Assignment - 2

Q.1.

$$\vec{E} = \vec{E_0}\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

Given: propagation along X-axis $\Rightarrow \vec{k} \parallel \hat{x}$
polarization along Y-axis $\Rightarrow \vec{E_0} \parallel \hat{y}$

$$\therefore \vec{E} = E_0\, e^{i(kx - \omega t)}\, \hat{y}$$

Now, $\vec{B} \perp \vec{E}$ & $\vec{B} \perp \vec{k}$

$$\therefore \vec{B_0} \parallel \hat{z}$$

$$\therefore \vec{B} = B_0\, e^{i(kx - \omega t)}\, \hat{z}$$

To check that they satisfy the wave equation

$$\frac{\partial^2 \vec{A}}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2}$$

$$\frac{\partial^2 \vec{E}}{\partial x^2} = E_0 \hat{y}\,(ik)\,\frac{\partial}{\partial x} e^{i(kx-\omega t)} = -E_0 k^2 e^{i(kx-\omega t)}\,\hat{}$$

$$= -k^2 \vec{E}\ \hat{y}$$

$$\frac{\partial^2 \vec{E}}{\partial t^2} = -\omega^2 \vec{E}\ \hat{y}$$

Now, for an EM wave in vacuum

$$k = \frac{\omega}{c}$$

Thus, $$\frac{\partial^2 \vec{E}}{\partial x^2} = -\frac{\omega^2}{c^2}\vec{E} = -\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}$$

Similarly, $$\frac{\partial^2 \vec{B}}{\partial x^2} = -k^2 \vec{B}\,\hat{z}\ ;\ \frac{\partial^2 \vec{B}}{\partial t^2} = -\omega^2 \vec{B}\,\hat{z}$$

a)

To prove: $$\frac{\partial E_y}{\partial x} = \frac{\partial B_z}{\partial t}$$

Now, $\vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$ —— (1) (Maxwell's relation)

holds since both $\vec{E}$ & $\vec{B}$ satisfy the wave equations that are derived from Maxwell's relations

$$\vec{\nabla} \times \vec{E} = (\partial_x E_y - \partial_y E_x)\hat{z} + (\partial_z E_x - \partial_x E_z)\hat{y} + (\partial_y E_z - \partial_z E_y)\hat{x}$$

Now,
$$\vec{E} = E_x \hat{x} + E_y \hat{y} + E_z \hat{z}$$
$$= E_0 e^{i(kx - \omega t)} \hat{y}$$

$$\Rightarrow E_x = E_z = 0$$

$$\therefore \vec{\nabla} \times \vec{E} = \partial_x E_y \hat{z} - \partial_z E_y \hat{x}$$

But $\dfrac{\partial}{\partial z} E_y = \dfrac{\partial}{\partial z} E_0 e^{i(kx - \omega t)} = 0$

$$\therefore \vec{\nabla} \times \vec{E} = \frac{\partial}{\partial x} E_y \hat{z} \quad \text{—— (2)}$$

Now, $\dfrac{\partial \vec{B}}{\partial t} = \dfrac{\partial B_z}{\partial t} \hat{z}$ —— (3) $(\because B_x, B_y = 0)$

$\therefore$ from (1), (2) & (3) we have

$$\frac{\partial}{\partial x} E_y \hat{z} = -\frac{\partial B_z}{\partial t} \hat{z}$$

$$\Rightarrow \boxed{\frac{\partial}{\partial x} E_y = -\frac{\partial B_z}{\partial t}}$$

Now,
$$\frac{\partial}{\partial x} E_y = E_0 (ik) e^{i(kx - \omega t)} \hat{z}$$

&
$$\frac{\partial B_z}{\partial t} = B_0 (-i\omega) e^{i(kx - \omega t)} \hat{z}$$

$$\Rightarrow E_0 (ik) = -B_0 (i\omega)$$

$$\Rightarrow \quad \frac{E_0}{B_0} = \frac{\omega}{k} = c$$

$$\therefore \quad E_0 = c B_0$$

$$\Rightarrow \boxed{|\vec{E}| = c\,|\vec{B}|}$$

b)

$$Q = CV \quad \& \quad I = \frac{\Phi}{L}$$

Now,

energy stored in a capacitor : $U_E = \dfrac{Q^2}{2C}$

energy stored in an inductor : $U_B = \dfrac{1}{2} L I^2$

$$\& \quad |\vec{E}| = \frac{dV}{dx} \quad ; \quad C = \frac{A \varepsilon_0}{x} \quad \text{(in vacuum)}$$

$\therefore$ for a capacitor $V = \dfrac{Q}{A \varepsilon_0} \cdot x$

$$\Rightarrow |\vec{E}| = \frac{Q}{A \varepsilon_0}$$

$$\therefore \quad U_E = \frac{A^2 \varepsilon_0^2 |\vec{E}|^2}{2 \cdot A \varepsilon_0 / x} \Rightarrow u_E = \frac{1}{2} \varepsilon_0 |\vec{E}|^2 \cdot \left(\frac{A x}{\text{volume}} \right)$$

Similarly,

$$\Phi = BA \quad , \quad L = \frac{\mu_0 N^2 A}{x} \quad \text{(put } N = 1\text{)}$$

$$\therefore \quad U_B = \frac{1}{2} \frac{(LI)^2}{L} = \frac{1}{2} \frac{B^2 A^2}{\mu_0 A} \cdot x$$

$$\therefore \quad u_B = \frac{1}{2\mu_0} \cdot B^2 \quad \frac{(A \cdot x)}{\text{volume}}$$

