1 Fourier Series

1.1 Introduction to Fourier Series

This section provides an introduction to the Fourier series. We present three very powerful claims to motivate the developments of this section.

Big Claim

1. Any periodic function $f(t)$ with period T can be represented by the trigonometric series
$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$
where $\omega_0 = \frac{2\pi}{T}$.
2. There exists a formula to calculate the coefficients a_0, a_n, and b_n in the trigonometric series.
3. It is possible to calculate the most accurate approximation of $f(t)$.

If these three claims are true, then any periodic function can be easily modelled on a computer.

Definition 1.1. A periodic function is any function for which
$$f(t) = f(t+T) \tag{1.1.1}$$
for all t. The smallest constant T that satisfies (1.1) is called the period of the function.

Corollary 1.1.1. *If $f(t+T) = f(t)$, we have*
$$\int_{\alpha}^{\beta} f(t)\,\mathrm{d}t = \int_{\alpha+T}^{\beta+T} f(t)\,\mathrm{d}t \tag{1.1.2}$$
$$\int_{0}^{T} f(t)\,\mathrm{d}t = \int_{a}^{a+T} f(t)\,\mathrm{d}t \tag{1.1.3}$$
for any α, β, and a.

Proof. To prove the first equality, we use the substitution $u = t + T$ on the LHS to obtain $\int_{\alpha}^{\beta} f(t)\,\mathrm{d}t = \int_{\alpha+T}^{\beta+T} f(u-T)\,\mathrm{d}u$, which is equal to the RHS. Next, by (2), we have $\int_a^0 f(t)\,\mathrm{d}t = \int_{a+T}^{T} f(t)\,\mathrm{d}t$, which means the RHS can be rewritten as $\int_a^{a+T} f(t)\,\mathrm{d}t = \int_{a+T}^{T} f(t)\,\mathrm{d}t + \int_0^{a+T} f(t)\,\mathrm{d}t$, which is equal to the LHS. □

Theorem 1.1.2 (Fourier's Theorem)**.** *Any periodic function $f(t)$ with period T can be represented by the trigonometric series*
$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \tag{1.1.4}$$
where $\omega_0 = \frac{2\pi}{T}$.

Remark 1.1.3. We note that both sin and cos have a period of 2π, and thus the constant ω_0 changes the period of both to T.

Remark 1.1.4. The series in (4) can be rewritten as
$$f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t - \theta_n). \tag{1.1.5}$$

This is obtained by first setting $\frac{1}{2}a_0 = C_0$, then doing the following manipulations:

$$\begin{aligned}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) &= \sqrt{a_n^2 + b_n^2}\left(\frac{a_n}{\sqrt{a_n^2 + b_n^2}} \cos n\omega_0 t + \frac{b_n}{\sqrt{a_n^2 + b_n^2}} \sin n\omega_0 t\right) \\ &= C_n(\cos\theta_n \cos n\omega_0 t + \sin\theta_n \sin n\omega_0 t) \\ &= C_n \cos(n\omega_t - \theta_n)\end{aligned} \tag{1.1.6}$$

Definition 1.2. We now give names to several components of the trigonometric series.

1. The terms a_n, b_n are the Fourier coefficients of $f(t)$.
2. The component of frequency $\omega_n = n\omega_0$ is called the nth harmonic of the periodic function.
3. The first harmonic $C_1 \cos(\omega_0 t \theta_1)$ is called the fundamental component, as it has the same period as $f(t)$.
4. The constant ω_0 is called the fundamental angular frequency, and $f_0 = \frac{1}{T}$ is the fundamental frequency.
5. The coefficients C_n and angles θ_n are the harmonic amplitudes and phase angles respectively.

Definition 1.3. A set of functions $\{\phi_k(t)\}$ is orthogonal on an interval $a < t < b$ if, for any two functions $\phi_m(t)$ and $\phi_n(t)$ in the set $\{\phi_k(t)\}$, the relation (7) holds.

$$\int_a^b \phi_m(t)\phi_n(t)\,\mathrm{d}t = \begin{cases} 0 & \text{for } m \neq n \\ r_n & \text{for } m = n \end{cases} \tag{1.1.7}$$

Remark 1.1.5. Consider the following sinusoidal functions where $\omega_0 = \frac{2\pi}{T}$. They show that on the interval $-\frac{T}{2} < t < \frac{T}{2}$ the functions $\{1, \cos\omega_0 t, \cos 2\omega_0 t, \ldots, \cos n\omega_0 t, \ldots, \sin\omega_0 t, \sin 2\omega_0 t, \ldots, \sin n\omega_0 t, \ldots\}$ form an orthogonal set.

