Report for Exercise 3 from Group K

Tasks addressed:	5
Authors:	Yun Fei Hsu (03732414)
	Jan Watter (03665485)
	Yaling Shen (03734727)
Last compiled:	2021–01–29
Source code:	`https://github.com/ElsieSHEN/Dynamical-Systems-and-Bifurcation-Theory`

The work on tasks was divided in the following way:
Equally for all group members and tasks.

Yun Fei Hsu (03732414)	Task 1	100%
	Task 2	100%
	Task 3	100%
	Task 4	100%
	Task 5	100%
Jan Watter (03665485)	Task 1	100%
	Task 2	100%
	Task 3	100%
	Task 4	100%
	Task 5	100%
Yaling Shen (03734727)	Task 1	100%
	Task 2	100%
	Task 3	100%
	Task 4	100%
	Task 5	100%

Report on task 1, Vector fields, orbits, and visualization

Consider the following linear dynamical system, with state space $X = \mathbb{R}^2, I = \mathbb{R}$, parameter $\alpha \in \mathbb{R}$, and flow ϕ_α defined by:

$$\left.\frac{\mathrm{d}\phi_\alpha(t,x)}{\mathrm{d}t}\right|_{t=0} = A_\alpha x \tag{1}$$

As stated in Figure 2.5 in [[1], p.49] for hyperbolic equilibria on the plane, there are three types (five small types) of the feature of the two eigenvalues:

1. Two negative eigenvalues:
   (a) Both are real values. - **stable node**
   (b) With imaginary part. - **stable focus**
2. One positive eigenvalue and one negative. - **unstable saddle**
3. Two positive eigenvalues:
   (a) Both are real values. - **unstable node**
   (b) With imaginary part. - **unstable focus**

To construct a system that can have all above five types of solutions and to produce the plane portraits as shown in Figure 2.5, we set A_α as:

$$\begin{bmatrix} \alpha^2 & 2\alpha \\ -\alpha^2 & -\alpha \end{bmatrix} \tag{2}$$

Therefore, the eigenvalues of system 1 are the roots of equation:

$$\lambda^2 - (\alpha^2 - \alpha)\lambda + \alpha^3 = 0 \tag{3}$$

We adjust the value of α to plot the phase portraits of these five types, respectively.

Figure 1 shows type 1(a) **stable node** with $\alpha = 0.1$ and Figure 2 shows type 1(b) **stable focus** with $\alpha = 0.6$. Both of them have negative real part.

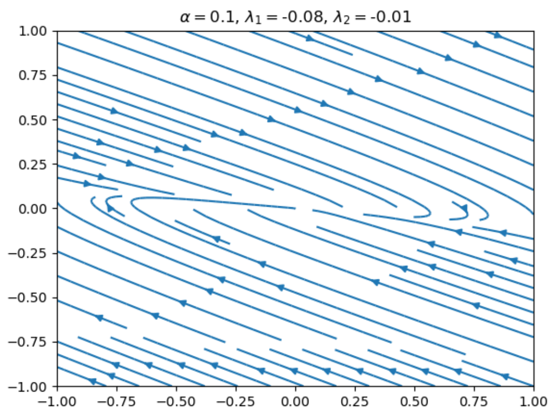

Figure 1: Two Negative Real Eigenvalues

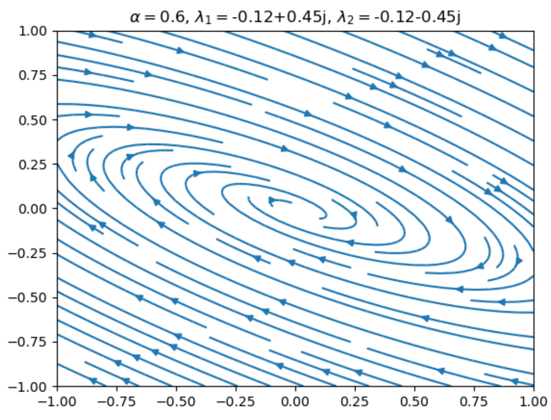

Figure 2: Two Imaginary Roots with Negative Real Part

The phase portrait of type 2 **unstable saddle** is shown in Figure 3. Here, $\alpha = -1$.

Figure 4 shows type 3(a) **unstable node**, where there are two positive eigenvalues with $\alpha = 8$ and Figure 5 shows type 3(b) **unstable focus** with $\alpha = 2$.

[[1], p.43ff] **Node-focus equivalence** has already shown that for one system the node and focus are topologically equivalent. We first consider two linear planar dynamical system, stable node (Figure 4) and stable focus (Figure 5). They are topologically equivalent. As for the unstable node (Figure 1) and unstable focus (Figure 2), they are topologically equivalent. The argument from [1] shows Node-focus equivalence for a stable

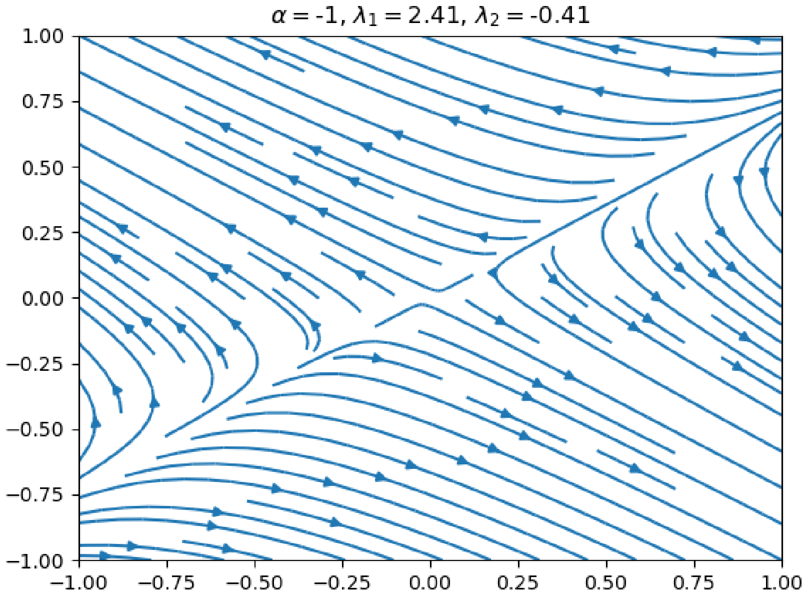

Figure 3: **Unstable Saddle:** One Positive and One Negative

equilibrium. The eigenvalues of the two systems (node and focus) are different but near the origin the differences or perturbations decay for both of the systems. This can be seen when one looks near the origin (by generating a closed disc around the origin). Now look at a point which starts at the center of this closed disc, imagine this as the start point of a particle which follows the vector field. For both system the particle will end up at the equilibrium point of 0 while on can construct a homeomorphism between the trajectories. This homeomorphism is a map that is continuous and invertible and lets you "switch" between the orbits. This means the two systems are topologically equivalent in the disc. To be fair, the figures from the mentioned source help in understanding the concept. Generally speaking: visualization help a lot with this topic.

