3. Localization and Mapping + SLAM

Table of Contents

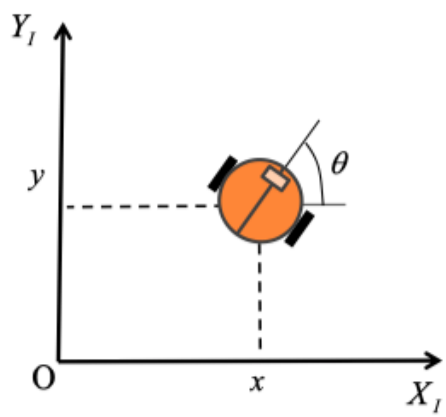
Localization

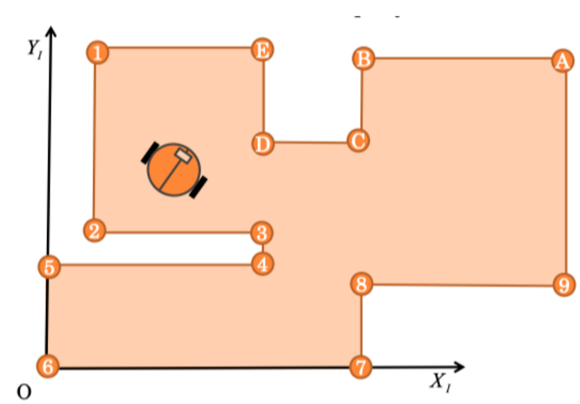
Formulation

- Localization:
  - using sensor info. to locate the vehicle in a known environment
- Formulation:
  - Given:
    - Control inputs and motion model
    - Sensor measurements and measurement model relative to environment
    - Environment model
  - Find:
    - Vehicle position
  - Problems:
    - I.C.
      - Local: Known initial position
        - Tracking position through motions with inputs and measurements
      - Global: Unknown initial positions
        - Finding position and then continuing to track
      - Kidnapped: Incorrect initial position

                - Correcting incorrect prior beliefs to recover true position and motion
        - Assumptions:
            - Known static env.
                - No moving obstacles, or other vehicles that cannot be removed from sensor measurements
            - Passive Estimation
                - Control law does not seek to minimize estimation error
            - Single Vehicle:
                - Only one measurement location is available

Feature-based Localization

- Feature-based localization
    - Most natural formulation of localization problem
        - Sensor measure bearing, range, relative position of features
        - Location based maps can be reduced to a set of measurable features
        - The more features tracked the better the solution
            - But the larger the matrix inverse at each timestep
- **Ex: Two-wheeled robot**
    - 



    - **Vehicle State, Inputs:**

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix} \quad \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} v \\ \omega \end{bmatrix}$$

    - **Motion Model:**

$$\begin{bmatrix} x_{1,t} \\ x_{2,t} \\ x_{3,t} \end{bmatrix} = g(x_{t-1}, u_t) = \begin{bmatrix} x_{1,t} + u_{1,t} \cos x_{3,t-1} dt \\ x_{2,t} + u_{1,t} \sin x_{3,t-1} dt \\ x_{3,t} + u_{2,t} dt \end{bmatrix}$$

    - **Feature Map:**

$m = \{m^1, \ldots, m^M\}. m^i = \{m^i_x, m^i_y\}$

- Assume all features are uniquely identifiable
- **Measurement Model:**
- Relative range and/or bearing to closest feature m^i, regardless of heading
- Assume measurement of closest feature only

$$\begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} = h(x_t) = \begin{bmatrix} \tan^{-1} \frac{m^i_y - x_{2,t}}{m^i_x - x_{1,t}} - x_{3,t} \\ \sqrt{(m^i_x - x_{1,t})^2 + (m^i_y - x_{2,t})^2} \end{bmatrix} \begin{matrix} \leftarrow \text{Bearing} \\ \leftarrow \text{Range} \end{matrix}$$

-
- **Two Approaches:**
  - 1) **EKF (UKF)** based localization:**
    - Fast computationally
    - Intuitive forumlation
    - Most frequently implemented
    - Possibility for **divergence** if nonlinearities are severe
    - Additive Gaussian noise:
      - $\epsilon_t \sim \mathcal{N}(\mu = 0,\ \sigma^2 = R_t)$ and $\delta_t \sim \mathcal{N}(\mu = 0,\ \sigma^2 = Q_t)$
  - 2) **Particle** Filter based localization:
    - Slightly cooler visualizations
    - More expensive computationally
    - More capable of handling extreme nonlinearities, constraints, discontinuities
- **EKF**:
  - Recall:
    - Prediction Update:

$$G_t = \frac{\partial}{\partial x_{t-1}} g(x_{t-1}, u_t)\Big|_{x_{t-1}=\mu_{t-1}}$$
$$\bar{\mu}_t = g(\mu_{t-1}, u_t)$$
$$\bar{\Sigma}_t = G_t \Sigma_{t-1} G_t^T + R_t$$

