Assignment - 2

Q.1.

$$\vec{E} = \vec{E_0}\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

Given: propagation along X-axis $\Rightarrow \vec{k} \parallel \hat{x}$
polarization along Y-axis $\Rightarrow \vec{E_0} \parallel \hat{y}$

$$\therefore \vec{E} = E_0\, e^{i(kx - \omega t)}\, \hat{y}$$

Now, $\vec{B} \perp \vec{E}$ & $\vec{B} \perp \vec{k}$

$$\therefore \vec{B_0} \parallel \hat{z}$$

$$\therefore \vec{B} = B_0\, e^{i(kx - \omega t)}\, \hat{z}$$

To check that they satisfy the wave equation

$$\frac{\partial^2 \vec{A}}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2}$$

$$\frac{\partial^2 \vec{E}}{\partial x^2} = E_0 \hat{y}\,(ik)\, \frac{\partial}{\partial x} e^{i(kx - \omega t)} = -E_0 k^2 e^{i(kx-\omega t)}\, \hat{}$$

$$= -k^2 \vec{E}\ \hat{y}$$

$$\frac{\partial^2 \vec{E}}{\partial t^2} = -\omega^2 \vec{E}\ \hat{y}$$

Now, for an EM wave in vacuum

$$k = \frac{\omega}{c}$$

Thus, $$\frac{\partial^2 \vec{E}}{\partial x^2} = -\frac{\omega^2}{c^2}\vec{E} = -\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}$$

Similarly, $$\frac{\partial^2 \vec{B}}{\partial x^2} = -k^2 \vec{B}\,\hat{z}\ ;\ \frac{\partial^2 \vec{B}}{\partial t^2} = -\omega^2 \vec{B}\,\hat{z}$$

a) To prove: $$\frac{\partial E_y}{\partial x} = \frac{\partial B_z}{\partial t}$$

Now, $\vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$ —— (1) (Maxwell's relation)

holds since both $\vec{E}$ & $\vec{B}$ satisfy the wave equations that are derived from Maxwell's relations

$$\vec{\nabla} \times \vec{E} = (\partial_x E_y - \partial_y E_x)\hat{z} + (\partial_z E_x - \partial_x E_z)\hat{y} + (\partial_y E_z - \partial_z E_y)\hat{x}$$

Now, $$\vec{E} = E_x \hat{x} + E_y \hat{y} + E_z \hat{z}$$
$$= E_0 e^{i(kx - \omega t)} \hat{y}$$

$$\Rightarrow E_x = E_z = 0$$

$$\therefore \vec{\nabla} \times \vec{E} = \partial_x E_y \hat{z} - \partial_z E_y \hat{x}$$

But $$\frac{\partial}{\partial z} E_y = \frac{\partial}{\partial z} E_0 e^{i(kx - \omega t)} = 0$$

$$\therefore \vec{\nabla} \times \vec{E} = \frac{\partial E_y}{\partial x} \hat{z} \quad \text{—— (2)}$$

Now, $$\frac{\partial \vec{B}}{\partial t} = \frac{\partial B_z}{\partial t} \hat{z} \quad \text{—— (3)} \quad (\because B_x, B_y = 0)$$

$\therefore$ from (1), (2) & (3) we have

$$\frac{\partial E_y}{\partial x} \hat{z} = -\frac{\partial B_z}{\partial t} \hat{z}$$

$$\Rightarrow \boxed{\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}}$$

Now, $$\frac{\partial E_y}{\partial x} = E_0 (ik) e^{i(kx - \omega t)} \hat{z}$$

& $$\frac{\partial B_z}{\partial t} = B_0 (-i\omega) e^{i(kx - \omega t)} \hat{z}$$

$$\Rightarrow E_0 (ik) = - B_0 (i\omega)$$

$$\Rightarrow \quad \frac{E_0}{B_0} = \frac{\omega}{k} = c$$

$$\therefore \quad E_0 = c B_0$$

$$\Rightarrow \boxed{|\vec{E}| = c\,|\vec{B}|}$$

b)

$$Q = CV \quad \& \quad I = \frac{\Phi}{L}$$

Now,

energy stored in a capacitor : $U_E = \frac{Q^2}{2C}$

energy stored in an inductor : $U_B = \frac{1}{2} L I^2$

$$\& \quad |\vec{E}| = \frac{dV}{dx} \quad ; \quad C = \frac{A \varepsilon_0}{x} \quad \text{(in vacuum)}$$

