

# Optimizing ROC Curves with a Sort-Based Surrogate Loss for Binary Classification and Changepoint Detection, arXiv:2107.01285

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## Problem Setting and Related Work

Proposed algorithm

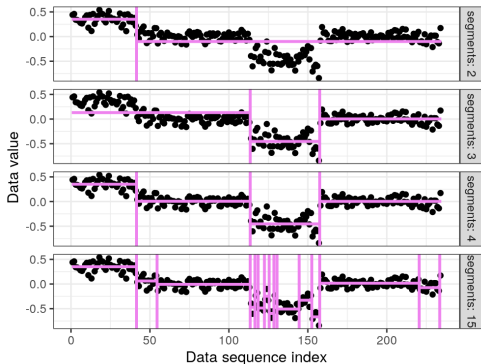
Empirical results

Discussion and Conclusions

# Problem: unsupervised changepoint detection

- ▶ Data sequence  $z_1, \dots, z_T$  at  $T$  points over time/space.
- ▶ Ex: DNA copy number data for cancer diagnosis,  $z_t \in \mathbb{R}$ .
- ▶ The penalized changepoint problem (Maidstone *et al.* 2017)

$$\arg \min_{u_1, \dots, u_T \in \mathbb{R}} \sum_{t=1}^T (u_t - z_t)^2 + \lambda \sum_{t=2}^T I[u_{t-1} \neq u_t].$$

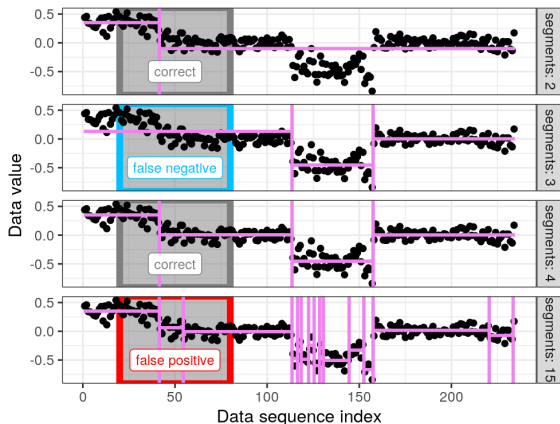


Larger penalty  $\lambda$   
results in fewer  
changes/segments.

Smaller penalty  
 $\lambda$  results in more  
changes/segments.

# Problem: weakly supervised changepoint detection

- ▶ First described by Hocking *et al.* ICML 2013.
- ▶ We are given a data sequence  $\mathbf{z}$  with labeled regions  $L$ .
- ▶ We compute features  $\mathbf{x} = \phi(\mathbf{z}) \in \mathbf{R}^p$  and want to learn a function  $f(\mathbf{x}) = -\log \lambda \in \mathbf{R}$  that minimizes label error.

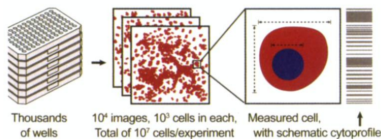


# Problem: supervised binary classification

- ▶ Given pairs of inputs  $\mathbf{x} \in \mathbb{R}^p$  and outputs  $y \in \{0, 1\}$  can we learn  $f(\mathbf{x}) = y$ ?
- ▶ Example: email,  $\mathbf{x}$  = bag of words,  $y$  = spam or not.
- ▶ Example: images. Jones *et al.* PNAS 2009.

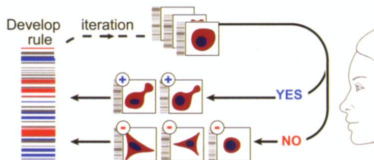
## A Automated Cell Image Processing

Cytoprofile of 500+ features measured for each cell



## B Iterative Machine Learning

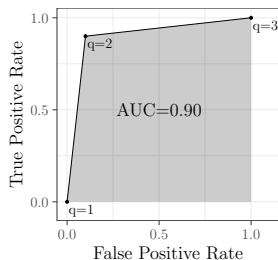
System presents cells to biologist for scoring, in batches



# Receiver Operating Characteristic (ROC) curve

Classic evaluation method from the signal processing literature (Egan and Egan, 1975).

