Universidad Autónoma Chapingo

Departamento de Mecánica Agrícola

Ingeniería Mecatrónica Agrícola

Informe 2

Asignatura:

Dinámica y Control de Robots

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Índice

Introducción

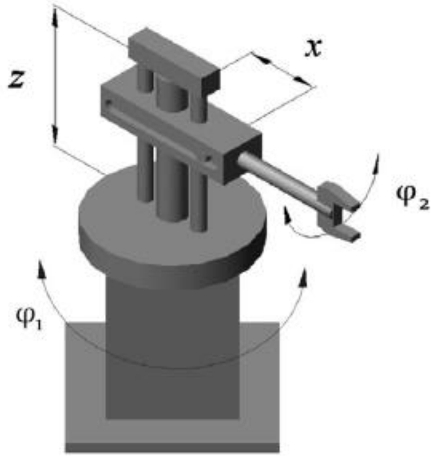
Un robot es cualquier estructura mecánica que opera con un cierto grado de autonomía, bajo el control de un computador, para la realización de una tarea, y que dispone de un sistema sensorial más o menos evolucionado para obtener información de su entorno.

Un robot está compuesto por una serie de elementos hardware, como son: una estructura mecánica, un sistema de actuación, un sistema sensorial interno, un sistema sensorial externo y un ordenador en el que se encuentra un software que gestiona el sistema sensorial y mueva la estructura mecánica para la realización de una determinada tarea.

En este informe se abarcará la obtención y deducción de las ecuaciones del Lagrangiano de cada uno de los robots

Desarrollo

Robot 1: Robot cilíndrico con dos articulaciones primaticas y dos rotacionales



1er eslabón:

De los parámetros de DH:

$$A_0^1 = \begin{bmatrix} C\theta_1 & -S\theta_1 & 0 & 0 \\ S\theta_1 & C\theta_1 & 0 & 0 \\ 0 & 0 & 1 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C\theta_1 & 0 & S\theta_1 & 0 \\ S\theta_1 & 0 & -C\theta_1 & 0 \\ 0 & 1 & 0 & l_1 + a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Posición y Velocidad

$$P_{x,y,z} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} ; v_1 = \frac{d}{dt} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \frac{d}{dt} \begin{bmatrix} 0 \\ 0 \\ l_1 + a_1 \end{bmatrix} = \dot{a}_1$$

Energía Cinética

$$K_1 = \frac{1}{2} m_1 v^t v = \frac{1}{2} m_1 {v_1}^2$$

$$\therefore K_1 = \frac{1}{2} m_1 {\dot{a}_1}^2$$

Energía potencial

$P_1 = m_1 g h_1$ si $h_1 = l_1 + a_1$ $\therefore P_1 = m_1 g [l_1 + a_1]$

2do eslabón:

$$A_0^2 = \begin{bmatrix} C\theta_1 & 0 & S\theta_1 & 0 \\ S\theta_1 & 0 & -C\theta_1 & 0 \\ 0 & 1 & 0 & l_1 + a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C\theta_1 & 0 & S\theta_1 & S\theta_1 a_2 \\ S\theta_1 & 0 & -C\theta_1 & C\theta_1 a_2 \\ 0 & 1 & 0 & l_1 + a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Posición y Velocidad

$$P_{x,y,z} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} ; v_1 = \frac{d}{dt} \begin{bmatrix} a_2 cos\theta_1 \\ a_2 sen\theta_1 \\ l_1 + a_1 \end{bmatrix} = \begin{bmatrix} \dot{a}_2 cos\theta_1 - a_2 \dot{\theta}_1 sen\theta_1 \\ \dot{a}_2 sen\theta_1 + a_2 \dot{\theta}_1 cos\theta_1 \\ \dot{a}_1 \end{bmatrix}$$

$$\upsilon_2{}^2 = \dot{x_2}{}^2 + \dot{y_2}{}^2 + \dot{z_2}{}^2$$

$$\upsilon_2{}^2 = \left(\dot{a_2}cos\theta_1 - a_2\dot{\theta}_1 sen\theta_1\right)^2 + \left(\dot{a_2}sen\theta_1 + a_2\dot{\theta}_1 cos\theta_1\right)^2 + (\dot{a_1})^2$$

Desarrollando los cuadrados:

$$\upsilon_2{}^2 = \left(\dot{a_2}{}^2cos^2\theta_1 - 2\dot{a_2}cos\theta_1 a_2\dot{\theta}_1 + a_2{}^2\dot{\theta}_1{}^2 sen^2\theta_1\right) + \left(\dot{a_2}{}^2sen^2\theta_1 + 2\dot{a_2}sen\theta_1 a_2\dot{\theta}_1 cos\theta_1 + a_2{}^2\dot{\theta}_1{}^2 cos^2\theta_1\right) + \left(\dot{a_1}{}^2\right)$$

$$\upsilon_2{}^2 = \dot{a_2}{}^2(sen^2\theta_1 + cos^2\theta_1) - 2\dot{a_2}cos\theta_1 a_2\dot{\theta}_1 + 2\dot{a_2}sen\theta_1 a_2\dot{\theta}_1 cos\theta_1 + a_2{}^2\dot{\theta}_1{}^2(sen^2\theta_1 + cos^2\theta_1) + \dot{a_1}{}^2$$

$$\upsilon_2{}^2 = \dot{a_2}{}^2 + a_2{}^2\dot{\theta}_1{}^2 + \dot{a_1}{}^2$$

Energía Cinética

$$\therefore K_2 = \frac{1}{2}m_2\left(\dot{a_2}{}^2 + a_2{}^2\dot{\theta}_1{}^2 + \dot{a_1}{}^2\right)$$

Energía Potencial

$si\ \ h_2 = l_1 + a_1$ $\qquad\qquad \therefore P_2 = m_2 g(l_1 + a_1)$

3er eslabón:

De los parámetros de DH:

$$A_0^3 = \begin{bmatrix} C\theta_1 & 0 & S\theta_1 & S\theta_1 a_2 \\ S\theta_1 & 0 & -C\theta_1 & C\theta_1 a_2 \\ 0 & 1 & 0 & l_1 + a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ l_3 \end{bmatrix} = \begin{bmatrix} a_2cos\theta_1 + l_3cos\theta_1 \\ a_2sen\theta_1 + l_3sen\theta_1 \\ l_1 + a_1 \end{bmatrix}$$

Posición y Velocidad

$$P_{x,y,z} = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}\ ; \upsilon_1 = \frac{d}{dt}\begin{bmatrix} a_2cos\theta_1 + l_3cos\theta_1 \\ a_2sen\theta_1 + l_3sen\theta_1 \\ l_1 + a_1 \end{bmatrix} = \begin{bmatrix} \dot{a_2}cos\theta_1 - a_2\dot{\theta}_1 sen\theta_1 - l_3\dot{\theta}_1 sen\theta_1 \\ \dot{a_2}sen\theta_1 + a_2\dot{\theta}_1 cos\theta_1 + l_3\dot{\theta}_1 cos\theta_1 \\ \dot{a_1} \end{bmatrix}$$

$$\upsilon_3{}^2 = \dot{x_3}{}^2 + \dot{y_3}{}^2 + \dot{z_3}{}^2$$

Sustituyendo datos:

$$\upsilon_3{}^2 = \left(\dot{a_2}cos\theta_1 - a_2\dot{\theta}_1 sen\theta_1 - l_3\dot{\theta}_1 sen\theta_1\right)^2 + \left(\dot{a_2}sen\theta_1 + a_2\dot{\theta}_1 cos\theta_1 + l_3\dot{\theta}_1 cos\theta_1\right)^2 + (\dot{a_1})^2$$

Desarrollando los cuadrados:

$$\dot{x_3}{}^2 = \dot{a_2}{}^2cos^2\theta_1 - 2\dot{a_2}cos\theta_1 a_2\dot{\theta}_1 sen\theta_1 - 2\dot{a_2}cos\theta_1 l_3\dot{\theta}_1 sen\theta_1 + 2a_2\dot{\theta}_1 sen\theta_1 l_3\dot{\theta}_1 sen\theta_1 + a_2{}^2\dot{\theta}_1{}^2 sen^2\theta_1 + l_3{}^2\dot{\theta}_1{}^2 sen^2\theta_1$$

$$\dot{y_3}{}^2 = \dot{a_2}{}^2sen^2\theta_1 + 2\dot{a_2}sen\theta_1 a_2\dot{\theta}_1 cos\theta_1 + 2\dot{a_2}sen\theta_1 l_3\dot{\theta}_1 cos\theta_1 + 2a_2\dot{\theta}_1 cos\theta_1 l_3\dot{\theta}_1 cos\theta_1 + a_2{}^2\dot{\theta}_1{}^2 cos^2\theta_1 + l_3{}^2\dot{\theta}_1{}^2 cos^2\theta_1$$

$$\dot{z_3}{}^2 = (\dot{a_1})^2$$

Simplificando:

$$\upsilon_3{}^2 = \dot{a_2}{}^2(sen^2\theta_1 + cos^2\theta_1) - 2\dot{a_2}cos\theta_1 a_2\dot{\theta_1}sen\theta_1 + 2\dot{a_2}sen\theta_1 a_2\dot{\theta_1}cos\theta_1 - 2\dot{a_2}cos\theta_1 l_3\dot{\theta_1}sen\theta_1 + 2\dot{a_2}sen\theta_1 l_3\dot{\theta_1}cos\theta_1 + 2a_2\dot{\theta_1}l_3\dot{\theta_1}(sen^2\theta_1 + cos^2\theta_1) + a_2{}^2\dot{\theta_1}{}^2(sen^2\theta_1 + cos^2\theta_1) + l_3{}^2\dot{\theta_1}{}^2(sen^2\theta_1 + cos^2\theta_1) + \dot{a_1}{}^2$$

$$\upsilon_3{}^2 = \dot{a_2}{}^2 + 2a_2\dot{\theta_1}{}^2 l_3 + a_2{}^2\dot{\theta_1}{}^2 + l_3{}^2\dot{\theta_1}{}^2 + \dot{a_1}{}^2$$

Energía Cinética

$$\therefore K_3 = \frac{1}{2}m_3\left(\dot{a_2}{}^2 + 2a_2\dot{\theta_1}{}^2 l_3 + a_2{}^2\dot{\theta_1}{}^2 + l_3{}^2\dot{\theta_1}{}^2 + \dot{a_1}{}^2\right)$$