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos(m\omega_0 t)\,\mathrm{d}t = 0 \text{ for } m \neq 0 \tag{1.1.8}$$

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \sin(m\omega_0 t)\,\mathrm{d}t = 0 \text{ for all } m \tag{1.1.9}$$

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos(m\omega_0 t)\cos(n\omega_0 t)\,\mathrm{d}t = \begin{cases} 0 & \text{for } m \neq n \\ \frac{T}{2} & \text{for } m = n \neq 0 \end{cases} \tag{1.1.10}$$

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \sin(m\omega_0 t)\sin(n\omega_0 t)\,\mathrm{d}t = \begin{cases} 0 & \text{for } m \neq n \\ \frac{T}{2} & \text{for } m = n \neq 0 \end{cases} \tag{1.1.11}$$

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \sin(m\omega_0 t)\cos(n\omega_0 t)\,\mathrm{d}t = 0 \text{ for all } m \text{ and } n \tag{1.1.12}$$

It is left to the reader to verify these calculations.

Corollary 1.1.6. *We can take advantage of orthogonality to evaluate the Fourier coefficients of (4).*

$$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cos(n\omega_0 t)\,\mathrm{d}t \tag{1.1.13}$$

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\sin(n\omega_0 t)\,\mathrm{d}t \tag{1.1.14}$$

$$\frac{1}{2}a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\,\mathrm{d}t \tag{1.1.15}$$

Proof. The proof techniques for (13) and (14) are similar, and involve expanding $f(t)$. Finally, (15) follows directly from setting $n = 0$ in (13). To prove (13), we begin by expanding the RHS.

$$\begin{aligned}
\frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cos(n\omega_0 t)\,\mathrm{d}t &= \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \left(\frac{1}{2}a_0 + \sum_{i=1}^{\infty}(a_i \cos i\omega_0 t + b_i \sin i\omega_0 t)\right)\cos(n\omega_0 t)\,\mathrm{d}t \\
&= \frac{a_0}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos(n\omega_0 t)\,\mathrm{d}t + \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \left(\sum_{i=1}^{\infty}(a_i \cos i\omega_0 t + b_i \sin i\omega_0 t)\right)\cos(n\omega_0 t)\,\mathrm{d}t \\
&= \frac{2}{T}\left(\sum_{i=1}^{\infty} a_i \int_{-\frac{T}{2}}^{\frac{T}{2}} \cos(i\omega_0 t)\cos(n\omega_0 t)\,\mathrm{d}t + \sum_{i=1}^{\infty} b_i \int_{-\frac{T}{2}}^{\frac{T}{2}} \sin(i\omega_0 t)\cos(n\omega_0 t)\,\mathrm{d}t\right) \\
&= \frac{2}{T}(a_n \frac{T}{2}) \\
&= a_n
\end{aligned}$$

□

We also note that $\frac{a_0}{2}$ is the average value of $f(t)$ over a period.

Remark 1.1.7. Finding the Fourier coefficients of a series is useful in finding the Fourier series for periodic functions.

Example 1. Find the Fourier series for the function $f(t)$ defined by

$$f(t) = \begin{cases} -1 & -\frac{T}{2} < t < 0 \\ 1 & 0 < t < \frac{T}{2} \end{cases}$$

and $f(t + T) = f(t)$.

Solution. First, we calculate a_n using (13), and realize that $a_n = 0$. Calculating b_n, we obtain $b_n = \frac{2}{n\pi}(1 - \cos n\pi)$. We then note that $\cos n\pi = (-1)^n$, and therefore obtain

$$b_n = \begin{cases} 0 & n \text{ even} \\ \frac{4}{n\pi} & n \text{ odd.} \end{cases}$$

Hence, $f(t) = \frac{4}{\pi}\left(\sin \omega_0 t + \frac{1}{3}\sin 3\omega_0 t + \frac{1}{5}\sin 5\omega_0 t + \ldots\right).$ □

Calculating approximations of the Fourier series of functions is often more efficient computing-wise than calculating stupid numbers of terms. The finite Fourier series

$$S_k(t) = \frac{a_0}{2} + \sum_{n=1}^{k}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \tag{1.1.16}$$

is the sum of the first $(2k + 1)$ terms of a Fourier series that represents $f(t)$ on $-\frac{T}{2} < t < \frac{T}{2}$. The function $\varepsilon_k(t)$ is the error between $f(t)$ and $S_k(t)$, and is simply

$$\varepsilon_k(t) = f(t) - S_k(t).$$

Definition 1.4. The mean-square error E_k is defined as

$$E_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} [\varepsilon_k(t)]^2\,\mathrm{d}t\,. \tag{1.1.17}$$

Claim.