Thus,

$$u_E = \frac{1}{2} \varepsilon_0 |\vec{E}|^2 \quad \& \quad u_B = \frac{1}{2\mu_0} |\vec{B}|^2$$

Now, for an EM wave

$$u_E = \frac{1}{2} \varepsilon_0 |\vec{E}|^2 = \frac{1}{2} \varepsilon_0 E_0^2$$

$$u_B = \frac{1}{2\mu_0} |\vec{B}|^2 = \frac{1}{2\mu_0} \cdot B_0^2$$

& $$E_0 = c B_0 = \frac{1}{\sqrt{\varepsilon_0 . \mu_0}} \cdot B_0$$

$$\therefore \quad u_E = \frac{1}{2} \cdot \varepsilon_0 \cdot \frac{1}{\varepsilon_0 \mu_0} B_0^2$$

$$\therefore \quad u_E = \frac{1}{2\mu_0} B_0^2 = u_B$$

$\therefore$ for an EM wave, the electric & magnetic field have same energy density.

(c)

$\vec{S}$ is the flux of energy flow.

Now, energy is flowing at the speed of light (radiation)

$\therefore$ Energy flux over a length $= U_E \times \frac{c}{x}$

$x \rightarrow$ length

$$\therefore \quad \vec{S} = U_E \times \frac{c}{x} \cdot \hat{x}$$ (flux vector of EM radiation)

Intensity will then be $\frac{|\vec{S}|}{\text{Area}}$

$$\therefore \quad I = \frac{U_E \times c}{x \times A} = \frac{1}{2} \varepsilon_0 E_0^2 \times \frac{(A.x) \times c}{x \times A}$$

$$\therefore \quad \boxed{I = \frac{1}{2} \varepsilon_0 c E_0^2}$$

Q. 2.

E_1 ——— $|1\rangle$

$h\nu = E_1 - E_0$

E_0 ——— $|0\rangle$

$N_0 \rightarrow$ populatⁿ of ground state

$N_1 \rightarrow$ populatⁿ of excited state

$$\frac{dN_1}{dt} = -AN_1 + B_{01}\rho N_0 - B_{10}\rho N_1$$

Einstein postulated than in equilibrium the two level system behaves like a black body

i.e. $$\rho_{eq^m} = \frac{8\pi h}{c^3} \cdot \nu^3 \cdot \frac{1}{e^{\beta h\nu} - 1} \quad \text{———} \quad (a)$$

& $$\left(\frac{dN_1}{dt}\right)_{eq^m} = 0 \implies \rho_{eq} = \frac{AN_1}{B_{01}N_0 - B_{10}N_1}$$

$$\therefore \quad \rho_{eq} = \frac{A}{B_{10}} \cdot \frac{1}{\frac{B_{01}}{B_{10}} \cdot \frac{N_0}{N_1} - 1}$$

Now, from Boltzmann Statistics we know the fraction of each energy state

$$\frac{N_0}{N_0 + N_1} = \frac{g_0 e^{-\beta E_0}}{Z} \quad ; \quad Z = \sum_{i=1}^{2} g_i e^{-\beta E_i}$$

where g_i is the multiplicity/degeneracy of E_i

$$\therefore \quad \frac{N_0}{N_1} = \frac{g_0 e^{-\beta E_0}}{g_1 e^{-\beta E_1}} = \frac{g_0}{g_1} e^{\beta(E_1 - E_0)}$$

let $h\nu = E_1 - E_0$

$$\therefore \quad \frac{N_0}{N_1} = \frac{g_0}{g_1} e^{\beta h\nu}$$

$$\therefore \quad \rho_{eq^m} = \frac{A}{B_{10}} \cdot \frac{1}{\left(\frac{g_0 B_{01}}{g_1 B_{10}}\right) e^{\beta h\nu} - 1} \quad \text{———} \quad (b)$$

$\therefore$ Comparing eqn (a) & (b)

$$\frac{A}{B_{10}} = \frac{8\pi h}{c^3} \cdot \nu^3 \quad \& \quad \frac{g_0 B_{01}}{g_1 B_{10}} = 1$$

$$\Longrightarrow \boxed{B_{01} = \frac{g_1}{g_0} B_{10}}$$

Q. 3.

$$\frac{dN_1}{dt} = -AN_1 + (B_{01}\rho N_0 - B_{10}\rho N_1)$$

$$\therefore \frac{dN_1}{dt} = -AN_1 - B_{10}N_1\rho + B_{01}\rho(N - N_1)$$

$$= -AN_1 - (B_{10} + B_{01})\rho N_1 + B_{01}\rho N$$

$$\therefore \frac{dN_1}{dt} + [A + (B_{10} + B_{01})\rho] N_1 = B_{01}\rho N$$

Now, $f_1(t) = \frac{N_1(t)}{N}$

$$\therefore \frac{d}{dt} f_1(t) + [A + (B_{10} + B_{01})\rho] f_1(t) = B_{01}\rho$$

$$\therefore IF = e^{\int A + (B_{10} + B_{01})\rho \, dt} = e^{[A + (B_{10} + B_{01})\rho]t} = g(t)$$

$$B_{10} + B_{01} = B_{10} + \frac{g_1}{g_0} B_{10} = \left(\frac{g_1 + g_0}{g_0}\right) B_{10}$$

$$\therefore f_1(t) e^{(\ )} \Big|_0^t = \int_0^t \exp\left\{\left[A + \left(\frac{g_1 + g_0}{g_0}\right) B_{10}\rho\right] t\right\} B_{01}\rho \, dt$$

$$= \frac{e^{(\ )} - 1}{A + \left(\frac{g_1 + g_0}{g_0}\right) B_{10}\rho} \times B_{01}\rho$$

$$\therefore f_1(t) = f_1(0) + \frac{B_{01}\rho}{A + \left(\frac{g_1 + g_0}{g_0}\right) B_{10}\rho} \left\{1 - e^{-\left[A + \left(\frac{g_1 + g_0}{g_0}\right) B_{10}\rho\right] t}\right\}$$