Energía potencial

$$h_3 = (l_1 + a_1) \qquad \therefore P_3 = m_3 g(l_1 + a_1)$$

Obtención de la ecuación de Lagrange

$$L = K - P$$

$$L = K_1 + K_2 + K_3 - P_1 - P_2 - P_3$$

$$L = \frac{1}{2}m_1\dot{a_1}{}^2 + \frac{1}{2}m_2\left(\dot{a_2}{}^2 + a_2{}^2\dot{\theta_1}{}^2 + \dot{a_1}{}^2\right) + \frac{1}{2}m_3\left(\dot{a_2}{}^2 + 2a_2\dot{\theta_1}{}^2 l_3 + a_2{}^2\dot{\theta_1}{}^2 + l_3{}^2\dot{\theta_1}{}^2 + \dot{a_1}{}^2\right) - m_1 g(l_1 + a_1) - m_2 g(l_1 + a_1) - m_3 g(l_1 + a_1)$$

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right] - \frac{\partial L}{\partial q} = \tau$$

Cálculo de las derivadas del Lagrangeano

$$\frac{\partial L}{\partial \dot{q}} = \begin{bmatrix} \frac{\partial L}{\partial \dot{a_1}} \\ \frac{\partial L}{\partial \dot{a_2}} \\ \frac{\partial L}{\partial \dot{\theta_1}} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial \dot{a_1}}\left[\frac{1}{2}m_1\dot{a_1}{}^2 + \frac{1}{2}m_2\dot{a_1}{}^2 + \frac{1}{2}m_3\dot{a_1}{}^2\right] \\ \frac{\partial}{\partial \dot{a_2}}\left[\frac{1}{2}m_2\dot{a_2}{}^2 + \frac{1}{2}m_3\dot{a_2}{}^2\right] \\ \frac{\partial}{\partial \dot{\theta_1}}\left[\frac{1}{2}m_2 a_2{}^2\dot{\theta_1}{}^2 + m_3 a_2\dot{\theta_1}{}^2 l_3 + \frac{1}{2}m_3 a_2{}^2\dot{\theta_1}{}^2 + \frac{1}{2}m_3 l_3{}^2\dot{\theta_1}{}^2\right] \end{bmatrix}$$

$$\frac{\partial L}{\partial \dot{q}} = \begin{bmatrix} m_1\dot{a_1} + m_2\dot{a_1} + m_3\dot{a_1} \\ m_2\dot{a_2} + m_3\dot{a_2} \\ m_2 a_2{}^2\dot{\theta_1} + 2m_3 a_2 l_3\dot{\theta_1} + m_3 a_2{}^2\dot{\theta_1} + m_3 l_3{}^2\dot{\theta_1} \end{bmatrix}$$

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right] = \begin{bmatrix} m_1\ddot{a}_1 + m_2\ddot{a}_1 + m_3\ddot{a}_1 \\ m_2\ddot{a}_2 + m_2\ddot{a}_2 \\ m_2\left(2a_2\dot{\theta}_1\dot{a}_2 + a_2{}^2\ddot{\theta}_1\right) + 2m_3l_3(\dot{a}_2\dot{\theta}_1 + a_2\ddot{\theta}_1) + m_3(2a_2\dot{\theta}_1\dot{a}_2 + a_2{}^2\ddot{\theta}_1) + m_3l_3{}^2\ddot{\theta}_1 \end{bmatrix}$$

$$\frac{\partial L}{\partial q} = \begin{bmatrix} \frac{\partial L}{\partial a_1} \\ \frac{\partial L}{\partial a_2} \\ \frac{\partial L}{\partial \theta_1} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial a_1}[-m_1ga_1 - m_2ga_1 - m_3ga_1] \\ \frac{\partial}{\partial a_2}\left[\frac{1}{2}m_2a_2{}^2\dot{\theta}_1{}^2 + m_3a_2\dot{\theta}_1{}^2l_3 + \frac{1}{2}m_3a_2{}^2\dot{\theta}_1{}^2\right] \\ \frac{\partial}{\partial \theta_1}[0] \end{bmatrix}$$

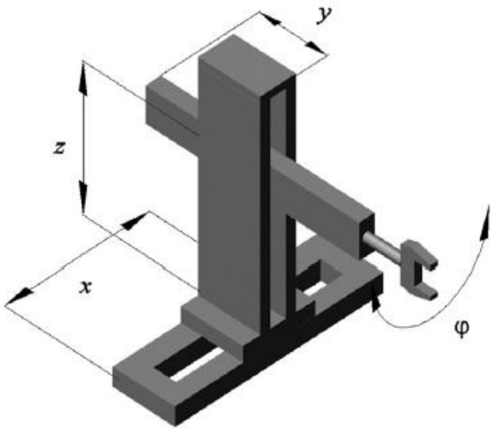
$$\frac{\partial L}{\partial q} = \begin{bmatrix} -m_1g - m_2g - m_3g \\ m_2a_2\dot{\theta}_1{}^2 + m_3\dot{\theta}_1{}^2l_3 + m_3a_2\dot{\theta}_1{}^2 \\ 0 \end{bmatrix}$$

Por lo que finalmente obtenemos:

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right] - \frac{\partial L}{\partial q} = \tau$$

$$\begin{bmatrix} m_1\ddot{a}_1 + m_2\ddot{a}_1 + m_3\ddot{a}_1 \\ m_2\ddot{a}_2 + m_2\ddot{a}_2 \\ m_2\left(2a_2\dot{\theta}_1\dot{a}_2 + a_2{}^2\ddot{\theta}_1\right) + 2m_3l_3\left(\dot{a}_2\dot{\theta}_1 + a_2\ddot{\theta}_1\right) + m_3\left(2a_2\dot{\theta}_1\dot{a}_2 + a_2{}^2\ddot{\theta}_1\right) + m_3l_3{}^2\ddot{\theta}_1 \end{bmatrix} - \begin{bmatrix} -m_1g - m_2g - m_3g \\ m_2a_2\dot{\theta}_1{}^2 + m_3\dot{\theta}_1{}^2l_3 + m_3a_2\dot{\theta}_1{}^2 \\ 0 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix}$$

Robot 2: Robot cartesiano con tres articulaciones prismáticas y una rotacional



1er eslabón:
De los parámetros de DH:

$$A_0^1 = \begin{bmatrix} 1 & 0 & 0 & a_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Posición y Velocidad

$$P_{x,y,z} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}; \ v_1 = \frac{d}{dt}\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} a_1 \\ 0 \\ 0 \end{bmatrix} = \dot{a}_1 \quad ; \quad v_1^2 = \dot{a}_1$$

Energía Cinética

$$\therefore K_1 = \frac{1}{2} m_1 \dot{a}_1{}^2$$

Energía potencial

$P_1 = m_1 g h_1$ si $h_1 = 0$ $\therefore P_1 = 0$

2do eslabón:
De los parámetros de DH:

$$A_0^2 = \begin{bmatrix} 1 & 0 & 0 & a_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ a_2 \\ 1 \end{bmatrix} = \begin{bmatrix} a_1 \\ 0 \\ a_2 \\ 1 \end{bmatrix}$$

Posición y Velocidad

$$P_{x,y,z} = \begin{bmatrix} x_2 \\ y_2 \\ z_0 \end{bmatrix} ; \ v_2 = \frac{d}{dt}\begin{bmatrix} x_2 \\ y_2 \\ z_0 \end{bmatrix} = \begin{bmatrix} a_1 \\ 0 \\ a_2 \end{bmatrix} = \dot{a}_1 + \dot{a}_2$$

$$v_2{}^2 = \dot{a}_1{}^2 + \dot{a}_2{}^2$$

Energía Cinética

$$\therefore K_2 = \frac{1}{2} m_2 (\dot{a}_1{}^2 + \dot{a}_2{}^2)$$

Energía potencial

$h_2 = a_2$ $\therefore P_2 = m_2 g a_2$

3er eslabón:
De los parámetros de DH:

$$A_0^3 = \begin{bmatrix} 1 & 0 & 0 & a_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ a_3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_3 \\ a_2 \\ 1 \end{bmatrix}$$

Posición y Velocidad

$$P_{x,y,z} = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} \;\; ; \;\; v_3 = \frac{d}{dt}\begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_3 \\ a_2 \end{bmatrix} = \dot{a_1} + \dot{a_3} + \dot{a_2}$$

$$v_3{}^2 = \dot{a_1}^2 + \dot{a_2}^2 + \dot{a_3}^2$$

Energía Cinética

$$\therefore K_3 = \frac{1}{2} m_3(\dot{a_1}^2 + \dot{a_2}^2 + \dot{a_3}^2)$$

Energía potencial

Si $h_3 = a_2$ $\quad\quad \therefore P_3 = m_3 g a_2$

4to eslabón:
De los parámetros de DH:

$$A_0^3 = \begin{bmatrix} 1 & 0 & 0 & a_1 \\ 0 & 1 & 0 & a_3 \\ 0 & 0 & 1 & a_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\theta_1 & -S\theta_1 & 0 & 0 \\ S\theta_1 & C\theta_1 & 0 & l_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_3 + l_4 \\ a_2 \\ 1 \end{bmatrix}$$

Posición y Velocidad

$$P_{x,y,z} = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} \;\; ; \;\; v_4 = \frac{d}{dt}\begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_3 + l_4 \\ a_2 \end{bmatrix} = \dot{a_1} + \dot{a_3} + \dot{a_2}$$

$v_4{}^2 = \dot{a_1}^2 + \dot{a_2}^2 + \dot{a_3}^2$

Energía Cinética

$$\therefore K_4 = \frac{1}{2} m_4(\dot{a_1}^2 + \dot{a_2}^2 + \dot{a_3}^2)$$

Energía potencial

Si $\quad h_4 = a_2$ $\quad\quad \therefore P_4 = m_4 g a_2$

Obtención de la ecuación de Lagrange

$$L = K - P$$

$$L = K_1 + K_2 + K_3 - P_1 - P_2 - P_3$$

$$L = \frac{1}{2} m_1 \dot{a_1}^2 + \frac{1}{2} m_2(\dot{a_1}^2 + \dot{a_2}^2) + \frac{1}{2} m_3(\dot{a_1}^2 + \dot{a_2}^2 + \dot{a_3}^2) + \frac{1}{2} m_4(\dot{a_1}^2 + \dot{a_2}^2 + \dot{a_3}^2) - m_2 g a_2 - m_3 g a_2 - m_4 g a_2$$

$$L = \frac{1}{2} m_1 \dot{a_1}^2 + \frac{1}{2} m_2 \dot{a_1}^2 + \frac{1}{2} m_2 \dot{a_2}^2 + \frac{1}{2} m_3 \dot{a_1}^2 + \frac{1}{2} m_3 \dot{a_2}^2 + \frac{1}{2} m_3 \dot{a_3}^2 + \frac{1}{2} m_4 \dot{a_1}^2 + \frac{1}{2} m_4 \dot{a_2}^2 + \frac{1}{2} m_4 \dot{a_3}^2 - m_2 g a_2 - m_3 g a_2 - m_4 g a_2$$