- ▶ Binary classification algo gives predictions  $[\hat{y}_1, \hat{y}_2, \hat{y}_3, \hat{y}_4]$ .
- ▶ Each point on the ROC curve is the FPR/TPR if you add  $c$  to the predictions,  $[\hat{y}_1 + c, \hat{y}_2 + c, \hat{y}_3 + c, \hat{y}_4 + c]$ .
- ▶ Best point in ROC space is upper left (0% FPR, 100% TPR).
- ▶ Maximizing Area Under the ROC curve (AUC) is a common objective for binary classification, especially for imbalanced data (example: 99% positive, 1% negative labels).

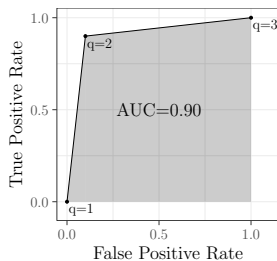
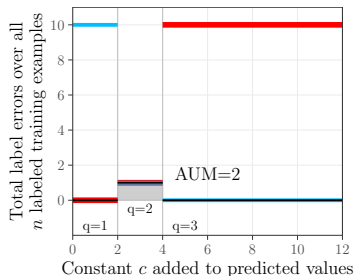


In binary classification, ROC curve is monotonic increasing.

- ▶ AUC=1 best.
- ▶ AUC=0.5 for constant prediction (usually worst).

# Area Under ROC curve, synthetic example

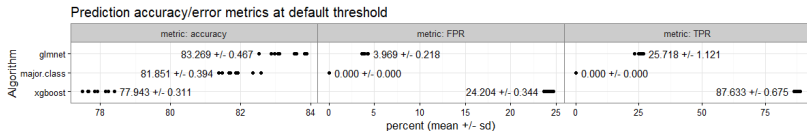
- ▶ Labels =  $[1, 0, 0, \dots, 1, 1, 0]$  (20 labels, 10 positive, 10 negative).
- ▶ Predictions =  $[-4, -4, -4, \dots, -2, -2, -2]$ .
- ▶ No constant added  $c = 0$ ,  $q = 1$ , everything predicted negative, so no false positives, but no true positives.
- ▶ Add  $c = 3 \Rightarrow [-1, -1, -1, \dots, 1, 1, 1]$ , 1 FP and 9 TP,  $q = 2$ .
- ▶ Add  $c = 5 \Rightarrow [1, 1, 1, \dots, 3, 3, 3]$ , all FP and TP,  $q = 3$ .



# Real data example when ROC curves are useful

Data from collaboration with SICCS professor Patrick Jantz, about predicting presence/absence of trees in different locations.

- ▶ glmnet: L1-regularized linear model.
- ▶ major.class: featureless baseline (ignores inputs, always predicts most frequent class label in train set)
- ▶ xgboost: gradient boosted decision trees.
- ▶ Which algorithm is the most accurate?



<https://bl.ocks.org/tdhock/raw/172d0f68a51a8de5d6f1bed7f23f5f82/>



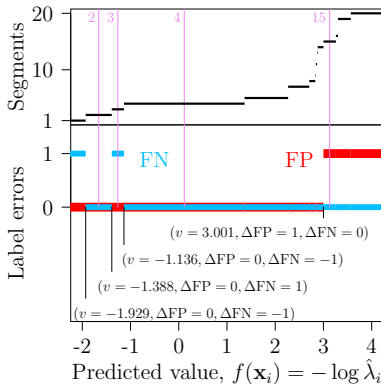
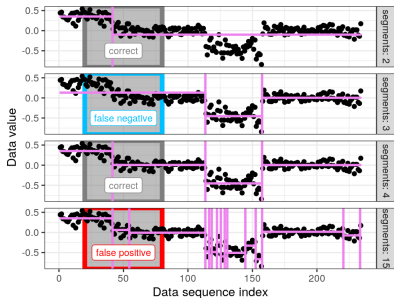
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Discussion and Conclusions

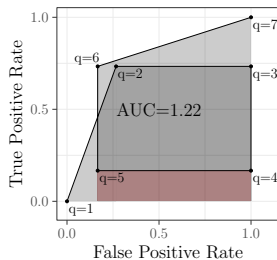
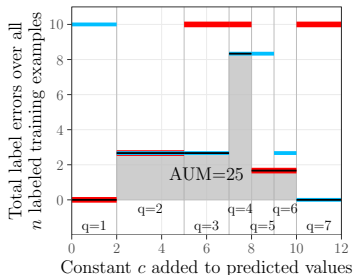
# Real changepoint problem with non-monotonic label error



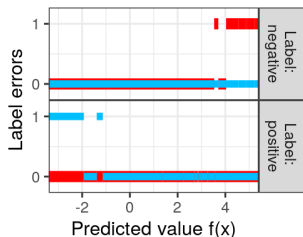
Optimal changepoint model may have non-monotonic error (for example FN), because changepoints at model size  $s$  may not be present in model  $s + 1$ .