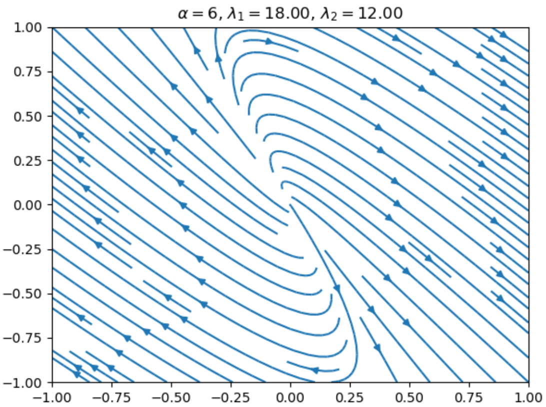

Figure 4: Two Positive Real Eigenvalues

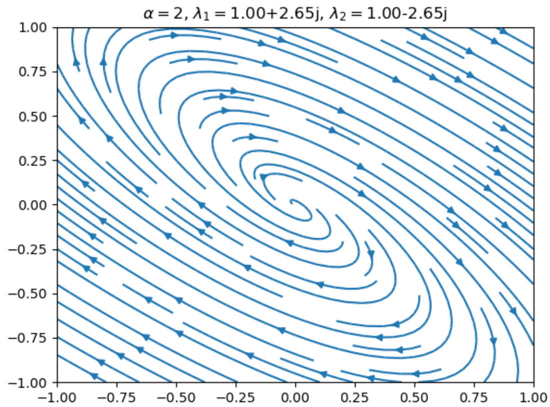

Figure 5: Two Imaginary Roots with Positive Real Part

Report on task 2, Common bifurcations in nonlinear systems

Consider a dynamical system on the real line $X = \mathbb{R}$, time $I = \mathbb{R}$, with the evolution described by

$$\dot{x} = \alpha - x^2 \tag{4}$$

Vary α in the range of $(-1, 1)$, we plot the bifurcation diagram for equation 4. As shown in Figure 6, for $\alpha > 0$, this system has two steady states at $x_0 = \pm\sqrt{\alpha}$, while for $\alpha < 0$, there are no steady states. We can also get this conclusion simply by solving equation

$$\dot{x} : \alpha - x^2 = 0 \tag{5}$$

and analyze whether it has real solutions with different α's. At $\alpha = 0$, there is one steady state at this point only, which means the bifurcation is a half-stable one. We can see that when $\alpha > 0$, there are two real roots, which are the two steady states. Rephrased one could say that at $\alpha = 0$ there is a critical point since this is the point where the system transitions from one fixed point to two fixed points, while having no fixed points when for $\alpha < 0$. When $\alpha < 0$, there are no real roots for this equation, therefore this system has no steady states. The graph looks very similar to a *pitchfork bifurcation* which gets its name from the looks of the graph. A similar bifurcation will be discussed in task 4.

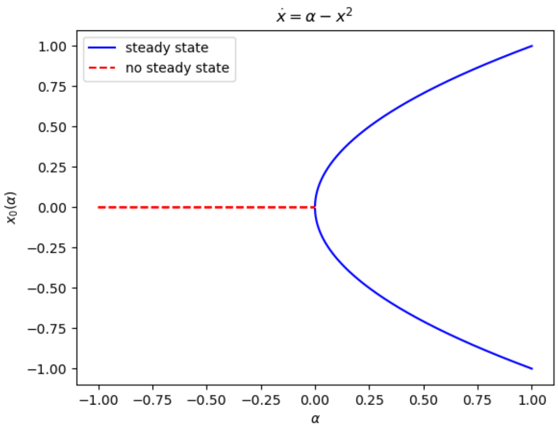

Figure 6: Bifurcation Diagram of System 4

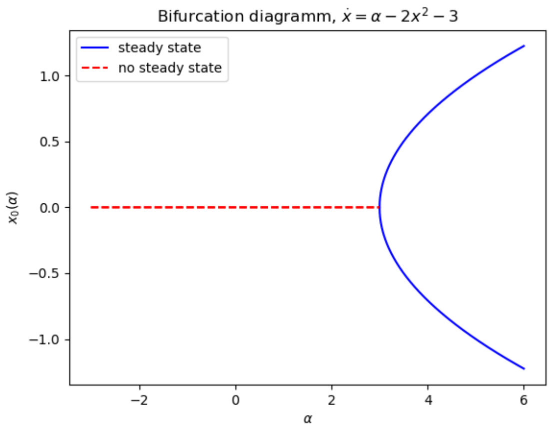

Figure 7: Bifurcation Diagram of System 6

For the second system:

$$\dot{x} = \alpha - 2x^2 - 3 \tag{6}$$

we can first analyze it from a mathematical view. If this equation has two real roots that correspond to two steady states, which is $x_0 = \pm\sqrt{\frac{\alpha-3}{2}}$, then $\alpha > 3$. Therefore, when $\alpha > 3$, this system has two steady states x_0, when $\alpha < 3$, there are no steady states. At $\alpha = 3$, there is one steady state and the bifurcation is stable.

When $\alpha = 1$, system 4 has two steady states while system 6 has no steady states. Therefore, they are not topologically equivalent. When $\alpha = -1$ both of the system have no steady states but share same bifurcation conditions making them have the same normal form. This means they are at least local topologically equivalent but not necessarily global topologically equivalent.