Simplificación de términos:

$$L = \frac{1}{2} \dot{a_1}^2(m_1 + m_2 + m_3 + m_4) + \frac{1}{2} \dot{a_2}^2(m_2 + m_3 + m_4) + \frac{1}{2} \dot{a_3}^2(m_3 + m_4) - g a_2(m_2 + m_3 + m_4)$$

Cálculo de derivadas del Lagrangeano

$$\left[\frac{\partial L}{\partial \dot{q}}\right] = \begin{bmatrix} \frac{\partial L}{\partial \dot{a_1}} \\ \frac{\partial L}{\partial \dot{a_2}} \\ \frac{\partial L}{\partial \dot{a_3}} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial \dot{a_1}}\left[\frac{1}{2}\dot{a_1}^2(m_1+m_2+m_3+m_4)\right] \\ \frac{\partial}{\partial \dot{a_2}}\left[\frac{1}{2}\dot{a_2}^2(m_2+m_3+m_4)\right] \\ \frac{\partial}{\partial \dot{a_3}}\left[\frac{1}{2}\dot{a_3}^2(m_3+m_4)\right] \end{bmatrix} = \begin{bmatrix} \dot{a_1}(m_1+m_2+m_3+m_4) \\ \dot{a_2}(m_2+m_3+m_4) \\ \dot{a_3}(m_3+m_4) \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} \dot{a_1}(m_1+m_2+m_3+m_4) \\ \dot{a_2}(m_2+m_3+m_4) \\ \dot{a_3}(m_3+m_4) \end{bmatrix} = \begin{bmatrix} \ddot{a_1}(m_1+m_2+m_3+m_4) \\ \ddot{a_2}(m_2+m_3+m_4) \\ \ddot{a_3}(m_3+m_4) \end{bmatrix}$$

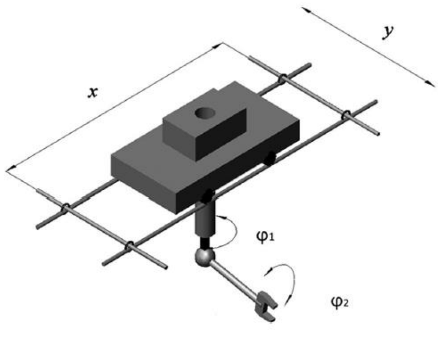
$$\frac{\partial L}{\partial q} = \begin{bmatrix} \frac{\partial L}{\partial a_1} \\ \frac{\partial L}{\partial a_2} \\ \frac{\partial L}{\partial a_3} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial a_1}[0] \\ \frac{\partial}{\partial a_2}[-ga_2(m_2+m_3+m_4)] \\ \frac{\partial}{\partial a_3}[0] \end{bmatrix} = \begin{bmatrix} 0 \\ -g(m_2+m_3+m_4) \\ 0 \end{bmatrix}$$

Por lo que finalmente obtenemos:

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right] - \frac{\partial L}{\partial q} = \tau$$

$$\begin{bmatrix} \ddot{a_1}(m_1+m_2+m_3+m_4) \\ \ddot{a_2}(m_2+m_3+m_4) \\ \ddot{a_3}(m_3+m_4) \end{bmatrix} - \begin{bmatrix} 0 \\ -g(m_2+m_3+m_4) \\ 0 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix}$$

Robot 3: Robot con dos articulaciones primaticas y 2 rotacionales



1er eslabón:
De los parámetros de DH:

$$A_0^1 = \begin{bmatrix} 1 & 0 & 0 & a_1 \\ 0 & 1 & 0 & a_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Posición y Velocidad

$$P_{x,y,z} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \;\; ; \;\; v_1 = \frac{d}{dt}\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} a_1 \\ a_2 \\ 0 \end{bmatrix} = \dot{a_1} + \dot{a_2}$$

$$v_1^2 = \dot{a_1}^2 + \dot{a_2}^2$$

Energía Cinética

$$\therefore K_1 = \frac{1}{2} m_1 \left(\dot{a_1}^2 + \dot{a_2}^2\right)$$

Energía potencial

$$h_1 = 0 \qquad\qquad \therefore P_1 = 0$$

2do eslabón:

De los parámetros de DH:

$$A_0^2 = \begin{bmatrix} 1 & 0 & 0 & a_1 \\ 0 & 1 & 0 & a_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -l_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & a_1 \\ 0 & 1 & 0 & a_2 \\ 0 & 0 & 1 & -l_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Posición y Velocidad

$$P_{x,y,z} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \;\; ; \;\; v_2 = \frac{d}{dt}\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} a_1 \\ a_2 \\ -l_3 \end{bmatrix} = \dot{a_1} + \dot{a_2}$$

$$v_2^2 = \dot{a_1}^2 + \dot{a_2}^2$$

Energía Cinética

$$K_2 = \frac{1}{2} m_2 \left(\dot{a_1}^2 + \dot{a_2}^2\right)$$

Energía potencial

$$h_2 = -l_3 \qquad\qquad \therefore P_2 = -m_2 g l_3$$

3er eslabón:

De los parámetros de DH:

$$A_0^3 = \begin{bmatrix} 1 & 0 & 0 & a_1 + l_4 C\theta_1 \\ 0 & 1 & 0 & a_2 + S\theta_1 \\ 0 & 0 & 1 & -l_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Posición y Velocidad

$$P_{x,y,z} = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} \;\; ; \;\; v_3 = \frac{d}{dt}\begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} a_1 + l_4 C\theta_1 \\ a_2 + l_4 S\theta_1 \\ -l_3 \end{bmatrix} = \left(\dot{a_1} - l_4 sen\theta_1 \dot{\theta}_1\right) + \left(\dot{a_2} + l_4 cos\theta_1 \dot{\theta}_1\right) + 0$$

$$v_3^2 = \dot{x_3}^2 + \dot{y_3}^2 + \dot{z_3}^2$$

$$v_3^2 = \left(\dot{a_1} - l_4 sen\theta_1 \dot{\theta}_1\right)^2 + \left(\dot{a_2} + l_4 cos\theta_1 \dot{\theta}_1\right)^2$$

$$\upsilon_3{}^2 = \dot{a_1}{}^2 - 2\dot{a_1}l_4 sen\theta_1\dot{\theta}_1 + l_4{}^2 sen^2\theta_1\dot{\theta}_1{}^2 + \dot{a_2}{}^2 + 2\dot{a_2}l_4 cos\theta_1\dot{\theta}_1 + l_4{}^2 cos^2\theta_1\dot{\theta}_1{}^2$$

$$\upsilon_3{}^2 = \dot{a_1}{}^2 - 2\dot{a_1}l_4 sen\theta_1\dot{\theta}_1 + \dot{a_2}{}^2 + 2\dot{a_2}l_4 cos\theta_1\dot{\theta}_1 + l_4{}^2\dot{\theta}_1{}^2\left(sen^2\theta_1 + l_4{}^2 cos^2\theta_1\right)$$

$$\upsilon_3{}^2 = \dot{a_1}{}^2 - 2\dot{a_1}l_4 sen\theta_1\dot{\theta}_1 + \dot{a_2}{}^2 + 2\dot{a_2}l_4 cos\theta_1\dot{\theta}_1 + l_4{}^2\dot{\theta}_1{}^2$$

Energía Cinética

$$K_3 = \frac{1}{2}m_3 * \upsilon_3{}^2$$

$$K_3 = \frac{1}{2}m_3\left(\dot{a_1}{}^2 - 2\dot{a_1}l_4 sen\theta_1\dot{\theta}_1 + \dot{a_2}{}^2 + 2\dot{a_2}l_4 cos\theta_1\dot{\theta}_1 + l_4{}^2\dot{\theta}_1{}^2\right)$$

Energía potencial

$$h_3 = -l_3 \qquad\qquad \therefore P_3 = -m_3 g l_3$$

Obtención de la ecuación de Lagrange

$$L = K_1 + K_2 + K_3 - P_1 - P_2 - P_3$$

$$\begin{aligned}L = \frac{1}{2}m_1\left(\dot{a_1}{}^2 + \dot{a_2}{}^2\right) + \frac{1}{2}m_2\left(\dot{a_1}{}^2 + \dot{a_2}{}^2\right)\\ + \frac{1}{2}m_3\left(\dot{a_1}{}^2 - 2\dot{a_1}l_4 sen\theta_1\dot{\theta}_1 + \dot{a_2}{}^2 + 2\dot{a_2}l_4 cos\theta_1\dot{\theta}_1 + l_4{}^2\dot{\theta}_1{}^2\right)\\ - (-m_2 g l_3) - (-m_3 g l_3)\end{aligned}$$

