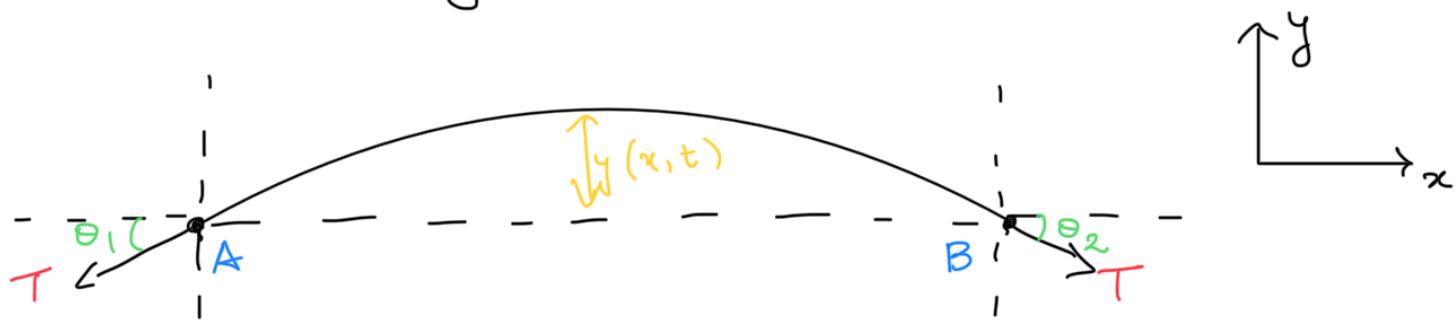
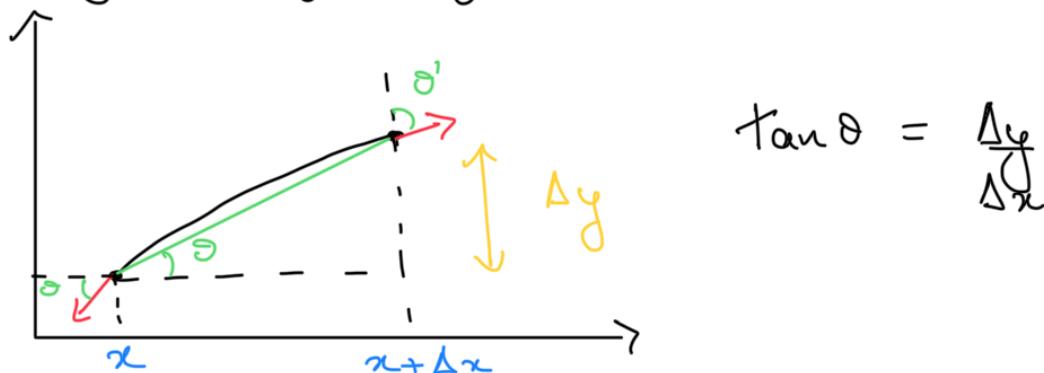


Q. 1

Consider a string as follows



To study the effect of a small displacement along Y



Now, the string has uniform mass density

$$\therefore \mu = \frac{\Delta m}{\Delta l} \quad \text{where } (\Delta l)^2 \approx (\Delta x)^2 + (\Delta y)^2$$

$$\therefore \Delta m = \mu \sqrt{(\Delta x)^2 + (\Delta y)^2} = \mu \Delta x \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}$$

Vertical forces: $T \sin \theta' - T \sin \theta$ Horizontal forces: $T \cos \theta' - T \cos \theta$ Since Δy is small, θ is also small

$$\therefore \sin \theta \approx \theta \approx \tan \theta \quad \& \quad \cos \theta \approx 1$$

$$\therefore F_y = T \tan \theta' - T \tan \theta \\ = T \left. \frac{dy}{dx} \right|_{x+\Delta x} - T \left. \frac{dy}{dx} \right|_x$$

$$F_x = T - T = 0$$

 \therefore Equating all the forces ($F = ma$)

$$T \left[\left. \frac{dy}{dx} \right|_{x+\Delta x} - \left. \frac{dy}{dx} \right|_x \right] = \Delta m \cdot \frac{d^2 y}{dt^2}$$

Now, for a small displacement $\Delta x \rightarrow dx$, $\Delta y \rightarrow dy$
 $\& \frac{\Delta y}{\Delta x}$ is small $\Rightarrow \left(\frac{dy}{dx}\right)^2$ is negligible

$$\therefore T \left[\frac{dy}{dx} \Big|_{x+dx} - \frac{dy}{dx} \Big|_x \right] = \mu dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \frac{d^2y}{dt^2}$$

$$\therefore \frac{\frac{\partial y}{\partial x}(x+dx, t) - \frac{\partial y}{\partial x}(x, t)}{dx} = \frac{\mu}{T} \frac{\partial^2 y}{\partial t^2}(x, t)$$

$$\therefore \frac{\partial^2 y}{\partial x^2}(x, t) = \frac{\mu}{T} \frac{\partial^2 y}{\partial t^2}(x, t)$$

$$\Rightarrow \frac{\partial^2 y}{\partial t^2}(x, t) = \frac{T}{\mu} \frac{\partial^2 y}{\partial x^2}(x, t)$$

