2025年电子科技大学自动化工程学院考研数学第二次模拟试卷答案

一、选择题

(1)【答案】 (D).

【解】 由$\lim\limits_{x\to 0}f(x)=\lim\limits_{x\to 0}\dfrac{e^x-1}{x}=1=f(0)$得$f(x)$在$x=0$处连续;

再由$\lim\limits_{x\to 0}\dfrac{f(x)-f(0)}{x}=\lim\limits_{x\to 0}\dfrac{\dfrac{e^x-1}{x}-1}{x}=\lim\limits_{x\to 0}\dfrac{e^x-1-x}{x^2}=\lim\limits_{x\to 0}\dfrac{e^x-1}{2x}=\dfrac{1}{2}$得

$f'(0)=\dfrac{1}{2}\neq 0$,应选(D).

(2)【答案】 (C).

【解】 $f(x+1,e^x)=x(x+1)^2$两边对x求导得

$$f'_1(x+1,e^x)+e^x f'_2(x+1,e^x)=(x+1)^2+2x(x+1),$$

取$x=0$得$f'_1(1,1)+f'_2(1,1)=1$;

$f(x,x^2)=2x^2\ln x$两边对x求导得

$$f'_1(x,x^2)+2xf'_2(x,x^2)=4x\ln x+2x,$$

取$x=1$得$f'_1(1,1)+2f'_2(1,1)=2$,

解得$f'_1(1,1)=0$,$f'_2(1,1)=1$,故$\mathrm{d}f(1,1)=\mathrm{d}y$,应选(C).

(3)【答案】 (A).

【解】 因为$f(x)=\dfrac{\sin x}{1+x^2}$为奇函数,所以$b=0$;

由$\sin x=x-\dfrac{x^3}{6}+o(x^3)$,$\dfrac{1}{1+x^2}=1-x^2+o(x^3)$则

$$f(x)=\frac{\sin x}{1+x^2}=x-\frac{7}{6}x^3+o(x^3),$$

应选(A).

(4)【答案】 (B).

【解】 $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}f\left(\dfrac{2k-1}{2n}\right)\dfrac{1}{n}=\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}f\left(\dfrac{2k}{2n}\right)\dfrac{1}{n}=\lim\limits_{n\to\infty}\dfrac{1}{n}\sum\limits_{k=1}^{n}f\left(\dfrac{k}{n}\right)=\int_0^1 f(x)\mathrm{d}x$,应选(B).

(5)【答案】 (B).

【解】 令$\boldsymbol{A}=\begin{pmatrix}0&1&1\\1&2&1\\1&1&0\end{pmatrix}$,$\boldsymbol{X}=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$,则$f=\boldsymbol{X}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{X}$,

$$由|\lambda\boldsymbol{E}-\boldsymbol{A}|=\begin{vmatrix}\lambda&-1&-1\\-1&\lambda-2&-1\\-1&-1&\lambda\end{vmatrix}=(\lambda+1)\begin{vmatrix}1&0&0\\-1&\lambda-2&-2\\-1&-1&\lambda-1\end{vmatrix}$$

$$=(\lambda+1)(\lambda^2-3\lambda)=0$$

得$\lambda_1=-1$,$\lambda_2=0$,$\lambda_3=3$,应选(B).

(6)【答案】 (A).

【解】 由施密特正交化得 $l_1=\dfrac{(\boldsymbol{\alpha}_3,\boldsymbol{\beta}_1)}{(\boldsymbol{\beta}_1,\boldsymbol{\beta}_1)}=\dfrac{5}{2}$,$l_2=\dfrac{(\boldsymbol{\alpha}_3,\boldsymbol{\beta}_2)}{(\boldsymbol{\beta}_2,\boldsymbol{\beta}_2)}=\dfrac{2}{4}=\dfrac{1}{2}$,应选(A).

