides with the next domino. This energy needs to be greater than the difference in potential energy between when the second domino is at rest and when it is also at its highest.

$$\frac{1}{2}mg\left(\sqrt{\left(\frac{h}{2}\right)^2+\left(\frac{d}{2}\right)^2}-\sqrt{\left(\frac{d}{2}\right)^2+\left(\frac{h}{2}\right)^2}\cos\alpha\right) \geq \frac{1}{2}mg\left(\sqrt{\left(\frac{h}{2}\right)^2+\left(\frac{d}{2}\right)^2}-\frac{h}{2}\right)$$
$$\sqrt{\left(\frac{d}{2}\right)^2+\left(\frac{h}{2}\right)^2}\cos\left(\arcsin\left(\frac{s}{h}\right)-\arctan\left(\frac{d}{h}\right)\right) \leq \frac{h}{2}$$
$$\sqrt{\left(\frac{d}{2}\right)^2+\left(\frac{h}{2}\right)^2}\cos\left(\arcsin\left(\frac{s}{h}\right)-\arctan\left(\frac{d}{h}\right)\right)-\frac{h}{2} \leq 0 \tag{1}$$

Where $\alpha = \theta_c - \phi$, are defined as in Figure 3. The above is simply a linear function in d and with some algebraic manipulation the limits on d are (with h and s always positive and s<h)

$$0 < d < \frac{h^2 - h^2\sqrt{\frac{(h-s)(h+s)}{h^2}}}{s} \tag{2}$$

Some models choose to ignore the thickness of dominoes ([5, 7, 8]) in the model asserting that it is much smaller than the height of the domino and thus will not have an important effect. While most others ([9–13]) include thickness as a relevant length to the geometry of the problem. Though all authors agree that the thickness of the domino is much less than the height.

Limits on Domino Spacing

Most models are only valid under certain assumptions for the spacing of dominoes. Clearly there is a strict upper limit which is $s < h \implies \frac{s}{h} < 1$. If $s \geq h$ then the dominoes would not hit each other and so there is no domino effect.

But even before this limit, the collision of one domino with the next would be too low to impart enough torque to rotate the next domino [10].

Both van Leeuwen [10] and Stronge [14] impose a lower bound on the spacing of the dominoes. Below this spacing, there would not be a steady wave speed and both claim that the wave would slow down until there is not enough momentum/energy for the dominoes to continue toppling.

For Stronge this lower limit is

$$s > d\,(1 + \cos 2\phi) \tag{3}$$

$$= d \cdot 2\left(\frac{dh}{d^2 + h^2}\right) \tag{4}$$

$$\implies \frac{s}{h} > \frac{2d^2}{d^2 + h^2} \tag{5}$$

$$= \frac{2\,(d/h)^2}{(d/h)^2 + 1} \tag{6}$$

d is the the thickness of dominoes. For van Leeuwen this limit was

$$\frac{s}{h} > \frac{2\,(d/h)^3}{\left[1 - (d/h)^2\right]} \tag{7}$$

These values do not match, with Stronge having a larger lower limit (0.065) than van Leeuwen (0.012) for the dimensions of standard dominoes ($\frac{d}{h} \approx 0.18$ [8]).

Stronge and Shu later imposed a different lower limit of

$$\frac{s}{h} > \frac{2\,(d/h)}{\left((h/d)^2 - 1\right)}$$

this value is roughly 0.012 for $\frac{d}{h} \approx 0.18$ which is much more similar to that of van Leeuwen previously. For a domino with a height of 5 cm, the separation would be 0.06 cm which is less than 1 mm. This separation is never experimentally observed.

This is likely because initially Stronge considered single collisions [14] whereas Stronge and Shu [13] considered dominoes sliding over one another which is the same as van Leeuwen's model.

For the upper limit, van Leeuwen [10] states that at large separations, the assumption of no slip at the point of rotation is not realistic. So for mathematical convenience and for physical reasons mentioned above, van Leeuwen imposed a stricter upper limit of

$$h^2 > (s+d)^2 - d^2$$

or

$$\frac{s}{h} < \sqrt{1 + (d/h)^2} - \frac{d}{h} \tag{8}$$

This is roughly 0.84 for a standard domino.

Considering the limits in the domino spacing is important when comparing models to experimental data. These limits are often made to satisfy mathematical equations and may not be reflective of real behaviour. Oftentimes, the models are not accurate even within the assumption domain.