(a)

$$y = A \sin(kx - \omega t)$$

$$L.H.S : \frac{\partial^2 y}{\partial t^2} = -A\omega^2 \sin(kx - \omega t)$$

$$R.H.S : \frac{I}{\mu} \frac{\partial^2 y}{\partial x^2} = - \frac{I}{\mu} A k^2 \sin(kx - \omega t)$$

$$\text{then } LHS = RHS \text{ if } \frac{\omega}{k} = \sqrt{\frac{I}{\mu}}$$

Clearly, $y = A \sin(kx - \omega t)$ is a solution

(c)

$$\frac{\omega}{k} = \sqrt{\frac{I}{\mu}} \Rightarrow \frac{2\pi v}{2\pi/\lambda} = \sqrt{\frac{I}{\mu}}$$

$$\Rightarrow v\lambda = \sqrt{\frac{I}{\mu}} \Rightarrow \text{speed of wave} = \sqrt{\frac{I}{\mu}}$$

$$\therefore \text{speed} \propto T^{1/2}$$

(d) For an electromagnetic wave

$$v = \frac{1}{\sqrt{\epsilon\mu}} = \frac{1}{\sqrt{\epsilon_0\mu_0} \cdot k} \quad (\mu \approx \mu_0)$$

Then, if $k \rightarrow 1$, $v \rightarrow c$

∴ In a dielectric medium, light travels slower than in vacuum.

Q. 2.

The classical picture of light matter interaction considers matter (atoms / molecules) as dipoles.

Then light is an oscillating electromagnetic field experienced by it.

$$\text{Now, } |\vec{E}| = c |\vec{B}| \Rightarrow \frac{|\vec{B}|}{|\vec{E}|} \sim 10^{-9} \text{ s/m}$$

∴ We only consider the effect of electric field

The forces acting are

$$F_{\text{ex}} = -Dx, F_{\text{dip}} = qE, F_{\text{damp}} = -b\dot{x}$$

Assumption: $\lambda > \gamma a$ (dipole length)

Then we can ignore the spatial variation of E

$$\therefore E = E_0 e^{i\omega t}$$

$$\therefore m\ddot{x} = -Dx + qE - b\dot{x}$$

$$\Rightarrow x(t) = x_0 e^{i\omega t}$$

$$\therefore m x_0 (-\omega^2) e^{i\omega t} = -Dx_0 e^{i\omega t} + qE_0 e^{i\omega t} - iwb e^{i\omega t} x_0$$

$$\Rightarrow x_0 = \frac{qE_0}{-m\omega^2 + D + iwb}$$

$$\therefore x_0 = -\frac{qE_0}{m} \cdot \frac{1}{(\omega^2 - \omega_0^2) - i\gamma\omega}$$

$$\text{where } D = \omega_0^2 m , \quad b = \gamma m$$

\therefore The induced dipole is

$$|\vec{P}| = q \left[\chi_0 e^{i\omega t} - \frac{q^2 E_0}{m} \cdot \frac{1}{\omega^2 - \omega_0^2 - i\gamma\omega} \right] e^{i\omega t}$$

$$\therefore |\vec{P}| = \frac{n}{V} |\vec{P}| \quad (\text{Polarizability})$$

$$\& \vec{P} \propto \vec{E}$$

$$\therefore |\vec{P}| = \epsilon_0 \chi |\vec{E}| \quad (|\vec{E}| = E_0 e^{i\omega t})$$

$$\Rightarrow \epsilon_0 \chi = \frac{q^2 N}{m} \cdot \frac{1}{\omega^2 - \omega_0^2 - i\gamma\omega}$$

$$\therefore \chi = \frac{q^2 N}{\epsilon_0 m} \cdot \frac{1}{\omega_0^2 - \omega^2 + i\gamma\omega}$$

$$= \frac{q^2 N}{\epsilon_0 m} \cdot \frac{(\omega_0^2 - \omega^2) - i\gamma\omega}{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}$$

$$\therefore \chi' = \frac{q^2 N}{\epsilon_0 m} \cdot \frac{\omega_0^2 - \omega^2}{[(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2]}$$

$$\chi'' = \frac{q^2 N}{\epsilon_0 m} \cdot \frac{\gamma\omega}{[(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2]}$$

Now,

$$n^2 = \epsilon \quad \& \quad \chi = \epsilon - 1$$

$$\Rightarrow n^2 = 1 + \chi$$

$$\therefore n = \sqrt{1+\chi} \sim 1 + \frac{1}{2}\chi$$

$$\therefore n'(\omega) = 1 + \frac{q^2 N}{2\epsilon_0 m} \cdot \frac{\omega_0^2 - \omega^2}{[(\omega^2 - \omega_0^2)^2 + (\gamma\omega)^2]}$$

$$\Im(\omega) = \frac{q^2 N}{2\epsilon_0 m} \cdot \frac{\gamma\omega}{[(\omega^2 - \omega_0^2)^2 + (\gamma\omega)^2]}$$

