The stress tensor

This is very much not examinable Part IB material. The concept of the stress tensor is properly introduced in Part II Fluids but a lot of you have been asking about forces, pressure and tangential stresses, and I think this is definitely the best way to look at them physically. Always answer questions with the Part IB approach – the purpose of this PDF is just to give you a little insight into some more foundational fluid mechanics.

Start by considering the velocity field in the neighbourhood of some fixed point $\boldsymbol{x_0}$. Everything we'll be doing here is linear, so take a first order Taylor series approximation to describe the flow locally,

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}(\boldsymbol{x_0}) + (\boldsymbol{x} - \boldsymbol{x_0}) \cdot \nabla \boldsymbol{u}|_{\boldsymbol{x}=\boldsymbol{x_0}} + \dots . \tag{1}$$

We decompose the velocity gradient $\nabla \boldsymbol{u}$ into symmetric and antisymmetric parts and write

$$(\nabla \boldsymbol{u})_{ij} = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{e_{ij}} + \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}_{\Omega_{ij}}, \tag{2}$$

where $\underline{\underline{\boldsymbol{e}}}$ is the **rate-of-strain tensor** and $\underline{\underline{\boldsymbol{\Omega}}}$ is the **vorticity tensor**. As an aside, the vorticity satisfies

$$(\boldsymbol{\omega} \wedge \boldsymbol{x})_i = 2\Omega_{ij} x_j, \tag{3}$$

so we can return to equation (1) and write

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}(\boldsymbol{x_0}) + \underline{\underline{\boldsymbol{e}}} \cdot (\boldsymbol{x} - \boldsymbol{x_0}) + \frac{1}{2}\boldsymbol{\omega} \wedge (\boldsymbol{x} - \boldsymbol{x_0}) + \dots . \tag{4}$$

In effect, what we've done here is decompose any flow into a straining part and a rotation. Importantly, note that incompressibility coupled with the definition of the rate-of-strain tensor means that $\operatorname{tr} \underline{\underline{\boldsymbol{e}}} = 0$, which is a useful fact for later.

Introducing the stress tensor

In continuum mechanics there are two types of forces – body forces like gravity which act per unit volume (denoted $\boldsymbol{f}$) and surface stresses $\boldsymbol{\tau}$ which act on surfaces and are stated 'per unit area'. Denote by $\boldsymbol{\tau}(\boldsymbol{n})$ the surface stress acting on a surface with outward (or inward, the choice is yours so long as you stay consistent) normal $\boldsymbol{n}$. An argument that you'll detail in the first couple of lectures of Part II Fluids[1] shows that we expect this normal stress to be related linearly to the normal vector. That means, in terms of tensors and index notation,

$$\tau_{ij} = \sigma_{ij} n_j \tag{5}$$

for some second-rank tensor σ_{ij}. The components of $\underline{\underline{\boldsymbol{\sigma}}}$ can therefore be found by considering the components of the normal stress if we set $\boldsymbol{n}$ to be the basis vectors $\boldsymbol{e}_j$ – i.e. $\sigma_{ij} = \tau_i(\boldsymbol{e}_j)$ (think about this for a couple of minutes and you should be able to convince yourself). We expect this stress tensor to have two parts:

- An **isotropic** part which acts normal to surfaces and has a magnitude independent of direction. If that sounds familiar to you, it's because that's exactly how pressure works. This part of the stress tensor, therefore, we expect to be the pressure of the fluid.
- A **deviatoric** part which acts both normally and tangentially and is due to relative motion and viscosity.

So we write $\sigma_{ij} = -p\delta_{ij} + \sigma_{ij}^{\text{dev}}$, and need σ_{ij}^{dev} to have zero trace (or else some of its values could be absorbed into the pressure term)[2]. This is where we make a few assumptions, starting with the fact that σ_{ij}^{dev} needs to be a function of $\nabla \boldsymbol{u}$, since we saw above that relative motion in the vicinity of a fluid particle can be described just by the velocity gradient (up to first order). The other two assumptions we make define a **Newtonian fluid**:

- The deviatoric stress is linear and related to the instantaneous value of $\nabla \boldsymbol{u}$ and thus has no historical dependence. Examples where this doesn't hold are things like toothpaste or mayonnaise, both very much nonlinear. This means we can write $\sigma_{ij}^{\text{dev}} = A_{ijkl}\partial u_k/\partial x_l$ for some fourth-rank tensor $\underline{\underline{\boldsymbol{A}}}$.

[1] if you want a "proof" of this, take a look at page 4 of `http://paul.metcalfe.googlepages.com/fluids-2b.pdf`.