Cálculo de las derivadas del Lagrangeano

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right] - \frac{\partial L}{\partial q} = \tau$$

$$\left[\frac{\partial L}{\partial \dot{q}}\right] = \begin{bmatrix}\left[\frac{\partial L}{\partial \dot{a_1}}\right]\\ \left[\frac{\partial L}{\partial \dot{a_2}}\right]\\ \left[\frac{\partial L}{\partial \dot{\theta}_1}\right]\end{bmatrix} = \begin{bmatrix}\frac{\partial}{\partial \dot{a_1}}\left[\frac{1}{2}m_1\dot{a_1}{}^2 + \frac{1}{2}m_2\dot{a_1}{}^2 + \frac{1}{2}m_3\dot{a_1}{}^2 - m_3\dot{a_1}l_4 sen\theta_1\dot{\theta}_1\right]\\ \frac{\partial}{\partial \dot{a_2}}\left[\frac{1}{2}m_1\dot{a_2}{}^2 + \frac{1}{2}m_2\dot{a_2}{}^2 + \frac{1}{2}m_3\dot{a_2}{}^2 + m_3\dot{a_2}l_4 cos\theta_1\dot{\theta}_1\right]\\ \frac{\partial}{\partial \dot{\theta}_1}\left[-m_3\dot{a_1}l_4 sen\theta_1\dot{\theta}_1 + m_3\dot{a_2}l_4 cos\theta_1\dot{\theta}_1 + \frac{1}{2}m_3 l_4{}^2\dot{\theta}_1{}^2\right]\end{bmatrix}$$

$$\left[\frac{\partial L}{\partial \dot{q}}\right] = \begin{bmatrix}m_1\dot{a_1} + m_2\dot{a_1} + m_3\dot{a_1} - m_3 l_4 sen\theta_1\dot{\theta}_1\\ m_1\dot{a_2} + m_2\dot{a_2} + m_3\dot{a_2} + m_3 l_4 cos\theta_1\dot{\theta}_1\\ -m_3\dot{a_1}l_4 sen\theta_1 + m_3\dot{a_2}l_4 cos\theta_1 + m_3 l_4{}^2\dot{\theta}_1\end{bmatrix}$$

$$\begin{aligned}&\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right]\\ &= \begin{bmatrix}m_1\ddot{a}_1 + m_2\ddot{a}_1 + m_3\ddot{a}_1 - m_3 l_4\left(cos\theta_1\dot{\theta}_1{}^2 + sen\theta_1\ddot{\theta}_1\right)\\ m_1\ddot{a}_2 + m_2\ddot{a}_2 + m_3\ddot{a}_2 + m_3 l_4\left(-sen\theta_1\dot{\theta}_1{}^2 + cos\theta_1\ddot{\theta}_1\right)\\ -m_3 l_4\left(\ddot{a}_1 sen\theta_1 + \dot{a}_1 cos\theta_1\dot{\theta}_1\right) + m_3 l_4\left(\ddot{a}_2 cos\theta_1 - \dot{a}_2 sen\theta_1\dot{\theta}_1\right) + m_3 l_4{}^2\ddot{\theta}_1\end{bmatrix}\end{aligned}$$

$$\frac{\partial L}{\partial q}=\begin{bmatrix}\frac{\partial L}{\partial a_1}\\ \frac{\partial L}{\partial a_2}\\ \frac{\partial L}{\partial \theta_1}\end{bmatrix}=\begin{bmatrix}\frac{\partial}{\partial a_1}[0]\\ \frac{\partial}{\partial a_2}[0]\\ \frac{\partial}{\partial \theta_1}\left[-m_3\dot{a}_1l_4sen\theta_1\dot{\theta}_1+m_3\dot{a}_2l_4cos\theta_1\dot{\theta}_1\right]\end{bmatrix}$$

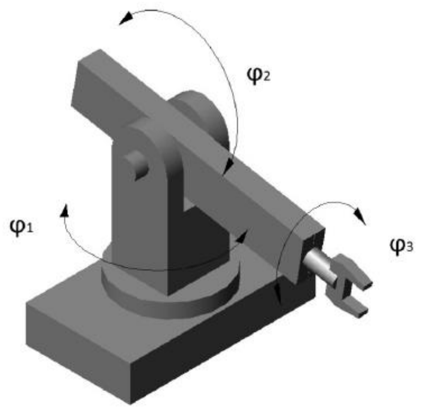
$$\frac{\partial L}{\partial q}=\begin{bmatrix}0\\0\\-m_3\dot{a}_1l_4cos\theta_1\dot{\theta}_1-m_3\dot{a}_2l_4sen\theta_1\dot{\theta}_1\end{bmatrix}$$

Finalmente obtenemos:

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right]-\frac{\partial L}{\partial q}=\tau$$

$$\begin{bmatrix}m_1\ddot{a}_1+m_2\ddot{a}_1+m_3\ddot{a}_1-m_3l_4\left(cos\theta_1\dot{\theta}_1{}^2+sen\theta_1\ddot{\theta}_1\right)\\ m_1\ddot{a}_2+m_2\ddot{a}_2+m_3\ddot{a}_2+m_3l_4\left(-sen\theta_1\dot{\theta}_1{}^2+cos\theta_1\ddot{\theta}_1\right)\\ -m_3l_4\left(\ddot{a}_1sen\theta_1+\dot{a}_1cos\theta_1\dot{\theta}_1\right)+m_3l_4\left(\ddot{a}_2cos\theta_1-\dot{a}_2sen\theta_1\dot{\theta}_1\right)+m_3l_4{}^2\ddot{\theta}_1\end{bmatrix}-\begin{bmatrix}0\\0\\-m_3\dot{a}_1l_4cos\theta_1\dot{\theta}_1-m_3\dot{a}_2l_4sen\theta_1\dot{\theta}_1\end{bmatrix}=\begin{bmatrix}\tau_1\\ \tau_2\\ \tau_3\end{bmatrix}$$

Robot 4: Robot esférico con tres articulaciones rotacionales



1er eslabón:
De los parámetros de DH:

$$A_0^1=\begin{bmatrix}C\theta_1 & 0 & S\theta_1 & 0\\ S\theta_1 & 0 & -C\theta_1 & 0\\ 0 & 1 & 0 & a_1\\ 0 & 0 & 0 & 1\end{bmatrix}$$

Posición y Velocidad

$$P_{x,y,z}=\begin{bmatrix}x_1\\y_1\\z_1\end{bmatrix}\ \ ;\ \ v_1=\frac{d}{dt}\begin{bmatrix}x_1\\y_1\\z_1\end{bmatrix}=\frac{d}{dt}\begin{bmatrix}0\\0\\a_1\end{bmatrix}=0$$

a_1 es constante. Por lo caul $v_1^2 = 0$

Energía Cinética

$$\therefore K_1 = 0$$

Energía potencial

$$h_1 = a_1 \qquad \therefore \mathrm{P}_1 = m_1 g a_1$$

2do eslabón:

De los parámetros de DH:

$$A_0^2 = \begin{bmatrix} C\theta_1 & 0 & S\theta_1 & 0 \\ S\theta_1 & 0 & -C\theta_1 & 0 \\ 0 & 1 & 0 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\theta_2 & -S\theta_2 & 0 & a_2 C\theta_2 \\ S\theta_2 & C\theta_2 & 0 & a_2 S\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} C\theta_1 C\theta_2 & -C\theta_1 S\theta_2 & S\theta_1 & a_2 C\theta_2 C\theta_1 \\ C\theta_2 S\theta_1 & -S\theta_2 C\theta_1 & -C\theta_1 & a_2 C\theta_2 S\theta_1 \\ S\theta_2 & C\theta_2 & 0 & a_2 S\theta_2 + a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Posición y Velocidad

$$P_{x,y,z} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \; ; \; v_2 = \frac{d}{dt}\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \dot{x_2} + \dot{y_2} + \dot{z_2}$$

$$x_2 = a_2 cos\theta_2 cos\theta_1 \,, \dot{x_2} = a_2(-sen\theta_2 \dot{\theta}_2 cos\theta_1 - cos\theta_2 sen\theta_1 \dot{\theta}_1)$$
$$y_2 = a_2 cos\theta_2 sen\theta_1, \dot{y_2} = a_2(-sen\theta_2 \dot{\theta}_2 sen\theta_1 + cos\theta_2 cos\theta_1 \dot{\theta}_1)$$
$$z_2 = a_1 + a_2 sen\theta_2 \;, \; \dot{z_2} = a_2 cos\theta_2 \dot{\theta}_2$$

$$v_2{}^2 = \dot{x_2}{}^2 + \dot{y_2}{}^2 + \dot{z_2}{}^2$$

$$v_2{}^2 = \left(-\mathrm{a}_2 sen\theta_2 \dot{\theta}_2 \cos\theta_1 - \mathrm{a}_2 cos\theta_2 sen\theta_1 \dot{\theta}_1\right)^2 + \left(-\mathrm{a}_2 sen\theta_2 \dot{\theta}_2 sen\theta_1 + \mathrm{a}_2 cos\theta_2 cos\theta_1 \dot{\theta}_1\right)^2 + a_2{}^2 cos^2 \theta_2 \dot{\theta}_2{}^2$$

$$v_2{}^2 = \mathrm{a}_2{}^2 sen^2\theta_2 \dot{\theta}_2{}^2 \cos^2\theta_1 + 2\mathrm{a}_2 sen\theta_2 \dot{\theta}_2 \cos\theta_1 \mathrm{a}_2 cos\theta_2 sen\theta_1 \dot{\theta}_1 + \mathrm{a}_2{}^2 cos^2\theta_2 sen^2\theta_1 \dot{\theta}_1{}^2 + \mathrm{a}_2{}^2 sen^2\theta_2 \dot{\theta}_2{}^2 sen^2\theta_1 - 2\mathrm{a}_2 sen\theta_2 \dot{\theta}_2 sen\theta_1 \mathrm{a}_2 cos\theta_2 cos\theta_1 \dot{\theta}_1 + \mathrm{a}_2{}^2 cos^2\theta_2 cos^2\theta_1 \dot{\theta}_1{}^2 + a_2{}^2 cos^2 \theta_2 \dot{\theta}_2{}^2$$

$$v_2{}^2 = \mathrm{a}_2{}^2 sen^2\theta_2 \dot{\theta}_2{}^2 (\cos^2\theta_1 + sen^2\theta_1) + \mathrm{a}_2{}^2 cos^2\theta_2 \dot{\theta}_1{}^2 (sen^2\theta_1 + cos^2\theta_1) + a_2{}^2 cos^2 \theta_2 \dot{\theta}_2{}^2$$

$$v_2{}^2 = \mathrm{a}_2{}^2 sen^2\theta_2 \dot{\theta}_2{}^2 + \mathrm{a}_2{}^2 cos^2\theta_2 \dot{\theta}_1{}^2 + a_2{}^2 cos^2 \theta_2 \dot{\theta}_2{}^2$$

$$v_2{}^2 = \mathrm{a}_2{}^2 \dot{\theta}_2{}^2 (sen^2\theta_2 + cos^2\theta_2) + \mathrm{a}_2{}^2 cos^2\theta_2 \dot{\theta}_1{}^2$$

$$\therefore v_2{}^2 = \mathrm{a}_2{}^2 \dot{\theta}_2{}^2 + \mathrm{a}_2{}^2 cos^2\theta_2 \dot{\theta}_1{}^2$$

Energía Cinética

$$\therefore K_2 = \frac{1}{2} m_2 \left(a_2{}^2 \dot{\theta_2}^2 + a_2{}^2 cos^2\theta_2 \dot{\theta_1}^2 \right)$$

Energía potencial

$$h_2 = a_1 + a_2 sen\theta_2 \qquad \therefore \mathrm{P}_2 = m_2 g(a_1 + a_2 sen\theta_2)$$