(b)

An electromagnetic wave inside a dielectric medium has the form $\vec{E}(r, t) = \vec{E}_0 e^{i(kr - \omega t)}$

$$\text{But } k = \frac{n(\omega) \cdot \omega}{c}; n(\omega) = n'(\omega) + i\kappa(\omega)$$

$$\therefore e^{ikr} \sim e^{in'(\omega) \frac{\omega r}{c}} \cdot e^{-\kappa(\omega) \frac{\omega r}{c}}$$

dispersion \hookleftarrow \hookrightarrow absorption

The dispersion term shows how the frequency of the EM wave affects its behaviour inside the dielectric material. There is no decay, but the oscillation frequency follows a dispersion.

The absorption term shows that if the wave travels over a large distance in the dielectric medium, it exponentially decays.

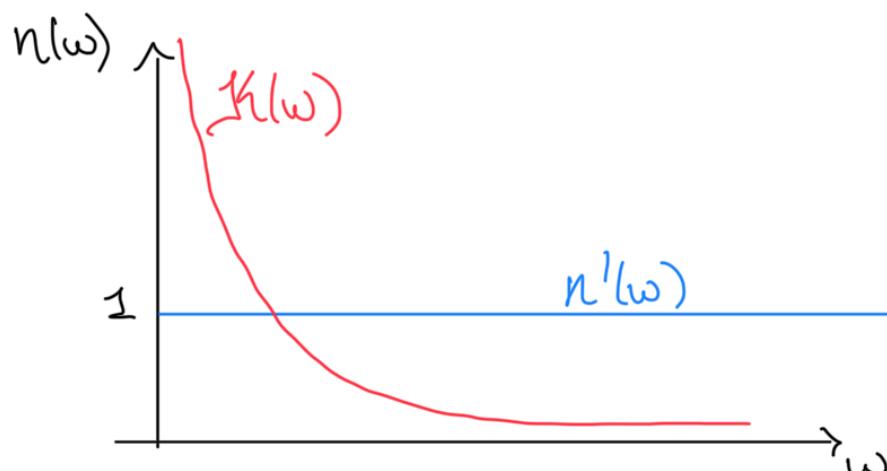
(c)

$$n'(\omega) = 1 + \omega_p^2 \cdot \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}$$

$$\kappa(\omega) = \omega_p^2 \cdot \frac{\gamma\omega}{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}$$

for $\omega \rightarrow \omega_0$ (s.t. $|\omega - \omega_0| \ll \omega_0$)

$$\begin{aligned} n'(\omega) &\rightarrow 1 \\ \kappa(\omega) &\rightarrow \frac{\omega_p^2}{\gamma\omega} \end{aligned}$$



(d) $n'(\omega)$ drops slightly below 1 as it approaches resonance.

The phase velocity thus appears to become greater than the speed of light. ($v_p = c/n'(\omega)$) However, since phase velocity is not the actual speed at which information is travelling, this is not troublesome.

The group velocity is given by

$$v_g = \frac{dw}{dk} = \frac{1}{\left(\frac{dk}{dw} \right)} = \frac{c}{n(\omega) + w \frac{dn'(\omega)}{dw}}$$

Even though $n'(\omega) < 1$, the dispersion takes care of it such that $v_g \leq c$.

One might argue that $\frac{dn'(\omega)}{dw} < 1$ close to

the resonance but this is not worrisome because in real systems there are more than one resonant frequencies. Thus, the dispersion is actually a summation. The additional terms then take care of the situation.

Q. 3.

Consider a cube of side length L . We want to find the no. of EM modes that can be possible inside it.

At the boundaries, the amplitude of these modes must vanish, so the wavelength is now an integer multiple of $2L$

$$\therefore \lambda = \frac{2L}{n}; n \in \{1, 2, \dots\}$$

But, we have to consider s & p-polarizations as two independent modes.

Consider a small interval v to $v+dv$

$$v = \frac{c}{\lambda} = \frac{nc}{2L}$$

The smallest interval on the cube is $\frac{c}{2L}$

\therefore Volume of this smallest cube is $\left(\frac{c}{2L}\right)^3$

Each point is shared by 8 small cubes, so to avoid extra counting, we divide

$$\therefore \text{no. of modes} = 2 \times \frac{1}{8} \times \text{small interval in } v \text{ to } v+dv \text{ in 3D}$$

The small volume element is $4\pi v^2 dv$ (spherical)

$$\therefore \text{density of these modes} = \frac{\frac{1}{8} \times 4\pi v^2 dv}{\left(\frac{c}{2L}\right)^3}$$

$$\therefore g(v) dv = \frac{8\pi V}{c^3} v^2 dv$$

$V = L^3$ (if we consider $\varepsilon = hv$, then this is energy density)

However, for a large ensemble we must consider the statistics at finite temperature