Report on task 3, Bifurcations in higher dimensions

The Andronov-Hopf bifurcation [[1], p.57] is an important bifurcation for systems with one parameter exists for two-dimensional state spaces, with the vector field, its normal form is:

$$\begin{aligned} \dot{x_1} &= \alpha x_1 - x_2 - x_1(x_1^2 + x_2^2), \\ \dot{x_2} &= x_1 + \alpha x_2 - x_2(x_1^2 + x_2^2). \end{aligned} \tag{7}$$

We first draw three phase diagrams with $\alpha = -0.5, 0, 0.5$ in Figure 8, 9, and 10, respectively.

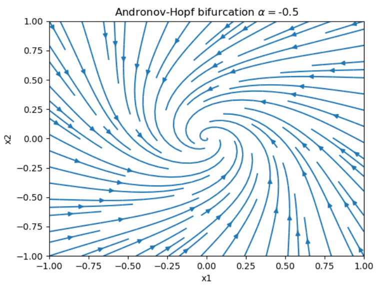

Figure 8: $\alpha = -0.5$

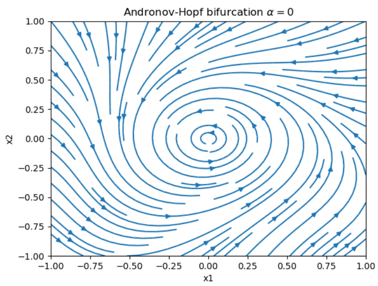

Figure 9: $\alpha = 0$

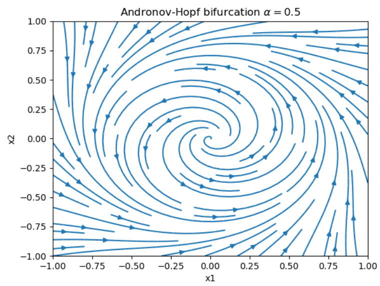

Figure 10: $\alpha = 0.5$

For $\alpha = 1$, we then numerically compute and visualize two orbits of system 13 forward in time. Figure 11 shows the orbits with starting point $(2, 0)$ while for Figure 12, the starting point is $(0.5, 0)$.

We use Euler's method to find the coordinate for the next position (x_1^{n+1}, x_2^{n+1}) by:

$$\begin{aligned} x_1^{n+1} &= x_1^n + \delta + \dot{x_1^n}, \\ x_2^{n+1} &= x_2^n + \delta + \dot{x_2^n}. \end{aligned} \tag{8}$$

where δ is the step size, whose default is 0.1.

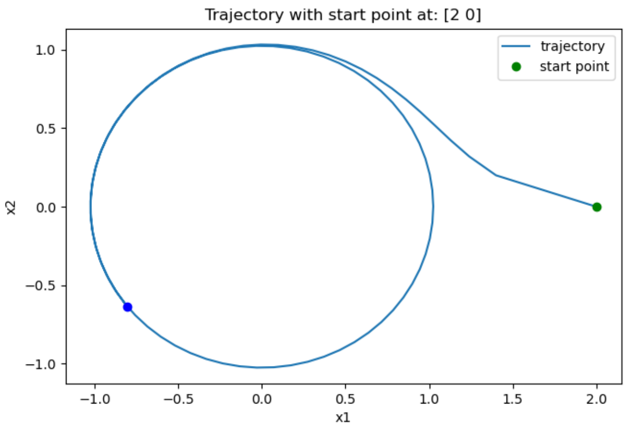

Figure 11: Orbit Starts at (2, 0)

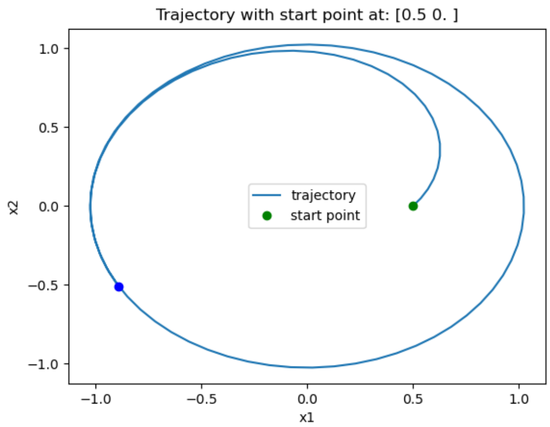

Figure 12: Orbit Starts at (0.5, 0)

The cusp bifurcation is another important bifurcation occurs in one state space dimension $X = \mathbb{R}$, but with two parameters $\alpha \in \mathbb{R}^2$ with normal form:

$$\dot{x} = \alpha_1 + \alpha_2 x - x^3 \tag{9}$$

We first visualize the bifurcation surface where $\dot{x} = 0$ and display it in Figure 13

In this 3D plot, all points satisfy the equation:

$$\dot{x} = \alpha_1 + \alpha_2 x - x^3 = 0 \tag{10}$$

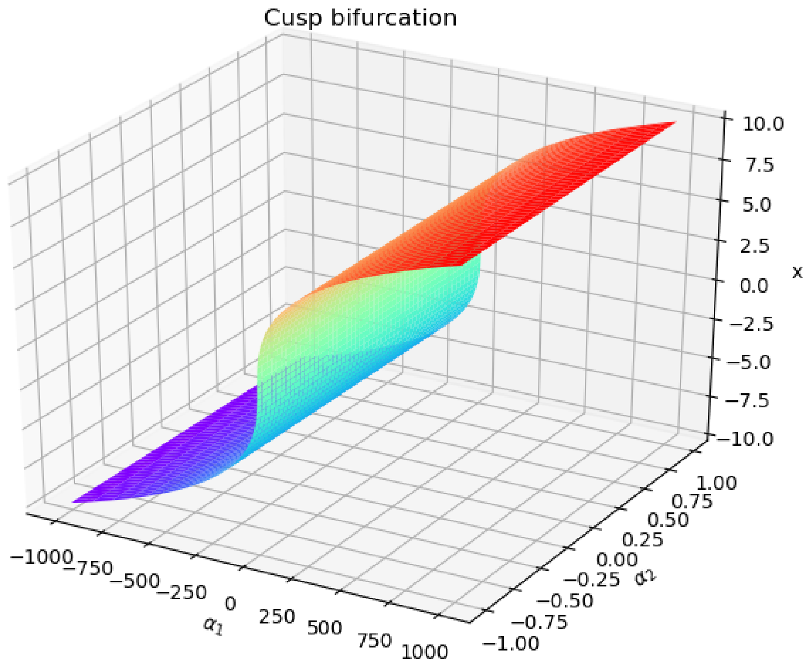

Figure 13: 3D Cusp Bifurcation

For implementation, we iterate x, α_2, and compute corresponding α_1 with this equation.