方法点评:将线性无关的向量组化为两两正交的规范向量组即施密特正交规范化,实对称矩阵的对角化的正交变换法需要将线性无关的特征向量进行正交化和单位化.
设 $\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3$ 线性无关,$\boldsymbol{\beta}_1=\boldsymbol{\alpha}_1$,$\boldsymbol{\beta}_2=\boldsymbol{\alpha}_2-l_1\boldsymbol{\beta}_1$,$\boldsymbol{\beta}_3=\boldsymbol{\alpha}_3-k_1\boldsymbol{\beta}_1-k_2\boldsymbol{\beta}_2$,且 $\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\boldsymbol{\beta}_3$ 线性无关,则 $l_1=\dfrac{(\boldsymbol{\alpha}_2,\boldsymbol{\beta}_1)}{(\boldsymbol{\beta}_1,\boldsymbol{\beta}_1)}$,$k_1=\dfrac{(\boldsymbol{\alpha}_3,\boldsymbol{\beta}_1)}{(\boldsymbol{\beta}_1,\boldsymbol{\beta}_1)}$,$k_2=\dfrac{(\boldsymbol{\alpha}_3,\boldsymbol{\beta}_2)}{(\boldsymbol{\beta}_2,\boldsymbol{\beta}_2)}$.

(7)【答案】 (C).

【解】 $r\begin{pmatrix}\boldsymbol{A} & \boldsymbol{O}\\ \boldsymbol{O} & \boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\end{pmatrix}=r(\boldsymbol{A})+r(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A})=2r(\boldsymbol{A})$;

由 $\begin{pmatrix}\boldsymbol{A} & \boldsymbol{AB}\\ \boldsymbol{O} & \boldsymbol{A}^{\mathrm{T}}\end{pmatrix}\xrightarrow{\text{列}}\begin{pmatrix}\boldsymbol{A} & \boldsymbol{O}\\ \boldsymbol{O} & \boldsymbol{A}^{\mathrm{T}}\end{pmatrix}$ 得 $r\begin{pmatrix}\boldsymbol{A} & \boldsymbol{AB}\\ \boldsymbol{O} & \boldsymbol{A}^{\mathrm{T}}\end{pmatrix}=2r(\boldsymbol{A})$;

由 $r\begin{pmatrix}\boldsymbol{A} & \boldsymbol{O}\\ \boldsymbol{BA} & \boldsymbol{A}^{\mathrm{T}}\end{pmatrix}\xrightarrow{\text{行}}r\begin{pmatrix}\boldsymbol{A} & \boldsymbol{O}\\ \boldsymbol{O} & \boldsymbol{A}^{\mathrm{T}}\end{pmatrix}$ 得 $r\begin{pmatrix}\boldsymbol{A} & \boldsymbol{O}\\ \boldsymbol{BA} & \boldsymbol{A}^{\mathrm{T}}\end{pmatrix}=2r(\boldsymbol{A})$,应选(C).

(8)【答案】 (D).

【解】 由 $P(A\mid B)=P(A)$ 得 $P(AB)=P(A)P(B)$,即事件 A,B 独立,

于是 $P(A\mid\overline{B})=\dfrac{P(A\overline{B})}{P(\overline{B})}=\dfrac{P(A)P(\overline{B})}{P(\overline{B})}=P(A)$;

由 $P(A\mid B)>P(A)$ 得 $P(AB)>P(A)P(B)$,

从而 $P(\overline{A}\mid\overline{B})=\dfrac{P(\overline{A}\,\overline{B})}{P(\overline{B})}=\dfrac{1-P(A)-P(B)+P(AB)}{1-P(B)}$

$$>\frac{1-P(A)-P(B)+P(A)P(B)}{1-P(B)}=1-P(A)=P(\overline{A});$$

由 $P(A\mid B)>P(A\mid\overline{B})$ 得 $\dfrac{P(AB)}{P(B)}>\dfrac{P(A)-P(AB)}{1-P(B)}$,整理得 $P(AB)>P(A)P(B)$,

则 $P(A\mid B)=\dfrac{P(AB)}{P(B)}>\dfrac{P(A)P(B)}{P(B)}=P(A)$,应选(D).