# Looping ROC curve, simple synthetic example

- ▶ Non-monotonic FP/FN can result in looping ROC curve.
- ▶ AUC can be greater than one (dark grey area double counted, red area negative counted).
- ▶ Loops have very sub-optimal points (large min error, for example  $q=4$ ), so do we want to maximize AUC?
- ▶ Minimize Area Under Min (AUM) instead, which encourages monotonic ROC curve with points in upper left (small min error, for example  $q=1,6,7$ ).



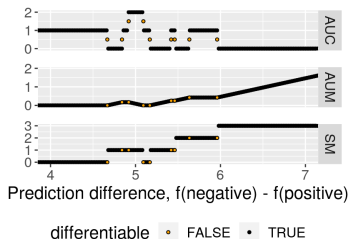
# Real data example with AUC greater than one



Error type

FN

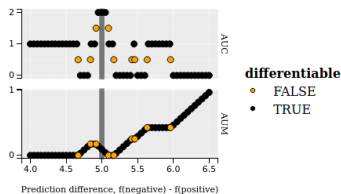
FP



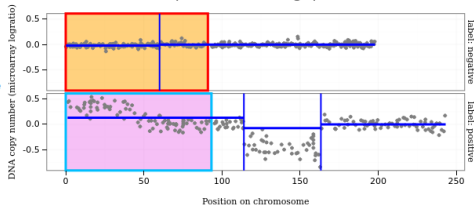
- ▶  $n = 2$  labeled changepoint problems.
- ▶  $\text{AUC} = 2$  when prediction difference = 5.
- ▶  $\text{AUM} = 0$  implies  $\text{AUC} = 1$ .
- ▶ AUM is smooth relaxation of non-convex Sum of Min (SM).
- ▶ AUM is differentiable almost everywhere.
- ▶ Main new idea: compute the gradient of this function and use it for learning.

# Real data example, interactive AUC/AUM demo

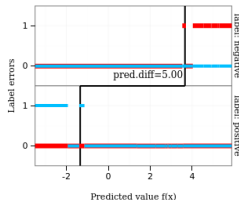
Overview, select difference



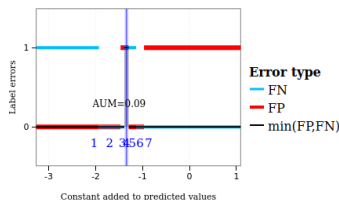
Data, labels, predicted changepoint models



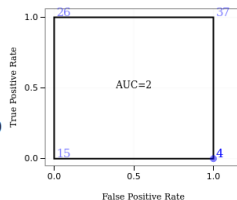
Example error functions



Total error, select interval



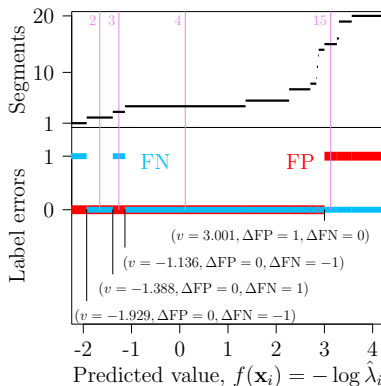
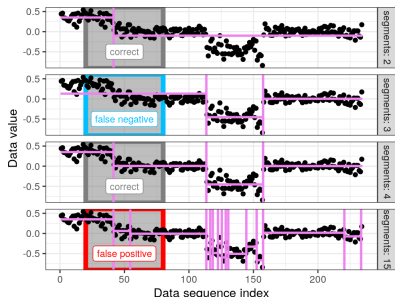
ROC curve, select point



<http://bl.ocks.org/tdhock/raw/e3f56fa419a6638f943884a3abe1dc0b/>

# Algorithm inputs: predictions and label error functions

- ▶ Each observation  $i \in \{1, \dots, n\}$  has a predicted value  $\hat{y}_i \in \mathbb{R}$ .
- ▶ Breakpoints  $b \in \{1, \dots, B\}$  used to represent label error via tuple  $(v_b, \Delta FP_b, \Delta FN_b, \mathcal{I}_b)$ .
- ▶ There are changes  $\Delta FP_b, \Delta FN_b$  at predicted value  $v_b \in \mathbb{R}$  in error function  $\mathcal{I}_b \in \{1, \dots, n\}$ .