[2] the reason for the minus sign in the pressure term can be seen by considering a unit cube in the fluid - the fluid exerts a pressure on the cube in the opposite direction to the normal pointing at the fluid, necessitating the sign change.

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- The fluid is isotropic, so $\underline{\underline{\boldsymbol{A}}}$ must be isotropic, taking the most general form for an isotropic fourth-rank tensor $A_{ijkl} = \mu' \delta_{ij}\delta_{kl} + \mu'' \delta_{ik}\delta_{jl} + \mu''' \delta_{il}\delta_{jk}$.

All of these assumptions mean

$$\sigma_{ij}^{\mathrm{dev}} = \mu' \frac{\partial u_l}{\partial x_l}\delta_{ij} + \mu'' \frac{\partial u_i}{\partial x_j} + \mu''' \frac{\partial u_j}{\partial x_i}. \tag{6}$$

The first term here has to be zero, by incompressibility, so we are left with only the second two. We also know that the overall stress tensor $\underline{\underline{\boldsymbol{\sigma}}}$ has to be symmetric[3], and $-p\delta_{ij}$ is symmetric, so $\sigma_{ij}^{\mathrm{dev}}$ needs to be symmetric as well. This means that $\mu'' = \mu''' = \mu$ and

$$\boxed{\sigma_{ij} = -p\delta_{ij} + 2\mu \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right] = -p\delta_{ij} + 2\mu e_{ij}.} \tag{7}$$

This is the stress tensor for a Newtonian fluid. As it turns out, many common fluids are Newtonian to a pretty good approximation, so this is the only stress tensor you'll work with in fluids until at least Part III. Note that the stress tensor depends on $\underline{\underline{\boldsymbol{e}}}$ but not on $\underline{\underline{\boldsymbol{\Omega}}}$: the reason for this is that rigid body motions, like the rotation described by that tensor, don't produce stress in the fluid, but straining motion does, as viscous fluid molecules glide past one another.

What does this mean for boundary conditions?

You've already seen that at an interface or solid boundary, $\boldsymbol{u} \cdot \boldsymbol{n}$ has to be continuous by mass conservation. In the case of a rigid wall, it has to be zero, for example. In order for the velocity gradient to not become singular at the interface, we also need the tangential component (i.e. $\boldsymbol{u} - (\boldsymbol{u}\cdot\boldsymbol{n})\,\boldsymbol{n}$) to be continuous. Both of these together imply that $\boldsymbol{u}$ is continuous across an interface. Again, you've implicitly seen this already, too – in the form of the 'no-slip' boundary condition for viscous fluids that requires $\boldsymbol{u} = \boldsymbol{0}$ on a rigid boundary.

So the net stress on a boundary is $\underline{\underline{\boldsymbol{\sigma}}} \cdot \boldsymbol{n}$. Let's think about a 2D case where there's a boundary at $y = 0$ with a fluid that has pressure p flowing in $y > 0$, with velocity $\boldsymbol{u} = u\,(y)\,\hat{\boldsymbol{x}}$. This describes pretty much all the situations you deal with in Part IB. The strain rate tensor for this flow comes out as

$$\underline{\underline{\boldsymbol{e}}} = \frac{1}{2}\begin{pmatrix} 0 & \partial u/\partial y & 0 \\ \partial u/\partial y & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{8}$$

so the stress tensor is

$$\underline{\underline{\boldsymbol{\sigma}}} = \begin{pmatrix} -p & \mu\partial u/\partial y & 0 \\ \mu\partial u/\partial y & -p & 0 \\ 0 & 0 & -p \end{pmatrix}, \tag{9}$$

Let's see what the normal and tangential components of the stress exerted **by** the fluid **on** the boundary[4] are.

$$\underline{\underline{\boldsymbol{\sigma}}} \cdot \boldsymbol{n} = -p\hat{\boldsymbol{y}} + \mu \frac{\partial u}{\partial y}\hat{\boldsymbol{x}}, \tag{10}$$

that is to say, we've got the fluid pressure p pushing down on the boundary, as we'd expect, and then the viscous stress acting along the boundary with the familiar expression. Both pressures and viscous stresses are unified here, which is pretty neat, and pretty much the only way to cope when your normal doesn't neatly align with the coordinate axes. But that's a job for next year...

[3]see the same PDF for a justification of this, or just take it on trust because it seems like it should be (not recommended exam advice for Part II).
[4]$\boldsymbol{n} = \hat{\boldsymbol{y}}$!

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