- Measurement Update:

$$
\begin{aligned}
H_t &= \frac{\partial}{\partial x_t} h(x_t)\Big|_{x_t=\mu_t} \\
K_t &= \bar{\Sigma}_t H_t^T (H_t \bar{\Sigma}_t H_t^T + Q_t)^{-1} \\
\mu_t &= \bar{\mu}_t + K_t (y_t - h(\bar{\mu}_t)) \\
\Sigma_t &= (1 - K_t H_t) \bar{\Sigma}_{t-1}
\end{aligned}
$$

- Linearization of Motion Model:

$$
\begin{bmatrix} x_{1,t} \\ x_{2,t} \\ x_{3,t} \end{bmatrix} = g(x_{t-1}, u_t) = \begin{bmatrix} x_{1,t} + u_{1,t} \cos x_{3,t-1} dt \\ x_{2,t} + u_{1,t} \sin x_{3,t-1} dt \\ x_{3,t} + u_{2,t} dt \end{bmatrix}
$$

  - =>

$$
\frac{\partial}{\partial x_{t-1}} g(x_{t-1}, u_t) = \begin{bmatrix} 1 & 0 & -u_{1,t} \cos x_{3,t-1} dt \\ 0 & 1 & u_{1,t} \sin x_{3,t-1} dt \\ 0 & 0 & 1 \end{bmatrix}
$$

- Linearization of Measurement Model:

$$
\begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} = h(x_t) = \begin{bmatrix} \tan^{-1}\left(\frac{m_y^i - x_{2,t}}{m_x^i - x_{1,t}}\right) - x_{3,t} \\ \sqrt{(m_x^i - x_{1,t})^2 + (m_y^i - x_{2,t})^2} \end{bmatrix} \quad \begin{matrix} \leftarrow \text{Bearing} \\ \leftarrow \text{Range} \end{matrix}
$$

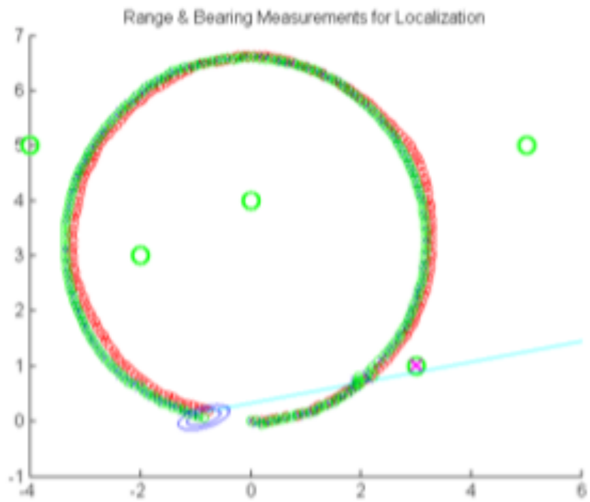
  - =>

$$
\frac{\partial}{\partial x_t} h(x_t) = \begin{bmatrix} \frac{(m_y^i - x_{2,t})}{q} & -\frac{(m_x^i - x_{1,t})}{q} & -1 \\ -\frac{(m_x^i - x_{1,t})}{\sqrt{q}} & -\frac{(m_y^i - x_{2,t})}{\sqrt{q}} & 0 \end{bmatrix}
$$

Where: $q = (m_x^i - x_{1,t})^2 + (m_y^i - x_{2,t})^2$

- **SIMULATION RESULT:**
  - Five features in a 2D world

- No confusion over which is which with correct correspondence
- Two wheeled robot (x, y, θ)
- Measurement to feature of Range, Bearing, both



- **Findings:**

```
- Moderate noise:
  - both measurements noisy, correct prior, large distubances
- Range only:
  - Elongate covariance erro elipse
  -
- Bearing only:
  - No idea how deep we are
- Bearing only, incorrect prior (Kidnapped):
  - till first feature, it correct the path
  - incorrect heading but consistent pathing
```

Mapping

- Types:
  - **Location based: Occupancy Grid**

$$m = \begin{bmatrix} m^1 & \dots & m^N \\ \vdots & \ddots & \vdots \\ m^{M-N+1} & \dots & m^M \end{bmatrix}$$

