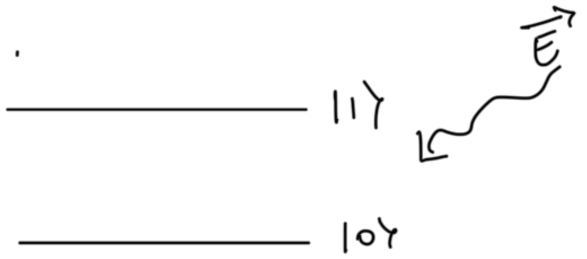
Q. 4.



$\langle t | 1, t \rangle = \psi_1(t)$

$\langle t | 0, t \rangle = \psi_0(t)$

$$\Psi(t) = a_0(t)\,\psi_0(t) + a_1(t)\,\psi_1(t)$$

$\left\{ \text{Bra-Ket notat}^{n} : \ |\psi_t\rangle = a_0(t)\,|0,t\rangle + a_1(t)\,|1,t\rangle \right\}$

Then $|a_0(t)|^2 \rightarrow$ probability of system being in ground state
 $|a_1(t)|^2 \rightarrow$ " " " " excited "

& $|a_0(t)|^2 + |a_1(t)|^2 = 1$

Electric field of incident radiation : $|\vec{E}| = \tilde{E}_0 \cos(\omega t)$

$\left(\dfrac{2\pi c}{\omega} = \lambda \gg d \implies \vec{E}(\vec{r},t) \sim \vec{E}(t) \right)$

Then, $\hat{H} = H_0 + H_1$

$H_0 = E_0\,|0\rangle\langle 0| + E_1\,|1\rangle\langle 1|$

$H_1 = -\hat{\vec{\mu}} \cdot \vec{E} = -\hat{\mu}\,\tilde{E}_0 \cos(\omega t)$

(Consider dipole type interact^{n} of light w/ the system)

To solve the Schrödinger eq^{n} $\quad H\Psi(t) = i\hbar \dfrac{\partial}{\partial t}\Psi(t)$

$$\frac{\partial}{\partial t}\Psi(t) = \dot{a}_0(t)\,\psi_0(t) + a_0(t)\,\dot{\psi}_0(t) + \dot{a}_1(t)\,\psi_1(t) + a_1(t)\,\dot{\psi}_1(t)$$

Since, $|0\rangle$ & $|1\rangle$ are eigenstates, we expect the time evolution as follows

$$\psi_0(t) = \psi_0\, e^{-i\frac{E_0 t}{\hbar}} \quad ; \quad \psi_1(t) = \psi_1\, e^{-i\frac{E_1 t}{\hbar}}$$

$$\therefore \frac{\partial}{\partial t}\Psi(t) = \dot{a}_0(t)\,.\,\psi_0 e^{-iE_0 t/\hbar} + a_0(t)\,\psi_0 \left(-i\frac{E_0}{\hbar}\right) e^{-i\frac{E_0 t}{\hbar}}$$
$$\dot{a}_1(t)\,.\,\psi_1 e^{-iE_1 t/\hbar} + a_0(t)\,\psi_1 \left(-i\frac{E_1}{\hbar}\right) e^{-iE_1 t/\hbar}$$

$$\therefore i\hbar \frac{\partial}{\partial t}\Psi(t) = i\hbar \dot{a}_0(t)\,\Psi_0 e^{-iE_0t/\hbar} + i\hbar \dot{a}_1(t)\,\Psi_0 e^{-iE_1t/\hbar}$$
$$+ a_0(t)\cdot E_0 \Psi_0 e^{-iE_0t/\hbar} + a_1(t)\cdot E_1 \Psi_0 e^{-iE_1t/\hbar}$$

Now,

$$H_0 \Psi(t) = a_0(t) \times E_0 \Psi_0(t) + a_1(t) \times E_1 \Psi_1(t)$$

& since $i\hbar \frac{\partial}{\partial t}\Psi(t) = H_0\Psi(t) + H_1\Psi(t)$

$$H_1\Psi(t) = i\hbar \dot{a}_0(t)\Psi_0 e^{-i\frac{E_0t}{\hbar}} + i\hbar \dot{a}_1(t)\,\Psi_1 e^{-iE_1t/\hbar}$$

