1) $\sin x = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n$

Suppose we use

$\sin x \simeq 0 + \frac{f^1(0)}{1!}x^1 + \frac{f^2(0)}{2!}x^2$

$\simeq 0 + \frac{1}{1}x + 0$

$\simeq x$

then, since x is very small $x \ll 1$ $\cos(x) \simeq \cos(0) = 1$

$\sin x = x + E_3$, where $E_3 = \frac{f^3(\xi)}{3!}x^3$ $0 \le \xi \le x$

$\frac{f^3(0)}{3!}x^3 = \frac{-\cos(0)}{6}x^3 = \frac{-x^3}{6}$

$\sin x - x = E_3$

Relative Error:

$\left|\frac{E_3}{\sin(x)}\right| = \frac{1}{2} \times 10^{-14}$

$\left|\frac{\sin(x) - x}{\sin(x)}\right| = \frac{1}{2} \times 10^{-14}$

$\left|\frac{E_3}{x + E_3}\right| = \frac{1}{2} \times 10^{-14}$

$\left|\frac{\frac{-x^3}{6}}{x - \frac{x^3}{6}}\right| = \frac{1}{2} \times 10^{-14}$

$\left|\frac{\frac{-x^2}{6}}{1 - \frac{x^2}{6}}\right| = \frac{1}{2} \times 10^{-14}$

$\frac{x^2}{6} = \frac{1}{2} \times 10^{-14}\left(1 - \frac{x^2}{6}\right)$

$x^2 = 3 \times 10^{-14}\left(1 - \frac{x^2}{6}\right)$

$0 = 3 \times 10^{-14} - \frac{1}{2} \times 10^{-14} - x^2$

$0 = 3 \times 10^{-14} - \left(1 + \frac{1}{2} \times 10^{-14}\right)x^2$

$x^2 = \frac{3 \times 10^{-14}}{\left(1 + \frac{1}{2} \times 10^{-14}\right)}$

$x = \sqrt{\frac{3 \times 10^{-14}}{1 + \frac{1}{2} \times 10^{-14}}}$

$x = \sqrt{3 \times 10^{-14}}$

$\boxed{x = \pm\sqrt{3} \times 10^{-7}}$

2) a) e^{x+2h}

$$f(x+h) = \sum_{k=0}^{n} \frac{f^{k}(x)}{k!} h^{k}$$

let $h = 2h$, $f(x+h) = e^{x+h}$

$$f(x+h) = \frac{f(x)}{0!} h^0 + \frac{f'(x)}{1!} h^1 + \frac{f^2(x)}{2!} h^2 + \frac{f^3(x)}{3!} h^3$$

$$= e^x + e^x(h)^1 + \frac{e^x}{2}(h^2) + \frac{e^x}{6}(h)^3$$

$$= e^x + e^x h + \frac{e^x}{2} h^2 + \frac{e^x}{6} h^3$$

Sub $2h$ back

$$e^{x+2h} = e^x + e^x(2h) + \frac{e^x}{2}(2h)^2 + \frac{e^x}{6}(2h)^3$$

$$\boxed{e^{x+2h} = e^x + 2e^x h + 2e^x h^2 + \frac{4}{3} e^x h^3}$$

2) b) $\sin(x-3h)$

$$f(x+h) = \sum_{k=0}^{n} \frac{f^{k}(x)}{k!} h^{k}$$

let $h = -3h$, $f(x+h) = \sin(x+h)$

$$f(x+h) = \sin(x) + \frac{\cos(x)}{1!} h^1 + \frac{-\sin(x)}{2!} h^2 + \frac{-\cos(x)}{3!} h^3$$

$$= \sin(x) + \cos(x) h - \frac{\sin(x)}{2} h^2 - \frac{\cos(x)}{6} h^3$$

Sub $-3h$ back

$$\sin(x-3h) = \sin(x) + \cos(x)(-3h) - \frac{\sin(x)}{2}(-3h)^2 - \frac{\cos(x)}{6}(-3h)^3$$

$$\boxed{\sin(x-3h) = \sin(x) - 3\cos(x) h - \frac{9}{2}\sin(x) h^2 + \frac{9}{2}\cos(x) h^3}$$

3) Maclaurin Series $c=0$:

$$f(x)=\sum_{k=0}^{\infty}\frac{f^{k}(0)}{k!}x^{k}$$

E_{n+1} is the error term

$E_{n+1}\leq 10^{-10}$ when $x=0.5$

$0\leq\xi\leq x$

$$\frac{e^{\xi}}{k!}x^{k}\leq 10^{-10}$$

Since any derivative of e^x is e^x so $f^{k}(x)=e^{x}$

$$\frac{e^{\xi}}{k!}(0.5)^{k}\leq 10^{-10}$$

$x=0.5$

$$\frac{e^{0.5}}{k!}(0.5)^{k}\leq 10^{-10}$$

by trial and error we get that $k=10$

$$\frac{e^{0.5}}{10!}(0.5)^{10}\leq 10^{-10}$$

$$4.437\times 10^{-10}\leq 10^{-10}$$

$\therefore$ we need $\underline{9}$ terms to get the error term($\underline{\underline{k=10}}$) to achieve (absolute) accuracy of 10^{-10}