    - Can be probablisitic in formulation with $m^i \in [0, 1]$
    - Scales poorly, but works well in 2D (Plannar Position)
  - **Feature based: Set of all features**
    - A feature is defined at a specific location, and may have a signature: $m^i = \{x^i, y^i, s^i\}$

$$M_n = \{m^1, \ldots, m^M\}$$

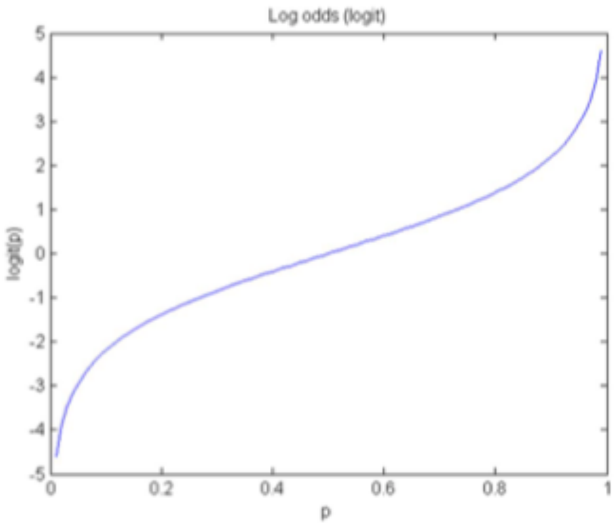
- Effective for localization
- Scales well to larger dimensions
- Hard to use for collision avoidance

Formulation

- Mapping:
  - Using sensor införmation known vehicle locations to define a map of the env.
- Given:
  - Vehicle location model
  - Sensor measurements and inverse measurement model
- Find:
  - Environment Map

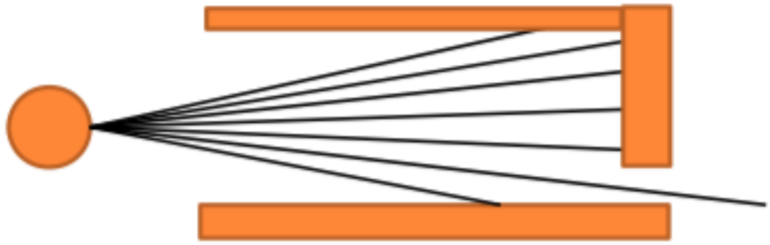
Occupancy Grid Mapping

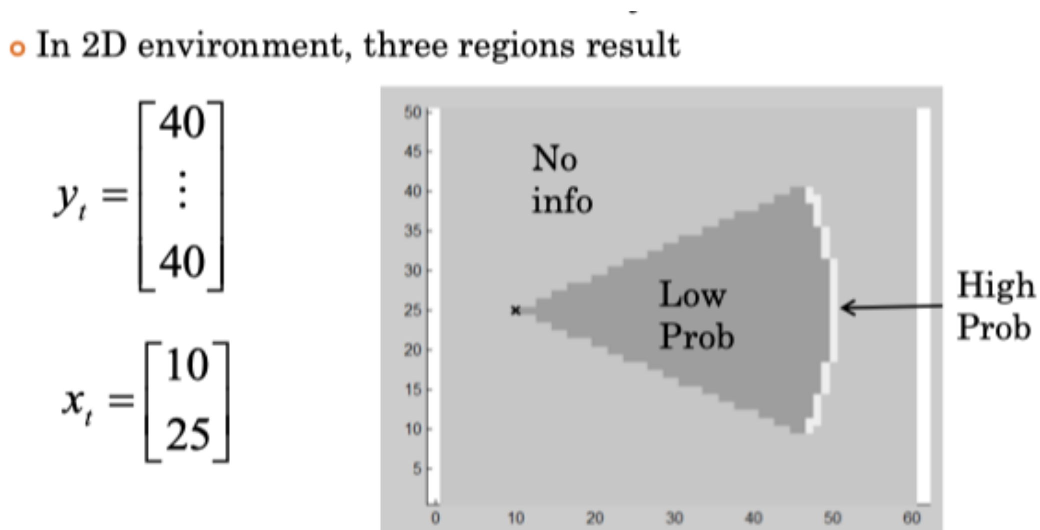
- Find probability at time t that each grid cell contains an obracle
  - $bel_t(m^i) = p(m^i | y_{1:t}, x_{1:t})$
  - Subscript t moved to emphasize that features are **static**
- **Assumptions:**
  - Static env.
  - Independence of cells
  - Known vehicle state at each time step
  - Sensor model is known
- [Recall Discrete **Bayes Filter Algorithm**]
  - Prediction update (Discrete Total probability)
    - $\bar{bel}(x_t) = \sum p(x_t | u_t, x_{t-1}) \, bel(x_{t-1})$
  - Measurement udpate (Bayes Theorem)
    - $bel(x_t) = \eta \, p(y_t | x_t) \, \bar{bel}(x_t)$
    - η is a normalizing constant that does not depend on the state (will become apparent in derivation)
- => **Bayes Filter** with **static** states
  - Since the cell contents do not move, the motion model is trivial
    - The predicted belief is simply the belief from the previous time step
      - $\bar{bel}_t(m) = bel_{t-1}(m)$
    - The prediction step is no longer needed, so we update with each new measurement regardless of vehilce motion
      - $bel_t(m) = \eta \, p(y_t | m) \, \bar{bel}_{t-1}(m)$
- **Log Odds Raio** ($\equiv$ **Logistic Regression** $\equiv$ **Logit Function**)
  - For easy computation (!= 0)
  - Instead of tracking the probability, we track the log odds ratio for each cell