$\therefore$ for a capacitor $V = \frac{Q}{A \varepsilon_0} \cdot x$

$$\Rightarrow |\vec{E}| = \frac{Q}{A \varepsilon_0}$$

$$\therefore \quad U_E = \frac{A^2 \varepsilon_0^2 |\vec{E}|^2}{2 \cdot A \varepsilon_0 / x} \Rightarrow u_E = \frac{1}{2} \varepsilon_0 |\vec{E}|^2 \cdot \left(\frac{A x}{\text{volume}}\right)$$

Similarly,

$$\Phi = BA \quad , \quad L = \frac{\mu_0 N^2 A}{x} \quad \text{(put } N = 1\text{)}$$

$$\therefore \quad U_B = \frac{1}{2} \frac{(LI)^2}{L} = \frac{1}{2} \frac{B^2 A^2}{\mu_0 A} \cdot x$$

$$\therefore \quad u_B = \frac{1}{2\mu_0} \cdot B^2 \quad \frac{(A \cdot x)}{\text{volume}}$$

Thus,

$$u_E = \frac{1}{2} \varepsilon_0 |\vec{E}|^2 \quad \& \quad u_B = \frac{1}{2\mu_0} |\vec{B}|^2$$

Now, for an EM wave

$$u_E = \frac{1}{2}\varepsilon_0 |\vec{E}|^2 = \frac{1}{2}\varepsilon_0 E_0^2$$

$$u_B = \frac{1}{2\mu_0}|\vec{B}|^2 = \frac{1}{2\mu_0} \cdot B_0^2$$

& $$E_0 = c B_0 = \frac{1}{\sqrt{\varepsilon_0 \cdot \mu_0}} \cdot B_0$$

$$\therefore \quad u_E = \frac{1}{2} \cdot \varepsilon_0 \cdot \frac{1}{\varepsilon_0 \mu_0} B_0^2$$

$$\therefore \quad u_E = \frac{1}{2\mu_0} B_0^2 = u_B$$

$\therefore$ for an EM wave, the electric & magnetic field have same energy density.

(c)

$\vec{S}$ is the flux of energy flow.

Now, energy is flowing at the speed of light (radiation)

$\therefore$ Energy flux over a length $= U_E \times \frac{c}{x}$ $x \rightarrow$ length

$$\therefore \quad \vec{S} = U_E \times \frac{c}{x} \cdot \hat{x} \quad \text{(flux vector of EM radiation)}$$

Intensity will then be $\dfrac{|\vec{S}|}{\text{Area}}$

$$\therefore \quad I = \frac{U_E \times c}{x \times A} = \frac{1}{2}\varepsilon_0 E_0^2 \times \frac{(A \cdot x) \times c}{x \times A}$$

$$\therefore \quad \boxed{I = \frac{1}{2}\varepsilon_0 c E_0^2}$$

Q. 2.

E_1 ________ $|1\rangle$

$\updownarrow$ $h\nu = E_1 - E_0$

E_0 ________ $|0\rangle$

$N_0 \rightarrow$ populat^n of ground state

$N_1 \rightarrow$ populat^n of excited state

$$\frac{dN_1}{dt} = -AN_1 + B_{01}\rho N_0 - B_{10}\rho N_1$$

Einstein postulated than in equilibrium the two level system behaves like a black body

i.e. $$\rho_{eq^m} = \frac{8\pi h}{c^3} \cdot \nu^3 \cdot \frac{1}{e^{\beta h\nu} - 1} \quad \text{———} \quad (a)$$

& $$\left(\frac{dN_1}{dt}\right)_{eq^m} = 0 \implies \rho_{eq} = \frac{AN_1}{B_{01}N_0 - B_{10}N_1}$$

$$\therefore \quad \rho_{eq} = \frac{A}{B_{10}} \cdot \frac{1}{\frac{B_{01}}{B_{10}} \cdot \frac{N_0}{N_1} - 1}$$