(9)【答案】 (C).

【解】 $\overline{X}\sim N\left(\mu_1,\dfrac{\sigma_1^2}{n}\right)$,$\overline{Y}\sim N\left(\mu_2,\dfrac{\sigma_2^2}{n}\right)$,

则$E(\hat{\theta})=E(\overline{X})-E(\overline{Y})=\mu_1-\mu_2=\theta$;

$$\begin{aligned}D(\hat{\theta})&=D(\overline{X}-\overline{Y})=D(\overline{X})+D(\overline{Y})-2\mathrm{Cov}(\overline{X},\overline{Y})\\&=\frac{\sigma_1^2}{n}+\frac{\sigma_2^2}{n}-\frac{2}{n}[\mathrm{Cov}(X_1,\overline{Y})+\mathrm{Cov}(X_2,\overline{Y})+\cdots+\mathrm{Cov}(X_n,\overline{Y})]\\&=\frac{\sigma_1^2}{n}+\frac{\sigma_2^2}{n}-\frac{2}{n^2}[\mathrm{Cov}(X_1,Y_1)+\mathrm{Cov}(X_2,Y_2)+\cdots+\mathrm{Cov}(X_n,Y_n)]\\&=\frac{\sigma_1^2}{n}+\frac{\sigma_2^2}{n}-\frac{2}{n^2}\cdot n\rho\sigma_1\sigma_2=\frac{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}{n},\text{应选(C).}\end{aligned}$$

(10)**【答案】** (B).

【解】 由题 $\overline{X}\sim N\left(11.5,\frac{1}{4}\right)$,或$\frac{\overline{X}-11.5}{\frac{1}{2}}\sim N(0,1)$,

犯第二类错误的概率为

$$P\{\overline{X}<11\}=P\left\{\frac{\overline{X}-11.5}{\frac{1}{2}}<-1\right\}=\Phi(-1)=1-\Phi(1),$$

应选(B).

二、填空题

(11)**【答案】** $\frac{\pi}{4}$.

【解】 $\int_0^{+\infty}\frac{\mathrm{d}x}{x^2+2x+2}=\int_0^{+\infty}\frac{\mathrm{d}(x+1)}{1+(x+1)^2}=\arctan(x+1)\Big|_0^{+\infty}=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$.

(12)**【答案】** $\frac{2}{3}$.

【解】 $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t}=\frac{4t\mathrm{e}^t+2t}{2\mathrm{e}^t+1}=2t$,$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}(2t)/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t}=\frac{2}{2\mathrm{e}^t+1}$,

则$\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{t=0}=\frac{2}{3}$.

(13)**【答案】** x^2.

【解】 令 $x=\mathrm{e}^t$,$D=\frac{\mathrm{d}}{\mathrm{d}t}$,则

$$xy'=Dy,x^2y''=D(D-1)y,$$

代入欧拉方程得

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2}-4y=0,$$

特征方程为$\lambda^2-4=0$,特征根为$\lambda_1=-2$,$\lambda_2=2$,

$\frac{\mathrm{d}^2y}{\mathrm{d}t^2}-4y=0$ 的通解为 $y=C_1\mathrm{e}^{-2t}+C_2\mathrm{e}^{2t}$,原方程的通解为

$$y=\frac{C_1}{x^2}+C_2x^2,$$

由 $y(1)=1$,$y'(1)=2$ 得 $C_1+C_2=1$,$-2C_1+2C_2=2$,解得 $C_1=0$,$C_2=1$,

故 $y=x^2$.

方法点评:形如

$$x^ny^{(n)}+a_{n-1}x^{n-1}y^{(n-1)}+\cdots+a_1xy'+a_0y=f(x)$$

的方程称为欧拉方程.