- 

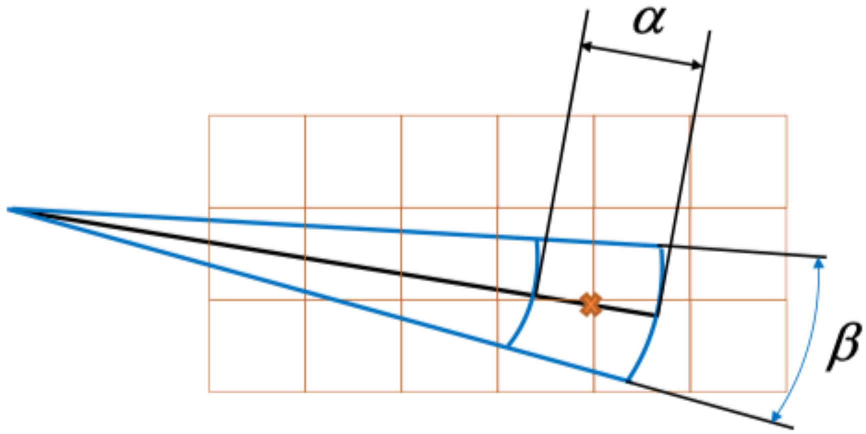
  $$\text{logit}(p) = \log\left(\frac{p}{1-p}\right)$$
- $logit(p) = log(\frac{p}{1-p})$
- We get simple addition instead of multiplication, but downside: we need to recompute models in logit space

- **=> Bayesion Log Odds Update**
  - Derivation:
    - For each cell, we have a measurement update (with the normalizer defined explicitly)
      - $p(m^i|y_{1:t}) = \frac{p(y_t|y_{1:t-1}, m^i)p(m^i|y_{1:t-1})}{p(y_t|y_{1:t-1})}$
    - We still trust in the Markov assumption:
      - $p(m^i|y_{1:t}) = \frac{p(y_t|m^i)p(m^i|y_{1:t-1})}{p(y_t|y_{1:t-1})}$
    - apply Bayes rule to measurement model:
      - $p(y_t|m^i) = \frac{p(m^i|y_t)\, p(y_t)}{p(m^i)}$
    - .... [Look at slides]
  - Shorhand of update rule:
    - $l_{t,i} = logit(p(m^i|y_t)) + l_{t-1,i} - l_{0,i}$
    - The log odd ratio at t is the sum of the ratio at $t-1+$ the inverse measurement ratio $-$the initial belief
    - To get the inverse measurement ratio, we need an inverse measurement model
      - Probability of a state given a certain measurement occurs
        - $p(m^i|y_t)$
      - Inverse conditional probability of the measurement models used to date
        - $p(y_t|m^i)$
- **Examples: Laser Scanner**
  -

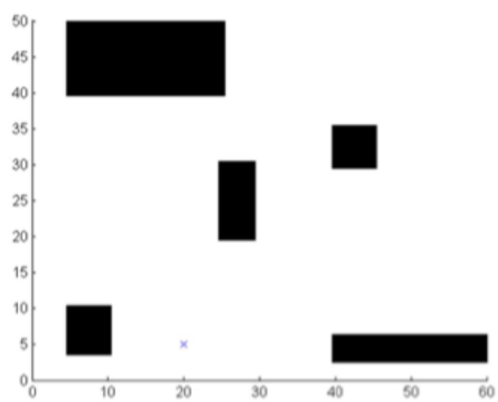


- Returns a range to the closest objects at a set of bearings relative to the vehicle heading
    - Scanner Bearings
        - $\phi^s = [-\phi^s_{max} \; \ldots \; \phi^s_{max}] \qquad \phi^s_j \in \phi^s$
    - Scanner Ranges
        - $r^s = [-r^s_1 \; \ldots \; r^s_J] \qquad r^s_j \in [0, r^s_{max}]$
- **Inverse measurement model**
    - 