Now, from Boltzmann Statistics we know the fraction of each energy state

$$\frac{N_0}{N_0 + N_1} = \frac{g_0 e^{-\beta E_0}}{Z} \quad ; \quad Z = \sum_{i=1}^{2} g_i e^{-\beta E_i}$$

where g_i is the multiplicity/degeneracy of E_i

$$\therefore \quad \frac{N_0}{N_1} = \frac{g_0 e^{-\beta E_0}}{g_1 e^{-\beta E_1}} = \frac{g_0}{g_1} e^{\beta(E_1 - E_0)}$$

let $h\nu = E_1 - E_0$

$$\therefore \quad \frac{N_0}{N_1} = \frac{g_0}{g_1} e^{\beta h\nu}$$

$$\therefore \quad \rho_{eq^m} = \frac{A}{B_{10}} \cdot \frac{1}{\left(\frac{g_0 B_{01}}{g_1 B_{10}}\right) e^{\beta h\nu} - 1} \quad \text{———} \quad (b)$$

$\therefore$ Comparing eqn (a) & (b)

$$\frac{A}{B_{10}} = \frac{8\pi h}{c^3} \cdot \nu^3 \quad \& \quad \frac{g_0 B_{01}}{g_1 B_{10}} = 1$$

$$\implies \boxed{B_{01} = \frac{g_1}{g_0} B_{10}}$$

Q. 3.

$$\frac{dN_1}{dt} = -AN_1 + (B_{01}\rho N_0 - B_{10}\rho N_1)$$

$$\therefore \frac{dN_1}{dt} = -AN_1 - B_{10}N_1\rho + B_{01}\rho(N - N_1)$$

$$= -AN_1 - (B_{10} + B_{01})\rho N_1 + B_{01}\rho N$$

$$\therefore \frac{dN_1}{dt} + [A + (B_{10} + B_{01})\rho] N_1 = B_{01}\rho N$$

Now, $f_1(t) = \frac{N_1(t)}{N}$

$$\therefore \frac{d}{dt} f_1(t) + [A + (B_{10} + B_{01})\rho] f_1(t) = B_{01}\rho$$

$$\therefore IF = e^{\int A + (B_{10} + B_{01})\rho\, dt} = e^{[A + (B_{10} + B_{01})\rho]t} = g(t)$$

$$B_{10} + B_{01} = B_{10} + \frac{g_1}{g_0} B_{10} = \left(\frac{g_1 + g_0}{g_0}\right) B_{10}$$

$$\therefore f_1(t)\, e^{(\ )} \Big|_0^t = \int_0^t \exp\left\{\left[A + \left(\frac{g_1 + g_0}{g_0}\right) B_{10}\rho\right] t\right\} B_{01}\rho\, dt$$

$$= \frac{e^{(\ )} - 1}{A + \left(\frac{g_1 + g_0}{g_0}\right) B_{10}\rho} \times B_{01}\rho$$

$$\therefore f_1(t) = f_1(0) + \frac{B_{01}\rho}{A + \left(\frac{g_1 + g_0}{g_0}\right) B_{10}\rho} \left\{1 - e^{-[A + (\frac{g_1 + g_0}{g_0}) B_{10}\rho]t}\right\}$$

Assume all molecules are in the ground state at $t = 0$ & assume no spontaneous emmission

$$\implies f_1(0) = 0 \quad \& \quad A = 0$$

$$\therefore \quad f_1(t) = \frac{g_0}{g_1+g_0} \times \frac{B_{01}}{B_{10}} \times \left[1 - \exp\left\{-B_{10}\rho\left(\frac{g_1+g_0}{g_0}\right)t\right\}\right]$$

$$\because \quad \frac{B_{01}}{B_{10}} = \frac{g_1}{g_0} \quad ; \quad \frac{g_0}{g_1+g_0}\cdot\frac{B_{01}}{B_{10}} = \frac{g_1}{g_1+g_0}$$

$$\therefore \quad \boxed{f_1(t) = \frac{g_1}{g_1+g_0}\cdot\left[1 - \exp\left\{-B_{10}\rho\left(\frac{g_1+g_0}{g_0}\right)t\right\}\right]}$$

This formula is valid under the assumptions that;

a) start with zero population of excited state
b) ignore spontaneous emmission effects

<plot attached below>