Assume all molecules are in the ground state at $t = 0$ & assume no spontaneous emmission

$$\Longrightarrow f_1(0) = 0 \quad \& \quad A = 0$$

$$\therefore \quad f_1(t) = \frac{g_0}{g_1+g_0} \times \frac{B_{01}}{B_{10}} \times \left[1 - \exp\left\{-B_{10}\rho\left(\frac{g_1+g_0}{g_0}\right)t\right\}\right]$$

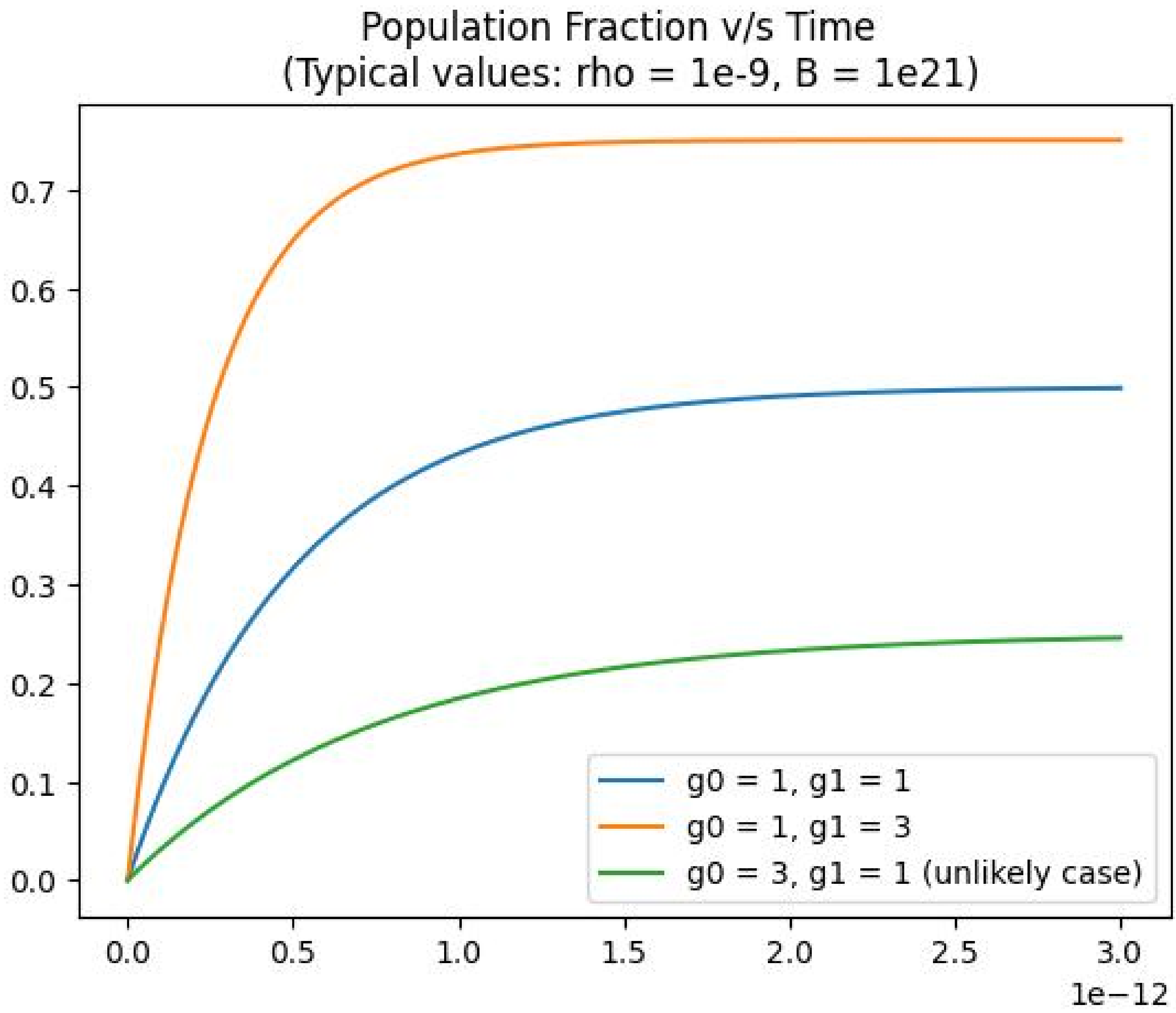
$$\because \quad \frac{B_{01}}{B_{10}} = \frac{g_1}{g_0} \quad ; \quad \frac{g_0}{g_1+g_0} \cdot \frac{B_{01}}{B_{10}} = \frac{g_1}{g_1+g_0}$$

$$\therefore \quad \boxed{f_1(t) = \frac{g_1}{g_1+g_0} \cdot \left[1 - \exp\left\{-B_{10}\rho\left(\frac{g_1+g_0}{g_0}\right)t\right\}\right]}$$

This formula is valid under the assumptions that;

