Indirect Regulation/Control/Herding

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Consider a tracking agent with dynamics governed by the model

$$\dot{y} = u, \tag{1}$$

where $y, \dot{y} \in \mathbb{R}^n$ denote the known position and velocity, $\ddot{y} \in \mathbb{R}^n$ denotes the unknown acceleration, and the function $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is unknown, representing modeling uncertainties. Consider a target agent with dynamics governed by the model

$$\dot{x} = g(x - y), \tag{2}$$

where $x \in \mathbb{R}^n$ denotes the known position , $\dot{x} \in \mathbb{R}^n$ denotes the velocity, and the constant g represents a known constant that will be replaced for a function in the future paper. We make the following assumptions on the dynamical system given by (1) and (2).

Assumption 1. The target agent is stable in the sense that there exist known constants $\overline{x}, \overline{\dot{x}} \in \mathbb{R}_{>0}$ such that $\|x(t)\| \leq \overline{x}$ and $\|\dot{x}(t)\| \leq \overline{\dot{x}}$, for all $t \in [0, \infty)$.

A. *Objective*

The primary objective is to develop a controller that enables the herding agent to regulate the target agent to some desired location $x_g : [0, \infty) \to \mathbb{R}^n$. To quantify the target regulation objective, we define the target regulation error $e_x : [0, \infty) \to \mathbb{R}^n$ as

$$e_x \triangleq x - x_g. \tag{3}$$

I. Control Synthesis

Since the regulation error dynamics in (3) do not explicitly contain a control input, a backstepping-based control strategy is developed to regulate the target agent to the desired trajectory, where the tracking agents state is used as a virtual control input. To facilitate the backstepping strategy, let $e_y : [0, \infty) \to \mathbb{R}^n$ denote the backstepping error, defined as

$$e_y \triangleq y_d - y, \tag{4}$$

where $y_d : [0, \infty) \to \mathbb{R}^n$ denotes the tracking agents desired trajectory.

Using (4), the derivative of (3) can be rewritten as

$$\dot{e}_x = g(x + e_y - y_d). \tag{5}$$

The virtual desired trjectory in (4) is defined as

$$y_d \triangleq x + \frac{1}{g}(k_1)\, e_x, \tag{6}$$

where $k_1 \in \mathbb{R}_{>0}$ is a user-selected gain. The derivative of (6) is

$$\dot{y}_d = \left(1 + \frac{k_1}{g}\right)(g e_y - k_1 e_x).$$

Using (6), (5) can be rewritten as

$$\dot{e}_x = g(e_y) - k_1 e_x. \tag{7}$$

Differentiating both sides of (4) and substituting (1) yields

$$\dot{e}_y = \left(1 + \frac{k_1}{g}\right)(g e_y - k_1 e_x) - u. \tag{8}$$

Based on the subsequent stability analysis the controller is designed as

$$u = k_2 e_y - \left(\frac{k_1^2}{g} + g\right) e_x, \tag{9}$$

where $k_2 \in \mathbb{R}_{>0}$ is a user selected gain. Your final gain conconditions will be

$$k_1 > 1, \tag{10}$$

$$k_2 > g + k_1. \tag{11}$$

The rest of the document just shows some math behind it, but I think you just need the dynamics [(1) & (2)], error systems [(3),(4),(7) & (8)], the virtual controlller/backstepping controller y_d [(6)], controller u [(9)], and the gain conditions [(10) & (11)].

II. Stability Analysis

Define a concatenated state vector $\nu \in \mathbb{R}^{\mathcal{N}}$ where $\nu \triangleq \begin{bmatrix} e_x^\top & e_y^\top \end{bmatrix}^\top$ and $\mathcal{N} \triangleq 2n$. Using (7) and (8) the closed-loop error system can be expressed as

$$\dot{\nu} = \begin{bmatrix} g e_y - k_1 e_x \\ \left(1 + \frac{k_1}{g}\right)(g e_y - k_1 e_x) - u \end{bmatrix}. \tag{12}$$

To facilitate the stability analysis, consider the Lyapunov candidate $V : \mathcal{N} \to \mathbb{R}_{\geq 0}$ defined as

$$V(z) \triangleq \frac{1}{2}\nu^\top \nu. \tag{13}$$

By the Rayleigh-Ritz theorem, (13) satisfies the inequality

$$\lambda_1 \|\nu\|^2 \leq V(\nu) \leq \lambda_2 \|\nu\|^2, \tag{14}$$

where $\lambda_1 \triangleq \frac{1}{2}\min\{1, 1\}$ and $\lambda_2 \triangleq \frac{1}{2}\max\{1, 1\}$. To facilitate the subsequent development, the following assumption is made.

Theorem 1. *Consider the dynamical systems described by (1) and (2). For any initial conditions of the states $\nu(t_0)$, the controller given by (9), ensures that ν exponentially converges in the sense that*

$$\|\nu(t)\| \leq \sqrt{\frac{\lambda_2}{\lambda_1}}\sqrt{\|z(t_0)\|^2 \,\mathrm{e}^{-\frac{\lambda_3}{\lambda_2}(t - t_0)}} \tag{15}$$

$\forall t \in [t_0, \infty)$, provided that the control gains k_2, and k_1 are selected such that $\lambda_3 \triangleq \min\{k_2 - k_1 - 1, k_1 - 1\} > 0$.

Proof: Substituting (12) into the time-derivative of (13) yields

$$\dot{V} = e_x^\top \left(ge_y - k_1 e_x\right) + e_y^\top \left(\left(1 + \frac{k_1}{g}\right)\left(ge_y - k_1 e_x\right) - u\right) \tag{16}$$

Substituting (9) into (16) yields

$$\dot{V} = e_x^\top \left(-k_1\right) e_x + e_x^\top g e_y + e_y^\top \left(g + k_1 - k_2\right) e_y + e_y^\top \left(-k_1 - \frac{k_1^2}{g}\right) e_x - e_y^\top \left(-k_1 - \frac{k_1^2}{g} + g\right) e_x. \tag{17}$$

Simplfying and upper bounding (17) yields

$$\dot{V} = -k_1 \left\|e_x\right\|^2 - \left(k_2 - g - k_1\right)\left\|e_y\right\|^2. \tag{18}$$

Applying the gain conditions for k_2, and k_1, using the definition of λ_3 and $\|\nu\|^2$, yields that (18) can be upper bounded as

$$\dot{V} \leq -\frac{\lambda_3}{\lambda_2} V(\nu). \tag{19}$$

Solving the differential inequality in (19) yields

$$V\left(z(t)\right) \leq V\left(z(t_0)\right) \mathrm{e}^{-\frac{\lambda_3}{\lambda_2}(t - t_0)}. \tag{20}$$

Since $e_x, e_y \in \mathcal{L}_\infty$, using (9) yields that $u \in \mathcal{L}_\infty$. ■