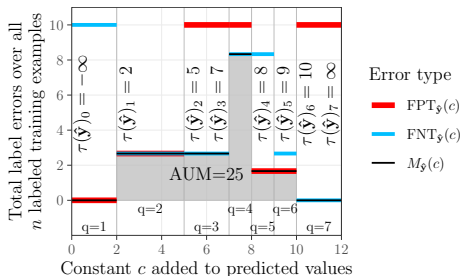


# Algorithm computes total FP and FN for each threshold/constant added to predicted values

- ▶ Breakpoint threshold,  $t_b = v_b - \hat{y}_{\mathcal{I}_b} = \tau(\hat{\mathbf{y}})_q$  for some  $q$ .
- ▶ Total error before/after each breakpoint can be computed via sort and modified cumsum:

$$\underline{\text{FP}}_b = \sum_{j: t_j < t_b} \Delta \text{FP}_j, \quad \overline{\text{FP}}_b = \sum_{j: t_j \leq t_b} \Delta \text{FP}_j,$$

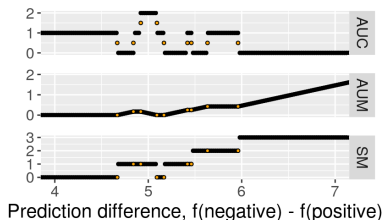
$$\underline{\text{FN}}_b = \sum_{j: t_j \geq t_b} -\Delta \text{FN}_j, \quad \overline{\text{FN}}_b = \sum_{j: t_j > t_b} -\Delta \text{FN}_j.$$



# Algorithm computes two directional derivatives

- ▶ Gradient only defined when function is differentiable, but AUM is not differentiable everywhere (see below).
- ▶ Directional derivatives defined everywhere.

$$\begin{aligned}\nabla_{\mathbf{v}(-1,i)}\text{AUM}(\hat{\mathbf{y}}) &= \sum_{b:\mathcal{I}_b=i} \min\{\overline{\text{FP}}_b, \overline{\text{FN}}_b\} - \min\{\overline{\text{FP}}_b - \Delta\text{FP}_b, \overline{\text{FN}}_b - \Delta\text{FN}_b\}, \\ \nabla_{\mathbf{v}(1,i)}\text{AUM}(\hat{\mathbf{y}}) &= \sum_{b:\mathcal{I}_b=i} \min\{\underline{\text{FP}}_b + \Delta\text{FP}_b, \underline{\text{FN}}_b + \Delta\text{FN}_b\} - \min\{\underline{\text{FP}}_b, \underline{\text{FN}}_b\}.\end{aligned}$$



Proposed learning algo uses mean of these two directional derivatives as “gradient.”



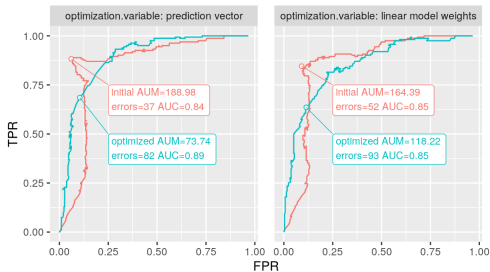
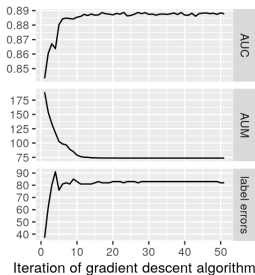
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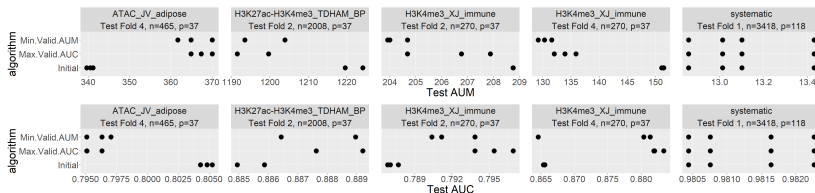
Discussion and Conclusions

# AUM gradient descent results in increased train AUC for a real changepoint problem



- ▶ Left/middle: changepoint problem initialized to prediction vector with min label errors, gradient descent on prediction vector.
- ▶ Right: linear model initialized by minimizing regularized convex loss (surrogate for label error, Hocking *et al.* ICML 2013), gradient descent on weight vector.