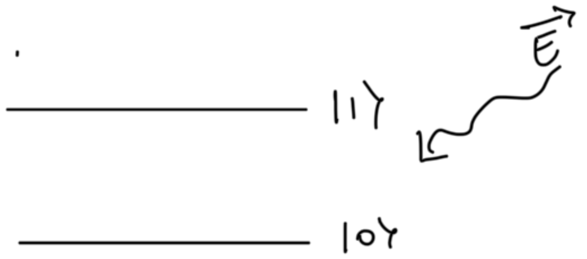
a) start with zero population of excited state
b) ignore spontaneous emmission effects

<plot attached below>



Population Fraction v/s Time
(Typical values: rho = 1e-9, B = 1e21)
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.0
0.5
1.0
1.5
2.0
2.5
3.0
1e−12
g0 = 1, g1 = 1
g0 = 1, g1 = 3
g0 = 3, g1 = 1 (unlikely case)

Q. 4.



$\langle t|1,t\rangle = \psi_1(t)$

$\langle t|0,t\rangle = \psi_0(t)$

$$\Psi(t) = a_0(t)\,\psi_0(t) + a_1(t)\,\psi_1(t)$$

$\{$Bra-Ket notat^n : $|\psi_t\rangle = a_0(t)\,|0,t\rangle + a_1(t)\,|1,t\rangle \}$

Then $|a_0(t)|^2 \rightarrow$ probability of system being in ground state
$|a_1(t)|^2 \rightarrow$ " " " " excited "

& $|a_0(t)|^2 + |a_1(t)|^2 = 1$

Electric field of incident radiation : $|\vec{E}| = \tilde{E}_0 \cos(\omega t)$

$\left(\frac{2\pi c}{\omega} = \lambda >> d \implies \vec{E}(\vec{r},t) \sim \vec{E}(t) \right)$

Then, $\hat{H} = H_0 + H_1$

$H_0 = E_0\,|0\rangle\langle 0| + E_1\,|1\rangle\langle 1|$

$H_1 = -\hat{\vec{\mu}} \cdot \vec{E} = -\hat{\mu}\,\vec{E}_0 \cos(\omega t)$

(Consider dipole type interact^n of light w/ the system)

To solve the Schrödinger eq^n $H\Psi(t) = i\hbar \frac{\partial}{\partial t}\Psi(t)$

$$\frac{\partial}{\partial t}\Psi(t) = \dot{a}_0(t)\,\psi_0(t) + a_0(t)\,\dot{\psi}_0(t) + \dot{a}_1(t)\,\psi_1(t) + a_1(t)\,\dot{\psi}_1(t)$$

Since, $|0\rangle$ & $|1\rangle$ are eigenstates, we expect the time evolution as follows

$$\psi_0(t) = \psi_0\, e^{-i\frac{E_0 t}{\hbar}} \; ; \; \psi_1(t) = \psi_1\, e^{-i\frac{E_1 t}{\hbar}}$$

$$\therefore \frac{\partial}{\partial t}\Psi(t) = \dot{a}_0(t) \cdot \psi_0 e^{-iE_0 t/\hbar} + a_0(t)\,\psi_0 \left(-i\frac{E_0}{\hbar}\right) e^{-i\frac{E_0 t}{\hbar}}$$
$$\dot{a}_1(t) \cdot \psi_1 e^{-iE_1 t/\hbar} + a_0(t)\,\psi_1 \left(-i\frac{E_1}{\hbar}\right) e^{-iE_1 t/\hbar}$$

$$\therefore\ i\hbar \frac{\partial}{\partial t}\Psi(t) = i\hbar\, \dot{a}_0(t)\, \Psi_0\, e^{-iE_0t/\hbar} + i\hbar\, \dot{a}_1(t)\, \Psi_0\, e^{-iE_1t/\hbar} + a_0(t)\cdot E_0\, \Psi_0\, e^{-iE_0t/\hbar} + a_1(t)\cdot E_1\, \Psi_0\, e^{-iE_1t/\hbar}$$

Now,

$$H_0\Psi(t) = a_0(t) \times E_0\, \Psi_0(t) + a_1(t) \times E_1\, \Psi_1(t)$$

& since $i\hbar \frac{\partial}{\partial t}\Psi(t) = H_0\Psi(t) + H_1\Psi(t)$

$$H_1\Psi(t) = i\hbar\, \dot{a}_0(t)\, \Psi_0\, e^{-i\frac{E_0t}{\hbar}} + i\hbar\, \dot{a}_1(t)\, \Psi_1\, e^{-iE_1t/\hbar}$$

Switching to Bra-Ket notation

$$\hat{H}_1|\Psi_t\rangle = i\hbar\, \dot{a}_0(t)\, e^{-i\frac{E_0t}{\hbar}}|0\rangle + i\hbar\, \dot{a}_1(t)\, \Psi_1\, e^{-iE_1t/\hbar}|1\rangle$$

Then,

$$e^{+iE_0t/\hbar}\langle 0|\hat{H}_1|\Psi_t\rangle = i\hbar\, \dot{a}_0(t)\langle 0|0\rangle + i\hbar\, \dot{a}_1(t)\, e^{-i\frac{(E_1-E_0)t}{\hbar}}\cancel{\langle 0|1\rangle}^{\,0}$$

where $|\alpha, t\rangle = e^{-i\frac{E_\alpha t}{\hbar}}|\alpha\rangle$

$$\therefore\ i\hbar\, \dot{a}_0(t) = -a_0(t)\langle 0|\hat{u}|0\rangle\, \tilde{E}_0\cos(\omega t) - a_1(t)\langle 0|\hat{u}|1\rangle\, e^{-i\frac{(E_1-E_0)t}{\hbar}}\, \tilde{E}_0\cos(\omega t)$$

$$\|^{ly}\quad i\hbar\, \dot{a}_1(t) = -a_0(t)\langle 1|\hat{u}|0\rangle\, e^{i\frac{(E_1-E_0)t}{\hbar}}\, \tilde{E}_0\cos(\omega t) - a_1(t)\langle 1|\hat{u}|1\rangle\, \tilde{E}_0\cos(\omega t)$$