令 $x=\mathrm{e}^t$,则$xy'=Dy=\frac{\mathrm{d}y}{\mathrm{d}t}$,$x^2y''=D(D-1)y=\frac{\mathrm{d}^2y}{\mathrm{d}t^2}-\frac{\mathrm{d}y}{\mathrm{d}t}$,

$$x^ny^{(n)}=D(D-1)\cdots(D-n+1)y,$$

代入原方程得高阶常系数线性微分方程,求出其通解,再将$t=\ln x$代入即可得原方程的通解.

(14)【答案】 4π.

【解】 设 Σ 所围成的几何体为 Ω,由高斯公式得

$$I=\oiint_{\Sigma} x^2\mathrm{d}y\mathrm{d}z+y^2\mathrm{d}z\mathrm{d}x+z\mathrm{d}x\mathrm{d}y=\iiint_{\Omega}(2x+2y+1)\mathrm{d}v,$$

由积分的奇偶性得

$$I=\iiint_{\Omega}\mathrm{d}v=2\iint_{D_{xy}}\mathrm{d}x\mathrm{d}y=2\cdot\pi\cdot 1\cdot 2=4\pi.$$

(15)【答案】 $\dfrac{3}{2}$.

【解】 $|\boldsymbol{A}|=2\begin{vmatrix}1 & a_{12} & a_{13}\\ 1 & a_{22} & a_{23}\\ 1 & a_{32} & a_{33}\end{vmatrix}=2(A_{11}+A_{21}+A_{31})=3$,则

$$A_{11}+A_{21}+A_{31}=\frac{3}{2}.$$

(16)【答案】 $\dfrac{1}{5}$.

【解】 (X,Y) 的可能取值为 $(0,0),(0,1),(1,0),(1,1)$,

$P\{X=0,Y=0\}=\dfrac{1}{2}\cdot\dfrac{3}{5}=\dfrac{3}{10}$,

$P\{X=0,Y=1\}=\dfrac{1}{2}\cdot\dfrac{2}{5}=\dfrac{1}{5}$,

$P\{X=1,Y=0\}=\dfrac{1}{2}\cdot\dfrac{2}{5}=\dfrac{1}{5}$,

$P\{X=1,Y=1\}=\dfrac{1}{2}\cdot\dfrac{3}{5}=\dfrac{3}{10}$,

由 $X\sim\begin{pmatrix}0 & 1\\ \dfrac{1}{2} & \dfrac{1}{2}\end{pmatrix}$ 得 $E(X)=\dfrac{1}{2},E(X^2)=\dfrac{1}{2},D(X)=\dfrac{1}{4}$;

由 $Y\sim\begin{pmatrix}0 & 1\\ \dfrac{1}{2} & \dfrac{1}{2}\end{pmatrix}$ 得 $E(Y)=\dfrac{1}{2},E(Y^2)=\dfrac{1}{2},D(Y)=\dfrac{1}{4}$;

由 $XY\sim\begin{pmatrix}0 & 1\\ \dfrac{7}{10} & \dfrac{3}{10}\end{pmatrix}$ 得 $E(XY)=\dfrac{3}{10}$,

$\mathrm{Cov}(X,Y)=E(XY)-E(X)E(Y)=\dfrac{3}{10}-\dfrac{1}{4}=\dfrac{1}{20}$,则 $\rho_{XY}=\dfrac{\dfrac{1}{20}}{\dfrac{1}{2}\cdot\dfrac{1}{2}}=\dfrac{1}{5}$.

三、解答题

(17)【解】 **方法一**

$$\lim_{x\to 0}\left(\frac{1+\int_0^x e^{t^2}\mathrm{d}t}{e^x-1}-\frac{1}{\sin x}\right)=\lim_{x\to 0}\frac{\left(1+\int_0^x e^{t^2}\mathrm{d}t\right)\sin x-e^x+1}{(e^x-1)\sin x}$$

$$=\lim_{x\to 0}\frac{\left(1+\int_0^x e^{t^2}dt\right)\sin x-e^x+1}{x^2}$$

$$=\lim_{x\to 0}\left(\frac{\sin x-x}{x^2}+\frac{\int_0^x e^{t^2}dt\cdot\sin x-e^x+1+x}{x^2}\right)$$

$$=\lim_{x\to 0}\frac{\int_0^x e^{t^2}dt\cdot\sin x-e^x+1+x}{x^2}$$

$$=\lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{\int_0^x e^{t^2}dt}{x}-\lim_{x\to 0}\frac{e^x-1-x}{x^2}$$

$$=\lim_{x\to 0}e^{x^2}-\lim_{x\to 0}\frac{e^x-1}{2x}=1-\frac{1}{2}=\frac{1}{2}.$$