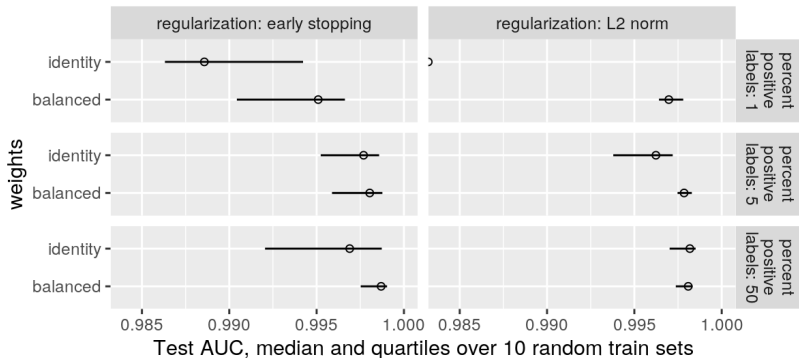
# Learning algorithm results in better test AUC/AUM for changepoint problems



- ▶ Five changepoint problems (panels from left to right).
- ▶ Two evaluation metrics (AUM=top, AUC=bottom).
- ▶ Three algorithms (Y axis), Initial=Min regularized convex loss (surrogate for label error, Hocking *et al.* ICML 2013), Min.Valid.AUM/Max.Valid.AUC=AUM gradient descent with early stopping regularization.
- ▶ Four points = Four random initializations.

# Standard logistic loss fails for highly imbalanced labels

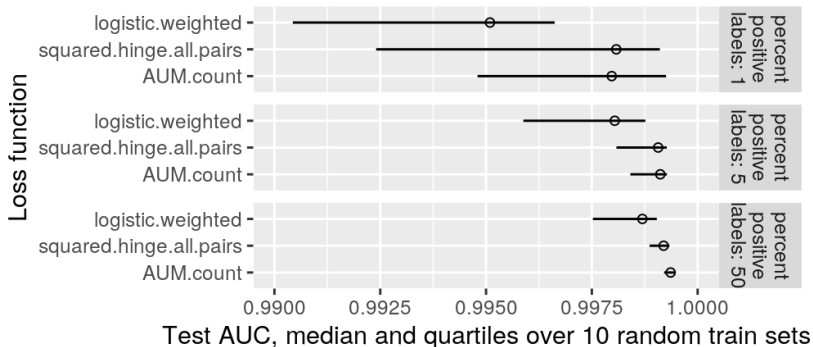
Comparing logistic regression models (control experiment)



- ▶ Subset of zip.train/zip.test data (only 0/1 labels).
- ▶ Test set size 528 with balanced labels (50%/50%).
- ▶ Train set size 1000 with variable class imbalance.
- ▶ Loss is  $\ell[f(x_i), y_i]w_i$  with  $w_i = 1$  for identity weights,  $w_i = 1/N_{y_i}$  for balanced, ex: 1% positive means  $w_i \in \{1/10, 1/990\}$ .

# Learning algorithm competitive for unbalanced binary classification

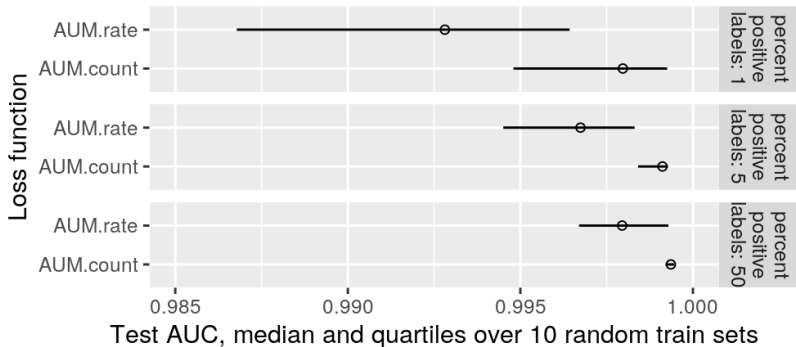
(b) AUM compared to baselines



- ▶ Squared hinge all pairs is a classic/popular surrogate loss function for AUC optimization. (Yan *et al.* ICML 2003)
- ▶ All linear models with early stopping regularization.