Now,

$\hat{u}$ is a dipole operator

$$\therefore\ \langle 0|\hat{u}|0\rangle = \langle 1|\hat{u}|1\rangle = 0$$

$$\&\quad \langle 0|\hat{u}|1\rangle = \langle 1|\hat{u}|0\rangle^* =$$

for simplicity we take $\hat{u}_{\alpha\beta}$ to be real [& equal to u]
($\because$ we have taken $\vec{E}$ as real too ($\cos(\omega t)$))

$$\therefore\ \dot{a}_0(t) = i\, a_1(t) \times \left(\frac{u\tilde{E}_0}{\hbar}\right)\cos(\omega t)\, e^{-i\frac{(E_1-E_0)t}{\hbar}}$$

$$\dot{a}_1(t) = i\, a_0(t) \times \left(\frac{u\tilde{E}_0}{\hbar}\right)\cos(\omega t)\, e^{+i\frac{(E_1-E_0)t}{\hbar}}$$

Let $\frac{E_1-E_0}{\hbar} = \omega_{10}$ & $\cos(\omega t) = \frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)$

$$\therefore\ \dot{a}_0(t) = \frac{i}{2}\, a_1(t)\, \omega_R\, e^{i\Delta t} + \tilde{c}_1(t)\, e^{i(\omega+\omega_{10})t}$$

$$\dot{a}_1(t) = \frac{i}{2} a_0(t)\, \omega_R\, e^{-i\Delta t} + \tilde{c}_0(t)\, e^{i(\omega+\omega_{10})t}$$

where $\omega_R = \frac{\mu \tilde{E}_0}{\hbar}$ & $\Delta = \omega - \omega_{10}$

Note: $\omega + \omega_{10} \gg \Delta$ since we look at $\omega \to \omega_{10}$ for transition

$\therefore$ when we integrate the differential equ. the terms containing $e^{i(\omega+\omega_{10})t}$ vanish since they average out to zero over many time periods

$$\boxed{\begin{aligned} \therefore\ \dot{a}_0(t) &= \frac{i}{2} a_1(t)\, \omega_R\, e^{i\Delta t} \\ \dot{a}_1(t) &= \frac{i}{2} a_0(t)\, \omega_R\, e^{-i\Delta t} \end{aligned}}$$

(b)

Solving the two differential equations

$$\dot{a}_1(t) = \frac{i}{2} a_0(t)\, \omega_R\, e^{-i\Delta t}$$

$$\therefore\ \dot{a}_1(t)\, e^{i\Delta t} = \frac{i}{2} a_0(t)\, \omega_R$$

$$\therefore\ \ddot{a}_1(t)\, e^{i\Delta t} + i\Delta\, \dot{a}_1(t)\, e^{i\Delta t} = \frac{i\omega_R}{2} \dot{a}_0(t)$$

$$\therefore\ \ddot{a}_1(t)\, e^{i\Delta t} + i\Delta\, \dot{a}_1(t)\, e^{i\Delta t} = \frac{i\omega_R}{2} \cdot \frac{i}{2} a_1(t)\, \omega_R\, e^{i\Delta t}$$

$$\therefore\ \ddot{a}_1(t) + i\Delta\, \dot{a}_1(t) + \frac{\omega_R^2}{4} a_1(t) = 0$$

$||^{ly}$ $$\ddot{a}_0(t) - i\Delta\, \dot{a}_0(t) + \frac{\omega_R^2}{4} a_0(t) = 0$$

General sol^n: $a_0(t) = A e^{i\alpha t} + B e^{i\beta t}$

$$\therefore\ \dot{a}_0(t) = A i\alpha\, e^{i\alpha t} + B i\beta\, e^{i\beta t}$$

But $\dot{a}_0(t) = \frac{i\omega_R}{2} a_1(t)\, e^{i\Delta t}$

$$\Rightarrow \quad a_1(t) = \frac{2}{i\omega_R} \cdot e^{-i\Delta t}\left(A i\alpha e^{i\alpha t} + B i\beta e^{i\beta t}\right)$$

Substitute the general solⁿ into the 2nd Order D.E.

$$A(i\alpha)(i\alpha)e^{i\alpha t} + B(i\beta)^2 e^{i\beta t} - i\Delta\left(A i\alpha e^{i\alpha t} + B i\beta e^{i\beta t}\right)$$

$$+ \frac{\omega_R^2}{4}\left(Ae^{i\alpha t} + Be^{i\beta t}\right) = 0$$

$$\therefore \quad A\left(-\alpha^2 + \Delta\alpha + \frac{\omega_R^2}{4}\right)e^{i\alpha t} + B\left(-\beta^2 + \Delta\beta + \frac{\omega_R^2}{4}\right)e^{i\beta t} = 0$$

Clearly,

$$\alpha^2 - \Delta\alpha - \frac{\omega_R^2}{4} = 0$$

$$\& \quad \beta^2 - \Delta\beta - \frac{\omega_R^2}{4} = 0$$

(for arbitrary values of A & B)