方法二

$$\lim_{x\to 0}\left(\frac{1+\int_0^x e^{t^2}dt}{e^x-1}-\frac{1}{\sin x}\right)=\lim_{x\to 0}\left(\frac{\int_0^x e^{t^2}dt}{e^x-1}+\frac{1}{e^x-1}-\frac{1}{\sin x}\right),$$

由$\lim\limits_{x\to 0}\dfrac{\int_0^x e^{t^2}dt}{e^x-1}=\lim\limits_{x\to 0}\dfrac{e^{x^2}}{e^x}=1$，

$$\lim_{x\to 0}\left(\frac{1}{e^x-1}-\frac{1}{\sin x}\right)=\lim_{x\to 0}\frac{\sin x-e^x+1}{(e^x-1)\sin x}=\lim_{x\to 0}\frac{\sin x-e^x+1}{x^2}$$

$$=\frac{1}{2}\lim_{x\to 0}\frac{\cos x-e^x}{x}=\frac{1}{2}\lim_{x\to 0}(-\sin x-e^x)=-\frac{1}{2},$$

得$\lim\limits_{x\to 0}\left(\dfrac{1+\int_0^x e^{t^2}dt}{e^x-1}-\dfrac{1}{\sin x}\right)=1-\dfrac{1}{2}=\dfrac{1}{2}$.

方法三

由泰勒公式得 $e^{t^2}=1+t^2+o(t^2)$，

从而$\int_0^x e^{t^2}dt=x+\dfrac{x^3}{3}+o(x^3)$，于是有

$$\lim_{x\to 0}\left(\frac{1+\int_0^x e^{t^2}dt}{e^x-1}-\frac{1}{\sin x}\right)=\lim_{x\to 0}\left[\frac{1+x+\frac{x^3}{3}+o(x^3)}{e^x-1}-\frac{1}{\sin x}\right]=\lim_{x\to 0}\left(\frac{1+x}{e^x-1}-\frac{1}{\sin x}\right)$$

$$=\lim_{x\to 0}\frac{x}{e^x-1}+\lim_{x\to 0}\left(\frac{1}{e^x-1}-\frac{1}{\sin x}\right)$$

$$=1+\lim_{x\to 0}\frac{\sin x-e^x+1}{(e^x-1)\sin x}=1+\lim_{x\to 0}\frac{\sin x-e^x+1}{x^2}$$

$$=1+\lim_{x\to 0}\frac{\cos x-e^x}{2x}=1+\lim_{x\to 0}\frac{-\sin x-e^x}{2}=\frac{1}{2}.$$

(18)**【解】** $\sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \mathrm{e}^{-nx} + \sum_{n=1}^{\infty} \frac{x^{n+1}}{n(n+1)}$,

当 $\lim_{n\to\infty} \frac{\mathrm{e}^{-(n+1)x}}{\mathrm{e}^{-nx}} = \mathrm{e}^{-x} < 1$ 即 $x > 0$ 时, $\sum_{n=1}^{\infty} \mathrm{e}^{-nx}$ 收敛;

再由 $\lim_{n\to\infty} \frac{\frac{1}{(n+1)(n+2)}}{\frac{1}{n(n+1)}} = 1$ 得 $\sum_{n=1}^{\infty} \frac{x^{n+1}}{n(n+1)}$ 的收敛半径为 $R = 1$,

当 $x = \pm 1$ 时, $\sum_{n=1}^{\infty} \left| \frac{(\pm 1)^{n+1}}{n(n+1)} \right| = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$,故 $\sum_{n=1}^{\infty} \frac{x^{n+1}}{n(n+1)}$ 的收敛域为$[-1,1]$,

故级数 $\sum_{n=1}^{\infty} u_n(x)$ 的收敛域为$(0,1]$.