Interaction Between Dominoes

Experimentally, it's been observed that dominoes display three types of behaviour during a collision [3,13]:

1. There is only one collision between neighbouring dominoes. This means that the only effect one domino has on the next domino is during the collision. Generally this collision is assumed to be instantaneous. (Banks [7], Bert [9] and Stronge in 1987 [14] make this assumption.)

2. After colliding the dominoes stay in sliding contact. In this case, the collision is definitely not elastic and one must consider the chain of dominoes prior to the current one when finding velocity. (van Leeuwen [10], Shaw [11], and Stronge and Shu [13] in 1988 make this assumption)

3. There are multiple collisions between neighbouring dominoes. The dominoes bounce on one another. (There are no models that use this assumption.)

The collision behaviour and its implications have the greatest effect on a model. When creating a model for the domino effect, there are really two main areas that require analysis.

1. How does one domino fall? Specifically what angular velocity does the domino have throughout the fall given an initial angular velocity. The inverse of the angular velocity gets integrated over the angular displacement in order to find the time it took for the domino to fall.

2. How is energy transferred during a collision? This determines the initial angular velocity of the following domino.

The nature of the collision between neighbouring dominoes affects both of the above, this will be made clear through the comparison of the models of Banks [7] and Shaw [11].

Theoretical Models

As was shown above from the assumptions, there are many variables that can be adjusted when modeling the domino effect. On top of these more explicit assumptions, many implicit assumptions and differences in derivation method will affect the model of the system.

As stated previously, the main goal of any model is to find the angular velocity over the duration of the fall of the leading domino, then how this angular momentum is transferred to the domino that gets collided against. The models varied wildly in the approaches they took in order to achieve this goal.

I will discuss two particular models that had different assumptions to compare and contrast them.

Elastic Single Collision

Both Banks and Bert [7, 9], considered elastic collisions where only one previous domino affects the following one rather than a chain of dominoes. The only significant difference between the two models is that Bert considered the thickness (d) of the domino for the moment of inertia about the tilting point. To simplify things slightly, Banks' model will be considered.

The model employs three steps to achieve the goals outlined above.

1. Use conservation of energy to find the angular velocity of the domino at any point as it falls.
2. Integrate the inverse of angular velocity with respect to the angular displacement to find the time it took to fall.
3. Use conservation of linear momentum during the collision to find the initial angular velocity of the next domino.

Step 1: Conservation of Energy

Conservation of energy gives

$$\omega_{n-1}(\theta_c) = \sqrt{\omega_m^2 - \frac{3g}{h}(1+\cos\theta_c)} \tag{9}$$

$$\omega_n(0) = \sqrt{\omega_m^2 - \frac{6g}{h}} \tag{10}$$

for the angular velocity of the $n-1$ domino at collision ($\omega_{n-1}(\theta)$) and for the next domino while standing ($\omega_n(0)$). It is assumed the a final asymptotic velocity has already been reached and hence all dominoes will have the same ω_m, which is the maximum possible angular velocity. If the domino were a physical pendulum with pivot at the center of its base, this would be the angular velocity when the domino is completely down.

Step 2: Integration over angular displacement to find time

Banks set up the integration to be the standard complete and incomplete elliptic integrals of the first kind.

$$t = \frac{1}{2}\omega_m \left(\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} - \int_0^{\phi_c} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} \right) \tag{11}$$

$$\phi_c = \frac{\pi - \theta_c}{2}, k = \frac{\sqrt{6g/h}}{\omega_m}$$

Step 3: Conservation of linear momentum

Momentum in the direction of propagation is conserved. This is a reasonable assumption because during the collision, only the x direction of the momentum affects the next domino as the next domino is standing perfectly vertical.

This gives

$$\omega_n (0) = \cos\theta_c \omega_{n-1} (\theta_c) \tag{12}$$

This relation was then used to find ω_m through equating equations 9 and 10. Giving

$$\omega_m = \frac{1}{\sin\theta_c} \sqrt{\frac{3g}{h} \left(2 - (1 + \cos\theta_c) \cos^2\theta_c\right)} \tag{13}$$

this value is used in step to to find the value of k.

Inelastic Collision with Sliding Dominoes

Many authors considered dominoes that stay in contact after collision. To contrast with Banks' model, Shaw's model will be used. Though it has a few limitations which will be discussed, it offers good insight into the core differences between single collision and sliding collision.

Similar to Banks, the model employs three steps to achieve the goals outlined above but there are a few key differences.

