Stokes drift through corals

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4th February 2020

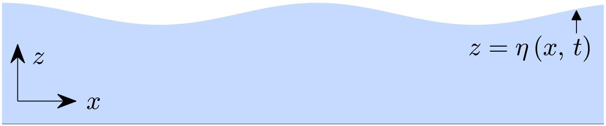
Stokes' work on waves

- In a now-famous paper of 1847, Stokes introduced a theoretical framework for water waves
  - Assume irrotational flow ($\nabla \times \boldsymbol{u} = 0$) $\Rightarrow$ $\boldsymbol{u} = \nabla\phi$.
  - Also assume incompressibility $\nabla \cdot \boldsymbol{u} = 0$ and thus $\nabla^2\phi = 0$.
- Making the assumption of waves at the surface taking the form

$$z = \eta\left(x,\ t\right) = A\exp\left(\mathrm{i}kx - \omega t\right)$$

  we can then take $\phi = \Phi\left(z\right)\exp\left(\mathrm{i}kx - \omega t\right),$. Plug in to Laplace to see that

$$\phi\left(x,\ z,\ t\right) = \alpha\cosh\left(kz + \beta\right)\exp\left(\mathrm{i}kx - \omega t\right).$$



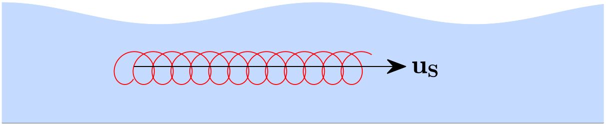
Stokes' work on waves

- We can describe fluid motion in two key ways:
  - **Eulerian:** a velocity field describes velocities at each point in the fluid
  - **Lagrangian:** the velocity is specified for an individual fluid 'particle' which we follow

$$\boldsymbol{x}\left(\boldsymbol{x_0},\, t\right) = \boldsymbol{x_0} + \int_0^t \boldsymbol{u_L}\left(\boldsymbol{x_0},\, s\right) \mathrm{d}s \quad \text{so}$$

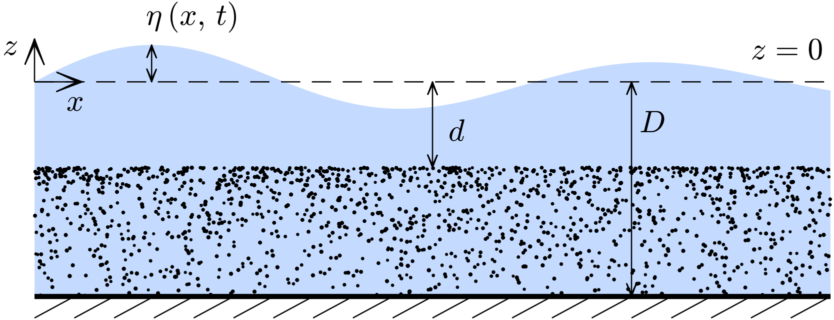
$$\boldsymbol{u_L}\left(\boldsymbol{x_0},\, t\right) = \boldsymbol{u_E}\left(\boldsymbol{x},\, t\right) = \boldsymbol{u_E}\left(\boldsymbol{x_0},\, t\right) + \underbrace{\left(\int_0^t \boldsymbol{u_L}\left(\boldsymbol{x_0},\, s\right) \mathrm{d}s\right) \cdot \nabla \boldsymbol{u}}_{\boldsymbol{u_S} \text{ when time-averaged}} + \dots$$

This gives rise to an $\mathcal{O}\left(\nabla \boldsymbol{u}\right)$ drifting term, which we denote $\boldsymbol{u_S} = \left\langle \left(\int_0^t \boldsymbol{u}\, \mathrm{d}s\right) \cdot \nabla \boldsymbol{u} \right\rangle$.