令 $S(x) = \sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \mathrm{e}^{-nx} + \sum_{n=1}^{\infty} \frac{x^{n+1}}{n(n+1)} = S_1(x) + S_2(x)$,

且 $S_1(x) = \sum_{n=1}^{\infty} \mathrm{e}^{-nx} = \frac{\mathrm{e}^{-x}}{1-\mathrm{e}^{-x}} = \frac{1}{\mathrm{e}^x - 1}$;

$$\begin{aligned} S_2(x) &= \sum_{n=1}^{\infty} \frac{x^{n+1}}{n(n+1)} = \sum_{n=1}^{\infty} \frac{x^{n+1}}{n} - \sum_{n=1}^{\infty} \frac{x^{n+1}}{n+1} = x\sum_{n=1}^{\infty} \frac{x^n}{n} - \sum_{n=1}^{\infty} \frac{x^n}{n} + x \\ &= (1-x)\ln(1-x) + x \quad (0 < x < 1), \end{aligned}$$

当 $x = 1$ 时,由 $S_1(1) = \frac{1}{\mathrm{e}-1}$, $S_2(1) = 1$ 得 $S(1) = \frac{1}{\mathrm{e}-1} + 1 = \frac{\mathrm{e}}{\mathrm{e}-1}$,

故
$$S(x) = \begin{cases} \frac{1}{\mathrm{e}^x - 1} + (1-x)\ln(1-x) + x, & 0 < x < 1, \\ \frac{\mathrm{e}}{\mathrm{e}-1}, & x = 1. \end{cases}$$

(19)**【解】** 设 $M(x,y,z) \in C$,点 M 到 xOy 坐标面的距离 $d = |z|$,

令 $F = z^2 + \lambda(x^2 + 2y^2 - z - 6) + \mu(4x + 2y + z - 30)$,

由 $\begin{cases} F'_x = 2\lambda x + 4\mu = 0, \\ F'_y = 4\lambda y + 2\mu = 0, \\ F'_z = 2z - \lambda + \mu = 0, \\ F'_\lambda = x^2 + 2y^2 - z - 6 = 0, \\ F'_\mu = 4x + 2y + z - 30 = 0 \end{cases}$ 得 $\begin{cases} x = 4, \\ y = 1, \\ z = 12, \end{cases}$ 或 $\begin{cases} x = -8, \\ y = -2, \\ z = 66, \end{cases}$

故 C 上的点$(-8,-2,66)$到 xOy 面的距离最大为 66.

(20)**【解】** (Ⅰ)显然 $I(D) = \iint_D (4 - x^2 - y^2)\mathrm{d}x\,\mathrm{d}y$ 取最大值的区域为 $4 - x^2 - y^2 \geqslant 0$,

即 $D_1 = \{(x,y) \mid x^2 + y^2 \leqslant 4\}$,则

$$\begin{aligned} I(D_1) &= \iint_{D_1} (4 - x^2 - y^2)\mathrm{d}x\,\mathrm{d}y = 2\pi\int_0^2 r(4 - r^2)\mathrm{d}r \\ &= 2\pi\int_0^2 (4r - r^3)\mathrm{d}r = 2\pi(8 - 4) = 8\pi; \end{aligned}$$