3er eslabón:
De los parámetros de DH:

$$A_0^3 = \begin{bmatrix} C\theta_1 C\theta_2 & -C\theta_1 S\theta_2 & S\theta_1 & a_2 C\theta_2 C\theta_1 \\ C\theta_2 S\theta_1 & -S\theta_2 C\theta_1 & -C\theta_1 & a_2 C\theta_2 S\theta_1 \\ S\theta_2 & C\theta_2 & 0 & a_2 S\theta_2 + a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_3 \\ 0 & C\theta_1 & S\theta_1 & 0 \\ 0 & S\theta_1 & C\theta_1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Posición y Velocidad

$$P_{x,y,z} = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} \;\; ; \;\; v_3 = \frac{d}{dt} \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \dot{x_3} + \dot{y_3} + \dot{z_3}$$

$$x_3 = a_{23} cos\theta_2 cos\theta_1 \;, \qquad \dot{x_3} = a_{23}\left(-sen\theta_2 \dot{\theta_2} cos\theta_1 - cos\theta_2 sen\theta_1 \dot{\theta_1}\right)$$

$$y_3 = a_{23} cos\theta_2 sen\theta_1 \;, \qquad \dot{y_3} = a_{23}\left(-sen\theta_2 \dot{\theta_2} sen\theta_1 + cos\theta_2 cos\theta_1 \dot{\theta_1}\right)$$

$$z_3 = a_1 + a_{23} sen\theta_2 \;, \qquad \dot{z_3} = a_{23} cos\theta_2 \dot{\theta_2}$$

$$v_3{}^2 = \dot{x_3}^2 + \dot{y_3}^2 + \dot{z_3}^2$$

$$v_3{}^2 = \left(-a_{23} sen\theta_2 \dot{\theta_2} cos\theta_1 - a_{23} cos\theta_2 sen\theta_1 \dot{\theta_1}\right)^2 + \left(-a_{23} sen\theta_2 \dot{\theta_2} sen\theta_1 + a_{23} cos\theta_2 cos\theta_1 \dot{\theta_1}\right)^2 + \left(a_{23} cos\theta_2 \dot{\theta_2}\right)^2$$

$$v_3{}^2 = a_{23}{}^2 sen^2\theta_2 \dot{\theta_2}^2 cos^2\theta_1 + 2a_{23} sen\theta_2 \dot{\theta_2} cos\theta_1 a_{23} cos\theta_2 sen\theta_1 \dot{\theta_1} + a_{23}{}^2 cos^2\theta_2 sen^2\theta_1 \dot{\theta_1}^2 + a_{23}{}^2 sen^2\theta_2 \dot{\theta_2}^2 sen^2\theta_1 - 2a_{23} sen\theta_2 \dot{\theta_2} sen\theta_1 a_{23} cos\theta_2 cos\theta_1 \dot{\theta_1} + a_{23}{}^2 cos^2\theta_2 cos^2\theta_1 \dot{\theta_1}^2 + a_{23}{}^2 cos^2\theta_2 \dot{\theta_2}^2$$

$$v_3{}^2 = a_{23}{}^2 sen^2\theta_2 \dot{\theta_2}^2 (cos^2\theta_1 + sen^2\theta_1) + a_{23}{}^2 cos^2\theta_2 \dot{\theta_1}^2 (sen^2\theta_1 + cos^2\theta_1) + a_{23}{}^2 cos^2\theta_2 \dot{\theta_2}^2$$

$$v_3{}^2 = a_{23}{}^2 sen^2\theta_2 \dot{\theta_2}^2 + a_{23}{}^2 cos^2\theta_2 \dot{\theta_1}^2 + a_{23}{}^2 cos^2\theta_2 \dot{\theta_2}^2$$

$$v_3{}^2 = a_{23}{}^2 \dot{\theta_2}^2 (sen^2\theta_2 + cos^2\theta_2) + a_{23}{}^2 cos^2\theta_2 \dot{\theta_1}^2$$

$$\therefore v_3{}^2 = a_{23}{}^2 \dot{\theta_2}^2 + a_{23}{}^2 cos^2\theta_2 \dot{\theta_1}^2$$

Energía Cinética

$$K_3 = \frac{1}{2} m_3 \left(a_{23}{}^2 \dot{\theta_2}^2 + a_{23}{}^2 cos^2\theta_2 \dot{\theta_1}^2 \right)$$

Energía potencial

$$h_3 = a_1 + a_{23} sen\theta_2 \qquad \mathrm{P}_3 = m_3 g(a_1 + a_{23} sen\theta_2)$$

Obtención de la ecuación de Lagrange

$$L = K_1 + K_2 + K_3 - P_1 - P_2 - P_3$$

$$L = \frac{1}{2}m_2\left(a_2{}^2\dot{\theta}_2{}^2 + a_2{}^2cos^2\theta_2\dot{\theta}_1{}^2\right) + \frac{1}{2}m_3\left(a_{23}{}^2\dot{\theta}_2{}^2 + a_{23}{}^2cos^2\theta_2\dot{\theta}_1{}^2\right) - m_1ga_1 - m_2g(a_1 + a_2sen\theta_2) - m_3g(a_1 + a_{23}sen\theta_2)$$

$$L = \frac{1}{2}m_2a_2{}^2\dot{\theta}_2{}^2 + \frac{1}{2}m_2a_2{}^2cos^2\theta_2\dot{\theta}_1{}^2 + \frac{1}{2}m_3a_{23}{}^2\dot{\theta}_2{}^2 + \frac{1}{2}m_3a_{23}{}^2cos^2\theta_2\dot{\theta}_1{}^2 - m_1ga_1 - m_2ga_1 - m_2ga_2sen\theta_2 - m_3ga_1 - m_3ga_{23}sen\theta_2$$

Cálculo de las derivadas del Lagrangeano

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right] - \frac{\partial L}{\partial q} = \tau$$

$$\left[\frac{\partial L}{\partial \dot{q}}\right] = \begin{bmatrix}\frac{\partial L}{\partial \dot{\theta}_1}\\ \frac{\partial L}{\partial \dot{\theta}_2}\end{bmatrix} = \begin{bmatrix}\frac{\partial}{\partial \dot{\theta}_1}\left[\frac{1}{2}m_2a_2{}^2cos^2\theta_2\dot{\theta}_1{}^2 + \frac{1}{2}m_3a_{23}{}^2cos^2\theta_2\dot{\theta}_1{}^2\right]\\ \frac{\partial}{\partial \dot{\theta}_2}\left[\frac{1}{2}m_2a_2{}^2\dot{\theta}_2{}^2 + \frac{1}{2}m_3a_{23}{}^2\dot{\theta}_2{}^2\right]\end{bmatrix}$$

$$\left[\frac{\partial L}{\partial \dot{q}}\right] = \begin{bmatrix}m_2a_2{}^2cos^2\theta_2\dot{\theta}_1 + m_3a_{23}{}^2cos^2\theta_2\dot{\theta}_1\\ m_2a_2{}^2\dot{\theta}_2 + m_3a_{23}{}^2\dot{\theta}_2\end{bmatrix}$$

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right] = \frac{d}{dt}\begin{bmatrix}m_2a_2{}^2cos^2\theta_2\dot{\theta}_1 + m_3a_{23}{}^2cos^2\theta_2\dot{\theta}_1\\ m_2a_2{}^2\dot{\theta}_2 + m_3a_{23}{}^2\dot{\theta}_2\end{bmatrix}$$
$$= \begin{bmatrix}-2m_2a_2{}^2cos\theta_2sen\theta_2\dot{\theta}_2\dot{\theta}_1 + m_2a_2{}^2cos^2\theta_2\ddot{\theta}_1 - 2m_3a_{23}{}^2cos\theta_2sen\theta_2\dot{\theta}_2\dot{\theta}_1 + m_3a_{23}{}^2cos^2\theta_2\ddot{\theta}_1\\ m_2a_2{}^2\ddot{\theta}_2 + m_3a_{23}{}^2\ddot{\theta}_2\end{bmatrix}$$

$$\frac{\partial L}{\partial q} = \begin{bmatrix}\frac{\partial L}{\partial \theta_1}\\ \frac{\partial L}{\partial \theta_2}\end{bmatrix} = \begin{bmatrix}\frac{\partial}{\partial \theta_1}[0]\\ \frac{\partial}{\partial \theta_2}\left[\frac{1}{2}m_2a_2{}^2cos^2\theta_2\dot{\theta}_1{}^2 + \frac{1}{2}m_3a_{23}{}^2cos^2\theta_2\dot{\theta}_1{}^2 - m_2ga_2sen\theta_2 - m_3ga_{23}sen\theta_2\right]\end{bmatrix}$$

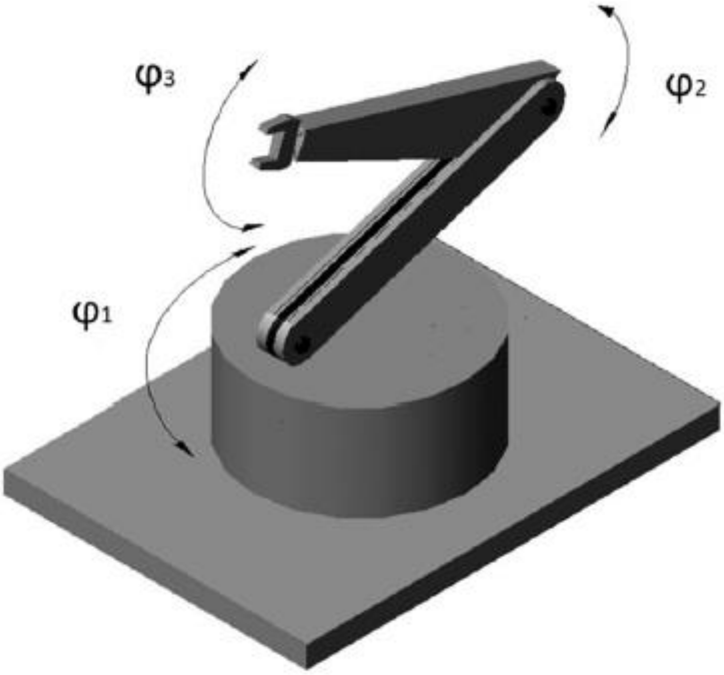
$$\frac{\partial L}{\partial q} = \begin{bmatrix}0\\ -m_2a_2{}^2cos\theta_2sen\theta_2\dot{\theta}_1{}^2 - m_3a_{23}{}^2cos\theta_2sen\theta_2\dot{\theta}_1{}^2 - m_2ga_2cos\theta_2 - m_3ga_{23}cos\theta_2\end{bmatrix}$$