1. Use conservation of energy of the **whole system of dominoes** to find the angular velocity of the foremost falling domino at any point as it falls.
2. Integrate the inverse of angular velocity with respect to the angular displacement to find the time it took to fall.
3. Use conservation of **angular** momentum during the collision to find the initial angular velocity of the next domino.

Step 1: Conservation of Energy

Rather than only considering two dominoes, the whole chain of dominoes that are leaning against each other must be taken into account when finding the energy. Thus conservation of energy becomes

$$E = \frac{1}{2} mg \left(h \sum_{i=1}^{n} \cos\theta_i + d \sum_{i=1}^{n} \sin\theta_i \right) + \frac{1}{2} I \sum_{i=1}^{n} \omega_i^2 (\theta_i) \tag{14}$$

Where θ_i is the angle of the i^{th} domino. The first term in the sum is the potential energy of all the dominoes and the second term is the kinetic energy of all the dominoes.

By using the following geometric relations

$$h \sin(\theta_i - \theta_{i+1}) = (s + d)\cos\theta_{i+1} - d \tag{15}$$

computationally, the values for U_i, V_i and W_i can be found (algebraically quite complicated)

$$\cos\theta_i = U_i \cos\theta_n \tag{16}$$

$$\sin\theta_i = V_i \sin\theta_n \tag{17}$$

$$\omega_i(\theta_i) = W_i \sin\theta_n \tag{18}$$

Then finally the angular velocity of the nth domino at any point of its fall is given by

$$\omega_n(\theta) = \left(\frac{2E - mg\left(h\cos\theta\sum_{i=1}^{n} U_i + a\sin\theta\sum_{i=1}^{n} V_i\right)}{I\sum_{i=1}^{n} W_i^2}\right)^{1/2} \tag{19}$$

Step 2: Integration over angular displacement to find time

The time integration step is quite straightforward once $\omega_n(\theta)$ is found.

$$t = \int_0^{\theta_c} \frac{d\theta}{\omega_n(\theta)} \tag{20}$$

Step 3: Conservation of angular momentum

Instead of linear momentum, conservation of angular momentum is more suitable since dominoes are leaning on one another at varying angles.

$$I\sum_{i=1}^{n} \omega_i(\theta_c) = I\sum_{i=1}^{n+1} \omega_i(0) \tag{21}$$

$\omega_i(\theta_c)$ is the angular velocity of the i^{th} domino when the n^{th} domino hits the $n+1$ domino. Similarly, $\omega_i(0)$ is the angular velocity of the i^{th} domino just after domino n+1 has been hit. Thus the initial angular velocity of domino $n+1$ is given by

$$\omega_{n+1}(0) = \left(\frac{\sum_1^n W_i(\theta_c)}{\sum_1^{n+1} W_i(0)}\right)\omega_n(\theta_c) \tag{22}$$

Comparison and Limitations

Despite the difference in assumptions it can be seen that the large strokes of the models are actually quite similar. It can be seen how much more involved the model for many dominoes is compared to two dominoes colliding.

In Shaw's model, all of the preceding dominoes are assumed to affect the leading domino. This has shown to be not true both experimentally and theoretically by van Leeuwen [10,13]. The number of dominoes that need to be considered depend greatly on their spacing. For smaller separations, more dominoes will affect the leading domino and for larger separations fewer dominoes will affect the leading domino. The number that mattered experimentally ranged anywhere between 1-6 [13].

Additionally this model does not holistically consider friction, instead opting to simply including a factor to the energy loss in order to better fit experimental data.

Neither of the models consider the effect of where the domino is hit (relative to its center of mass). Since torque depends on the length of the moment arm is variable. In van Leeuwen's model, the torque is considered when modeling the collision between dominoes and so it should be an overall more comprehensive model of the system. However due to the model's complexity it is more difficult to clearly compare the mathematics. This is where numerical comparisons are useful.

Numerical Modeling

All authors agree that there is no closed form solution for time and thus numerical integration must be performed. While all papers mention the implementation of some computer program to realize this, van Leeuwen [10] was the only author who included references for the actual code.

An attempt was made to decipher the code to translate it into a more readable form in python but ultimately this was unsuccessful. Despite the explicit nature of code, van Leeuwen took a few liberties when implementing it so the mathematics is somewhat obscured.

Van Leeuwen also implemented the model by Shaw and by Banks in his code. However, the results that Banks reported for his model and the results reported by van Leeuwen for Banks' model did not agree. Furthermore, the values reported by van Leeuwen in his paper (for Banks' model) could not be reproduced when the code was run.

