Image credit: John E. Simpson

Time to approach similarity

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Gravity currents

- Vast array of physical examples [Huppert (2006)], including
  - Pyroclastic flows
  - Honey on toast
  - Atmospheric flows
- Generally governed by time-dependent nonlinear partial differential equations
- Driven forwards **horizontally** by the (vertical) effect of gravity
- Therefore, generally rely on either numerical solutions or [hoped-for] similarity solutions

Figure 1: *Gravity currents are not always as obvious as this, but are important in many meteorological phenomena.*

Image credit: P.F. Linden

Similarity solutions

Considering the two-dimensional viscous gravity current described by

$$\frac{\partial h}{\partial t} - \beta \frac{\partial}{\partial x}\left(h^3 \frac{\partial h}{\partial x}\right) = 0, \tag{1}$$

x is horizontal distance, h height of the viscous current and β a constant ($\beta = g\Delta\rho/3\mu$). We're interested in the horizontal position of the nose of the current, $x_N(t)$.

Similarity solutions

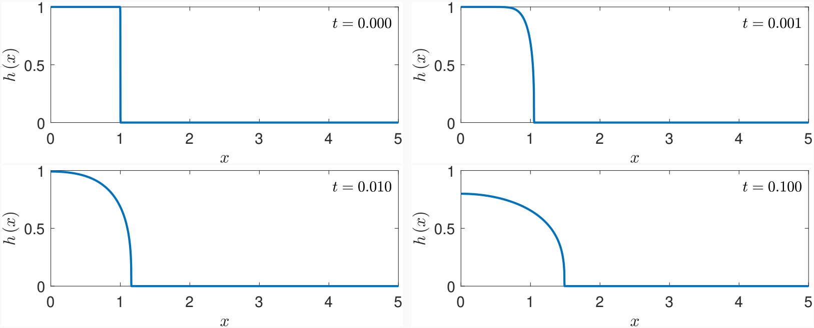


Figure 2: *Plots showing the evolution of the gravity current in a 'dam-break' problem. Produced with code based off that written by Prof John Lister, and later modified by Thomasina Ball.*

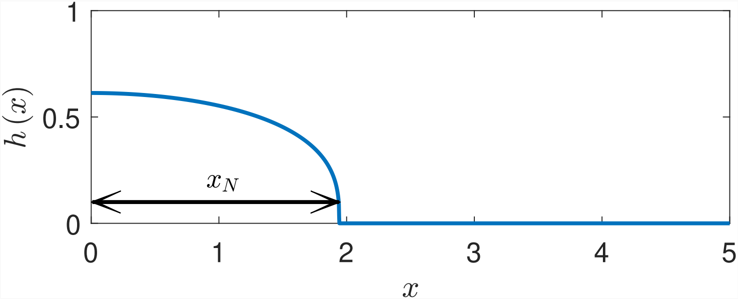


Figure 3: *A plot showing the horizontal extent of the flow, x_N.*

- Huppert (1982) derives

$$x_s(t) = \eta\left(\beta A^3 t\right)^{1/5}. \tag{2}$$

- Here, A is total area of the flow and η is a constant, ≈ 1.411.

Similarity solutions

- Note that this similarity solution has no dependence on the initial conditions
- We hope that $x_N(t) \to x_s(t)$ as $t \to \infty$, but how quickly?
- Speed of convergence clearly depends on initial conditions; plots support this.

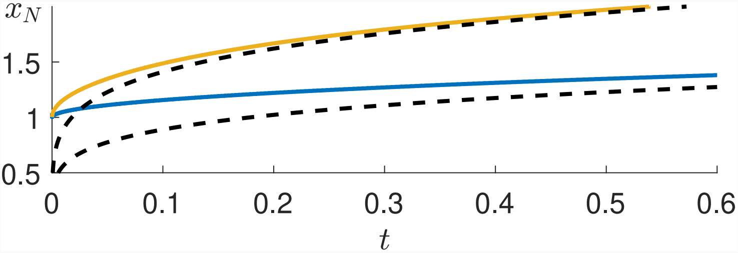


Figure 4: *Plots of x_N (solid lines) and x_s (dashed lines) for initial shape of a unit square — convergence is much faster for $\beta = 10$ (yellow) than for $\beta = 1$.*