Applying to coral reefs

For a coral reef, we consider the following simplified model



Taking the velocity field to be $\boldsymbol{u} = (u, v)$, we can use Stokes' velocity potential $\boldsymbol{u} = \nabla\phi$ above the reef. Impose conditions

- $v = 0$ at $z = -D$
- Continuity of v and pressure p at interface
- Bernoulli streamline theorem at surface and interface

Bernoulli's streamline theorem

Starting from the Navier-Stokes equations with a potential body force $\boldsymbol{f} = -\nabla\Phi$

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) = -\nabla\Phi - \nabla p + \mu\nabla^2\boldsymbol{u}, \tag{1}$$

let $\boldsymbol{u} = \nabla\phi$, and $\nabla^2\boldsymbol{u} = \nabla\left(\nabla^2\phi\right) = 0$, and $\frac{1}{2}\nabla\left(|\nabla\phi|^2\right) = (\nabla\phi\cdot\nabla)\nabla\phi$,

$$\nabla\left(\rho\frac{\partial\phi}{\partial t} + p + \Phi + \frac{\rho}{2}|\nabla\phi|^2\right) = 0. \tag{2}$$

The expression within the gradient is therefore independent of space, and is a function only of time. In our linearised treatment, we drop the $|\nabla\phi|^2$ term.

Darcy's law

Darcy proposed an empirical relation for the flow of a viscous fluid in a porous medium. Assuming isotropy,

$$\boldsymbol{u} = -\frac{\kappa}{\mu}\nabla p \tag{3}$$

where κ is the **permeability** of the porous layer, μ is the **dynamic viscosity** of the fluid. This relation has since been derived from Navier-Stokes.

At the interface between the reef and the ocean above, pressure has two components:

$$p_{\text{int}} = p_{\text{atm}} + \rho g d - \rho \left(\frac{\partial \phi}{\partial t}\right)\bigg|_{z=-d}. \tag{4}$$

Also note that $\nabla^2 p = 0$ as $\nabla \cdot \boldsymbol{u} = 0$.

Applying to coral reefs

Again taking $\eta(x, t) = A\exp(\mathrm{i}kx - \omega t)$ we can derive the velocity potential above the reef

$$\boxed{\phi(x, t) = \alpha\cosh(kz + \beta)e^{\mathrm{i}kx - \omega t} \quad \text{with} \quad \boldsymbol{u} = \nabla\phi,} \tag{5}$$

and the pressure field within the reef

$$\boxed{p(x, t) = \gamma\cosh[k(z + D)]e^{\mathrm{i}kx - \omega t} \quad \text{with} \quad \boldsymbol{u} = -\frac{\kappa}{\mu}\nabla p.} \tag{6}$$

Applying boundary and matching conditions gives the **dispersion relation**

$$\boxed{\kappa\omega\tanh[k(D - d)] = \mathrm{i}\nu\tanh\left[\operatorname{arctanh}\left(\frac{\omega^2}{gk}\right) - kd\right],} \tag{7}$$

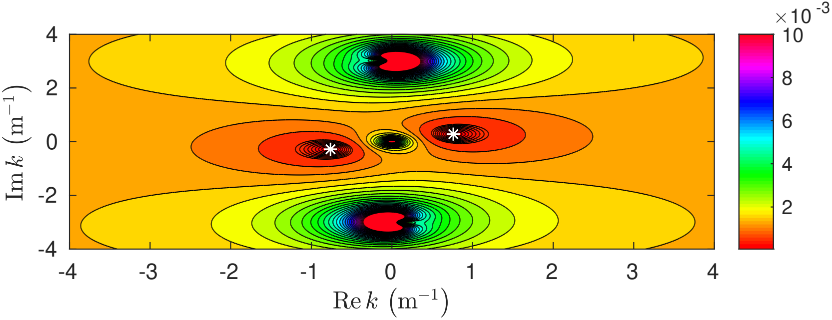
for $\nu = \mu/\rho$.

The dispersion relation

$$\boxed{\kappa\omega \tanh\left[k\left(D-d\right)\right] = \mathrm{i}\nu \tanh\left[\operatorname{arctanh}\left(\frac{\omega^2}{gk}\right) - kd\right]}$$

Classically, we'd expect $\omega^2 = gk \tanh kD$ – we can recover this in simple limits:

- $\boldsymbol{d \to D}$ – i.e. the limit where there is no porous layer.
- $\boldsymbol{\kappa \to 0}$ – i.e. the limit where the porous layer is impermeable (and we just get a water layer of total depth D).