To further explore the cusp bifurcation, we want to find the relationship between α_1, α_2. We first compute the derivative of function 9:

$$\frac{\partial \dot{x}}{\partial x} = \alpha_2 - 3x^2 \tag{11}$$

Solve x from setting equation 11 equals to zero, and apply the value of x with respect to α_2 into equation 10. We can compute the mathematical relationship between α_1 and α_2:

$$LP_{1,2} = \{(\alpha_1, \alpha_2) : \alpha_1 = \pm\frac{2}{3\sqrt{3}}\alpha_2^{3/2},\ \alpha_2 > 0\} \tag{12}$$

We then plot this equation in Figure 14. We can see that the cusp singularity shows up at the origin (0, 0). This may be the reason why it is called *cusp* bifurcation.

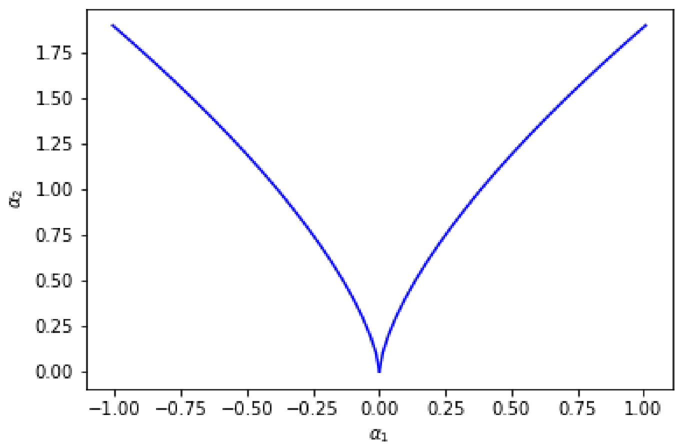

Figure 14: Cusp Bifurcation

Report on task 4, Chaotic dynamics

1. In this task we are asked to vary the given discrete map with different r values ranging from (0,2]. To tackle this task we defined a function which returns the next x value of the given map and called this function with different r values. We then plotted the resulting x values against their corresponding r values making it essentially a bifurcation diagram. Since the plot gets unclear fast, we just plotted the last 100 iterations of the discrete map. We started each iteration with $x_0 = 0.1$, which applies also for the following sub tasks. The result is shown in figure 15.
   This diagram plots the fixed points for the map against r and is therefore useful because we can extract all the information we need. We can see that for a $r < 1$ the map will converge to 0 and for bigger r values the fixed points become greater than 0. Since it will converge to one fixed point, we do not have any bifurcations. This implies that we also do not find any limit cycles.

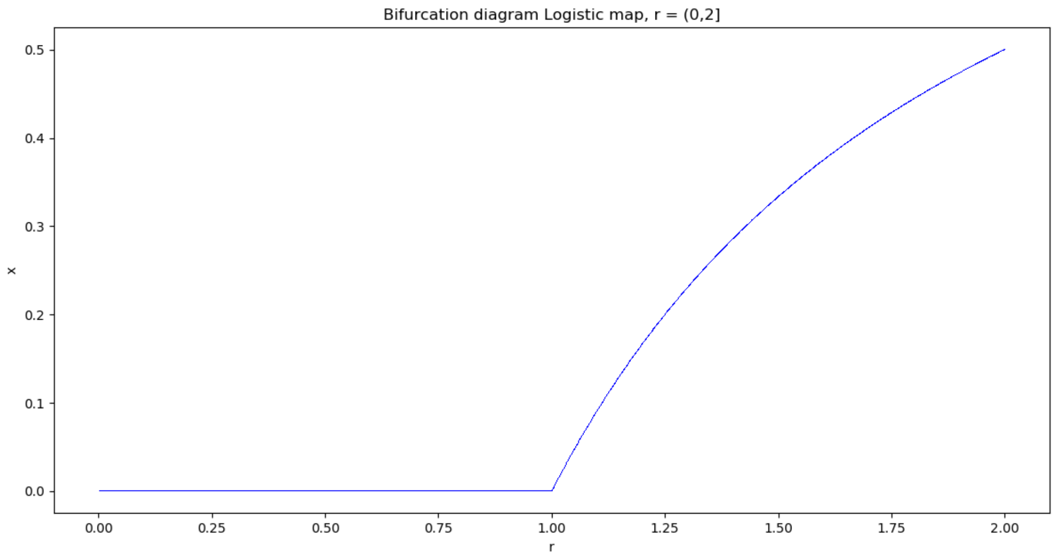

Figure 15: r = (0,2]

2. Now we are asked to vary r from 2 to 4. The result is shown in figure 16. The behaviour now is very unexpected and interesting. A first bifurcation occurs at $r = 3$, the next at around r = 3.44.. and then at r = 3.54... The amount that r has to increase to get to the next bifurcation or period doubling gets smaller for each bifurcation. The ratio between these bifurcation intervals is called Feigenbaum constant (to be more precise: it is the limiting ratio of the intervals). To sum it up so far: we have a single fixed point for $r < 3$ which loses stability at the critical value of $r = 3$ when a stable and attractive 2-cycle-period emerges. This cycle then also loses stability and is replaced by a stable and attractive 4-period-cycle. As seen in the figure this cycle does not remain stable for long and will be replaced by a stable 8-period-cycle and so on. Eventually at $r = 3.569..$ a seemingly chaotic behaviour emerges, since there is no pattern or repeating to be seen. However as r increases further, some windows a periodic behaviour occur, for example at about $r = 3.83$ before the behaviour becomes chaotic again.
   Another interesting observation is that the graph appears to show some self similar behaviour.