Por lo que finalmente tenemos:

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right] - \frac{\partial L}{\partial q} = \tau$$

$$\begin{bmatrix}-2m_2a_2{}^2cos\theta_2sen\theta_2\dot{\theta}_2\dot{\theta}_1 + m_2a_2{}^2cos^2\theta_2\ddot{\theta}_1 - 2m_3a_{23}{}^2cos\theta_2sen\theta_2\dot{\theta}_2\dot{\theta}_1 + m_3a_{23}{}^2cos^2\theta_2\ddot{\theta}_1\\ m_2a_2{}^2\ddot{\theta}_2 + m_3a_{23}{}^2\ddot{\theta}_2\end{bmatrix} - \begin{bmatrix}0\\ -m_2a_2{}^2cos\theta_2sen\theta_2\dot{\theta}_1{}^2 - m_3a_{23}{}^2cos\theta_2sen\theta_2\dot{\theta}_1{}^2 - m_2ga_2cos\theta_2 - m_3ga_{23}cos\theta_2\end{bmatrix} = \begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix}$$

Robot 5: Robot articulado con tres juntas rotacionales



1er eslabón:
De los parámetros de DH:

$$A_0^1 = \begin{bmatrix} C\theta_1 & -S\theta_1 & 0 & a_1C\theta_1 \\ S\theta_1 & C\theta_1 & 0 & a_1S\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Posición y Velocidad

$$P_{x,y,z} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \;\; ; \;\; v_1 = \frac{d}{dt}\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \dot{x_1} + \dot{y_1} + \dot{z_1}$$

$x_1 = a_1 cos\theta_1$ $\qquad\qquad$ $\dot{x_1} = -a_1 sen\theta_1\dot{\theta_1}$
$y_1 = 0$ $\qquad\qquad$ $\dot{y_1} = 0$
$z_1 = a_1 sen\theta_1$ $\qquad\qquad$ $\dot{z_1} = a_1 cos\theta_1\dot{\theta_1}$

$$v_1{}^2 = \dot{x_1}{}^2 + \dot{y_1}{}^2 + \dot{z_1}{}^1$$

$$v_1{}^2 = \left(-a_1 sen\theta_1\dot{\theta_1}\right)^2 + \left(a_1 cos\theta_1\dot{\theta_1}\right)^2$$

$$v_1{}^2 = a_1{}^2 sen^2\theta_1\dot{\theta_1}{}^2 + a_1{}^2 cos^2\theta_1\dot{\theta_1}{}^2 = a_1{}^2\dot{\theta_1}{}^2(sen^2\theta_1 + cos^2\theta_1)$$

$$\therefore v_1{}^2 = a_1{}^2\dot{\theta_1}{}^2$$

Energía Cinética

$$K_1 = \frac{1}{2} m_1 v_1{}^2 = \frac{1}{2} m_1 a_1{}^2 \dot{\theta}_1{}^2$$

Energía potencial

$$h_1 = a_1 sen\theta_1 \qquad\qquad P_1 = m_1 g a_1 sen\theta_1$$

2do eslabón:
De los parámetros de DH:

$$P_{x,y,z} = \begin{bmatrix} a_1 cos\theta_1 + a_2 \cos(\theta_1 + \theta_2) \\ 0 \\ a_1 sen\theta_1 + a_2 sen(\theta_1 + \theta_2) \end{bmatrix}$$

Posición y Velocidad

$$P_{x,y,z} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \;\; ; \;\; v_2 = \frac{d}{dt} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \dot{x_2} + \dot{y_2} + \dot{z_2}$$

$x_2 = a_1 cos\theta_1 + a_2 cos(\theta_1 + \theta_2)$ $\dot{x_2} = -a_1 sen\theta_1 \dot{\theta}_1 - a_2 sen(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2)$
$y_2 = 0$ $\dot{y_2} = 0$
$z_2 = a_1 sen\theta_1 + a_2 sen(\theta_1 + \theta_2)$ $\dot{z_2} = a_1 cos\theta_1 \dot{\theta}_1 + a_2 cos(\theta_1 + \theta_2)\left(\dot{\theta}_1 + \dot{\theta}_2\right)$

$$v_2{}^2 = \dot{x_2}{}^2 + \dot{y_2}{}^2 + \dot{z_2}{}^2$$

$$v_2{}^2 = \left(-a_1 sen\theta_1 \dot{\theta}_1 - a_2 sen(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2)\right)^2 + \left(a_1 cos\theta_1 \dot{\theta}_1 + a_2 cos(\theta_1 + \theta_2)\left(\dot{\theta}_1 + \dot{\theta}_2\right)\right)^2$$

$$v_2{}^2 = a_1{}^2 sen^2\theta_1 \dot{\theta}_1{}^2 + 2a_1 sen\theta_1 \dot{\theta}_1 a_2 sen(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2) + a_2{}^2 sen^2(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2)^2 + a_1{}^2 cos^2\theta_1 \dot{\theta}_1{}^2 + 2a_1 cos\theta_1 \dot{\theta}_1 a_2 cos(\theta_1 + \theta_2)\left(\dot{\theta}_1 + \dot{\theta}_2\right) + a_2{}^2 cos^2(\theta_1 + \theta_2)\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2$$

$$v_2{}^2 = a_1{}^2 \dot{\theta}_1{}^2 (sen^2\theta_1 + cos^2\theta_1) + 2a_1 sen\theta_1 \dot{\theta}_1 a_2 sen(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2) + 2a_1 cos\theta_1 \dot{\theta}_1 a_2 cos(\theta_1 + \theta_2)\left(\dot{\theta}_1 + \dot{\theta}_2\right) + a_2{}^2 \left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 \left(sen^2(\theta_1 + \theta_2) + cos^2(\theta_1 + \theta_2)\right)$$

$$v_2{}^2 = a_1{}^2 \dot{\theta}_1{}^2 + a_2{}^2 \left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 + 2a_1 sen\theta_1 \dot{\theta}_1 a_2 sen(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2) + 2a_1 cos\theta_1 \dot{\theta}_1 a_2 cos(\theta_1 + \theta_2)\left(\dot{\theta}_1 + \dot{\theta}_2\right)$$

$$v_2{}^2 = a_1{}^2 \dot{\theta}_1{}^2 + a_2{}^2 \left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 + 2a_1 sen\theta_1 \dot{\theta}_1 a_2 \left(\dot{\theta}_1 + \dot{\theta}_2\right)(sen\theta_1 cos\theta_2 + cos\theta_1 sen\theta_2) + 2a_1 cos\theta_1 \dot{\theta}_1 a_2 \left(\dot{\theta}_1 + \dot{\theta}_2\right)(cos\theta_1 cos\theta_2 - sen\theta_1 sen\theta_2)$$

$$\upsilon_2{}^2 = a_1{}^2\dot{\theta}_1{}^2 + a_2{}^2(\dot{\theta}_1 + \dot{\theta}_2)^2 + 2a_1 sen\theta_1\dot{\theta}_1 a_2(\dot{\theta}_1 + \dot{\theta}_2)sen\theta_1 cos\theta_2$$
$$+ 2a_1 sen\theta_1\dot{\theta}_1 a_2(\dot{\theta}_1 + \dot{\theta}_2)cos\theta_1 sen\theta_2$$
$$+ 2a_1 cos\theta_1\dot{\theta}_1 a_2(\dot{\theta}_1 + \dot{\theta}_2)cos\theta_1 cos\theta_2$$
$$- 2a_1 cos\theta_1\dot{\theta}_1 a_2(\dot{\theta}_1 + \dot{\theta}_2)sen\theta_1 sen\theta_2$$

$$\upsilon_2{}^2 = a_1{}^2\dot{\theta}_1{}^2 + a_2{}^2(\dot{\theta}_1 + \dot{\theta}_2)^2 + 2a_1\dot{\theta}_1 a_2(\dot{\theta}_1 + \dot{\theta}_2)cos\theta_2(sen^2\theta_1 + cos^2\theta_1)$$

$$\upsilon_2{}^2 = a_1{}^2\dot{\theta}_1{}^2 + a_2{}^2(\dot{\theta}_1 + \dot{\theta}_2)^2 + 2a_1\dot{\theta}_1 a_2(\dot{\theta}_1 + \dot{\theta}_2)cos\theta_2$$

Energía Cinética

$$K_2 = \frac{1}{2}m_2\upsilon_2{}^2 = \frac{1}{2}m_2\left(a_1{}^2\dot{\theta}_1{}^2 + a_2{}^2(\dot{\theta}_1 + \dot{\theta}_2)^2 + 2a_1\dot{\theta}_1 a_2(\dot{\theta}_1 + \dot{\theta}_2)cos\theta_2\right)$$

Energía potencial

$$h_2 = a_1 sen\theta_1 + a_2 sen(\theta_1 + \theta_2)$$

Por lo tanto:

$$P_2 = m_2 g\left(a_1 sen\theta_1 + a_2 sen(\theta_1 + \theta_2)\right)$$

3er eslabón:
De los parámetros de DH:

$$P_{x,y,z} = \begin{bmatrix} a_1 cos\theta_1 + (a_2 + a_3)cos(\theta_1 + \theta_2) \\ 0 \\ a_1 sen\theta_1 + (a_2 + a_3)sen(\theta_1 + \theta_2) \end{bmatrix}$$

Posición y Velocidad

$$P_{x,y,z} = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} \; ; \; v_3 = \frac{d}{dt}\begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \dot{x_3} + \dot{y_3} + \dot{z_3}$$

$x_3 = a_1 cos\theta_1 + a_{23}cos(\theta_1 + \theta_2) \quad \dot{x_3} = -a_1 sen\theta_1\dot{\theta}_1 - a_{23}sen(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2)$