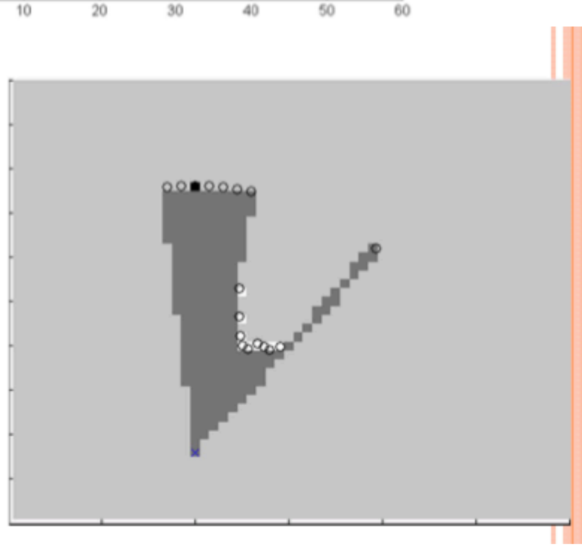
    - Define relative range and bearing to each cell
        - $$\begin{bmatrix} \phi^i \\ r^i \end{bmatrix} = \begin{bmatrix} \tan^{-1}\left(\frac{m^i_y - x_{2,t}}{m^i_x - x_{1,t}}\right) - x_{3,t} \\ \sqrt{(m^i_x - x_{1,t})^2 + (m^i_y - x_{2,t})^2} \end{bmatrix}$$
    - Find relevant range measurement for that cell
        - Closest bearing of a measurement
            - $k = argmin(|\phi^i - phi^s_j|)$
    - Identify each of the three regions and assign correct probability of object
        - if $r^i > \min(r^s_{max}, r^s_k)$ or $|\phi^i - \phi^s_k| > \beta/2$

        - $\Rightarrow$ then no info.
    - else if $r_k^s < r_{max}^s$ and $|r^i - r_k^s| < \alpha/2$
        - $\Rightarrow$ then high probability of an object
    - else if $r^i < r_k^s$
        - $\Rightarrow$ then low probability of an object
- α and β defines the extent of the region to be updated:
    - 

- **Example: Simple Motion**
    - Simple motion
        - Move up until stuck
        - Turn right
        - Repeat
        - Rotate scanner at each timestep
    - Fixed map

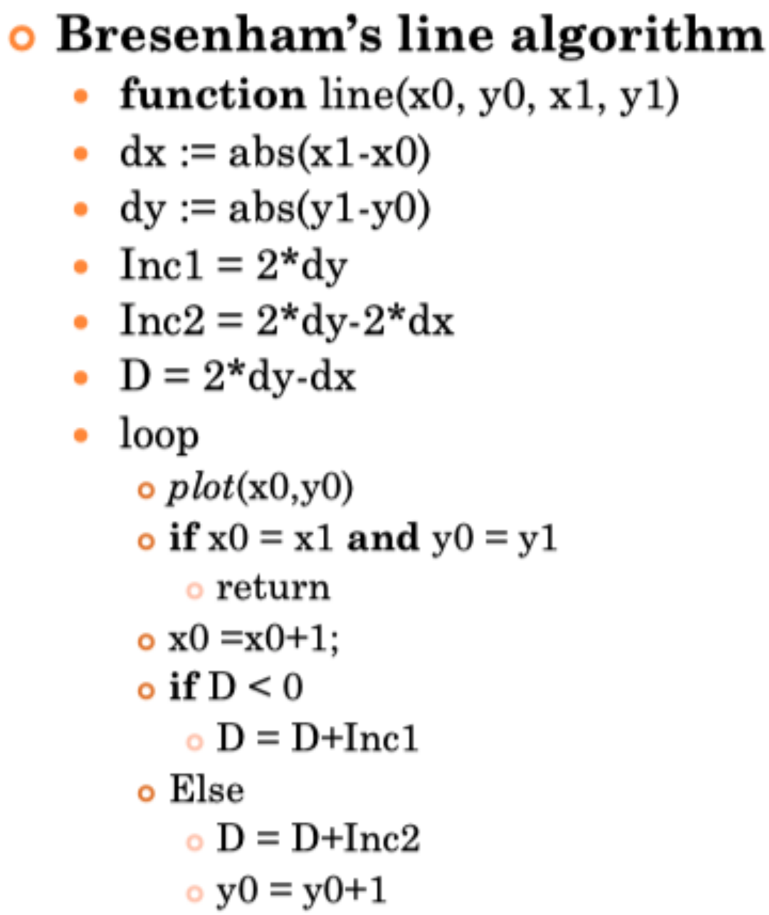


- Example
    - 17 Measurements
        - 46 degree FOV
        - 30 m max range
        - 1 set of measurements per time step
        - Probability of object at scan range: 0.6
        - Probability of no object in front: 0.4



- **Bresenham's line algorithm**

- Instead of updating each cell once for a complete scan
- Perform one udpate per range measurement
- Converted ray tracing into integer math update
- **[See details in Slides]**