They both give the same two solutions

$$\alpha = \frac{\Delta}{2} \pm \frac{\sqrt{\Delta^2 + \omega_R^2}}{2}$$

$$\therefore \quad \alpha = \frac{\Delta}{2} + \frac{\Omega}{2}, \quad \beta = \frac{\Delta}{2} - \frac{\Omega}{2}$$

$$\therefore \quad a_0(t) = A e^{\frac{i\Delta t}{2}} e^{\frac{i\Omega t}{2}} + B e^{\frac{i\Delta t}{2}} e^{-i\frac{\Omega}{2}t}$$

$$\therefore \quad a_1(t) = \frac{2}{\omega_R} e^{-\frac{i\Delta t}{2}}\left(A\alpha e^{i\frac{\Omega}{2}t} + B\beta e^{-i\frac{\Omega}{2}t}\right)$$

Let $a_0(0) = 1$ [ground state at $t = 0$]

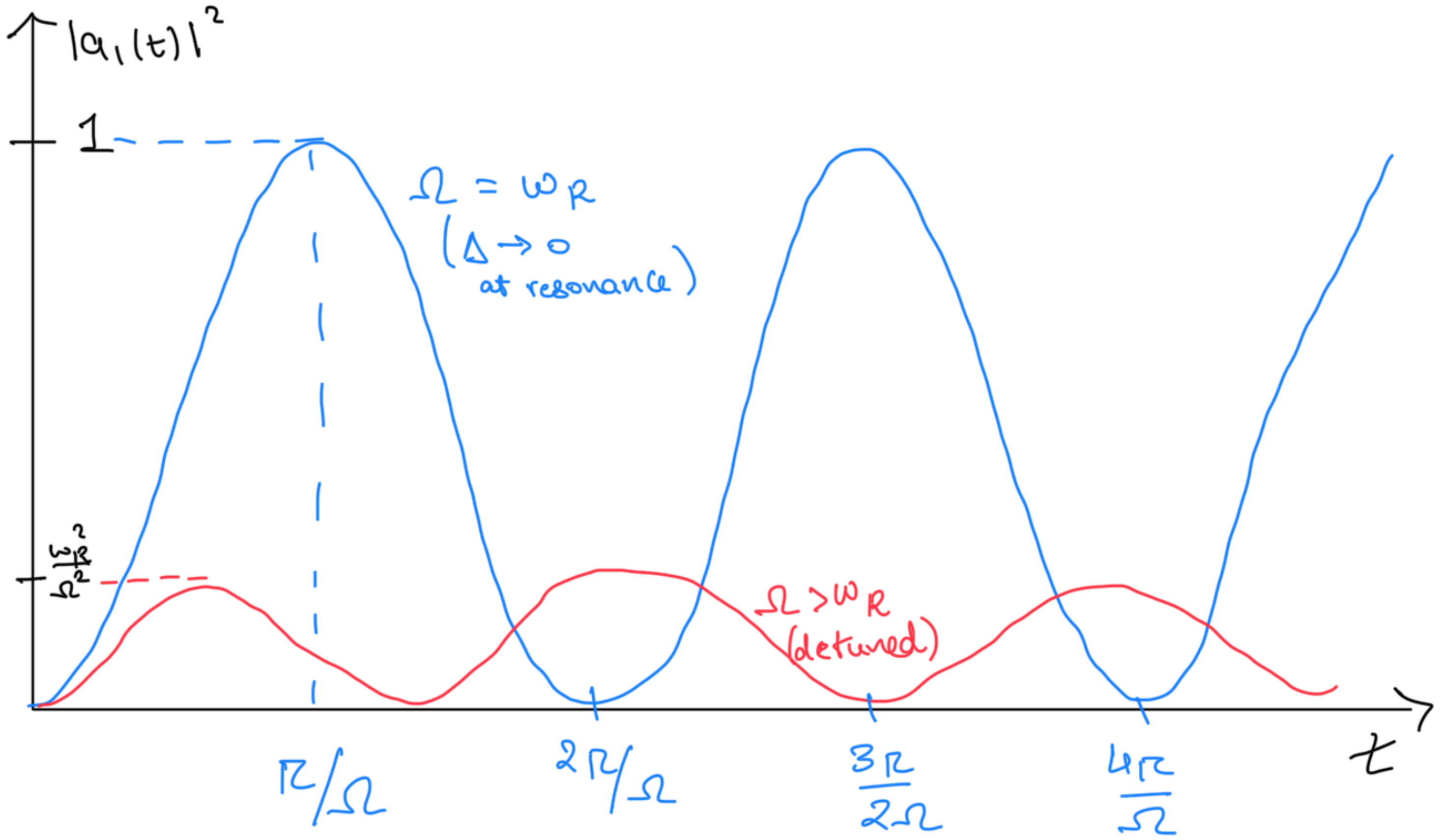
$$\therefore \quad 1 = A + B$$

Put $A = 1/2$ then $B = -1/2$

Then

$$a_1(t) = i\left(\frac{\omega_R}{\Omega}\right)\sin\left(\frac{\Omega t}{2}\right)e^{-i\Delta t/2}$$

$$\Rightarrow \quad |a_1(t)|^2 = \frac{\omega_R^2}{\Omega^2}\sin^2\left(\frac{\Omega t}{2}\right)$$



$|a_1(t)|^2$
1
$\frac{\omega_R^2}{\Omega^2}$
$\Omega = \omega_R$
($\Delta \to 0$
at resonance)
$\Omega > \omega_R$
(detuned)
π/Ω
$2\pi/\Omega$
$\frac{3\pi}{2\Omega}$
$\frac{4\pi}{\Omega}$
t

Q.5.