(Ⅱ)令 $L_0: x^2+4y^2=r^2(r>0, L_0$ 在 L 内,取逆时针),设 ∂D_1 与 L_0^- 所围成的区域为 D_0,L_0 围成的区域为 D_2,则

$$\int_{\partial D_1}\frac{(xe^{x^2+4y^2}+y)dx+(4ye^{x^2+4y^2}-x)dy}{x^2+4y^2}$$

$$=\oint_{\partial D_1+L_0^-}\frac{(xe^{x^2+4y^2}+y)dx+(4ye^{x^2+4y^2}-x)dy}{x^2+4y^2}+$$

$$\int_{L_0}\frac{(xe^{x^2+4y^2}+y)dx+(4ye^{x^2+4y^2}-x)dy}{x^2+4y^2},$$

而 $\oint_{\partial D_1+L_0^-}\frac{(xe^{x^2+4y^2}+y)dx+(4ye^{x^2+4y^2}-x)dy}{x^2+4y^2}=\iint_{D_0}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy=0$,

$$\int_{L_0}\frac{(xe^{x^2+4y^2}+y)dx+(4ye^{x^2+4y^2}-x)dy}{x^2+4y^2}$$

$$=\frac{1}{r^2}\int_{L_0}(xe^{x^2+4y^2}+y)dx+(4ye^{x^2+4y^2}-x)dy$$

$$=\frac{1}{r^2}\iint_{D_2}(8xye^{x^2+4y^2}-1-8xye^{x^2+4y^2}-1)dx\,dy$$

$$=\frac{-2}{r^2}\iint_{D_2}dx\,dy=\frac{-2}{r^2}\cdot\pi\cdot r\cdot\frac{r}{2}=-\pi.$$

故 $\int_{\partial D_1}\frac{(xe^{x^2+4y^2}+y)dx+(4ye^{x^2+4y^2}-x)dy}{x^2+4y^2}=-\pi$.

(21)**【解】**(Ⅰ)由

$$|\lambda\boldsymbol{E}-\boldsymbol{A}|=\begin{vmatrix}\lambda-a & -1 & 1\\ -1 & \lambda-a & 1\\ 1 & 1 & \lambda-a\end{vmatrix}=\begin{vmatrix}\lambda-a+1 & -(\lambda-a+1) & 0\\ -1 & \lambda-a & 1\\ 1 & 1 & \lambda-a\end{vmatrix}$$

$$=(\lambda-a+1)\begin{vmatrix}1 & -1 & 0\\ -1 & \lambda-a & 1\\ 1 & 1 & \lambda-a\end{vmatrix}$$

$$=(\lambda-a+1)\begin{vmatrix}1 & 0 & 0\\ -1 & \lambda-a-1 & 1\\ 1 & 2 & \lambda-a\end{vmatrix}$$

$$=(\lambda-a+1)^2(\lambda-a-2)=0,$$

得 $\lambda_1=\lambda_2=a-1, \lambda_3=a+2$,

由 $(a-1)\boldsymbol{E}-\boldsymbol{A}=\begin{pmatrix}-1 & -1 & 1\\ -1 & -1 & 1\\ 1 & 1 & -1\end{pmatrix}\to\begin{pmatrix}1 & 1 & -1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$ 得 $\lambda_1=\lambda_2=a-1$ 对应的线性无关的特征向量为 $\boldsymbol{\alpha}_1=\begin{pmatrix}-1\\ 1\\ 0\end{pmatrix}, \boldsymbol{\alpha}_2=\begin{pmatrix}1\\ 0\\ 1\end{pmatrix}$;

由$(a+2)\boldsymbol{E}-\boldsymbol{A}=\begin{pmatrix}2&-1&1\\-1&2&1\\1&1&2\end{pmatrix}\rightarrow\begin{pmatrix}1&-2&-1\\0&1&1\\0&0&0\end{pmatrix}\rightarrow\begin{pmatrix}1&0&1\\0&1&1\\0&0&0\end{pmatrix}$得$\lambda_3=a+2$对应的