Assume T is large enough such that Boltzmann Statistics describe it well enough

then

$$f_v(T) dv = \frac{8\pi v^2}{c^3} \times V \times \frac{\bar{z}}{V}$$

where \bar{z} is the avg. energy of each mode/oscillator

$\therefore \frac{\bar{z}}{V}$ is the avg. energy density

$$\therefore f_v(T) dv = \frac{8\pi v^2}{c^3} \times \frac{hv}{e^{hv/k_B T} - 1}$$

(comes from the partition function)

$$\therefore f_v(T) = \frac{8\pi h}{c^3} \cdot \frac{v^3}{\exp\left\{\frac{hv}{k_B T}\right\} - 1}$$

Q. 4.

< code attached below >

Q. 5.

< code attached below >

from the plots, we see that λ_{max} corresponding to each temperature is too close to resolve using solid state detectors.

However, there is a significant difference in the peak intensity

We can design a detector such that it has a series of good conductors that absorb the radiation from the human body (at 310K & 311K)

Depending on how much heat they have absorbed (which is proportional to intensity of radiation), they will show a corresponding increase in temperature.

Thus, if we then connect thermocouples to them (two wires with different resistances, typically Nickel-Chrome & Chromium) we can measure the temperature as a function of voltage. Taking the average will give us the answer we require.

Then, we just have to calibrate this with known values of temperatures.

* We would have to hold the detector quite close to the body so as to avoid the difference caused by solid angle.

```
In [18]: from scipy.constants import pi, c, h, k
from math import e
import matplotlib.pyplot as plt
import numpy as np

#defining the prefactor of spectral energy density in SI units; Joules per metre

rho_0 = (8*pi*h)/(c**3) #value = 6.180644726185984e-58

#defining the constant in the exponent (h/k) in SI units: Kelvin per Hertz

d = h/k #value = 4.799243073366221e-11

#defining the spectral energy density function

def rho(T,v):
    return ((rho_0)*(v**3))*(1/((e**((d*v)/T))-1))
```

```
In [19]: X = np.arange(1,4e14,0.01e14) #frequency points
T1 = 100; T2 = 300; T3 = 600; T4 = 1000; T5 = 1200; T6 = 1500 #according to assi
T = [T1,T2,T3,T4,T5,T6] #storing the temp. values in a list

#plotting the spectral energy density at different temperatures

F = [] #will store the energy density values for each temp. (6 arrays)

for i in range(len(T)):
    f = rho(T[i],X)
    F.append(f.copy())

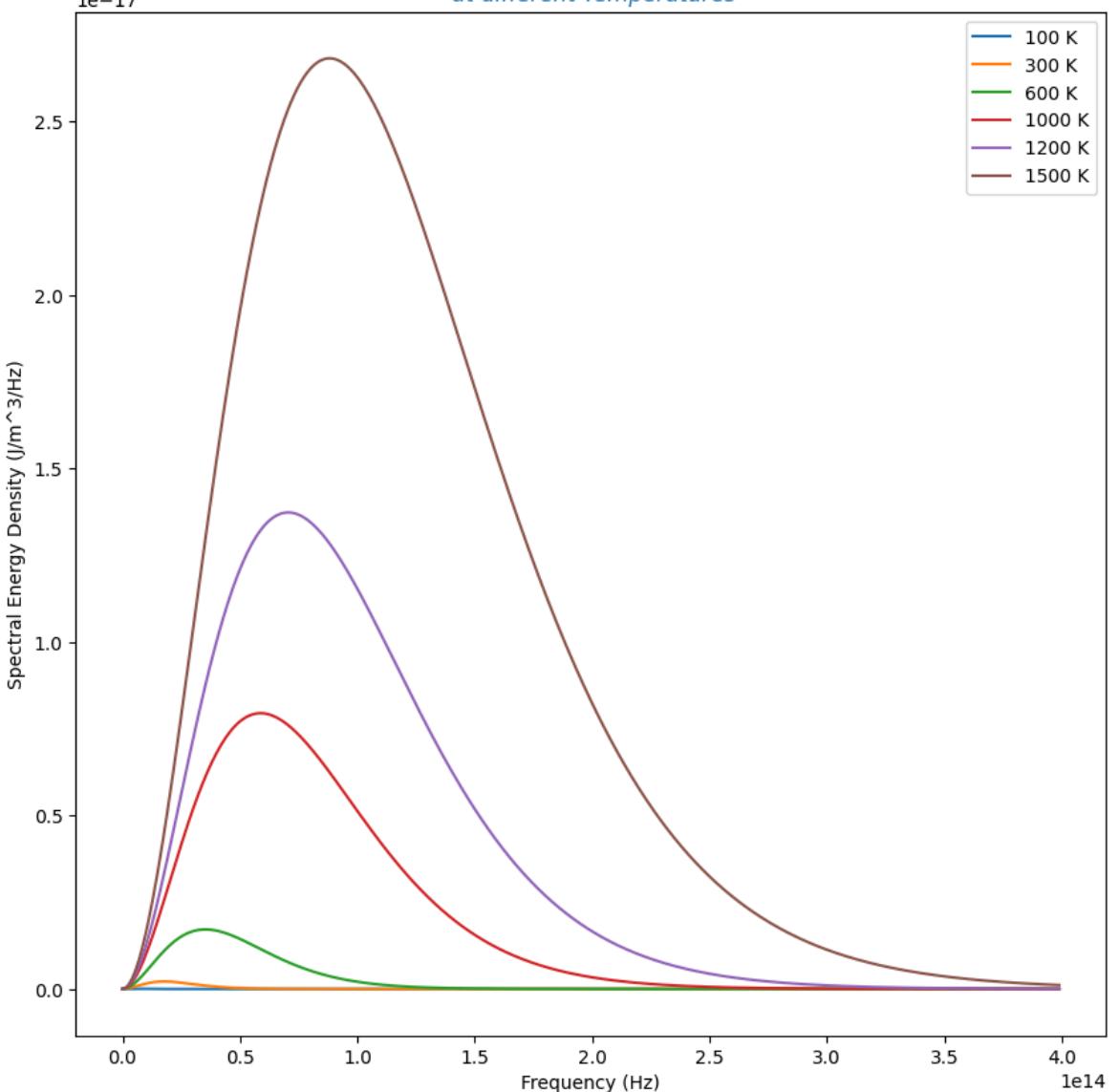
plt.figure(figsize=(10,10))

for i in range(len(T)): #adding each curve to the plot (6 in total)
    plt.plot(X,F[i])

plt.xlabel("Frequency (Hz)")
plt.ylabel("Spectral Energy Density (J/m^3/Hz)")
plt.title(label="Spectral Energy Density v/s Frequency \n at different Temperatu
plt.legend(["100 K","300 K","600 K","1000 K","1200 K","1500 K"],loc="upper right")

plt.show()
```