3. Here we show the plot for the whole range of $r = (0, 4]$. Note that we thought about plotting other visualization like a cobweb diagram which shows the map for one specific r value in the tasks above. But we decided that the bifurcation diagram is more useful as our focus is on showing bifurcations and we can illustrate our analysis better with these diagrams.

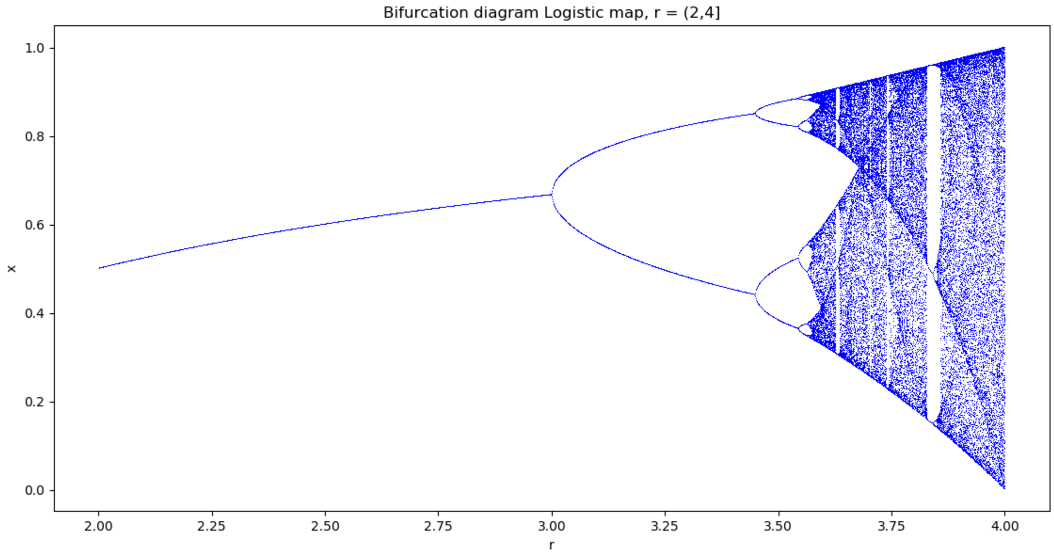

Figure 16: r = (2,4]

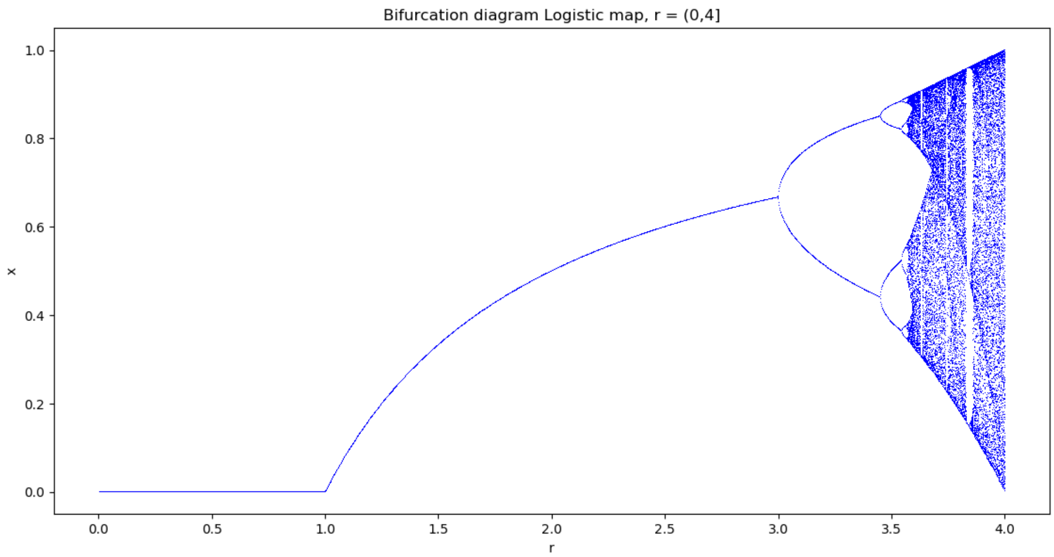

Figure 17: r = (0,4]

In the second part of the exercise we are asked to visualize a trajectory of the Lorenz system with a given set of parameters and a specific starting point.
The Lorenz equations are defined as follows:

$$\begin{aligned} x' &= \sigma(y - x) \\ y' &= x(\rho - z) - y \\ z' &= xy - \beta z, \end{aligned}$$

and were developed to simulate atmospheric convection [2]. Furthermore we are asked to investigate the chaotic nature of the system. For this we plotted the same trajectory of the system but a with slightly perturbed initial condition in the same figure so one can examine the difference. Figure 18 shows the plot where the trajectory in red visualizes the original initial values and the trajectory in black visualizes the perturbed initial values. Note that the line width for this plot is set to be rather thin.
The resulting trajectories seem to circle around two attractors and the term *wings of the butterfly* or *butterfly effect* with regards to chaos theory comes to mind. Apparently this here is the origin of this terminology.
Furthermore one can clearly see that the two trajectories diverge. We implemented a function that calculates the euclidean distance between the trajectories and got the result that at timestep = 640 the difference is initially greater than one.