4) $1-a^2 \approx 1$ by IEEE 754 double precision

So, $a^2 \approx 0$ which means we want a^2 to be less than ε_{mach}

Since any value $< \varepsilon_{mach} \approx 0$

$a^2 < \frac{\varepsilon_{mach}}{2}$ (double)

$a^2 < \frac{2^{-52}}{2}$

$a^2 < 2^{-53}$

$a < \sqrt{2^{-53}}$

$a < 2^{-\frac{53}{2}} \approx 1.05 \times 10^{-8}$

Assuming it's rounding to the nearest

$$\boxed{\therefore\ a^2 < 2^{-53} \text{ or } a < 2^{-\frac{53}{2}}}$$

5) a) $(a+b)+c \neq a+(b+c)$ $\qquad B=10,\ t=5,\ L=-4,\ U=-4$

$a = 1.5\times10^{4}$
$b = 2.3\times10$
$c = 8\times10^{-1}$

$(a+b)+c$
$=(1.5023\times10^{4})+8\times10^{-1}$
$=(1.5023\times10^{4})$

$a+(b+c)$
$=1.5\times10^{4}+(2.38\times10)$
$=1.5024\times10^{4}$

b) $(a*b)*c \neq a*(b*c)$ $\qquad B=10,\ t=5,\ L=-12,\ U=12$

$a = 1.5\times10^{6}$
$b = 3\times10^{-9}$
$c = 2\times10^{-9}$

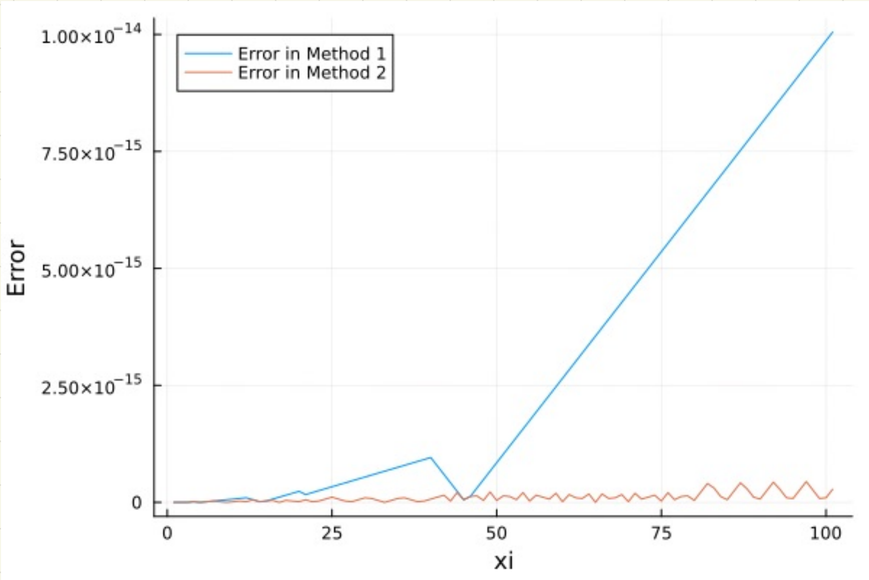
$(a*b)*c$
$=(4.5\times10^{-3})*(2\times10^{-9})$
$=9\times10^{-12}$

$a*(b*c)$
$=1.5\times10^{4}*(6\times10^{-18}\approx 0)$
$=0$

$6\times10^{-18} < \varepsilon_{mach} = 1\times10^{-16}$
$\varepsilon_{mach} = 1.0001\times10^{-12} - 1.0000\times10^{-12}$
$= 0.0001\times10^{-12} = 1\times10^{-16}$

6) a) The method 2 $|x_i - \tilde{x}_i|$ is more accurate than method 1 because in method 2. It contains one source error from the multiplication of $i*h$, but in method 1 it have error from the addition and since it uses previous term. So the error could be carried and result a larger error. Also, in method 1 it can contain roundoff error if the $h < \epsilon_{mach}$ but in method 2 there is no concern about this.

b)