*Spectral Energy Density v/s Frequency
at different Temperatures*



```
In [20]: from scipy.integrate import quad #simple numerical integration

Rho = [] #storing the density 'functions' in a list for 6 temperatures (as per a

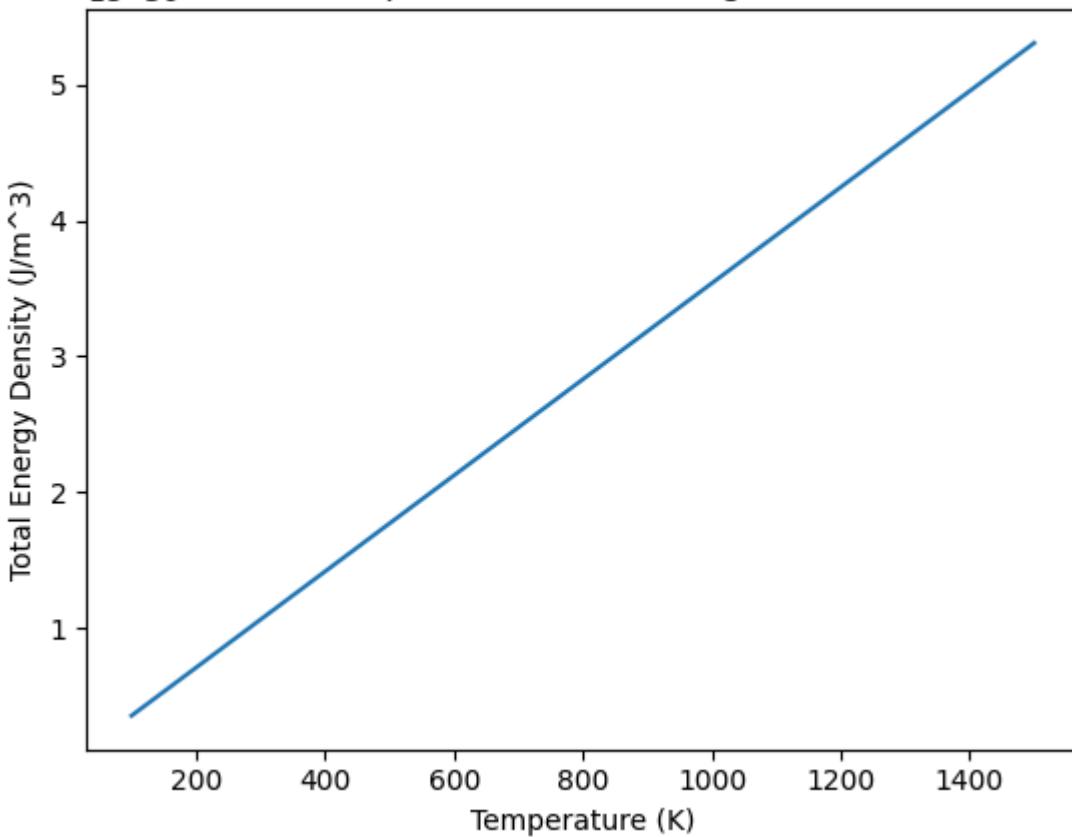
for i in range(len(T)):
    def Rho_T(v):
        return rho(T[i],v)
    Rho.append(Rho_T)

Tot_Dens = []

for i in range(len(T)):
    I, err = quad(Rho[i],0.01,np.inf)
    Tot_Dens.append(I)

plt.plot(T,Tot_Dens)
plt.xlabel("Temperature (K)")
plt.ylabel("Total Energy Density ( $\text{J}/\text{m}^3$ )")
plt.title("Should vary as  $T^4$  \n (incorrect plot - numerical integration error?)")
plt.show() #it gives linear behaviour which is wrong, we know it should vary as
```