Similarity solutions

- Ball and Huppert (2019) propose a method of finding the time taken for a given percentage agreement, $\tau_{p\%}$
- What can this 'equilibration time' depend on?
  - β (dimension $L^{-1}T^{-1}$)
  - x_0 and h_0, the initial size (dimension L)
  - p, the percentage (dimensionless)
- For simplicity, replace x_0, h_0 with dimensionless $\gamma_0 = x_0/h_0$ and $A = x_0 h_0$, with dimensions L^2
- Change variables to try to find dependence on γ_0

$$x = x_0 X \qquad h = h_0 H \qquad t = T/\left(\beta A^{1/2}\right), \tag{3a}$$

$$\Rightarrow H_T - \gamma_0^{5/2}\left(H^3 H_X\right)_X = 0. \tag{3b}$$

Similarity solutions

Then, we claim that

$$\tau_{p\%} = \beta^{-1} A^{-1/2} \gamma_0^{-5/2} f\,(p, \text{shape})\,. \tag{4}$$

For the purposes of this exposition, we assume the initial shape to be a rectangle.

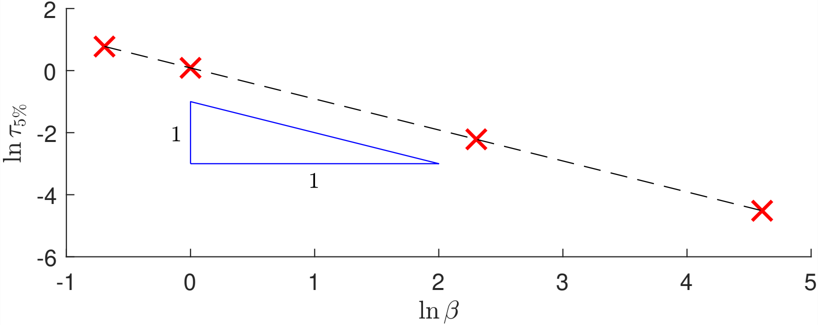


Figure 5: *A scatterplot indicating* $\tau_{5\%} \propto 1/\beta$*.*

The form of $f\,(p, \textbf{shape})$

- Ball and Huppert (2019) remark that the convergence is dominated by the $1/p$ term
- Postulate a form $f\,(p, \text{shape}) = p^{-1} f_a\,(p, \text{shape})$

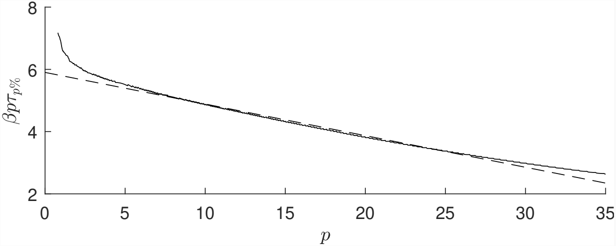


Figure 6: *A first-order (in p) approximation for f_a.*

$$f_a\,(p, \text{rectangle}) \approx 5.9\,[1 - 0.017p]\,, \tag{5a}$$

$$f_a\,(p, \text{quarter-ellipse}) \approx 5.2\,[1 - 0.016p]\,. \tag{5b}$$

Why bother?

We have applied this reasoning successfully to other gravity currents — including axisymmetric ones, and those in porous media.

- However, the equations satisfied by high-Reynolds-number flows are different
- In fact, Grundy and Rottman (1985) suggest that convergence is not monotonic, but oscillatory

The 2D shallow-water equations

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + h\frac{\partial u}{\partial x} = 0 \tag{6a}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g'\frac{\partial h}{\partial x} = 0 \tag{6b}$$

where $g' = (\rho_1 - \rho_0)/\rho_1$ and heavy fluid volume is conserved.

Shallow water equations

$$u(0,\ t) = 0 \qquad x_s(t) = C\left(g'A\right)t^{2/3}$$

$$u(x_N,\ t) = \dot{x}_N = 0 \qquad C = \left[27\mathrm{Fr}^2/\left(12 - 2\mathrm{Fr}^2\right)\right]^{1/3}$$

- $u(x,\ 0) = 0$ and $h(x,\ 0) = h_0$ for $0 \le x \le x_0$ and zero for $x > x_0$ — i.e. the initial shape is a rectangle
- $\dot{x}_N^2 = \mathrm{Fr}^2 g' h(x_N,\ t)$
  - Some uncertainty in value of Fr to choose. **Grundy and Rottman (1985) suggest 1 in this context**
  - von Kármán (1940), Benjamin (1968) suggest (theoretically) $\sqrt{2}$
  - Huppert and Simpson (1980) suggest (experimentally) 1.19

Shallow water equations

Restrict attention to monotonic section of convergence — can we apply the same theory?