- Mapping: **Computation issues**
  - Grid size
    - Calculation grows as resolution of grid increases
    - Topological approximations possible
  - Measurement model pre-caching
    - Model does depend on state, but does not change, so entire model can be pre-calculated
  - Sensor subsampling
    - Not all measurements need be applied, may be significant overlap in scans
  - Selective updating
    - Only update cells for which significant new information is available

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SLAM : Simultaneous Localization And Mapping

Formulation

- Given:
  - Motion model
  - Measurement model
  - Uniquely identifiable static features
  - Vehicle inputs, u_t
  - Measurements to some features, y_t
- Find:
  - Vehicle State, x_t^T
  - Feature Locations, m^i
- Relative calculation, coord. Sys determinedupon init.
- Significantly larger estimation problem than straight localization

SLAM Types

Online SLAM

- Estimates the current state and the map given all information to date
  - $p(x_t^r, m|y_{1:t}, u_{1:t})$
- Most useful for a moving vehicle that needs to estimate its state relative to env. in real time
- usually run online

Full SLAM

- Estimates the entire state history and the map given all information
  - $p(x_{1:t}^r, m|y_{1:t}, u_{1:t})$
- Most useful for creating maps froms sensor data after the fact
- Usually run in batch mode

SLAM Algo. (4 Main in Thrun - Probabilistic Robotics)

EKF/UKF SLAM

- Extension of EKF localization to online SLAM problem
- Very commonly used, especially for improving vehicle state estimation when static features are available

GraphSLAM

- Solves the full SLAM problem by storing data as a set of constraints between variables
- Can create maps based on 1000s of features, not possible with EKF due to matrix inversion limitations
- Many variations, all boild down to a nonlinear optimization that needs to be fast to be useful
- (Predominant area of research over the last decade)
- Super-impressive results

Sparse Extended Information Filter SLAM

- Approximate application of Extended Information Filter to SLAM problem
- Can create a sparse (nearly diagonal) information matrix, which also enables tracking many features, constant time udpates

FastSLAM

- Solves the online SLAM problem simultaneously by combining particles and EKFs
    - Rao-Blackwellized Particle Filters
- Can track multiple correspondences with different particles
- Show robustness to incorrect correspondence
- **Most active area** of research, large scale mapping
- => **Occupancy Grid SLAM** : FastSLAM with mapping by each pixel

Main Focus

- EKF SLAM
    - quick SLAM solution, great for improving vehicle state estimation from information about the environment
    - Not too robust to incorrect feature correspondence
- FastSLAM
    - A more robust approach, particularly with respect to feature correspondece
    - Computationally more expensive, especially with higher dimension vehicle state
- Occupancy Grid SLAM

Note

- Attempting to estimate $nT + fM$ states with $MT, 2MT, 3MT$ Measurements, depending on sensor
    - T: number of time steps
    - M: number of features
    - n: number of vehicle state variables
    - f: number of map feature variables
- Always use many sensors as possible: Wheel encoders + Lidar + IMU

EKF SLAM

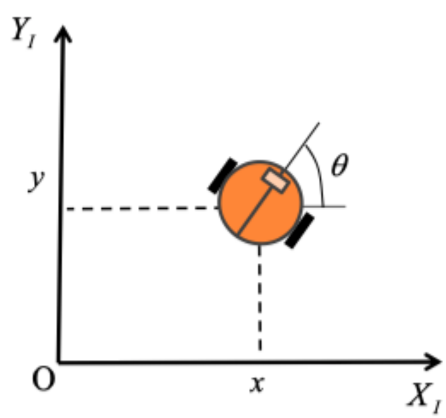
- Variables
    - Full State
        - Vehicle States
        - Feature locations
        - Signatures (Not included here)
        -

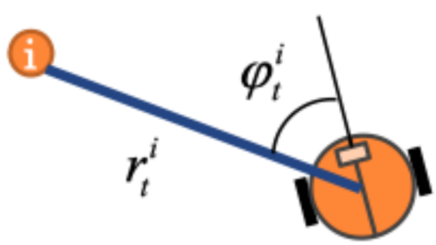
$$x_t = \begin{bmatrix} x_t^r \\ m \end{bmatrix} = \begin{bmatrix} x_t^r \\ m_x^1 \\ m_y^1 \\ \vdots \\ m_x^M \\ m_y^M \end{bmatrix}$$

- Brief: Full state mean and covariance
  - Components for vehicle state and map state
  -

$$\mu_t = \begin{bmatrix} \mu_t^r \\ \mu_t^m \end{bmatrix} \begin{matrix} \leftarrow Robot \\ \leftarrow Map \end{matrix} \qquad \Sigma_t = \begin{bmatrix} \Sigma_t^{rr} & \Sigma_t^{rm} \\ \Sigma_t^{mr} & \Sigma_t^{mm} \end{bmatrix}$$