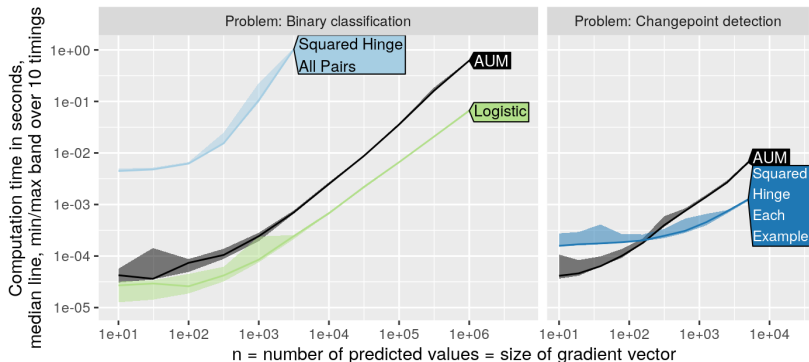
# Error rate loss is not as useful as error count loss

(a) Comparing AUM variants



- ▶ AUM.count is as described previously: error functions used to compute  $\text{Min}(\text{FP}, \text{FN})$  are absolute label counts.
- ▶ AUM.rate is a variant which uses normalized error functions,  $\text{Min}(\text{FPR}, \text{FNR})$ .
- ▶ Both linear models with early stopping regularization.

# Comparable computation time to other loss functions



- ▶ Logistic  $O(n)$ .
- ▶ AUM  $O(n \log n)$ . (proposed)
- ▶ Squared Hinge All Pairs  $O(n^2)$ . (Yan *et al.* ICML 2003)
- ▶ Squared Hinge Each Example  $O(n)$ . (Hocking *et al.* ICML 2013)

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# Conclusions, Pre-print arXiv:2107.01285

- ▶ ROC curves are used to evaluate binary classification and changepoint detection algorithms.
- ▶ In changepoint detection there can be loops in ROC curves, so maximizing AUC may not be desirable.
- ▶ Instead we propose to minimize a new loss,  $AUM = \text{Area Under Min}(FP, FN)$ .
- ▶ We propose new algorithm for efficient AUM and directional derivative computation.
- ▶ Empirical results provide evidence that learning using AUM minimization results in AUC maximization.
- ▶ Future work: sort-based surrogates for all pairs loss functions (binary classification, information retrieval).

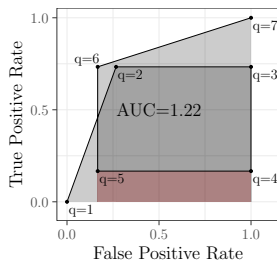
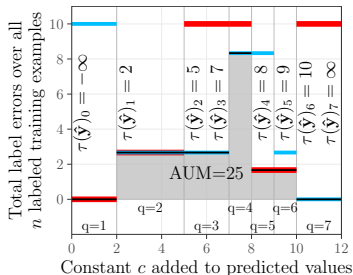
Thanks to co-author Jonathan Hillman! (second from left)



Contact: [toby.hocking@nau.edu](mailto:toby.hocking@nau.edu)

## More notation

- First let  $\{(\text{fpt}(\hat{\mathbf{y}})_q, \text{fnt}(\hat{\mathbf{y}})_q, \tau(\hat{\mathbf{y}})_q)\}_{q=1}^Q$  be a sequence of  $Q$  tuples, each of which corresponds to a point on the ROC curve (and an interval on the fn/fp error plot).
- For each  $q$  the  $\text{fpt}(\hat{\mathbf{y}})_q, \text{fnt}(\hat{\mathbf{y}})_q$  are false positive/negative totals at that point (in that interval) whereas  $\tau(\hat{\mathbf{y}})_q$  is the upper limit of the interval.
- The limits are increasing,  $-\infty = \tau(\hat{\mathbf{y}})_0 < \dots < \tau(\hat{\mathbf{y}})_Q = \infty$ .
- Then we define  $m(\hat{\mathbf{y}})_q = \min\{\text{fpt}(\hat{\mathbf{y}})_q, \text{fnt}(\hat{\mathbf{y}})_q\}$  as the min of fp and fn totals in that interval.