Switching to Bra-Ket notation

$$\hat{H}_1 |\Psi_t\rangle = i\hbar \dot{a}_0(t) e^{-i\frac{E_0 t}{\hbar}} |0\rangle + i\hbar \dot{a}_1(t)\Psi_1 e^{-iE_1t/\hbar}|1\rangle$$

Then,

$$e^{+iE_0t/\hbar} \langle 0|\hat{H}_1|\Psi_t\rangle = i\hbar \dot{a}_0(t)\langle 0|0\rangle + i\hbar \dot{a}_1(t) e^{-i\frac{(E_1-E_0)t}{\hbar}} \cancel{\langle 0|1\rangle}^{0}$$

where $|\alpha, t\rangle = e^{-i\frac{E_\alpha t}{\hbar}}|\alpha\rangle$

$$\therefore i\hbar \dot{a}_0(t) = -a_0(t)\langle 0|\hat{u}|0\rangle \tilde{E}_0\cos(\omega t) - a_1(t)\langle 0|\hat{u}|1\rangle e^{-i\frac{(E_1-E_0)t}{\hbar}} \tilde{E}_0\cos(\omega t)$$

$||^{ly}$ $$i\hbar \dot{a}_1(t) = -a_0(t)\langle 1|\hat{u}|0\rangle e^{i\frac{(E_1-E_0)t}{\hbar}} \tilde{E}_0\cos(\omega t) - a_1(t)\langle 1|\hat{u}|1\rangle \tilde{E}_0\cos(\omega t)$$

Now,

$\hat{u}$ is a dipole operator

$$\therefore \langle 0|\hat{u}|0\rangle = \langle 1|\hat{u}|1\rangle = 0$$
$$\& \quad \langle 0|\hat{u}|1\rangle = \langle 1|\hat{u}|0\rangle^* =$$

for simplicity we take $\hat{u}_{\alpha\beta}$ to be real [& equal to u]
($\because$ we have taken $\vec{E}$ as real too $(\cos(\omega t))$)

$$\therefore \dot{a}_0(t) = i\, a_1(t) \times \left(\frac{u\tilde{E}_0}{\hbar}\right)\cos(\omega t)\; e^{-i\frac{(E_1-E_0)t}{\hbar}}$$

$$\dot{a}_1(t) = i\, a_0(t) \times \left(\frac{u\tilde{E}_0}{\hbar}\right)\cos(\omega t)\; e^{+i\frac{(E_1-E_0)t}{\hbar}}$$

Let $\frac{E_1-E_0}{\hbar} = \omega_{10}$ & $\cos(\omega t) = \frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)$

$$\therefore \dot{a}_0(t) = \frac{i}{2} a_1(t)\,\omega_R\, e^{i\Delta t} + \tilde{c}_1(t)\, e^{i(\omega+\omega_{10})t}$$

$$\dot{a}_1(t) = \frac{i}{2} a_0(t)\, \omega_R\, e^{-i\Delta t} + \tilde{c}_0(t)\, e^{i(\omega+\omega_{10})t}$$

where $\omega_R = \frac{\mu \tilde{E}_0}{\hbar}$ & $\Delta = \omega - \omega_{10}$

Note: $\omega + \omega_{10} >> \Delta$ since we look at $\omega \to \omega_{10}$ for transition

$\therefore$ when we integrate the differential equ. the terms containing $e^{i(\omega+\omega_{10})t}$ vanish since they average out to zero over many time periods

$$\boxed{\begin{aligned} \therefore\ \dot{a}_0(t) &= \frac{i}{2} a_1(t)\, \omega_R\, e^{i\Delta t} \\ \dot{a}_1(t) &= \frac{i}{2} a_0(t)\, \omega_R\, e^{-i\Delta t} \end{aligned}}$$

(b)