Stokes drift velocities

Recalling that the Stokes drift velocity is given by

$$\boldsymbol{u_S} = \left\langle \left(\int_0^t \boldsymbol{u} \, \mathrm{d}s \right) \cdot \nabla \boldsymbol{u} \right\rangle .$$

combine with our expressions for velocities in the two layers and the values of the constants to get

$$\boldsymbol{u_S}^{\text{above}} = \frac{|k|^2 \, |\alpha|^2 \, e^{-2k_I x}}{2\omega} \left(k_R \cosh\left[2\mathrm{Re}\left(kz + \beta\right)\right], \; k_I \sinh\left[2\mathrm{Re}\left(kz + \beta\right)\right] \right) \tag{8}$$

and

$$\boldsymbol{u_S}^{\text{within}} = \frac{|k|^2 \, |\gamma|^2 \, \kappa^2 e^{-2k_I x}}{2\omega} \left(k_R \cosh\left[2\mathrm{Re}\left(kz + \beta\right)\right], \; k_I \sinh\left[2\mathrm{Re}\left(kz + \beta\right)\right] \right) \tag{9}$$

Stokes drift velocities

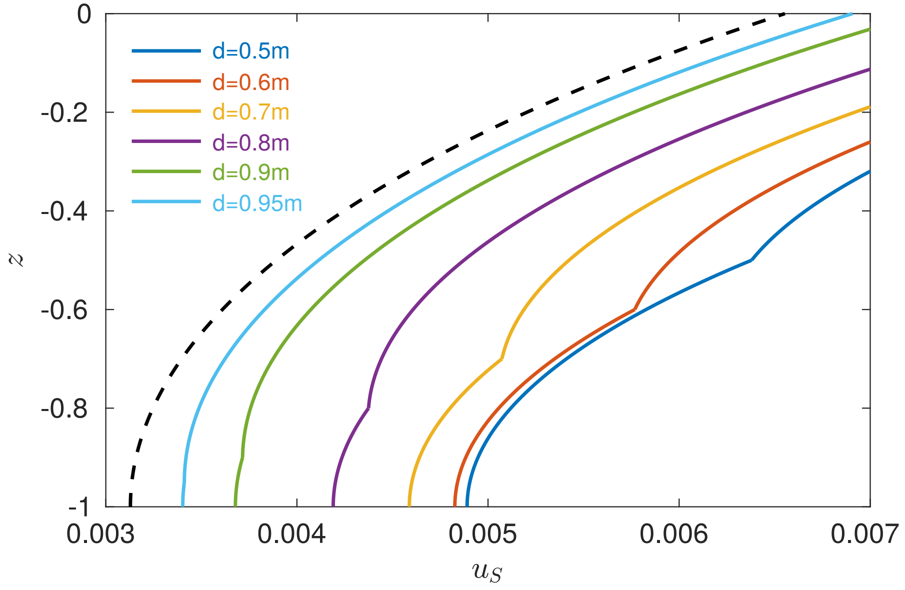
$$\boldsymbol{u_S}^{\text{above}} = \frac{|k|^2 \, |\alpha|^2 \, e^{-2k_I x}}{2\omega} \left(k_R \cosh\left[2\text{Re}\left(kz + \beta\right)\right], \; k_I \sinh\left[2\text{Re}\left(kz + \beta\right)\right] \right)$$

- Note the dependence on $|\alpha|^2$, suggesting a relation $\boldsymbol{u_S} \sim$ (amplitude)^2.
- The damping of the waves is captured in the $exp\left(-k_I x\right)$ term.
- With no porous layer and $\phi = \alpha \cosh\left(kz + \beta\right) \exp\left(\mathrm{i}kx - \omega t\right)$,
  - $\beta = kD$ to impose no-penetration at the lower boundary.
  - $k\alpha \sinh\left(kD\right) = -\omega A$ by matching $\partial\phi/\partial z$ with $\partial\eta/\partial t$ at $z = 0$.
  - $\omega^2 = gk \tanh\left(kD\right)$ as seen before.