$y_3 = 0 \qquad \dot{y_3} = 0$

$z_3 = a_1 sen\theta_1 + a_{23}sen(\theta_1 + \theta_2) \quad \dot{z_3} = a_1 cos\theta_1\dot{\theta}_1 + a_{23}cos(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2)$

$$\upsilon_3{}^2 = \dot{x_3}{}^2 + \dot{z_3}{}^2$$

$$\upsilon_3{}^2 = \left(-a_1 sen\theta_1\dot{\theta}_1 - a_{23}sen(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2)\right)^2$$
$$+ \left(a_1 cos\theta_1\dot{\theta}_1 + a_{23}cos(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2)\right)^2$$

$$\upsilon_3{}^2 = a_1{}^2 sen^2\theta_1\dot{\theta_1}{}^2 + 2a_1 sen\theta_1\dot{\theta_1} a_{23} sen(\theta_1+\theta_2)(\dot{\theta_1}+\dot{\theta_2}) + a_{23}{}^2 sen^2(\theta_1+\theta_2)(\dot{\theta_1}+\dot{\theta_2})^2 + a_1{}^2 cos^2\theta_1\dot{\theta_1}{}^2 + 2a_1 cos\theta_1\dot{\theta_1} a_{23} cos(\theta_1+\theta_2)(\dot{\theta_1}+\dot{\theta_2}) + a_{23}{}^2 cos^2(\theta_1+\theta_2)(\dot{\theta_1}+\dot{\theta_2})^2$$

$$\upsilon_3{}^2 = a_1{}^2\dot{\theta_1}{}^2 + 2a_1 sen\theta_1\dot{\theta_1} a_{23} sen(\theta_1+\theta_2)(\dot{\theta_1}+\dot{\theta_2}) + 2a_1 cos\theta_1\dot{\theta_1} a_{23} cos(\theta_1+\theta_2)(\dot{\theta_1}+\dot{\theta_2}) + a_{23}{}^2(\dot{\theta_1}+\dot{\theta_2})^2$$

$$\upsilon_3{}^2 = a_1{}^2\dot{\theta_1}{}^2 + a_{23}{}^2(\dot{\theta_1}+\dot{\theta_2})^2 + 2a_1 sen\theta_1\dot{\theta_1} a_{23}(\dot{\theta_1}+\dot{\theta_2})(sen\theta_1 cos\theta_2 + cos\theta_1 sen\theta_2) + 2a_1 cos\theta_1\dot{\theta_1} a_{23}(\dot{\theta_1}+\dot{\theta_2})(cos\theta_1 cos\theta_2 - sen\theta_1 sen\theta_2)$$

$$\upsilon_3{}^2 = a_1{}^2\dot{\theta_1}{}^2 + a_{23}{}^2(\dot{\theta_1}+\dot{\theta_2})^2 + 2a_1\dot{\theta_1} a_{23}(\dot{\theta_1}+\dot{\theta_2}) sen^2\theta_1 cos\theta_2 + 2a_1 sen\theta_1\dot{\theta_1} a_{23}(\dot{\theta_1}+\dot{\theta_2}) cos\theta_1 sen\theta_2 + 2a_1\dot{\theta_1} a_{23}(\dot{\theta_1}+\dot{\theta_2}) cos^2\theta_1 cos\theta_2 - 2a_1 cos\theta_1\dot{\theta_1} a_{23}(\dot{\theta_1}+\dot{\theta_2}) sen\theta_1 sen\theta_2$$

$$\upsilon_3{}^2 = a_1{}^2\dot{\theta_1}{}^2 + a_{23}{}^2(\dot{\theta_1}+\dot{\theta_2})^2 + 2a_1\dot{\theta_1} a_{23}(\dot{\theta_1}+\dot{\theta_2}) cos\theta_2(sen^2\theta_1 + cos^2\theta_1)$$

$$\upsilon_3{}^2 = a_1{}^2\dot{\theta_1}{}^2 + a_{23}{}^2(\dot{\theta_1}+\dot{\theta_2})^2 + 2a_1\dot{\theta_1} a_{23}(\dot{\theta_1}+\dot{\theta_2}) cos\theta_2$$

Energía Cinética

$$K_3 = \frac{1}{2} m_3\left(a_1{}^2\dot{\theta}_1{}^2 + a_{23}{}^2(\dot{\theta}_1+\dot{\theta}_2)^2 + 2a_1\dot{\theta}_1 a_{23}(\dot{\theta}_1+\dot{\theta}_2) cos\theta_2\right)$$

Energía potencial

$$h_1 = a_1 sen\theta_1 + a_{23} sen(\theta_1+\theta_2)$$

Entonces

$$P_3 = m_3 g\left(a_1 sen\theta_1 + a_{23} sen(\theta_1+\theta_2)\right)$$

Obtención de la ecuación de Lagrange

$$L = K_1 + K_2 + K_3 - P_1 - P_2 - P_3$$

$$L = \frac{1}{2} m_1 a_1{}^2\dot{\theta}_1{}^2 + \frac{1}{2} m_2 a_1{}^2\dot{\theta}_1{}^2 + \frac{1}{2} m_2 a_2{}^2(\dot{\theta}_1+\dot{\theta}_2)^2 + m_2 a_1\dot{\theta}_1 a_2(\dot{\theta}_1+\dot{\theta}_2) cos\theta_2 + \frac{1}{2} m_3 a_1{}^2\dot{\theta}_1{}^2 + \frac{1}{2} m_3 a_{23}{}^2(\dot{\theta}_1+\dot{\theta}_2)^2 + m_3 a_1\dot{\theta}_1 a_{23}(\dot{\theta}_1+\dot{\theta}_2) cos\theta_2 - m_1 g a_1 sen\theta_1 - m_2 g\left(a_1 sen\theta_1 + a_2 sen(\theta_1+\theta_2)\right) - m_3 g\left(a_1 sen\theta_1 + a_{23} sen(\theta_1+\theta_2)\right)$$

$$L = \frac{1}{2} m_1 a_1{}^2\dot{\theta}_1{}^2 + \frac{1}{2} m_2 a_1{}^2\dot{\theta_1}{}^2 + \frac{1}{2} m_2 a_2{}^2\left(\dot{\theta}_1{}^2 + 2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2{}^2\right) + m_2 a_1\dot{\theta_1} a_2 cos\theta_2(\dot{\theta_1}+\dot{\theta_2}) + \frac{1}{2} m_3 a_1{}^2\dot{\theta_1}{}^2 + \frac{1}{2} m_3 a_{23}{}^2\left(\dot{\theta}_1{}^2 + 2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2{}^2\right) + m_3 a_1\dot{\theta_1} a_{23} cos\theta_2(\dot{\theta_1}+\dot{\theta_2}) - m_1 g a_1 sen\theta_1 - m_2 g\left(a_1 sen\theta_1 + a_2 sen(\theta_1+\theta_2)\right) - m_3 g\left(a_1 sen\theta_1 + a_{23} sen(\theta_1+\theta_2)\right)$$

$$L = \frac{1}{2}m_1a_1{}^2\dot{\theta}_1{}^2 + \frac{1}{2}m_2a_1{}^2\dot{\theta}_1{}^2 + \frac{1}{2}m_2a_2{}^2\dot{\theta}_1{}^2 + m_2a_2{}^2\dot{\theta}_1\dot{\theta}_2 + \frac{1}{2}m_2a_2{}^2\dot{\theta}_2{}^2 + m_2a_1a_2cos\theta_2\dot{\theta}_1{}^2 + m_2a_1a_2cos\theta_2\dot{\theta}_1\dot{\theta}_2 + \frac{1}{2}m_3a_1{}^2\dot{\theta}_1{}^2 + \frac{1}{2}m_3a_{23}{}^2\dot{\theta}_1{}^2 + m_3a_{23}{}^2\dot{\theta}_1\dot{\theta}_2 + \frac{1}{2}m_3a_{23}{}^2\dot{\theta}_2{}^2 + m_3a_1a_{23}cos\theta_2\dot{\theta}_1{}^2 + m_3a_1a_{23}cos\theta_2\dot{\theta}_1\dot{\theta}_2 - m_1ga_1sen\theta_1 - m_2ga_1sen\theta_1 - m_2ga_2sen\theta_1cos\theta_2 - m_2ga_2cos\theta_1sen\theta_2 - m_3ga_1sen\theta_1 - m_3ga_{23}sen\theta_1cos\theta_2 - m_3ga_{23}cos\theta_1sen\theta_2$$

Cálculo de las derivadas del Lagrangeano

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right] - \frac{\partial L}{\partial q} = \tau$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = \frac{\partial}{\partial \dot{\theta}_1}\left[\frac{1}{2}m_1a_1{}^2\dot{\theta}_1{}^2 + \frac{1}{2}m_2a_1{}^2\dot{\theta}_1{}^2 + \frac{1}{2}m_2a_2{}^2\dot{\theta}_1{}^2 + m_2a_2{}^2\dot{\theta}_1\dot{\theta}_2 + m_2a_1a_2cos\theta_2\dot{\theta}_1{}^2 + m_2a_1\dot{\theta}_1a_2cos\theta_2\dot{\theta}_2 + \frac{1}{2}m_3a_1{}^2\dot{\theta}_1{}^2 + \frac{1}{2}m_3a_{23}{}^2\dot{\theta}_1{}^2 + m_3a_{23}{}^2\dot{\theta}_1\dot{\theta}_2 + m_3a_1\dot{\theta}_1{}^2a_{23}cos\theta_2 + m_3a_1\dot{\theta}_1a_{23}cos\theta_2\dot{\theta}_2\right]$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = m_1a_1{}^2\dot{\theta}_1 + m_2a_1{}^2\dot{\theta}_1 + m_2a_2{}^2\dot{\theta}_1 + m_2a_2{}^2\dot{\theta}_2 + 2m_2a_1\dot{\theta}_1a_2cos\theta_2 + m_2a_1a_2cos\theta_2\dot{\theta}_2 + m_3a_1{}^2\dot{\theta}_1 + m_3a_{23}{}^2\dot{\theta}_1 + m_3a_{23}{}^2\dot{\theta}_2 + 2m_3a_1\dot{\theta}_1a_{23}cos\theta_2 + m_3a_1a_{23}cos\theta_2\dot{\theta}_2$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = \frac{\partial}{\partial \dot{\theta}_2}\left[m_2a_2{}^2\dot{\theta}_1\dot{\theta}_2 + \frac{1}{2}m_2a_2{}^2\dot{\theta}_2{}^2 + m_2a_1\dot{\theta}_1a_2cos\theta_2\dot{\theta}_2 + m_3a_{23}{}^2\dot{\theta}_1\dot{\theta}_2 + \frac{1}{2}m_3a_{23}{}^2\dot{\theta}_2{}^2 + m_3a_1\dot{\theta}_1a_{23}cos\theta_2\dot{\theta}_2\right]$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2a_2{}^2\dot{\theta}_1 + m_2a_2{}^2\dot{\theta}_2 + m_2a_1\dot{\theta}_1a_2cos\theta_2 + m_3a_{23}{}^2\dot{\theta}_1 + m_3a_{23}{}^2\dot{\theta}_2 + m_3a_1\dot{\theta}_1a_{23}cos\theta_2$$