Should vary as T^4
1e-36 (incorrect plot - numerical integration error?)



```
In [21]: Tn = 310; Tf = 311 #according to Q5 (n-normal, f-fever)

Xa = np.arange(0.1e12,0.7e14,0.01e14)
Yna = rho(Tn,Xa); Yfa = rho(Tf,Xa)

plt.subplot(1,2,1)
plt.plot(Xa,Yna)
plt.plot(Xa,Yfa)
plt.xlabel("Frequency (Hz)")
plt.ylabel("Spectral Energy Density (J/m^3/Hz)")
plt.legend(["310 K","311 K"])
plt.title(label="Full Plot",loc="right",fontstyle="italic")

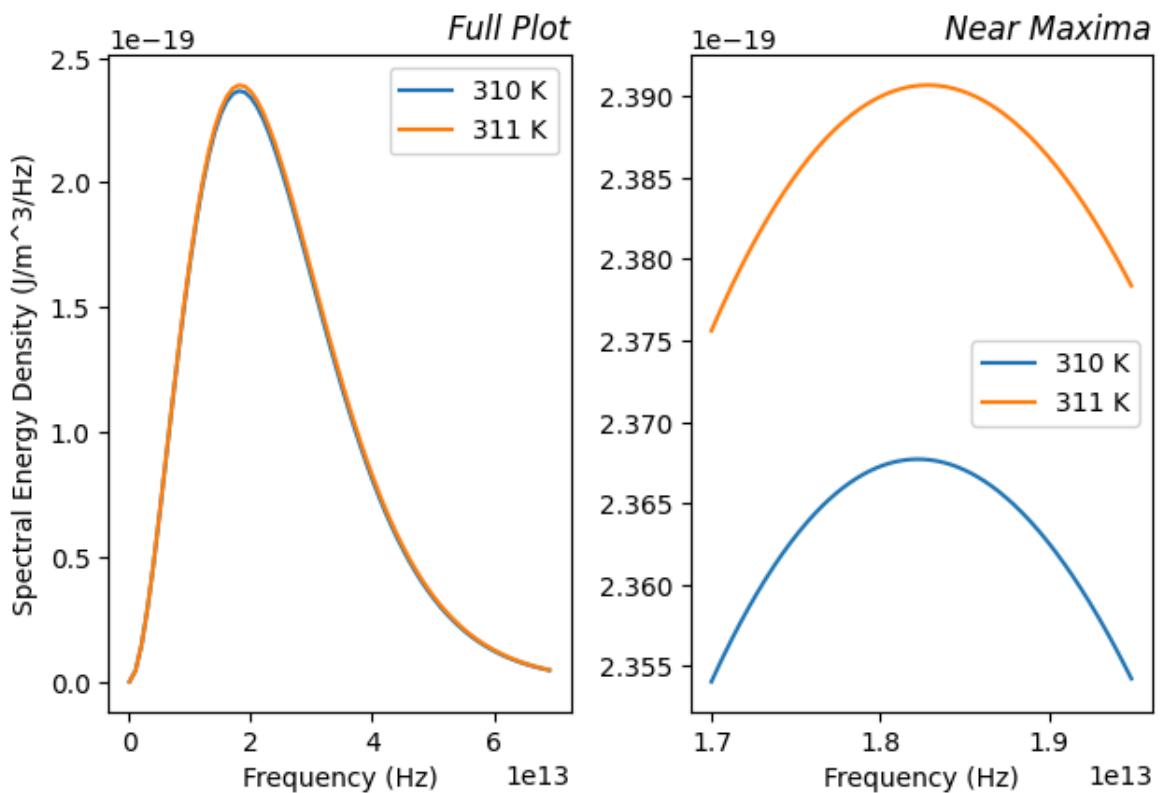
Xb = np.arange(1.7e13,1.95e13,0.01e12)
Ynb = rho(Tn,Xb); Yfb = rho(Tf,Xb)

plt.subplot(1,2,2)
plt.plot(Xb,Ynb)
plt.plot(Xb,Yfb)
plt.xlabel("Frequency (Hz)")
plt.legend(["310 K","311 K"])
plt.title(label="Near Maxima",loc="right",fontstyle="italic")

plt.suptitle(t="Spectral Energy Density v/s Frequency",fontstyle="italic",color="red")
plt.tight_layout()

plt.show()
```