A program was written to implement Banks' model (check appendix), using standard good coding practices as well as staying true to the mathematics in the paper. When executed, this model gave very similar results to what was reported by Banks [7].

At small separations ($s/h < 0.12$), Banks' plot showed a chance in the curvature of the graph, going from a positive curvature to a negative one. It is unclear why there is this disagreement as Banks does not provide details for how the plot was generated.

Numerical Comparison of two models

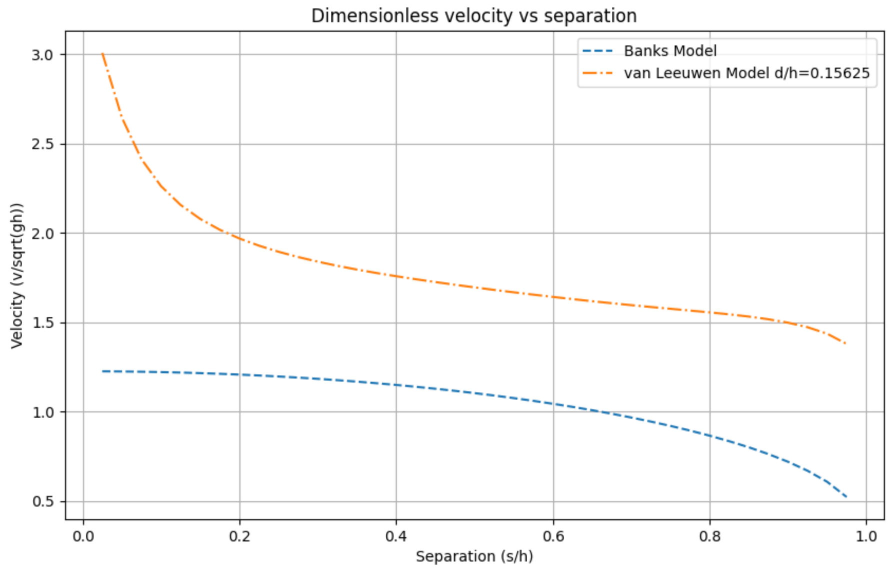

Figure 4: Comparison of model by Banks (generated from the program written by the present author) and model by van Leeuwen (generated from his own program) with dominoes of ratio $d/h = 0.15625$ to match experiment. Vertical axis it the dimensionless asymptotic velocity $v/\sqrt{gh}$ and the horizontal axis is the dimensionless separation s/h.

When the values from the two models were plotted against one another, it can be seen that their values are obviously different. Qualitatively, the biggest difference is when the separation ratio is less than 0.2 and greater than 0.8. Since van Leeuwen's model considered the chain of dominoes all leaning and affecting on the leading domino, it makes sense that overall the velocity would be greater due to the extra energy in the system.

Comparison of Models with Experiment

In van Leeuwen's model, friction would be an important factor since many dominoes are sliding against one another as they fall. When taking into account friction the following plot was produced.

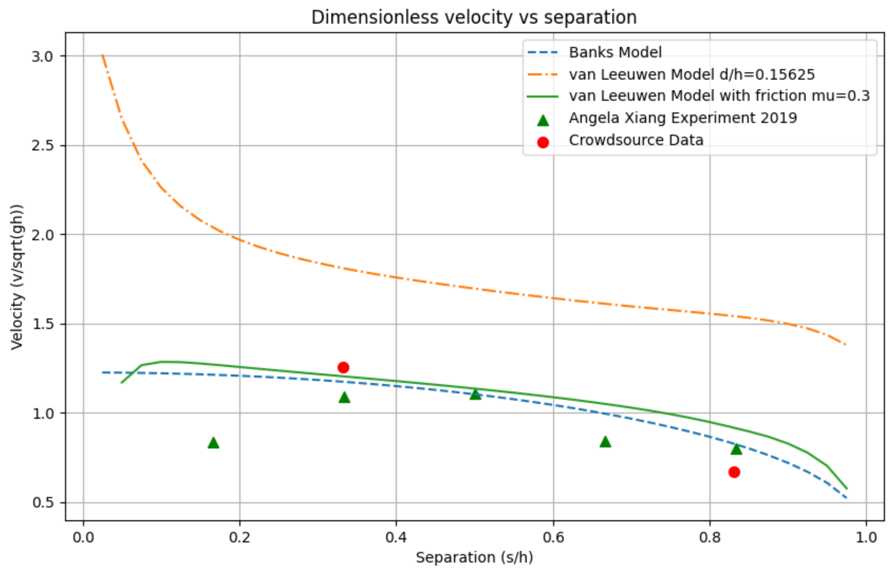

Figure 5: Comparison of model by Banks (generated from the program written by the present author) and model by van Leeuwen with and without frction(generated from his own program) with experimental data. One set of experimental data was gathered by me in 2019 and the error is no more than 10%. There are two points from the crowdsourced data which is from by Destin Sandlin in a YouTube video [3]. This data was gathered by many different groups and an average was taken. There are only two separations available because the video was mainly concerned with the difference between different types of surfaces rather than the spacing between dominoes. Axis are the same as in fig 4.