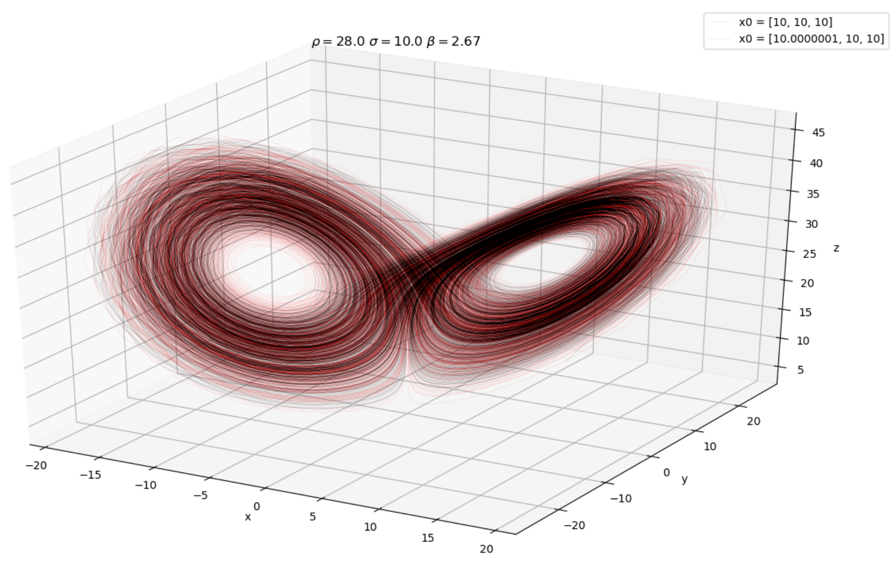

Figure 18: Lorenz system of two trajectories

Next we are asked to change the parameter ρ to the value 0.5 and plot the two trajectories again. The result is shown in figure 19. Here the black trajectory is visualized as points. The system now converges to 0. Since the trajectories are equal this time, one would otherwise only see one trajectory although there are two in the figure. So the system seems to be less sensitive to initial conditions. We confirmed this by calculating the difference of the trajectories for the corresponding points. The result is that the difference starts with 1e−7 but this time decreases to 2e−17. In the long run it settles at about 1e−14.
Here we have a nonlinear system and its behaviour is dependent on a set of parameters. As we changed one of the parameters, the resulting behaviour changed dramatically. For a $\rho = 0.5$ we observed a fixed point at 0, whereas before we observed, what is called strange attractors. In this sense, we would argue that there indeed is a bifurcation. However the bifurcation definition varies and can also be defined as when a small, and smooth change of the parameters causes sudden or topological changes of the behaviour. This is what we analyzed in

the logistic map task where we definitely witnessed many bifurcation. For the Lorenz system we only made one rather big change in a parameter. We argue therefore that in order to get a better understanding of bifurcations of the Lorenz system, one should do more simulations. For example changing the parameters by small values and plot bifurcation diagrams.

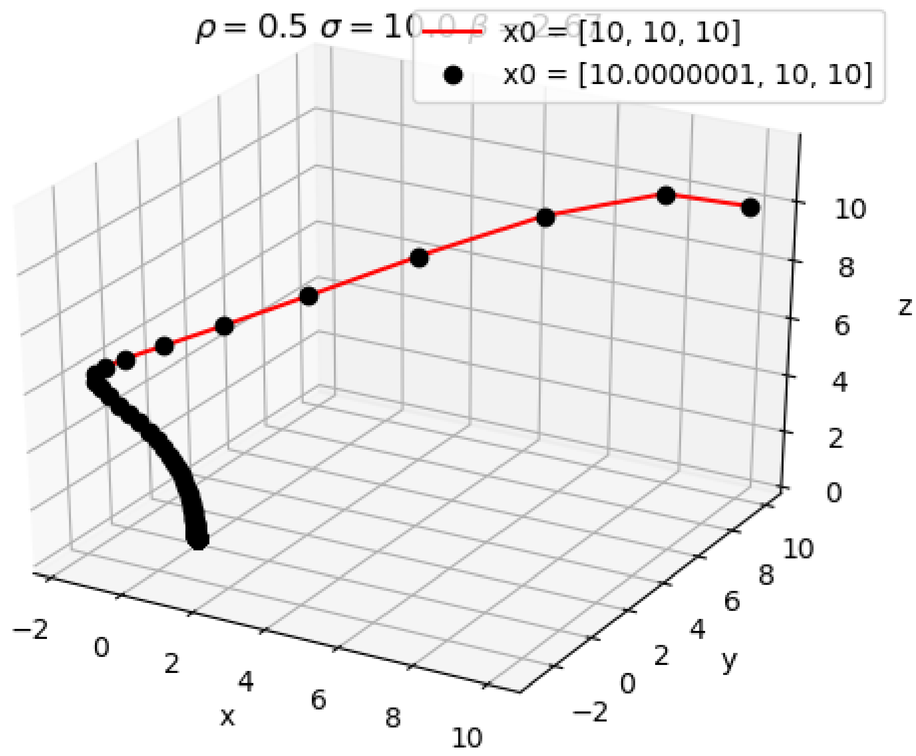

Figure 19: Lorenz system with two trajectories and $\rho = 0.5$

Report on task 5, Analysis and visualization of results

1. In this task, we are asked to implement the visualization of SIR model. We download the example Jupyter notebook from Moodle and implement it.

2. To complete the SIR model, we implement the differential equation in the model function, which returns $\frac{\mathrm{d}S}{\mathrm{d}t}$, $\frac{\mathrm{d}I}{\mathrm{d}t}$ and $\frac{\mathrm{d}R}{\mathrm{d}t}$.

3. We set t_end to 30000 and change b from 0.01 to 0.03 with 0.001 increments. The trajectory starting at (195.3, 0.052, 4.4) spirals inward to a stable focus when the parameter b <= 0.022, as shown in Figures 20 21. As b > 0.022, the trajectory spirals outward and ends up to a fixed point which closes to a disease free equilibrium $E_0 = (\frac{A}{d}, 0, 0)$ as shown in Figure 22 28.
For the trajectory starting at (195.7, 0.03, 3.92), it spirals inward to a stable focus (black point) as b < 0.022 as shown in Figure 23, but spiraling outward to a limit cycle (black line) when b = 0.022 as shown in Figure 24. As b > 0.022, the trajectory keeps spiraling outward and ends up at the fixed point (black point) as shown in Figure 25 28.
And for the trajectory starting at (193, 0.08, 6.21), it first spirals inward to a stable focus as b < 0.022 as shown in Figure 26, and spirals inward to a limit cycle (black line) when b = 0.022 as shown in Figure 27. As b > 0.022, the trajectory converges to the fixed point (black point) as the green line shows in Figure 28.