and hence

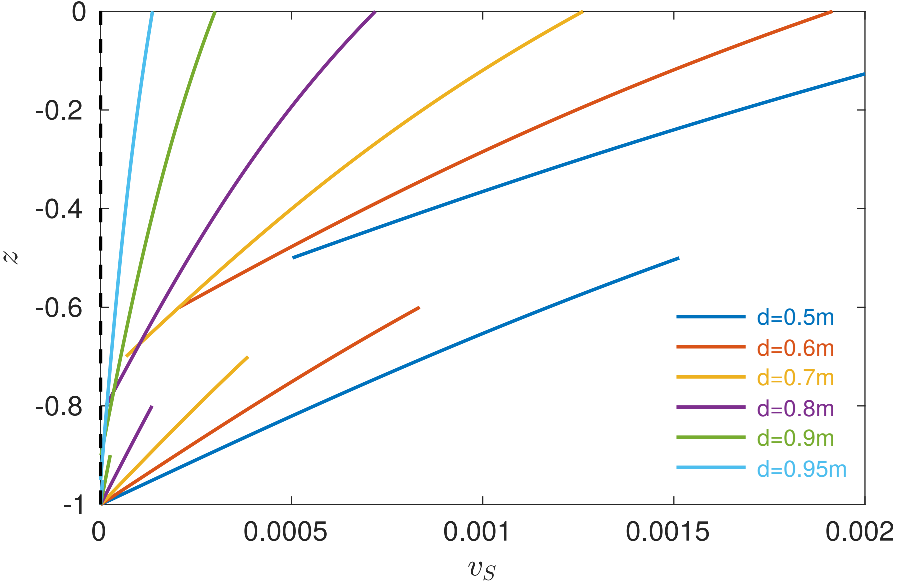
$$u_S = \underbrace{\frac{A^2 k\omega}{2 \sinh^2\left(kD\right)} \cosh\left[2k\left(z + D\right)\right]}_{\text{Stokes' formula!}} \tag{10}$$

Stokes drift velocities - horizontal



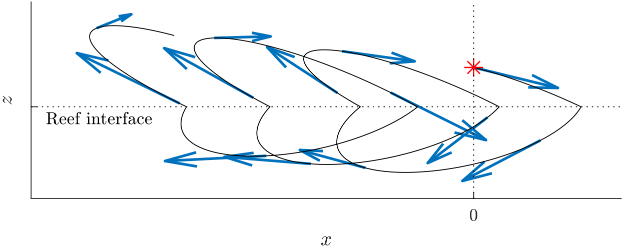
Stokes drift velocities - vertical

A novel vertical drift effect arises from the damping of the waves.



Potential limitations

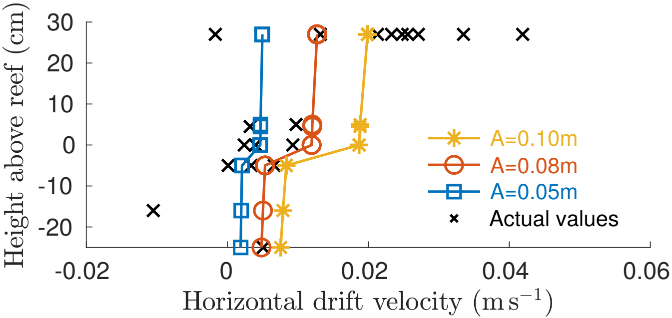
- There are some limitations apparent with the model, namely around the interface between the porous layer and the overlying water.
  - The Stokes drift velocities of particles crossing between the two flow regimes are not accurately described by our analytic expressions – direct numerical results needed.