# L1 relaxation interpretation

Our proposed loss function is

$$\text{AUM}(\hat{\mathbf{y}}) = \sum_{q=2}^{Q-1} [\tau(\hat{\mathbf{y}})_q - \tau(\hat{\mathbf{y}})_{q-1}] m(\hat{\mathbf{y}})_q.$$

It is an L1 relaxation of the following non-convex **Sum of Min(FP,FN)** function,

$$\text{SM}(\hat{\mathbf{y}}) = \sum_{q=2}^{Q-1} I[\tau(\hat{\mathbf{y}})_q \neq \tau(\hat{\mathbf{y}})_{q-1}] m(\hat{\mathbf{y}})_q = \sum_{q: \tau(\hat{\mathbf{y}})_q \neq \tau(\hat{\mathbf{y}})_{q-1}} m(\hat{\mathbf{y}})_q.$$

## Definition of data set, notations

- ▶ Let there be a total of  $B$  breakpoints in the error functions over all  $n$  labeled training examples.
- ▶ Each breakpoint  $b \in \{1, \dots, B\}$  is represented by the tuple  $(v_b, \Delta FP_b, \Delta FN_b, \mathcal{I}_b)$ , where the  $\mathcal{I}_b \in \{1, \dots, n\}$  is an example index, and there are changes  $\Delta FP_b, \Delta FN_b$  at predicted value  $v_b \in \mathbb{R}$  in the error functions.
- ▶ For example in binary classification, there are  $B = n$  breakpoints (same as the number of labeled training examples); for each breakpoint  $b \in \{1, \dots, B\}$  we have  $v_b = 0$  and  $\mathcal{I}_b = b$ . For breakpoints  $b$  with positive labels  $y_b = 1$  we have  $\Delta FP = 0, \Delta FN = -1$ , and for negative labels  $y_b = -1$  we have  $\Delta FP = 1, \Delta FN = 0$ .
- ▶ In changepoint detection we have more general error functions, which may have more than one breakpoint per example.

# Proposed algorithm uses sort to compute AUM and directional derivatives

- 1: **Input:** Predictions  $\hat{\mathbf{y}} \in \mathbb{R}^n$ , breakpoints in error functions  $v_b, \Delta FP_b, \Delta FN_b, \mathcal{I}_b$  for all  $b \in \{1, \dots, B\}$ .
  - 2: Zero the  $AUM \in \mathbb{R}$  and directional derivatives  $\mathbf{D} \in \mathbb{R}^{n \times 2}$ .
  - 3:  $t_b \leftarrow v_b - \hat{y}_{\mathcal{I}_b}$  for all  $b$ .
  - 4:  $s_1, \dots, s_B \leftarrow \text{SORTEDINDICES}(t_1, \dots, t_B)$ .
  - 5: Compute  $\underline{FP}_b, \overline{FP}_b, \underline{FN}_b, \overline{FN}_b$  for all  $b$  using  $s_1, \dots, s_B$ .
  - 6: **for**  $b \in \{2, \dots, B\}$  **do**
  - 7:    $AUM += (t_{s_b} - t_{s_{b-1}}) \min\{\underline{FP}_b, \overline{FN}_b\}$ .
  - 8: **for**  $b \in \{1, \dots, B\}$  **do**
  - 9:    $\mathbf{D}_{\mathcal{I}_b, 1} += \min\{\overline{FP}_b, \overline{FN}_b\} - \min\{\overline{FP}_b - \Delta FP_b, \overline{FN}_b - \Delta FN_b\}$
  - 10:    $\mathbf{D}_{\mathcal{I}_b, 2} += \min\{\underline{FP}_b + \Delta FP_b, \underline{FN}_b + \Delta FN_b\} - \min\{\underline{FP}_b, \underline{FN}_b\}$
  - 11: **Output:** AUM and matrix  $\mathbf{D}$  of directional derivatives.
- Overall  $O(B \log B)$  time due to sort.