Solving the two differential equations

$$\dot{a}_1(t) = \frac{i}{2} a_0(t)\, \omega_R\, e^{-i\Delta t}$$

$$\therefore\ \dot{a}_1(t)\, e^{i\Delta t} = \frac{i}{2} a_0(t)\, \omega_R$$

$$\therefore\ \ddot{a}_1(t)\, e^{i\Delta t} + i\Delta\, \dot{a}_1(t)\, e^{i\Delta t} = \frac{i\omega_R}{2} \dot{a}_0(t)$$

$$\therefore\ \ddot{a}_1(t) e^{i\Delta t} + i\Delta \dot{a}_1(t) e^{i\Delta t} = \frac{i\omega_R}{2} \cdot \frac{i}{2} a_1(t)\, \omega_R\, e^{i\Delta t}$$

$$\therefore\ \ddot{a}_1(t) + i\Delta\, \dot{a}_1(t) + \frac{\omega_R^2}{4} a_1(t) = 0$$

$$||^{ly}\quad \ddot{a}_0(t) - i\Delta\, \dot{a}_0(t) + \frac{\omega_R^2}{4} a_0(t) = 0$$

General sol^n: $a_0(t) = A e^{i\alpha t} + B e^{i\beta t}$

$$\therefore\ \dot{a}_0(t) = A i\alpha e^{i\alpha t} + B i\beta e^{i\beta t}$$

But $\dot{a}_0(t) = \frac{i\omega_R}{2} a_1(t) e^{i\Delta t}$

$$\Rightarrow \quad a_1(t) = \frac{2}{i\omega_R} \cdot e^{-i\Delta t}\left(Ai\alpha e^{i\alpha t} + Bi\beta e^{i\beta t}\right)$$

Substitute the general solⁿ into the 2nd Order D.E.

$$A(i\alpha)(i\alpha)e^{i\alpha t} + B(i\beta)^2 e^{i\beta t} - i\Delta\left(Ai\alpha e^{i\alpha t} + Bi\beta e^{i\beta t}\right)$$

$$+ \frac{\omega_R^2}{4}\left(Ae^{i\alpha t} + Be^{i\beta t}\right) = 0$$

$$\therefore \quad A\left(-\alpha^2 + \Delta\alpha + \frac{\omega_R^2}{4}\right)e^{i\alpha t} + B\left(-\beta^2 + \Delta\beta + \frac{\omega_R^2}{4}\right)e^{i\beta t} = 0$$

Clearly,

$$\alpha^2 - \Delta\alpha - \frac{\omega_R^2}{4} = 0$$

$$\& \quad \beta^2 - \Delta\beta - \frac{\omega_R^2}{4} = 0 \qquad \text{(for arbitrary values of } A \,\&\, B\text{)}$$

They both give the same two solutions

$$x = \frac{\Delta}{2} \pm \frac{\sqrt{\Delta^2 + \omega_R^2}}{2}$$

$$\therefore \quad \alpha = \frac{\Delta}{2} + \frac{\Omega}{2}, \quad \beta = \frac{\Delta}{2} - \frac{\Omega}{2}$$

$$\therefore \quad a_0(t) = A e^{\frac{i\Delta t}{2}} e^{\frac{i\Omega t}{2}} + B e^{\frac{i\Delta t}{2}} e^{-i\frac{\Omega}{2}t}$$

$$\therefore \quad a_1(t) = \frac{2}{\omega_R} e^{-\frac{i\Delta t}{2}}\left(A\alpha e^{i\frac{\Omega}{2}t} + B\beta e^{-i\frac{\Omega}{2}t}\right)$$

Let $a_0(0) = 1$ [ground state at $t=0$]