- **[Recall 2-wheel robot] Models:**





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$$\begin{bmatrix} x_1^r \\ x_2^r \\ x_3^r \end{bmatrix} = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix} \qquad \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} v \\ \omega \end{bmatrix}$$

  - Motion models:
    -

$$\begin{bmatrix} x^r_{1,t} \\ x^r_{2,t} \\ x^r_{3,t} \end{bmatrix} = g(x^r_{t-1}, u_t, \epsilon_t) = \begin{bmatrix} x^r_{1,t} + u_{1,t} \cos x^r_{3,t-1} dt \\ x^r_{2,t} + u_{1,t} \sin x^r_{3,t-1} dt \\ x^r_{3,t} + u_{2,t} dt \end{bmatrix} + \epsilon_t \sim \mathcal{N}(\mu = 0, \sigma^2 = R_t)$$

- Measurement models:
  - Relative range and / or bearing to numerous features m^i in field of view
  - Define $\delta x^i_t = m^i_x - x_{1,t}$, $\delta y^i_t = m^i_y - x_{2,t}$
    - $r^i_t = \sqrt{(\delta x^i_t)^2 + (\delta y^i_t)^2}$
  - Then:

$$\begin{bmatrix} y^i_{1,t} \\ y^i_{2,t} \end{bmatrix} = h^i(x_t, \delta_t) = \begin{bmatrix} \phi^i_t \\ r^i_t \end{bmatrix} = \begin{bmatrix} \tan^{-1}\left(\frac{\delta y^i_t}{\delta x^i_t}\right) - x^r_{3,t} \\ \sqrt{(\delta x^i_t)^2 + (\delta y^i_t)^2} \end{bmatrix} + \delta_t \sim \mathcal{N}(\mu = 0, \sigma^2 = Q_t) \qquad \begin{matrix} \textit{Beaing} \\ \textit{Range} \end{matrix}$$

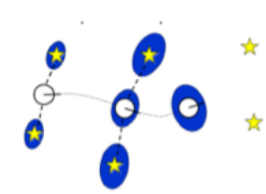
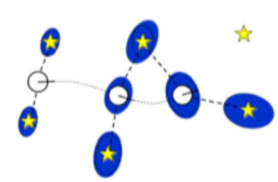
- **Vehicle Prior**
  - In localization or mapping, coordinate system was clearly defined
    - Localization relative to fixed map
    - Mapping relative to known vehicle motion
  - In pure SLAM, neither is known, so coordinate system is arbitrary choice
    - Assume vehicle starts at origin with zero heading
    - Know this with absolute certainty
      - $x^r_0 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T \quad \Sigma^{rr}_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

- **Map Prior**
  - No clue where any of the features are:
    - Theoretically we may say:

$$x^m_0 = \begin{bmatrix} 0 & 0 & \dots & 0 \end{bmatrix}^T \quad \Sigma^{rr}_0 = \begin{bmatrix} \infty & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \infty \end{bmatrix}$$

    - In Practice, **not very useful:**
      - Linearization with all features assumed to be at the origin performs very poorly

        - Inversion with infinite diagonal numerically difficult
- **Preferred Method:**
    - Initialize each feature location based on first set of measurements
        - Measurements must uniquely define feature position
        - Bearing and range + vehicle state required
            - $\mu_t^i = \begin{bmatrix} x_{1,t}^r + y_{2,t}^i \cos(y_{1,t}^i + x_{3,t}^r) \\ x_{2,t}^r + y_{2,t}^i \cos(y_{1,t}^i + x_{3,t}^r) \end{bmatrix}$
        - Can define covariance based on measurement noise and vehicle state uncertainty, or predefine explicitly
    - If initial measurements are insufficient, can accumulate multiple measurements before initialization
        - Bearing only SLAM (for vision data)
- **Sketch:**

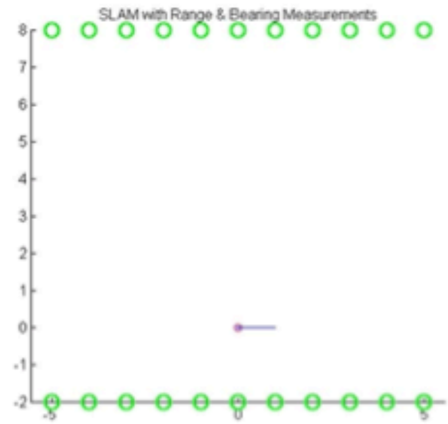
Description	Sketch
1. A vehicle and a set of features, perfect knowledge of vehicle location initially	
2. The vehicle measures the location of two featuers and moves one time step forward - Measurement and motion uncertainty	
3. At the next time step: two new features are observed with more uncertainty - Combination of vehicle and measurement uncertainty - Motion uncertainty continues to grow	
4. The next set of measurements includes a feature that has already observed . - The vehicle uncertainty can be reduced . - The additional features are not as uncertain	