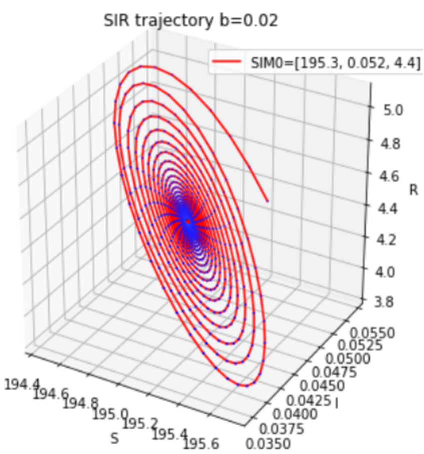

Figure 20: $b = 0.02$

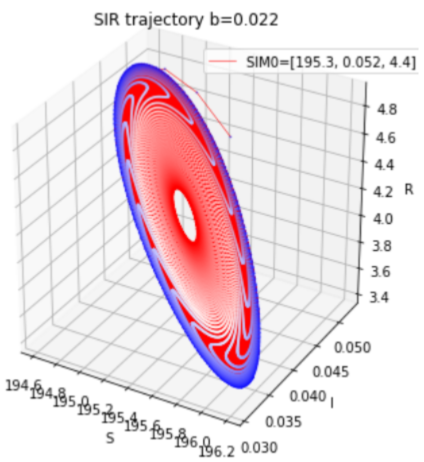

Figure 21: $b = 0.022$

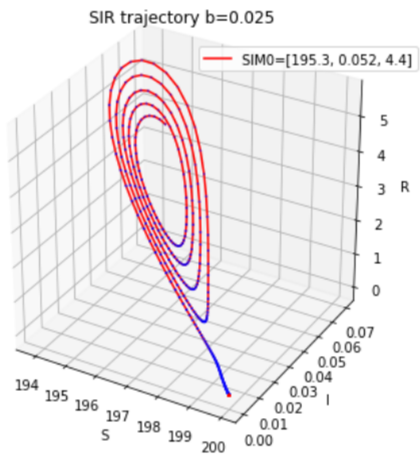

Figure 22: $b = 0.025$

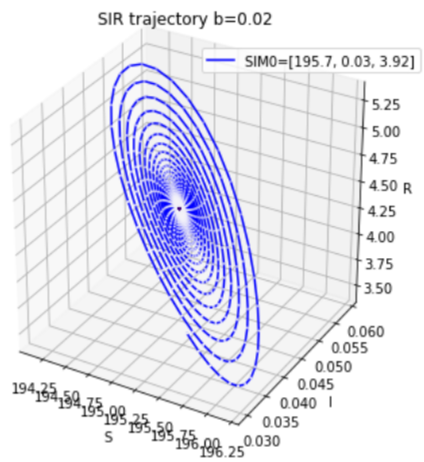

Figure 23: $b = 0.02$

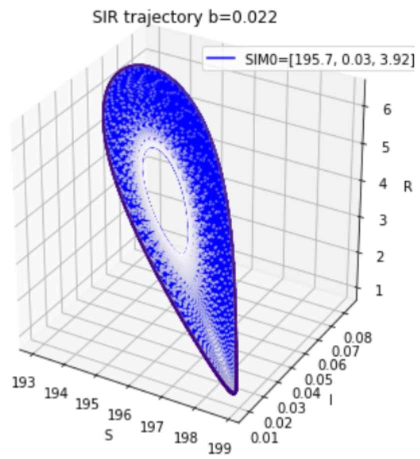

Figure 24: $b = 0.022$

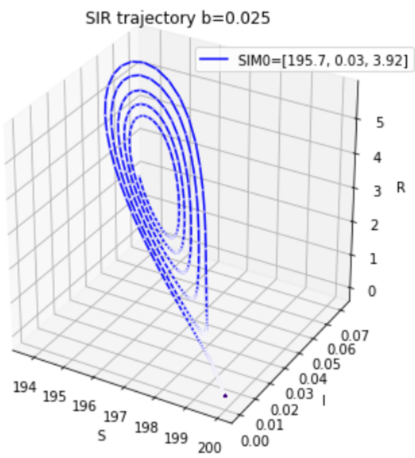

Figure 25: $b = 0.025$

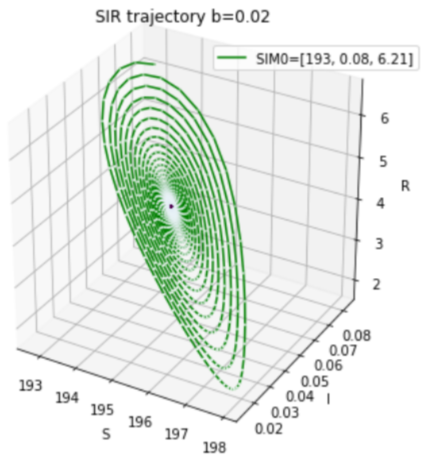

Figure 26: $b = 0.02$

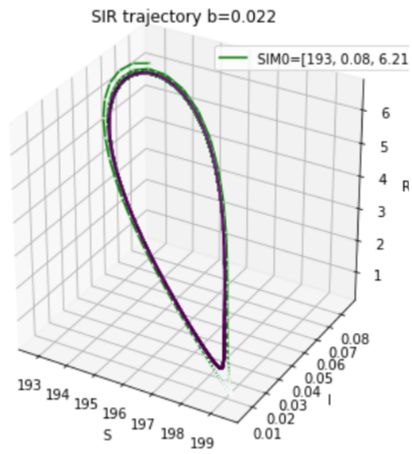

Figure 27: $b = 0.022$

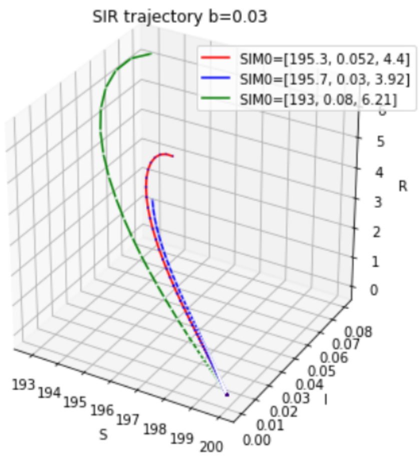

Figure 28: $b = 0.03$

4. As the trajectories shown in Figure 29 , we can see a Andronov-Hopf bifurcations happens when the value of the parameter b is equal to 0.022. The normal form of the bifurcations is:

$$\begin{aligned} \dot{x_1} &= \alpha x_1 - x_2 - x_1(x_1^2 + x_2^2), \\ \dot{x_2} &= x_1 + \alpha x_2 - x_2(x_1^2 + x_2^2). \end{aligned} \tag{13}$$

which the 2-D space x_1, x_2 refer to S, I, and the parameter α refers to b in the SIR model.