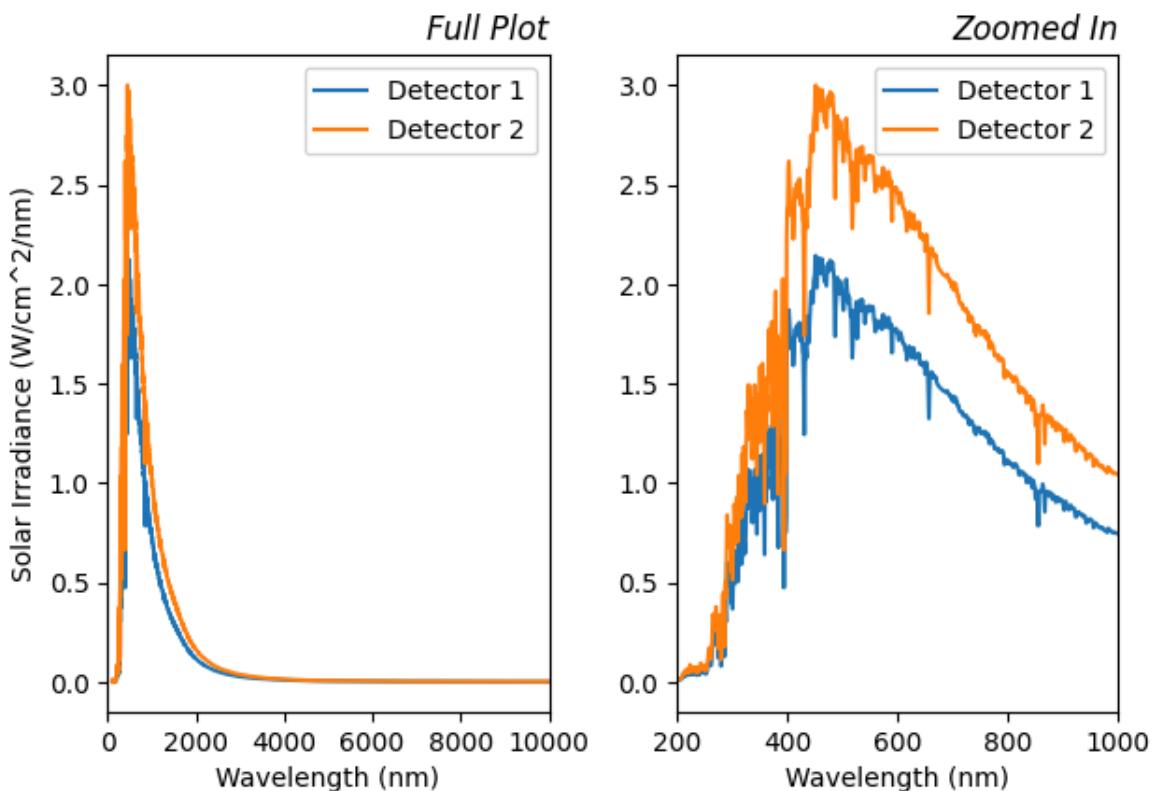
Spectral Energy Density v/s Frequency



```
In [159...]  
sol_spec = open('solar_spectrum_export.txt','r') #data file  
  
data = sol_spec.readlines() #reading each line of the file  
  
wl = []; det1 = []; det2 = [] #creating lists to store data  
  
for line in data:  
    list = line.split()  
    wl.append(float(list[0]))  
    det1.append(float(list[1]))  
    det2.append(float(list[2]))  
  
sol_spec.close()
```

```
In [160...]  
import matplotlib.pyplot as plt  
  
plt.subplot(1,2,1)  
plt.plot(wl,det1)  
plt.plot(wl,det2)  
plt.xlim(0,10000)  
plt.xlabel("Wavelength (nm)")  
plt.ylabel("Solar Irradiance (W/cm^2/nm)")  
plt.legend(["Detector 1","Detector 2"])  
plt.title(label="Full Plot",loc="right",fontstyle="italic")  
  
plt.subplot(1,2,2)  
plt.plot(wl,det1)  
plt.plot(wl,det2)  
plt.xlim(200,1000)  
plt.xlabel("Wavelength (nm)")  
plt.legend(["Detector 1","Detector 2"])  
plt.title(label="Zoomed In",loc="right",fontstyle="italic")  
  
plt.suptitle(t="Solar Irradiance v/s Wavelength",fontstyle="italic",color="tab:b  
plt.tight_layout()  
  
plt.show()
```

Solar Irradiance v/s Wavelength



```
In [161...]: from scipy.constants import c, h, k
from math import e

B0 = 2*h*(c**2); a0 = (h*c)/k #defining some constants

def B(T,l): #defining the solar irradiance as a function of temperature (T) & wavelength (l)
    return (B0/((l*1e-09)**5))*(1/((e**(a0/(l*T*1e-09))-1))*1e-13 #in nanometer
```