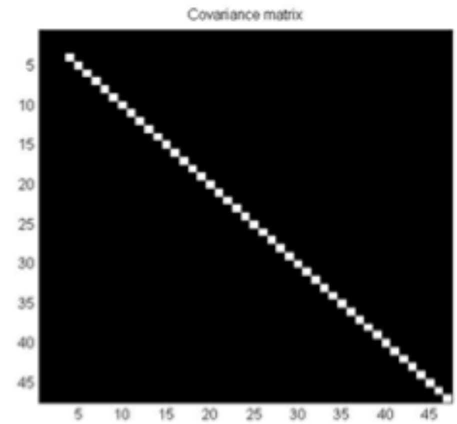
    - The result: as old features are discarded and new features are added, uncertainty grows

EKF SLAM Algorithm

- Prediction step
  - Only vehicle states and covariance change
  - Map states and covariance are unaffected
  - Quick 3x3 update
    - $$\begin{aligned} G_t &= \frac{\partial}{\partial x_{t-1}^r} g(x_{t-1}^r, u_t)\Big|_{x_{t-1}^r=\mu_{t-1}^r} \\ \bar{\mu}_t^r &= g(\mu_{t-1}^r, u_t) \\ \bar{\Sigma}_t^{rr} &= G_t \Sigma_{t-a}^{rr} G_t^T + R_t \end{aligned}$$
- Linearization of Motion Model, as before:
  - $$G_t = \frac{\partial}{\partial x_{t-1}^r} g(x_{t-1}^r, u_t) = \begin{bmatrix} 1 & 0 & -u_{1,t} \cos x_{3,t-1}^r dt \\ 0 & 1 & u_{1,t} \sin x_{3,t-1}^r dt \\ 0 & 0 & 1 \end{bmatrix}$$
- Measurement Update, for feature i
  - Since each measurement pair depends on one feature,
  - independence means updates can be performed one feature at a time
  - $$\begin{aligned} H_t^i &= \frac{\partial}{\partial x_t} h^i(x_t)\Big|_{x_t=\mu_t} \\ K_t^i &= \bar{\Sigma}_t (H_t^i)^T (H_t^i \bar{\Sigma}_t (H_t^i)^T + Q_t)^{-1} \\ \mu_t &= \bar{\mu}_t + K_t^i (y_t - h(\bar{\mu}_t)) \\ \Sigma_t &= (1 - K_t H_t^i) \bar{\Sigma}_{t-1} \end{aligned}$$
- Linearization of measurement Model:
  - $$H_t^i = \frac{\partial}{\partial x_t} h(x_t) = \begin{bmatrix} \frac{dy_t^i}{r^2} & \frac{-dx_t^i}{r^2} & -1 & 0 & \dots & 0 & \frac{-dy_t^i}{r^2} & \frac{dx_t^i}{r^2} & 0 & \dots & 0 \\ \frac{-dx_t^i}{r^2} & \frac{-dy_t^i}{r^2} & 0 & 0 & \dots & 0 & \frac{dx_t^i}{r^2} & \frac{dy_t^i}{r^2} & 0 & \dots & 0 \end{bmatrix}$$
  - Derivatives w.r.t. m^i in appropriate columns
- **Example:**

- Example
  - 22 features in two lines
  - Same circular motion as for localization example
  - Field of view similar to camera
    - +/- 45 degrees
    - 5 m range





Discussion

- Vehicle state error correlates feature estimates
  - If vehicle state known exactly (mapping) features could be estimated independently
  - Knowing more about one feature improves estimates about entire map
- Covariance matrix divided in 3x3 structure
  - Vehicle state and two sets of features
  - Each row of features strongly connected
  - Rows weakly connected by uncertain multiple time step motion
- Growth in state uncertainty without loop closure
  - When first feature is re-observed, all estimates improve
  - Correction information carried in covariance matrix
- Wrong correspondence can be catastrophic
  - Linearization about wrong point can cause deterioration of estimate, divergence of covariance
- Strategies:
  - Provisional Feature List
    - Features on the list are tracked identically to other features
    - Not used to update vehicle state or vehicle/map covariance
    - Once trace of covariance drops below threshold, incorporate feature into map
  - Feature Selection
    - Features are selected so as to avoid correspondence issues
      - Spatially distributed
      - Distinct signatures
  - Feature Tracking and Windowed Correspondence
    - Features can be expected to move in a consistent way from frame to frame, so only a subset of features need be considered for matches