  - Potential issues with matching the (inviscid) flow above the reef with the (viscous) flow within need to be considered.
    see Levy & Sanchez-Palencia (1975)

Comparing with field data

There is some uncertainty here – we don't have measurements of A, κ, or a reliable value for D. So we only look for approximate agreement.

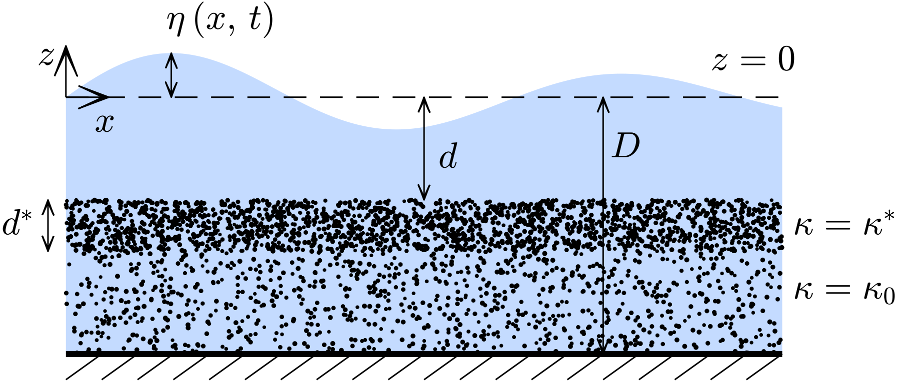


Data from Koehl, M. A. R. & Hadfield, M. G. Soluble settlement cue in slowly moving water within coral reefs induces larval adhesion to surfaces, J. Mar. Sys. **49** pp 75-78 (2004).

Allowing for depth-varying permeability

If we let κ be a function of z as opposed to just a constant, we can model reefs where permeability changes.

- Piecewise constant $\kappa(z)$ can model algal turf (see diagram below).
- Can also model reefs where the coral density increases with depth.

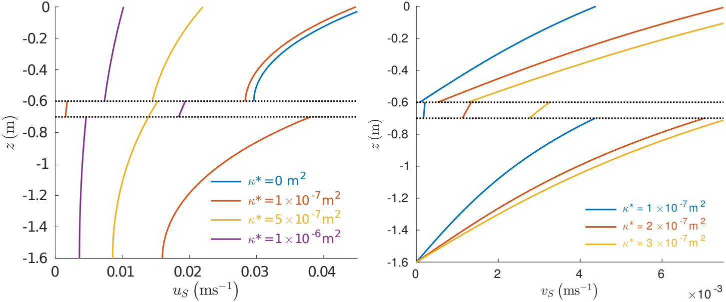


Allowing for depth-varying permeability

- Taking the same basic model as before, matching conditions depend only on the value of $\kappa\,(-d)$.
- This means that wavenumber and all constants involved in our expressions 'only care about' the interface region of the reef, up to drag effects not considered here.
- Obviously this breaks down in edge cases (taking $d^* \to 0$, for example) where consideration of the viscous drag at the interface becomes very important.
  - In cases like this, it may be more profitable to consider the additional layer as a jump condition when matching at the interface.

An example: thin algal layer

As the wavenumber only depends on the value of the permeability at the interface, so Stokes drift velocities are heavily dependent on this value, and not so strongly on the thickness d^* of the layer.



Note the different values of κ^* – the effect of changing κ^* is much stronger on vertical drifts.

Conclusions

- We've considered deriving expressions for the Stokes drift velocities in a two-layer system, with a porous medium underlying water.
- This can be extended to cases where the porous medium has a varying permeability, as is often the case in coral reefs.
- **Surprising results are seen:**
  - Drift occurs both vertically and horizontally
  - Properties of drift only affected strongly by value of permeability at the interface
  - Further research needed into the importance of viscous drag at the interface between layers
- Results appear to agree relatively well with measurements made of coral reefs

Acknowledgements

- Prof. Herbert Huppert — project supervisor
- Prof. Mimi Koehl
- Heilbronn Fund at Trinity College

Image courtesy of Koehl Lab, UC Berkeley (https://ib.berkeley.edu/labs/koehl/)

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