$\tau_{co} = 10\,ms$, $\omega_R = \dfrac{\mu \tilde{E}_0}{\hbar}$, $\mu_{co} = 0.12$ Debye

(a) Continuous Laser Beam

Area of the laser beam $= \dfrac{\pi d^2}{2}$

$\therefore\ I = \dfrac{P}{A}$ & $I = \dfrac{1}{2} c \epsilon_0 E_0^2$ (already derived)

$$\Rightarrow E_0 = \sqrt{\frac{2P}{c\epsilon_0 A}}$$

for 10 mW laser $E_0^1 = 3.1$ kV/m

for 1 W laser $E_0^2 = 31.0$ kV/m

Now

1 Debye $= 3.3356 \times 10^{-30}$ C-m

$\therefore\ \mu = 4.0028 \times 10^{-31}$ C-m

$\therefore$

for 10 mW laser

$$\omega_R = \frac{(4.0028 \times 10^{-31})(3.1 \times 10^3)}{1.05 \times 10^{-34}}$$

$\therefore\ \omega_R^1 = 1.18 \times 10^8$ rad/s

for 1 W laser

$\omega_R^2 = 1.18 \times 10^9$ rad/s

(b) Pulsed Laser Beam

Area $= \dfrac{\pi d^2}{2}$, $P = \dfrac{\text{Energy}}{\text{time}} = \dfrac{10\,mJ}{10\,ns} = 10^6$ W

$\therefore\ E_0 = 4.9 \times 10^7$ V/m

$\therefore \omega_R = 1.88 \times 10^{11}$ rad/s

(c) Broddening

$$\Gamma = \frac{1}{\tau} = \frac{1}{10 \times 10^{-3}} = 100 \text{ Hz}$$

Now,

$$\Delta E \, \Delta t \geq \frac{\hbar}{2}$$

if $\Delta t \sim \tau$ then $\Delta E \geq \frac{\Gamma \hbar}{2}$

$$\therefore \hbar \Delta\omega \geq \frac{\Gamma \hbar}{2} \Rightarrow \Delta\omega_{natural} = \frac{\Gamma}{2}$$

$$\therefore \Delta\omega_{natural} = 50 \text{ rad/s}$$

($\because$ spontaneous emission is not an issue as it completes many Rabi Oscillations before decaying)

Q.6.

$\Delta\nu \sim \tau^{-1}$, wavenumber $= \Delta\nu / c$ (cm^{-1})

- 10 fs pulse

$$\Delta\nu = 10^{14} \text{ Hz} = 10^{8} \text{ MHz}$$
$$\Delta\bar{\lambda} = 3.33 \times 10^{3} \text{ cm}^{-1}$$

- 10 ns pulse

$$\Delta\nu = 100 \text{ MHz}$$
$$\Delta\bar{\lambda} = 3.33 \times 10^{-3} \text{ cm}^{-1}$$

- 10 μs pulse

$$\Delta\nu = 0.1 \text{ MHz}$$
$$\Delta\bar{\lambda} = 3.33 \times 10^{-6} \text{ cm}^{-1}$$

Q. 7.

Maxwell's distribution of gas molecules with velocity in $v+dv$ is

$$g(v) \propto \exp\{-\beta \times K.E.\}$$

where $KE = \frac{1}{2}mv^2$, $\beta = \frac{1}{k_B T}$

$\therefore$ Normalizing, we get

$$g(v) = \left(\frac{\beta m}{2\pi}\right)^{1/2} \exp\left\{-\beta \frac{mv^2}{2}\right\}$$

Now,

Doppler shift is given by

$$\Delta\omega = \vec{k_0} \cdot \vec{v}$$

$$\therefore \quad \omega - \omega_0 = \frac{2\pi}{\lambda_0} \hat{r} \cdot \vec{v}$$

(for simplicity assume 1D i.e. polarized light)

$$\therefore 2\pi(\nu - \nu_0) = \frac{2\pi\nu_0}{c}(v)$$

$$\Rightarrow \nu = \nu_0\left(1 + \frac{v}{c}\right)$$

$\therefore$ The frequency distribution is now Maxwellian

$$\therefore \quad v = c\left(\frac{\nu - \nu_0}{\nu_0}\right)$$

$$\Rightarrow g(\nu) \propto \exp\left\{-\beta \frac{mc^2}{2}\left(\frac{\nu-\nu_0}{\nu_0}\right)^2\right\}$$

again, normalizing $\left[A = \left(\frac{\beta mc^2}{2\nu_0^2} \times \frac{1}{\pi}\right)^{1/2}\right.$

$$\therefore g(\nu) = \frac{1}{\nu_0}\left(\frac{\beta mc^2}{2\pi}\right)^{1/2} \exp\left\{-\beta \frac{mc^2(\nu-\nu_0)^2}{2\nu_0^2}\right\}$$

re-writing

$$g(\nu) = \frac{1}{\nu_0}\left(\frac{mc^2}{2\pi k_B T}\right)^{1/2} \exp\left\{-\frac{mc^2(\nu-\nu_0)^2}{2k_B T \nu_0^2}\right\}$$

a) To find the FWHM

$$g(\nu_1) = \frac{1}{2}\cdot g(\nu_0) \quad \& \quad g(\nu_2) = \frac{1}{2}\cdot g(\nu_0)$$

$$\text{FWHM} = |\nu_1 - \nu_2|$$

Let

$$g(\nu) = \frac{1}{\sqrt{2\pi}\cdot\sigma} \exp\left\{-\frac{(\nu-\nu_0)^2}{2\sigma^2}\right\}$$