7) a) $f'(x) \approx \dfrac{f(x+h) - f(x-h)}{2h}$

Sub $f(x+h)$ and $f(x-h)$ into $f'(x)$ eq

$$\frac{f(x+h)-f(x-h)}{2h} = \frac{\cancel{f(x)} + f'(x)h + \cancel{\frac{f^2(x)}{2}h^2} + \frac{f^3(x)}{6}h^3 - \left[\cancel{f(x)} + f'(x)(-h) + \cancel{\frac{f^2(x)}{2}h^2} + \frac{f^3(x)}{6}(-h^3)\right]}{2h}$$

$$= \frac{\cancel{2}f'(x)\cancel{h} + \cancel{2}f^3(x)h^3\left(\frac{1}{6}\right)}{\cancel{2}\cancel{h}}$$

$$\frac{f(x+h)-f(x-h)}{2h} = f'(x) + f^3(x)h^2\left(\frac{1}{6}\right)$$

$$f'(x) = \frac{f(x+h)-f(x-h)}{2h} \underbrace{- \frac{f^3(x)h^2}{6}}_{E_{n+1}}$$

$$f(x+h) = \sum_{k=0}^{n} \frac{f^{k}(x)}{k!}h^k$$

$$= \frac{f(x)}{0!}h^0 + \frac{f'(x)}{1!}h^1 + \frac{f^2(x)}{2!}h^2 + \frac{f^3(x)}{3!}h^3$$

$$= f(x) + f'(x)h + \frac{f^2(x)}{2}h^2 + \frac{f^3(x)}{6}h^3$$

$$f(x-h) = \frac{f(x)}{0!}(-h)^0 + \frac{f'(x)}{1!}(-h)^1 + \frac{f^2(x)}{2!}(-h)^2 + \frac{f^3(x)}{3!}(-h)^3$$

$$= f(x) + f'(x)(-h) + \frac{f^2(x)}{2}(h^2) + \frac{f^3(x)}{6}(-h^3)$$

$\therefore$ The truncation error of Eq. 3 is

$$-\frac{f^3(\xi)h^2}{6}$$

7) b)

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + \frac{\delta_1 - \delta_2}{2h}$$

δ_1 : error from $f(x+h)$

δ_2 : error from $f(x-h)$

$$\left| f'(x) - \frac{f_1 - f_2}{2h} \right| = \left| -\frac{f^3(\xi)h^2}{6} - \frac{\delta_1 - \delta_2}{2h} \right|$$

$$\leq \left| -\frac{f^3(\xi)h^2}{6} \right| + \left| \frac{\delta_1 - \delta_2}{2h} \right|$$

Let M be $f^3(\xi)$ and $|\delta_1|, |\delta_2| \leq \varepsilon_{mach}$

$$\leq \frac{Mh^2}{6} + \frac{2\varepsilon_{mach}}{2h}$$

$$\leq \frac{Mh^2}{6} + \frac{\varepsilon_{mach}}{h}$$

To find smallest we want to find min of $\frac{Mh^2}{6} + \frac{\varepsilon_{mach}}{h}$ and we do this by using derivative

$$f(h) = \frac{Mh^2}{6} + \frac{\varepsilon_{mach}}{h}$$

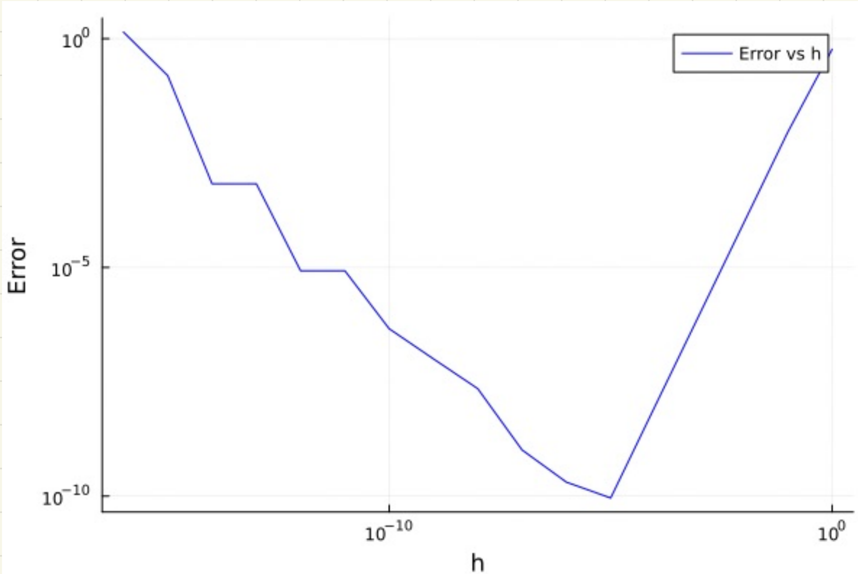
$$f'(h) = \frac{Mh}{3} - \frac{\varepsilon_{mach}}{h^2}$$

min $f'(h) = 0$

$$0 = \frac{Mh}{3} - \frac{\varepsilon_{mach}}{h^2}$$

$$\boxed{h = \sqrt[3]{\frac{3\varepsilon_{mach}}{M}} \quad \text{or} \quad \sqrt[3]{\frac{3\varepsilon_{mach}}{f^3(\xi)}}}$$