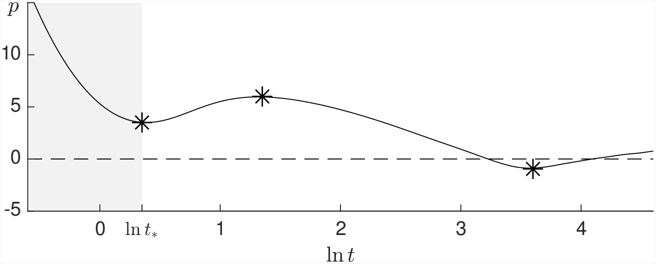


Figure 7: *A plot which shows the oscillatory nature of convergence.*

- Possible parameters are x_0, h_0, g'
- Postulate $\tau \propto x_0 \left(h_0 g'\right)^{-1/2}$ - **interesting**

Equilibration time for inviscid flow

$$\tau_{p\%} = x_0 \left(h_0 g'\right)^{-1/2} f\left(p, \text{shape}\right) \tag{8}$$

- Dependence on x_0 and h_0, unlike before
- $f \not\propto 1/p$ for small p in this case — somewhat close to $p^{-2/3}$

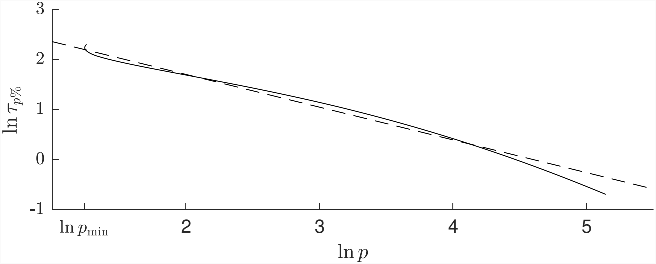


Figure 8: *Attempts to find $f(p, shape)$ - dotted line has gradient $-2/3$.*

Convergence and oscillation

We can consider convergence at other times, seeking, say $\tau_{0\%}$. As it turns out, the same approaches can still be applied with no modification.

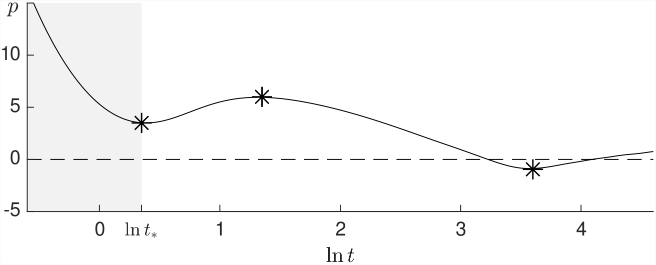


Figure 9: *Considering convergence at later times.*

The easiest way to encapsulate all of this is to let $f\left(p, \text{shape}\right)$ be discontinuous.

Recap

- Gravity currents have important applications in atmospheric dynamics, geology, and breakfast
- Equations are (generally) nonlinear, but we applied the methods outlined to find out how quickly solutions reach similarity solutions, **which are known**
- The same reasoning applies for inviscid currents, but the equilibration time depends more delicately on the initial conditions
- Can we do any better than the similarity solutions we have?
    - It's clear that the biggest issue is no dependence on initial conditions
    - Can we combine initial conditions with similarity solution in some manner?

Time-shifted similarity solutions

Find t_0 such that $x_s(t) = x_0$, then consider $x_s(t + t_0)$.

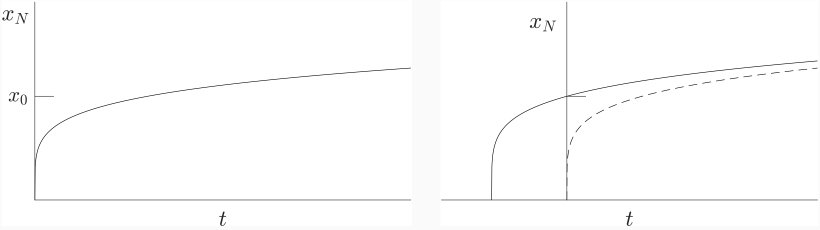


Figure 10: *Illustrating the time-shifting method we use to match the initial conditions.*

- Obviously better for small times (zero percentage difference at $t = 0$)
- Also no concern over difference as $t \to \infty$
- Is the shifted case a better solution in between?