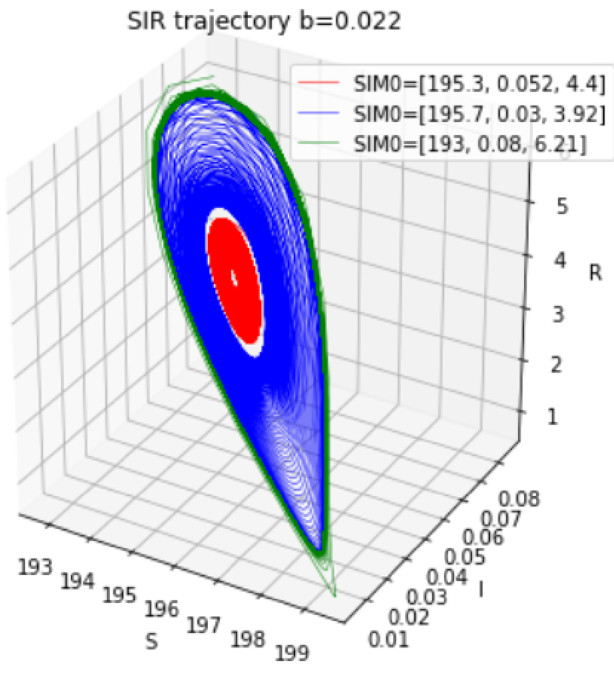

Figure 29: Trajectories of b = 0.022

5. The equation of the reproduction rate $\mathbb{R}_0$ is:

$$\mathbb{R}_0 = \frac{\beta}{d + \nu + \mu_1}, \tag{14}$$

where β is the average number of adequate contacts per unit time with infectious individuals, d is the per capita natural death rate, ν is the per capita disease-induced death rate and μ_1 is the maximum recovery

rate based on the number of available beds.
If the parameter β increases, the reproduction rate will increase as well, which means an infected individual can infect more susceptible individuals. If the $\mathbb{R}_0$ exceeds 1, each infected person produces more than one new infection, this raises the number of infected people. Conversely, the decrease of β reduces the reproduction rate means that an infected individual will infect less susceptible individuals. If the $\mathbb{R}_0$ is less than 1, each infected individual produces less than one new infection, the number of infected people cannot grow over the infectious period.

6. The disease free equilibrium E_0 at $\mathbb{R}_0 < 1$ is an attracting node, which means that the equilibrium E_0 is a stable node, the eigenvalues of Jacobian at E_0 are all negative. For the values of (S, I, R) close to E_0, the trajectories which pass through those values will be pulled toward the E_0 as shown in Figure 29,

7. By fixing the parameter b at 0.012 and changing the parameter μ_1, we found a Homoclinic bifurcation occurs at $\mu_1 = 10.57$. The parameters are set to the following values:

$$A = 20, d = 0.1, \nu = 1, \mu_0 = 10, \beta = 11.5, b = 0.012 \tag{15}$$

As μ_1 decrease from 10.6, the reproduction rate $\mathbb{R}_0$ increase from 0.982. As shown in Figure 31, the trajectory starting at (198.042, 0.018, 1.767) spirals inward to a stable focus when the parameter μ_1 drops to 10.57 and the $\mathbb{R}_0$ raises to 0.985, which is where the Homoclinic bifurcation occur. As μ_1 keeps decreasing, all the trajectories spiral inward and converge to the new fixed point as shown in Figure 32.

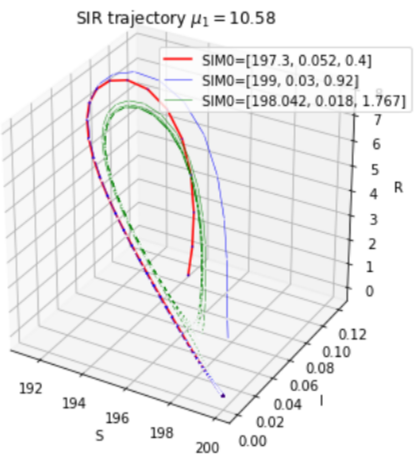

Figure 30: $\mu = 10.58$

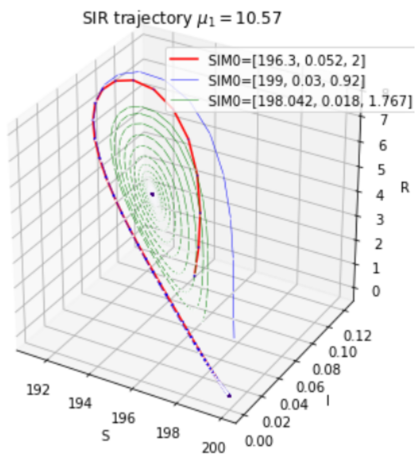

Figure 31: $\mu = 10.57$

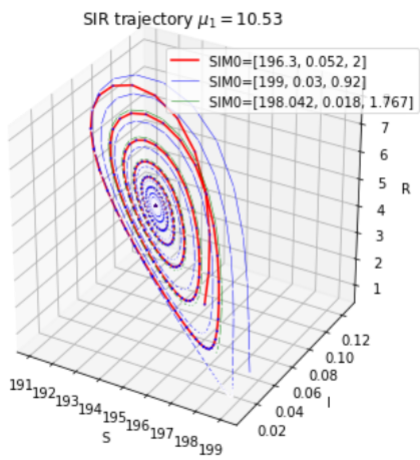

Figure 32: $\mu = 10.53$

References

[1] Yuri A Kuznetsov. *Elements of applied bifurcation theory*. Vol. 112. Springer Science & Business Media, 2013.

[2] Edward N Lorenz. “Deterministic nonperiodic flow”. In: *Journal of the atmospheric sciences* 20.2 (1963), pp. 130–141.