7) c)

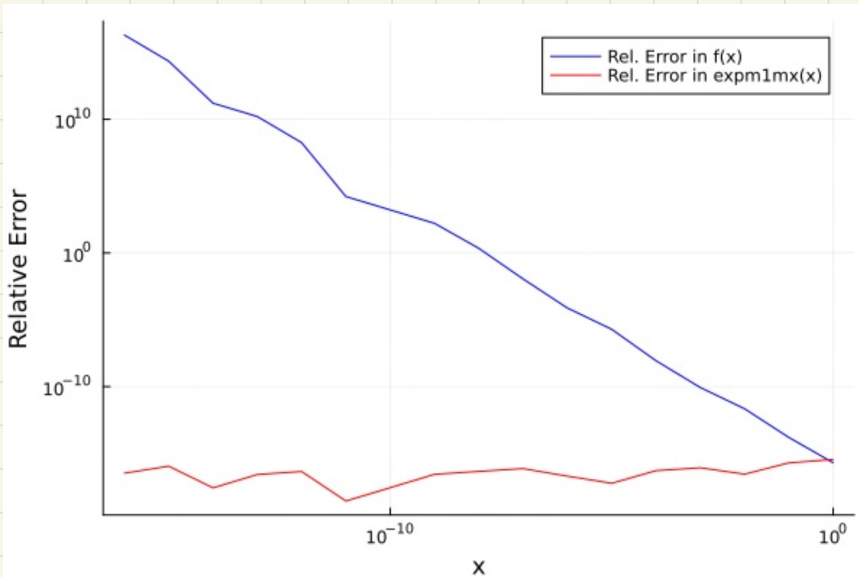


8) a) As x gets closer to 0, $x \approx 0$

$$f(x) = \frac{e^0 - 0 - 1}{0^2} = \frac{0}{0} \approx (NaN)$$

This will give us a NaN value which is very large relative error.

b)



9) $g(x) = \frac{e^x - 1 - x}{x^2}$ $h(x) = \frac{e^x - x - 1}{x^2}$

The difference between the g(x) and h(x) is due to the order of operations that cause the rounding errors which is the differences shown in the output.

Since that in the code we are using the default double precission which only contains 16 significant digits. So the value after the 16 digits gets round off / truncated. Based on the operations we do, there will be some roundoff / truncation in the final value.

Taking $x = 1\times10^{-10}$

g(x) does $e^x - 1$ first which is

$= e^x - 1$

$= e^{1\times10^{-10}} - 1$

$= \underbrace{1.0\ldots}\,82740371\times10^{-10}$

There are less 0s between the first and next none zero digit so its able keep some of its value left compared to h(x). Then when we do $(e^x - 1) - x$ it will just give us a none-zero small number

h(x) does $e^x - x$ first which is

$= e^x - x$

$= e^{1\times10^{-16}} - 1\times10^{-16}$

$= 1.\underbrace{000\ldots}_{\text{during } e^x - x}0827$

there are too much 0s >16 digits between the 1st digit to next none zero digit it got round to 1.0. So when we do $(e^x - x) - 1$ this gives us 0

Since we divide it by a way smaller number it will give us a large value

$= \frac{e^x - 1 - x}{x^2}$

$= \frac{8.27\ldots\times10^{-16}}{(1\times10^{-10})^2}$

$\boxed{= 827\ldots}$

9) (continue)

Now taking $x = 2^{-33}$

$g(x) = e^x - 1$

$= e^{2^{-33}} - 1$

$= 1.000\ldots1164153 - 1$

$= 1.164\ldots \times 10^{-10}$

$= 1.164\ldots \times 10^{-10} - x$ ← next step

$= 1.164\ldots \times 10^{-10} - 1.164 \times 10^{-10}$

$= 0$

No roundoff error this time

Now the issue is that since the first 16 digits are identical and it doesn't contains the digits beyond 16 it will just cancel out and result a final value of 0.

$h(x) = e^x - x$

$= e^{2^{-33}} - 2^{-33}$

$= 1.000\ldots1164153 - 1.16\ldots81 \times 10^{-10}$

$= 1.00\ldots0 - 1$

$= 0$

Since this time we have the next significant digit before 16 digits so not a lot of rounding error but since the x and the decimals after the 1 in e^x are identical for first 16 digits it will cancel out and give us 1. Then when we substract 1 it will give us 0.