Justifying time-shift

We reconsider the case $h_t - \beta\left(h^3 h_x\right)_x = 0$. Recall that $x_s\left(t\right) = \eta\left(\beta A^3 t\right)^{1/5}$, so we shift to

$$t + t_0 = t + x_0^5 / \left(\beta A^3 \eta^5\right) \tag{9}$$

and instead consider $x_s\left(t + t_0\right)$. This appears a better solution for all time:

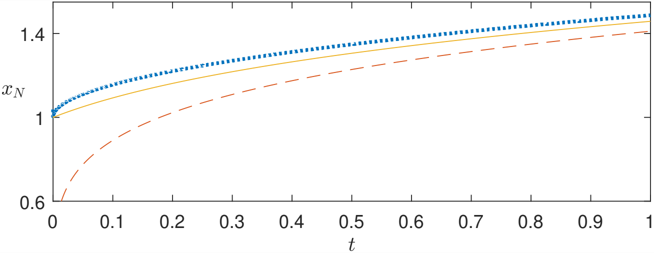


Figure 11: *Plot of the numerical solution (dotted line), similarity solution (dashed line) and shifted similarity solution (solid line).*

Justifying time-shift

We cannot, however, prove that this is always a better solution — merely that it is better provided that

$$\left[1+\frac{0.06\left(A^{1/2}\beta\gamma_0^{5/2}\right)^{-1}}{t}\right]\left[1+\left(\frac{t}{t+t_0}\right)^{1/5}\right]>2. \qquad (10)$$

Derivation provided in accompanying paper — Webber, J.J. and Huppert, H.E., J. Math. Anal. Appl. (to be submitted).

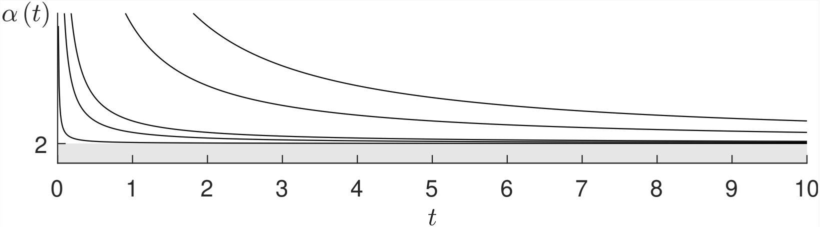
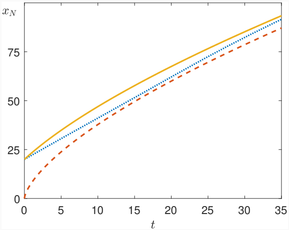
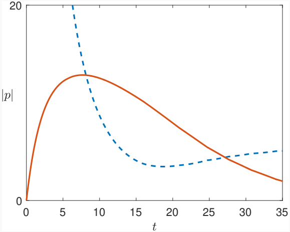


Figure 12: *Plots of the function above, showing viability of our approach.*

Remarks

- In the case of shallow water equations, intersections between x_N and x_s mean that the shifted solution is not always a better approximation over time
- Plotting percentage difference between x_N and the shifted solution in the original case shows a maximum percentage difference over time
  - This is around 5.9% in our case
  - It is intrinsic to the geometry of the initial conditions, not dimensions





x_N — dotted line, x_s — dashed line, x_s, time-shifted — solid line

Remarks

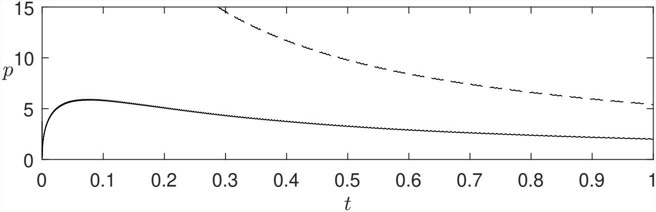


Figure 13: *A plot showing the percentage difference between the similarity solutions (both shifted and not) and the numerical result.*

Shape	2D viscous	2D in porous medium	Axisymmetric in porous medium
Rectangle / cylinder	5.9%	13.0%	9.8%
Quarter-ellipse / ellipsoid	2.4%	5.6%	4.3%
Inverted triangle / cone	17.6%	21.2%	12.9%
Boxcar / ring	23.1%	24.0%	12.8%

Stokes drift

The second of two projects finished this summer, briefly summarised. More details in JFM paper.