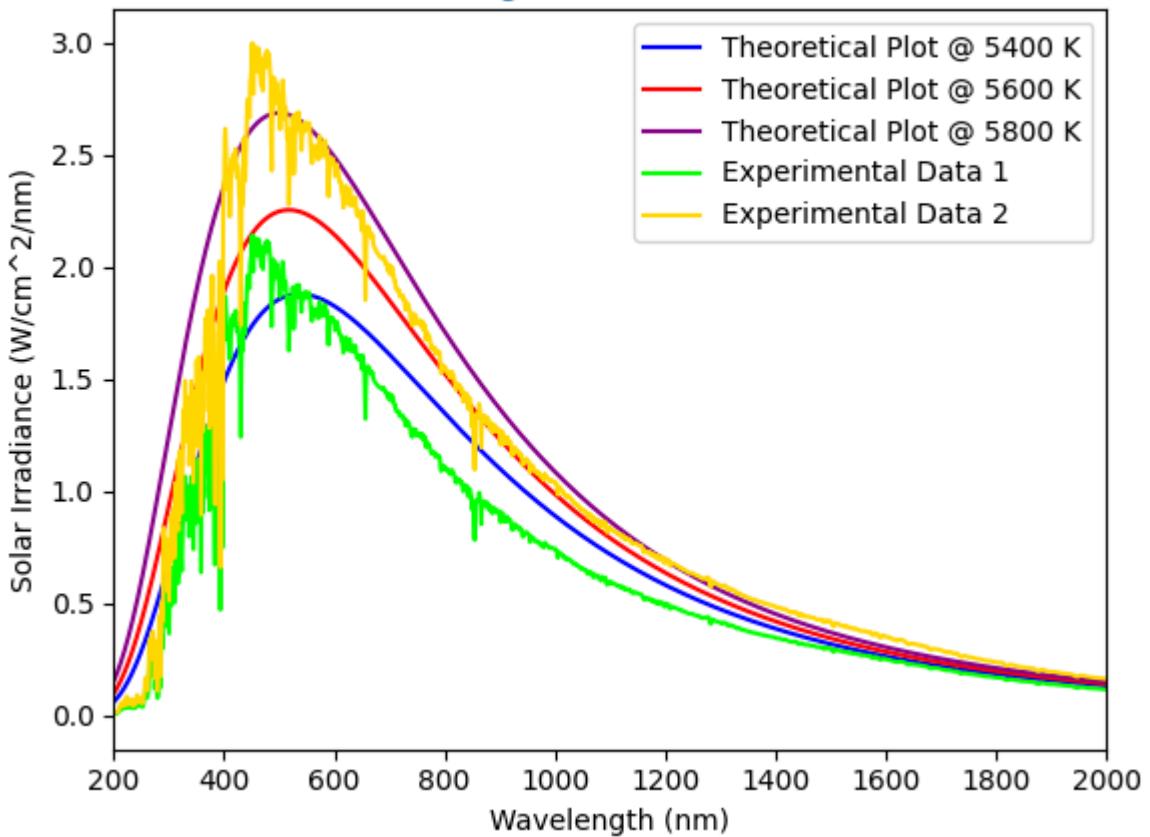
```
In [162...]: import numpy as np

X = np.arange(200,2000,0.1)
Y1 = B(5400,X)
Y2 = B(5600,X)
Y3 = B(5800,X)

plt.plot(X,Y1,color="blue")
plt.plot(X,Y2,color="red")
plt.plot(X,Y3,color="darkmagenta")
plt.plot(wl,det1,color="lime")
plt.plot(wl,det2,color="gold")
plt.xlim(200,2000)
plt.legend(["Theoretical Plot @ 5400 K","Theoretical Plot @ 5600 K","Theoretical Plot @ 5800 K","Experimental Data 1","Experimental Data 2"])
plt.xlabel("Wavelength (nm)")
plt.ylabel("Solar Irradiance (W/cm^2/nm)")
plt.title(label="Solar Irradiance v/s Wavelength \n (Fitting Planck's Formula)",

plt.show()
```

*Solar Irradiance v/s Wavelength
(Fitting Planck's Formula)*



Q. 6.

(Code attached below)

(a) The measurements follow the same behaviour but appear to be shifted along the Y-axis.

This indicates the presence of a systematic error in each measurement.

One reason could be due to misalignment.

The intensity / solar irradiance depends on the solid angle as well. We just take $\Omega = 4\pi$ for theoretical studies. However, any disparity in alignment would change the solid angle which would show up in the data.

(b) From the approximate fit (manual fit) it appears that the temperature of the sun is around 5,6700 K

Known value : 5,780 K

∴ We are right upto an error of 3%

(c)

These dips correspond to selective absorbance (which is NOT a characteristic of black bodies).

This means that on its route from the sun to the detector, the radiation encounters species that have discrete absorbance.

This could be molecules in the atmosphere, space dust etc.

These peaks are very sharp & could correspond to the discrete modes of gas molecules (rotational, vibrational, electronic etc.)