where, $\sigma = \left(\frac{\nu_0^2}{\beta mc^2}\right)^{1/2}$

then,

$$g(\nu_0)\cdot\frac{1}{2} = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(\nu_1-\nu_0)^2}{2\sigma^2}\right\}$$

$$\therefore \frac{1}{\sqrt{2\pi}\cdot\sigma}\cdot\frac{1}{2} = \frac{1}{\sqrt{2\pi}\cdot\sigma} \exp\left\{-\frac{(\nu_1-\nu_0)^2}{2\sigma^2}\right\}$$

$$\Rightarrow \ln\left(\frac{1}{2}\right) = -\frac{(\nu_1-\nu_0)^2}{2\sigma^2}$$

$$\therefore (\nu_1-\nu_0)^2 = 2\sigma^2 \ln(2)$$

$$\therefore \nu_1 = \nu_0 \pm \sigma\sqrt{2\ln(2)}$$

$$\therefore \Delta\nu = 2\sigma\sqrt{2\ln(2)}$$

$$\therefore \text{FWHM} = 2\nu_0\left(\frac{k_B T \sqrt{2\ln 2}}{mc^2}\right)^{1/2}$$

Clearly, $\Delta\nu \propto \sqrt{T}$

(b)

$\lambda_0 = 589.1 \text{ nm}$, $\tau = 16 \text{ ns}$, $m_{Na} = 39 \times 10^{-27} \text{ kg}$

$$\therefore \quad \nu_0 = \frac{3 \times 10^8}{589.1 \times 10^{-9}} \text{ Hz} = 5.093 \times 10^{14} \text{ Hz}$$

$$\therefore \quad \Delta\nu = 10.186 \left(\frac{k_B T \sqrt{2\ln 2}}{m_{Na} \, c^2} \right)^{1/2} \times 10^{14} \text{ Hz}$$

$$= 2.403 \times 10^9 \text{ Hz} \quad \text{at} \quad 300 \text{ K}$$

$$= 4.387 \times 10^9 \text{ Hz} \quad \text{at} \quad 1000 \text{ K}$$

Natural linewidth: $\frac{\Gamma}{2} = \frac{1}{2\tau}$

$$\therefore \quad \Delta\nu_{nat} = \frac{1}{4\pi\tau} = 4.974 \times 10^6 \text{ Hz}$$

$\therefore$ Doppler broadening is 3 times more than natural brodening.

Q. 8.

Assume that the molecule interacts with the beam for a finite time (say τ)

Then the distribution is given as the Fourier Transform :

$$g(\omega) = \int_{-\tau/2}^{\tau/2} E_0 \cos(\omega_0 t) \, e^{-i\omega t} \, dt$$

$$\therefore \quad g(\omega) = \text{Re} \left\{ \frac{E_0}{2} \int_{-\tau/2}^{\tau/2} \left(e^{i\omega_0 t} + e^{-i\omega_0 t} \right) e^{-i\omega t} \, dt \right\}$$

$$= \text{Re} \left\{ \frac{E_0}{2} \int_{-\tau/2}^{\tau/2} e^{-i(\omega - \omega_0) t} \, dt \right\} + F(\omega + \omega_0)$$

we consider a near resonance case so $|\omega - \omega_0| \ll \omega_0$

& $\omega + \omega_0 \approx 2\omega_0$

Assume $2\omega_0$ to be large enough such than many oscillations are completed in the interval 'τ'

Then we can average out the second term to

$$\therefore \quad g(\omega) = \frac{E_0}{2} \, Re \left\{ \int_{-\tau/2}^{\tau/2} e^{-i(\omega-\omega_0)t} \, dt \right\}$$

$$= \frac{E_0}{2} \frac{\sin\left[(\omega-\omega_0)\tau/2 \right]}{(\omega-\omega_0)} \times 2$$

$$\therefore \quad g(\omega) = E_0 \frac{\sin\left[(\omega-\omega_0)\tau/2 \right]}{(\omega-\omega_0)}$$

Q. 9.

$$\frac{I}{I_0} = \exp\left\{ -\sigma (N_0 - N_1) L \right\}$$

$\sigma = 4 \times 10^{-22} \ cm^{-1}$, $L = 1m = 100 \ cm$

to calculate $\frac{N_1}{N_0} = e^{-\beta(E_1 - E_0)}$

$$\Delta E = h\nu = hc\bar{\lambda} \qquad (\bar{\lambda} = 5500 \ cm^{-1})$$
$$= 1.089 \times 10^{-19} \ J$$

$$\therefore \frac{N_1}{N_0} = 5.1 \times 10^{-12} \implies 1 - \frac{N_1}{N_0} \approx 1$$

Now, N_0 can be calculated from the molar fraction using the partial pressure

$$P_{NO} = x_{NO} P_{atm}$$

$$\therefore \quad x_{NO} = \frac{10^{-6} \times 10^{-3} \ Bar}{1 \ Bar}$$

$$\therefore \quad N_0 = 6.023 \times 10^{14} \qquad (\text{in } 1 \text{ mol.})$$

$$\therefore \quad \frac{I}{I_0} = \exp\left\{ -\sigma N_0 \left\{ 1 - \frac{N_1}{N_0} \right\} L \right\}$$

$$\sim \exp\left\{ -10^{-10} \right\} \sim 1$$

$$\therefore \quad I \approx I_0 \implies \text{almost no absorption}$$

To increase the absorption, we have to vary the parameters σ, ΔN, L

We can only change L. Thus, we have to find a way to increase L.

We can do this by constructing an Optical Parametric Amplifier type cavity which effictively increases the path length manifold.