The cited coefficient of kinetic friction for plastic on plastic is around 0.2-0.4 and it can be seen that van Leeuwen's model when accounting for friction (with $\mu = 0.3$) is much closer to Bank's model and experimental data. Qualitatively, the models are also more similar with friction.

Shaw [11] also found that without considering friction the model gave much higher velocities than experimental values. So when models consider sliding after collision, it is much more important to also consider the effects of friction in order to produce a realistic model.

Experimental Design

The experimental setup to measure the speed of dominoes is very accessible and can easily be executed in by students. However it is worth considering and evaluating various approaches. Since all models rely on the perfect tilting of dominoes, all experimental setups require a rough surface, commonly sandpaper.

Measuring the start and end time

Both McLachlan et al. [8] and Shaw [11] only looked at the overall time for a chain of dominoes to fall rather than considering individual collisions.

For Shaw, dominoes with fixed thickness and height are place in a line at a fixed spacing (as seen in figure 2). A photocell gate is positioned at the beginning and at the end of the chain. The first is positioned to be triggered when the first domino collides with the second, the second gate is triggered when the last domino (that is being considered in this trial) hits the following domino. The total time was recorded.

It was specified that the chain was set off by tipping the first domino until it becomes unstable and it is released with negligible initial angular velocity.

Experimentally, Shaw only had values for N (number of dominoes) smaller than 20. Unfortunately it was shown by Stronge and Shu [13] that the collisions are accelerating for the first 6-15 blocks. Thus the values for time obtained by Shaw cannot be reasonably compared to other models as most of them are concerned with the long-term behaviour of the chain as the number of dominoes go to infinity.

McLachlan et al. [8] simply used a stopwatch, starting when the first domino was hit (by hand) and ending when the last domino fell. The chain of dominoes was around 100 long for their experiments.

It is immediately obvious that this method is problematic due to the acceleration of dominoes observed for the first 20 or so in the chain [3, 13]. So this is likely not an accurate representation of the asymptotic speed of individual domino collisions. This problem could be minimized for very long chains of dominoes ($N >> 20$) but it is unrealistic to set up this many dominoes.

Slow motion videos

Stronge's experimental procedure allowed for the analysis of the time for individual collisions [13,14]. The dominoes were set up on a rough surface and the chain was set off by a "slow-speed collision at the centre-of-percussion" [13]. The collisions were filmed with a high-speed cine camera at 600-1200 frames per second. It is unclear how the film was subsequently analyzed to obtain the speed.

This method allows for a more accurate propagation speed to be obtained as the initial few collisions can be ignored when calculating the steady state velocity. The film also allows for analysis of the qualitative behaviour of the system while toppling. This is how they were able to describe the existence of multiple collisions or bouncing of dominoes after the initial collision.

With a long chain of dominoes, this method could prove challenging as one would have to ensure all of the dominoes are in frame. Additionally, the moment of collision could be difficult to pinpoint due to parallax. Destin Sandlin mitigated these issues by mounting a slow motion camera on a rig with wheels so the camera can follow the dominoes as they fall [3].

While this method could be the most accurate, it is resource intensive (slow motion captures very large files) and may not be feasible in a normal laboratory setting.

Audio Recording

The last experimental setup of interest is that of Larham [6]. Rather than recording video to be analyzed, sound was recorded instead. Then the frequency of domino collisions was extracted from this audio signal. This setup does not allow for as much qualitative analysis since there is only indication that a collision happened and not anything else about the nature of the collision.

Additionally, since it is possible that there are multiple collisions, using this method it could be difficult to discern which is the initial collision when so many dominoes are bouncing on one another. However this method is very accessible and may be valuable as a tool in conjunction with a video recording.