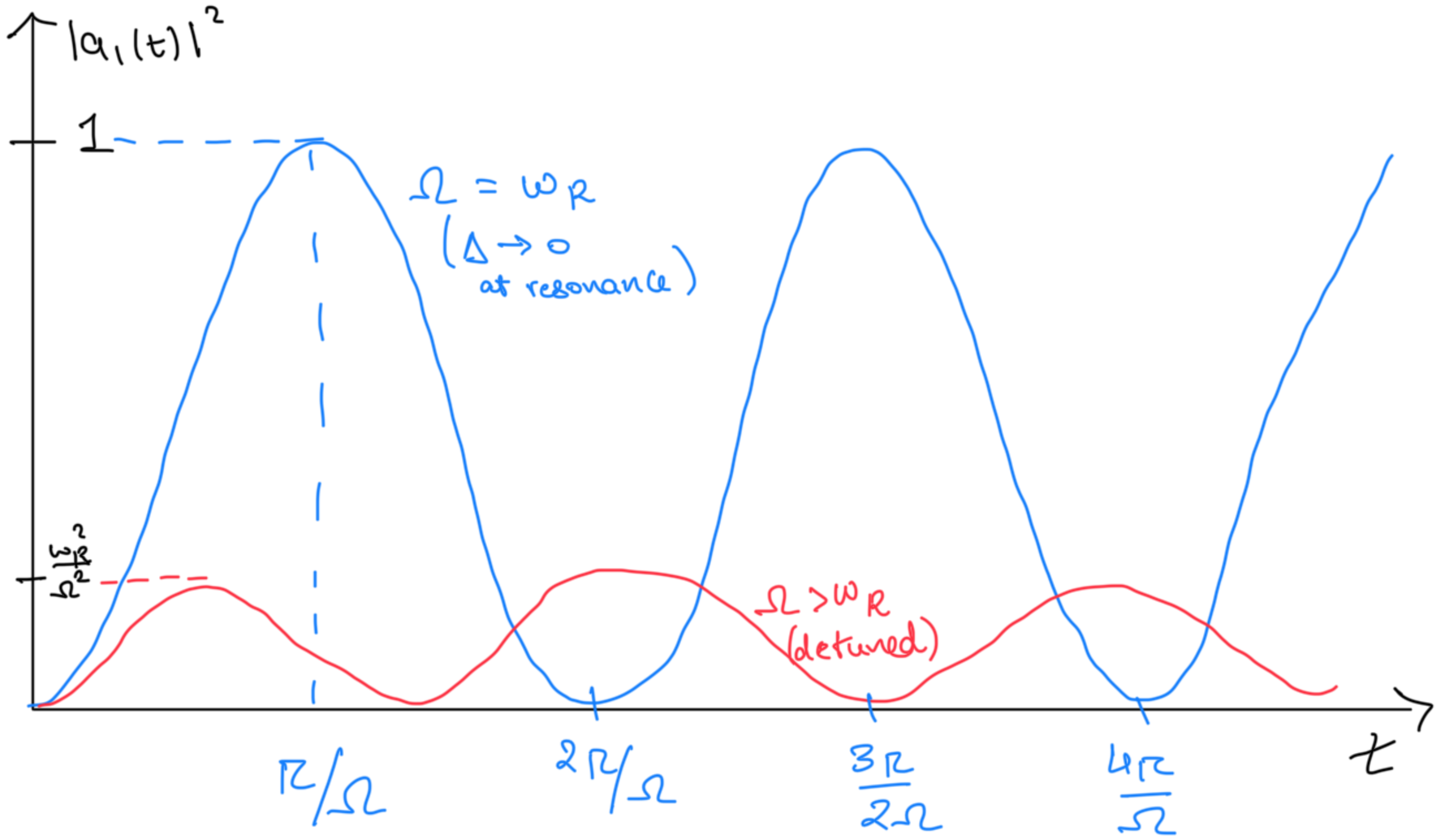
$$\therefore \quad 1 = A + B$$

Put $A = 1/2$ then $B = -1/2$

Then

$$a_1(t) = i\left(\frac{\omega_R}{\Omega}\right)\sin\left(\frac{\Omega t}{2}\right)e^{-i\Delta t/2}$$

$$\Rightarrow \quad |a_1(t)|^2 = \frac{\omega_R^2}{\Omega^2}\sin^2\left(\frac{\Omega t}{2}\right)$$



|a_1(t)|^2
1
Ω = ω_R
(Δ → 0
at resonance)
ω_R^2/Ω^2
Ω > ω_R
(detuned)
π/Ω
2π/Ω
3π/2Ω
4π/Ω
t