$$E_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} [f(t)]^2\,\mathrm{d}t - \frac{a_0^2}{4} - \frac{1}{2}\sum_{n=1}^{k}(a_n^2 + b_n^2). \tag{1.1.18}$$

Proof. This proof is very long and tedious, so we rewrite certain parts first. Let $\alpha := \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} [f(t)]^2 \,\mathrm{d}t$ and let $p := a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$. We start from the definition of E_k and expand from there.

$$\begin{aligned}
E_k &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} [\varepsilon_k(t)]^2 \,\mathrm{d}t \\
&= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} [f(t)]^2 - 2f(t)S_k(t) + [S_k(t)]^2 \,\mathrm{d}t \\
&= \alpha - \left(\frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)S_k(t)\,\mathrm{d}t\right) + \left(\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} [S_k(t)]^2 \,\mathrm{d}t\right)
\end{aligned}$$

Now, we work on simplifying the second term. Several of the identities are heavily abused here, namely (13)-(15).

$$\begin{aligned}
\frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)S_k(t)\,\mathrm{d}t &= \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\left(\frac{a_0}{2} + \sum_{n=1}^{k} p\right)\mathrm{d}t \\
&= \frac{a_0}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\,\mathrm{d}t + \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\left(\sum_{n=1}^{k} p\right)\mathrm{d}t \\
&= \frac{a_0^2}{2} + \sum_{n=1}^{k} a_n \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cos(n\omega_0 t)\,\mathrm{d}t + \sum_{n=1}^{k} b_n \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\sin(n\omega_0 t)\,\mathrm{d}t \\
&= \frac{a_0^2}{2} + \sum_{n=1}^{k} a_n^2 + \sum_{n=1}^{k} b_n^2 \\
&= \frac{a_0^2}{2} + \sum_{n=1}^{k} \left(a_n^2 + b_n^2\right)
\end{aligned}$$

Next, we simplify the third term, abusing (8)-(12).

$$\begin{aligned}
\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} [S_k(t)]^2 \,\mathrm{d}t &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \left(\frac{a_0}{2} + \sum_{n=1}^{k} p\right)\left(\frac{a_0}{2} + \sum_{n=1}^{k} p\right)\mathrm{d}t \\
&= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{a_0^2}{4} + a_0\sum_{n=1}^{k} p + \left(\sum_{n=1}^{k} p\right)^2 \mathrm{d}t \\
&= \frac{a_0^2}{4} + \frac{a_0}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \sum_{n=1}^{k} p\,\mathrm{d}t + \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \left(\sum_{n=1}^{k} p\right)^2 \mathrm{d}t \\
&= \frac{a_0^2}{4} + \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \left(\sum_{n=1}^{k} p\right)^2 \mathrm{d}t \\
&= \frac{a_0^2}{4} + \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \sum_{n=1}^{k} a_n^2\cos^2(n\omega_0 t) + \sum_{n=1}^{k} b_n^2 \sin^2(n\omega_0 t) + R\,\mathrm{d}t \\
&= \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{k}(a_n^2 + b_n^2)
\end{aligned}$$

Putting all three terms back together, we obtain

$$E_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} [f(t)]^2 \,\mathrm{d}t - \frac{a_0^2}{4} - \frac{1}{2}\sum_{n=1}^{k}(a_n^2 + b_n^2).$$

The R in the second last line of the working is the remainder after expanding the square of the summation, and its integral is 0, as noted by (12). $\square$

Claim. The approximation of $f(t)$ by the finite Fourier series $S_k(t)$ has the least mean-square error property, i.e. $S_k(t)$ is the most accurate approximation of $f(t)$.

Proof. The mean-square error E_k can be considered as a function of a_0, a_n, and b_n. To minimize E_k, its partial derivatives with respect to all three variables must be zero. Therefore, if we are able to show that $\frac{\partial E_k}{\partial a_0} = \frac{\partial E_k}{\partial a_n} = \frac{\partial E_k}{\partial b_n} = 0$, we are done. After doing some differentiation and rearranging of terms, we are done. □

Lemma 1.1.8 (Parseval's Identity)**.** *If a_0, a_n, and b_n are the coefficients in the Fourier expansion of a periodic function $f(t)$ with period T, then*

$$\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} [f(t)]^2 \,\mathrm{d}t = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) \tag{1.1.19}$$

Proof. From (17), we know that E_k is non-negative, and from (18) we know that

$$E_{k+1} = E_k - \frac{1}{2}\left(a_{k+1}^2 + b_{k+1}^2\right) \tag{1.1.20}$$

Thus, the sequence $\{E_k\}$ is decreasing, and therefore converges as it is also non-negative. In other words, $\lim_{k\to\infty} E_k = 0$, and we obtain (19) from (18). □