Possible Experimental Designs for Undergraduate Labs

A full experimental procedure will not be developed in the present paper but some suggestions will be made on how to create an experiment that is suitable for the undergraduate lab. In order to record useful data about the asymptotic velocity of dominoes falling, there must be more than 20 dominoes set up to avoid the issue of accelerating dominoes. A realistic recommendation is around fifty to one hundred dominoes.

For recording, the camera can be set up to only capture the dominoes after the first twenty if the interest is only in the long-term velocity of the dominoes falling. If possible record both video and audio together for the same experiment to analyze separately and compare.

Students should be encouraged to make their own adjustments to the experiment in order to best mitigate the challenges presented by the existing designs.

Pedagogical Considerations

The problem of how fast do dominoes fall is quite complex but offers many avenues for simplifications, allowing students at many levels to approach the problem. At the most basic level, an experiment can be conducted and compared with existing models, with code provided to students for the models.

At a more advanced level, students should be encouraged to critically analyze existing models in order to numerically implement the models themselves. This offers a first step in mathematical modeling and is another area of coding, other than data analysis, that can be developed. Additionally, students can attempt to create their own model by adapting existing ones and adding or even removing assumptions.

Conclusion and Next Steps

Over the last few decades, many different models have been proposed for the propagation of a disturbance through evenly spaced dominoes and they have varied widely. Though the basic principles and steps were oftentimes similar, it is clear that subtle assumptions and differing method caused large differences in the final model.

Surprisingly, there did not appear to be any comparisons of the different models in a meaningful way. Up to now, the only comparisons have been between experimental data. This paper visualized the difference between two models that employed vastly different assumptions. The models of van Leeuwen and Banks were found to be quite similar when friction was taken into account for the former.

The other area that has not been explored by the current literature is the effect of varying friction at the contact point between the domino and the ground. This paper discussed the challenges with conducting such an analysis. Despite the potential difficulties, a very interesting next step to better understand the three dimensional domino effect may be to model 2D dominoes that are allowed to slip.

While complex, this problem is able to be simplified in many ways for students to approach both through experiment and modeling. Tackling the problem offers students and opportunity to make decisions and grow as scientists.

References

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Appendix

```
import math as m
import scipy.integrate as integrate

# Python implementation of Banks model for the domino effect
# This model only considers the effect of one domino on the next and does not
# consider the whole chain of dominoes. As well the collision is elastic.

ratio_d = 0.17363  # d/h aspect ratio of dominoes
# ratio_s = 2/6  # s/h ratio of distance between dominoes to the height of
# the domino
h = 4.445  # cm height of domino
d = ratio_d * h  # cm width/depth of domino
# sep = ratio_s * h  # cm separation between dominoes
g = 9.81  # m/s^2 gravity


def omega_max(sep, g, h):
    # maximum value for angular velocity. This would be the angular velocity
    # when the physical pendulum is at the bottom of a swing
    sintheta = sep / h
    costheta = m.sqrt(h ** 2 - sep ** 2) / h
    const = (3 * g) / h
    return (1 / sintheta) * m.sqrt(const * (2 - (1 + costheta) * costheta **
2))


def time(max_omega, theta_c, g, h):
    # compute the time it takes for a domino to go from vertical to collision
    # angle theta_c
    k = m.sqrt(6 * g / h) / max_omega
    phi_c = (m.pi - theta_c) / 2
    comp_ell_int = ell_int(k, 0, m.pi / 2)  # K
    incomp_ell_int = ell_int(k, 0, phi_c)  # F
    return (2 / max_omega) * (comp_ell_int[0] - incomp_ell_int[0])


def ell_func(phi, k):
    """ Function we want to integrate
    """
    # test = k ** 2 * m.sin(phi) ** 2
    value = 1 / m.sqrt(1 - k ** 2 * m.sin(phi) ** 2)
    return value
```

```python
def ell_int(k, a, b):
    """ Elliptical integral integration
    phi is the variable we are integrating
    """
    return integrate.quad(ell_func, a, b, args=k)


num_sep = 40
for i in range(1, num_sep):
    ratio_s = i / num_sep
    sep = ratio_s * h
    theta_c = m.asin(ratio_s)  # ratio_s is s/h
    # print(theta_c)
    max_angular_velocity = omega_max(sep, g, h)
    dimensionless_vel = ((sep / time(max_angular_velocity, theta_c, g, h)) /
                         m.sqrt(g * h))
    formatted_vel = "{:.10f}".format(dimensionless_vel)
    formatted_ratio = "{:.10f}".format(ratio_s)
    print(formatted_ratio, '\t', formatted_vel)
```