$$\frac{d}{dt}\begin{bmatrix} m_1a_1{}^2\dot{\theta}_1 + m_2a_1{}^2\dot{\theta}_1 + m_2a_2{}^2\dot{\theta}_1 + m_2a_2{}^2\dot{\theta}_2 + 2m_2a_1\dot{\theta}_1a_2cos\theta_2 + m_2a_1a_2cos\theta_2\dot{\theta}_2 + m_3a_1{}^2\dot{\theta}_1 + m_3a_{23}{}^2\dot{\theta}_1 + m_3a_{23}{}^2\dot{\theta}_2 + 2m_3a_1\dot{\theta}_1a_{23}cos\theta_2 + m_3a_1a_{23}cos\theta_2\dot{\theta}_2 \\ m_2a_2{}^2\dot{\theta}_1 + m_2a_2{}^2\dot{\theta}_2 + m_2a_1\dot{\theta}_1a_2cos\theta_2 + m_3a_{23}{}^2\dot{\theta}_1 + m_3a_{23}{}^2\dot{\theta}_2 + m_3a_1\dot{\theta}_1a_{23}cos\theta_2 \end{bmatrix}$$
$$= \begin{bmatrix} m_1a_1{}^2\ddot{\theta}_1 + m_2a_1{}^2\ddot{\theta}_1 + m_2a_2{}^2\ddot{\theta}_1 + m_2a_2{}^2\ddot{\theta}_2 + 2m_2a_1a_2(-sen\theta_2\dot{\theta}_1\dot{\theta}_2 + \cos\theta_2\ddot{\theta}_1) + m_2a_1a_2(-sen\theta_2\dot{\theta}_2{}^2 + \cos\theta_2\ddot{\theta}_2) + m_3a_1{}^2\ddot{\theta}_1 + m_3a_{23}{}^2\ddot{\theta}_1 + m_3a_{23}{}^2\ddot{\theta}_1 + 2m_3a_1a_{23}(-sen\theta_2\dot{\theta}_1\dot{\theta}_2 + cos\theta_2\ddot{\theta}_1) + m_3a_1a_{23}(-sen\theta_2\dot{\theta}_2{}^2 + cos\theta_2\ddot{\theta}_2) \\ m_2a_2{}^2\ddot{\theta}_1 + m_2a_2{}^2\ddot{\theta}_2 + m_2a_1a_2(-sen\theta_2\dot{\theta}_1\dot{\theta}_2 + \cos\theta_2\ddot{\theta}_1) + m_3a_{23}{}^2\ddot{\theta}_1 + m_3a_{23}{}^2\ddot{\theta}_2 + m_3a_1a_{23}(-sen\theta_2\dot{\theta}_1\dot{\theta}_2 + cos\theta_2\ddot{\theta}_1) \end{bmatrix}$$

$$\frac{\partial L}{\partial q} = \begin{bmatrix} \frac{\partial L}{\partial \theta_1} \\ \frac{\partial L}{\partial \theta_2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial}{\partial\theta_1}[-m_1 g a_1 sen\theta_1 - m_2 g a_1 sen\theta_1 - m_2 g a_2 sen\theta_1 cos\theta_2 - m_2 g a_2 cos\theta_1 sen\theta_2 - m_3 g a_1 sen\theta_1 - m_3 g a_{23} sen\theta_1 cos\theta_2 - m_3 g a_{23} cos\theta_1 sen\theta_2] \\ \frac{\partial}{\partial\theta_2}\left[m_2 a_1 a_2 cos\theta_2 \dot\theta_1^{\,2} + m_2 a_1 a_2 cos\theta_2 \dot\theta_1\dot\theta_2 + m_3 a_1 a_{23} cos\theta_2 \dot\theta_1^{\,2} + m_3 a_1 a_{23} cos\theta_2\dot\theta_1\dot\theta_2 - m_2 g a_2 sen\theta_1 cos\theta_2 - m_2 g a_2 cos\theta_1 sen\theta_2 - m_3 g a_{23} sen\theta_1 cos\theta_2 - m_3 g a_{23} cos\theta_1 sen\theta_2\right] \end{bmatrix}$$

$$\frac{\partial L}{\partial q} = \begin{bmatrix} -m_1 g a_1 cos\theta_1 - m_2 g a_1 cos\theta_1 - m_2 g a_2 cos\theta_1 cos\theta_2 + m_2 g a_2 sen\theta_1 sen\theta_2 - m_3 g a_1 cos\theta_1 - m_3 g a_{23} cos\theta_1 cos\theta_2 + m_3 g a_{23} sen\theta_1 sen\theta_2 \\ -m_2 a_1 a_2 sen\theta_2 \dot\theta_1^{\,2} - m_2 a_1 a_2 sen\theta_2 \dot\theta_1\dot\theta_2 - m_3 a_1 a_{23} sen\theta_2 \dot\theta_1^{\,2} - m_3 a_1 a_{23} sen\theta_2 \dot\theta_1\dot\theta_2 + m_2 g a_2 sen\theta_1 sen\theta_2 - m_2 g a_2 cos\theta_1 cos\theta_2 + m_3 g a_{23} sen\theta_1 sen\theta_2 - m_3 g a_{23} cos\theta_1 cos\theta_2 \end{bmatrix}$$

Por lo que finalmente tenemos:

$$\frac{\partial L}{\partial q} = \begin{bmatrix} \frac{\partial L}{\partial \theta_1} \\ \frac{\partial L}{\partial \theta_2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial}{\partial\theta_1}[-m_1 g a_1 sen\theta_1 - m_2 g a_1 sen\theta_1 - m_2 g a_2 sen\theta_1 cos\theta_2 - m_2 g a_2 cos\theta_1 sen\theta_2 - m_3 g a_1 sen\theta_1 - m_3 g a_{23} sen\theta_1 cos\theta_2 - m_3 g a_{23} cos\theta_1 sen\theta_2] \\ \frac{\partial}{\partial\theta_2}\left[m_2 a_1 a_2 cos\theta_2 \dot\theta_1^{\,2} + m_2 a_1 a_2 cos\theta_2 \dot\theta_1\dot\theta_2 + m_3 a_1 a_{23} cos\theta_2 \dot\theta_1^{\,2} + m_3 a_1 a_{23} cos\theta_2\dot\theta_1\dot\theta_2 - m_2 g a_2 sen\theta_1 cos\theta_2 - m_2 g a_2 cos\theta_1 sen\theta_2 - m_3 g a_{23} sen\theta_1 cos\theta_2 - m_3 g a_{23} cos\theta_1 sen\theta_2\right] \end{bmatrix}$$

$$\frac{\partial L}{\partial q} = \begin{bmatrix} -m_1 g a_1 cos\theta_1 - m_2 g a_1 cos\theta_1 - m_2 g a_2 cos\theta_1 cos\theta_2 + m_2 g a_2 sen\theta_1 sen\theta_2 - m_3 g a_1 cos\theta_1 - m_3 g a_{23} cos\theta_1 cos\theta_2 + m_3 g a_{23} sen\theta_1 sen\theta_2 \\ -m_2 a_1 a_2 sen\theta_2 \dot\theta_1^{\,2} - m_2 a_1 a_2 sen\theta_2 \dot\theta_1\dot\theta_2 - m_3 a_1 a_{23} sen\theta_2 \dot\theta_1^{\,2} - m_3 a_1 a_{23} sen\theta_2 \dot\theta_1\dot\theta_2 + m_2 g a_2 sen\theta_1 sen\theta_2 - m_2 g a_2 cos\theta_1 cos\theta_2 + m_3 g a_{23} sen\theta_1 sen\theta_2 - m_3 g a_{23} cos\theta_1 cos\theta_2 \end{bmatrix}$$

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right]-\frac{\partial L}{\partial q}=\tau$$

$$\begin{bmatrix} m_1a_1{}^2\ddot{\theta}_1+m_2a_1{}^2\ddot{\theta}_1+m_2a_2{}^2\ddot{\theta}_1+m_2a_2{}^2\ddot{\theta}_2+2m_2a_1a_2(-sen\theta_2\dot{\theta}_1\dot{\theta}_2+\cos\theta_2\ddot{\theta}_1)+m_2a_1a_2\left(-sen\theta_2\dot{\theta}_2{}^2+\cos\theta_2\ddot{\theta}_2\right)+m_3a_1{}^2\ddot{\theta}_1+m_3a_{23}{}^2\ddot{\theta}_1+m_3a_{23}{}^2\ddot{\theta}_1+2m_3a_1a_{23}(-sen\theta_2\dot{\theta}_1\dot{\theta}_2+cos\theta_2\ddot{\theta}_1)+m_3a_1a_{23}\left(-sen\theta_2\dot{\theta}_2{}^2+cos\theta_2\ddot{\theta}_2\right) \\ m_2a_2{}^2\ddot{\theta}_1+m_2a_2{}^2\ddot{\theta}_2+m_2a_1a_2(-sen\theta_2\dot{\theta}_1\dot{\theta}_2+\cos\theta_2\ddot{\theta}_1)+m_3a_{23}{}^2\ddot{\theta}_1+m_3a_{23}{}^2\ddot{\theta}_2+m_3a_1a_{23}(-sen\theta_2\dot{\theta}_1\dot{\theta}_2+cos\theta_2\ddot{\theta}_1) \end{bmatrix}$$

$$-\begin{bmatrix} -m_1ga_1cos\theta_1-m_2ga_1cos\theta_1-m_2ga_2cos\theta_1cos\theta_2+m_2ga_2sen\theta_1sen\theta_2-m_3ga_1cos\theta_1-m_3ga_{23}cos\theta_1cos\theta_2+m_3ga_{23}sen\theta_1sen\theta_2 \\ -m_2a_1a_2sen\theta_2\dot{\theta}_1{}^2-m_2a_1a_2sen\theta_2\dot{\theta}_1\dot{\theta}_2-m_3a_1a_{23}sen\theta_2\dot{\theta}_1{}^2-m_3a_1a_{23}sen\theta_2\dot{\theta}_1\dot{\theta}_2+m_2ga_2sen\theta_1sen\theta_2-m_2ga_2cos\theta_1cos\theta_2+m_3ga_{23}sen\theta_1sen\theta_2-m_3ga_{23}cos\theta_1cos\theta_2 \end{bmatrix}=\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix}$$