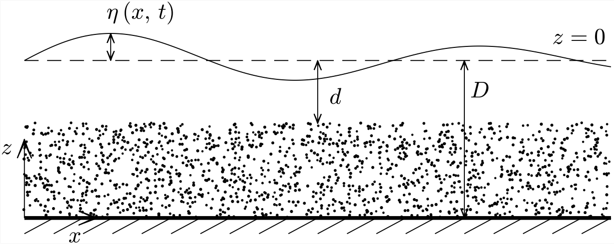


Figure 14: *Diagram to outline the physical situation.*

- We seek the difference between the Eulerian and Lagrangian velocities due to waves
- This produces a drift through the coral reef layer, important for the ecosystem within

Stokes drift

- Unlike without a porous bed, the elliptical paths of fluid particles don't 'join' in the vertical direction — due to damping effect of the layer
- This introduces a vertical drift as well as a horizontal one
- We seek a method to calculate velocity profiles in this case

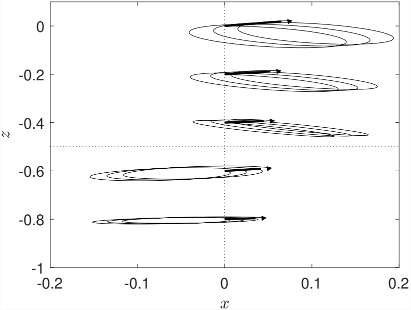
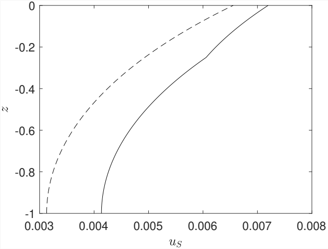
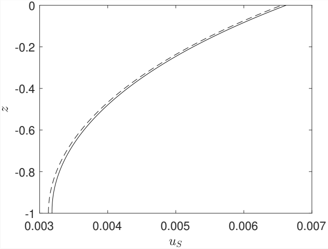


Figure 15: *The paths of fluid particles released at $x = 0$ at different depths, with $d = 0.5$, $D = 1.0$.*

Stokes drift

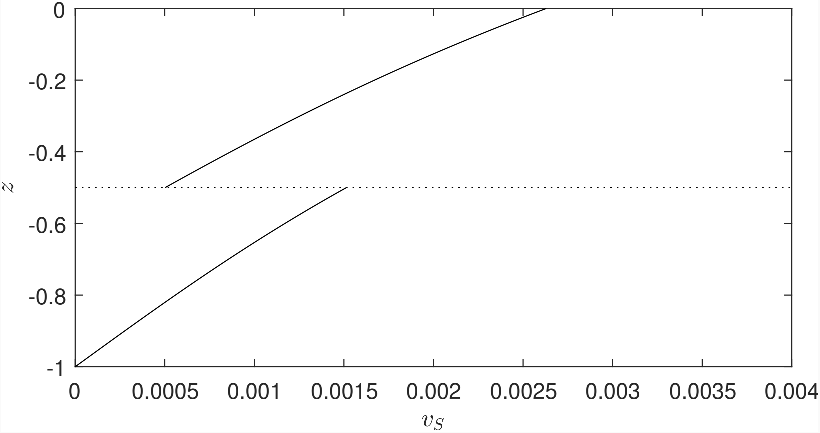
- By using a velocity potential approach (where $\nabla^2\phi = 0$) in the upper layer, and Darcy's Law in the lower layer, we can find expressions for the Stokes drift velocity
- Horizontally, we see that this velocity approaches that of a fluid with no porous layer (dotted line on the graphs), as the relative thickness of the layer decreases
- Plots show (with $D = 1.0$) $d = 0.25$ followed by $d = 0.99$





Vertical Stokes drift

The damping effect of the reef leads to a complex wavenumber k. This means, in turn, that we get an additional drift in the vertical direction, graphed below in the case $D = 1.0$ and $d = 0.5$.



Acknowledgements

- Prof. Herbert Huppert — project supervisor
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- Prof. Mimi Koehl for introducing the research idea of drift in coral reefs
- Heilbronn Fund at Trinity College

Webber, J.J. and Huppert, H.E. **Time to approach similarity** *Journal of Mathematical Analysis and Applications* — to be submitted

Webber, J.J. and Huppert, H.E. **Stokes drift through corals** *Journal of Fluid Mechanics* — to be submitted