特征向量为$\boldsymbol{\alpha}_3=\begin{pmatrix}-1\\-1\\1\end{pmatrix}$,

令$\boldsymbol{\beta}_1=\boldsymbol{\alpha}_1=\begin{pmatrix}-1\\1\\0\end{pmatrix}$,$\boldsymbol{\beta}_2=\boldsymbol{\alpha}_2-\dfrac{(\boldsymbol{\alpha}_2,\boldsymbol{\beta}_1)}{(\boldsymbol{\beta}_1,\boldsymbol{\beta}_1)}\boldsymbol{\beta}_1=\begin{pmatrix}1\\0\\1\end{pmatrix}+\dfrac{1}{2}\begin{pmatrix}-1\\1\\0\end{pmatrix}=\dfrac{1}{2}\begin{pmatrix}1\\1\\2\end{pmatrix}$,$\boldsymbol{\beta}_3=\boldsymbol{\alpha}_3=\begin{pmatrix}-1\\-1\\1\end{pmatrix}$,

再令$\boldsymbol{\gamma}_1=\dfrac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\\0\end{pmatrix}$,$\boldsymbol{\gamma}_2=\dfrac{1}{\sqrt{6}}\begin{pmatrix}1\\1\\2\end{pmatrix}$,$\boldsymbol{\gamma}_3=\dfrac{1}{\sqrt{3}}\begin{pmatrix}-1\\-1\\1\end{pmatrix}$,

得正交矩阵$\boldsymbol{P}=\begin{pmatrix}-\dfrac{1}{\sqrt{2}}&\dfrac{1}{\sqrt{6}}&-\dfrac{1}{\sqrt{3}}\\\dfrac{1}{\sqrt{2}}&\dfrac{1}{\sqrt{6}}&-\dfrac{1}{\sqrt{3}}\\0&\dfrac{2}{\sqrt{6}}&\dfrac{1}{\sqrt{3}}\end{pmatrix}$,

使得$\boldsymbol{P}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{P}=\begin{pmatrix}a-1&0&0\\0&a-1&0\\0&0&a+2\end{pmatrix}$.

(Ⅱ)由$\boldsymbol{P}^{\mathrm{T}}[(a+3)\boldsymbol{E}-\boldsymbol{A}]\boldsymbol{P}=\begin{pmatrix}4&0&0\\0&4&0\\0&0&1\end{pmatrix}$得

$$(a+3)\boldsymbol{E}-\boldsymbol{A}=\boldsymbol{P}\begin{pmatrix}4&0&0\\0&4&0\\0&0&1\end{pmatrix}\boldsymbol{P}^{\mathrm{T}}=\boldsymbol{P}\begin{pmatrix}2&0&0\\0&2&0\\0&0&1\end{pmatrix}\boldsymbol{P}^{\mathrm{T}}\cdot\boldsymbol{P}\begin{pmatrix}2&0&0\\0&2&0\\0&0&1\end{pmatrix}\boldsymbol{P}^{\mathrm{T}},$$

令$\boldsymbol{C}=\boldsymbol{P}\begin{pmatrix}2&0&0\\0&2&0\\0&0&1\end{pmatrix}\boldsymbol{P}^{\mathrm{T}}$,

$$=\begin{pmatrix}-\dfrac{1}{\sqrt{2}}&\dfrac{1}{\sqrt{6}}&-\dfrac{1}{\sqrt{3}}\\\dfrac{1}{\sqrt{2}}&\dfrac{1}{\sqrt{6}}&-\dfrac{1}{\sqrt{3}}\\0&\dfrac{2}{\sqrt{6}}&\dfrac{1}{\sqrt{3}}\end{pmatrix}\begin{pmatrix}2&0&0\\0&2&0\\0&0&1\end{pmatrix}\begin{pmatrix}-\dfrac{1}{\sqrt{2}}&\dfrac{1}{\sqrt{2}}&0\\\dfrac{1}{\sqrt{6}}&\dfrac{1}{\sqrt{6}}&\dfrac{2}{\sqrt{6}}\\-\dfrac{1}{\sqrt{3}}&-\dfrac{1}{\sqrt{3}}&\dfrac{1}{\sqrt{3}}\end{pmatrix}$$

$$=\frac{1}{3}\begin{pmatrix}5&-1&1\\-1&5&1\\1&1&5\end{pmatrix}.$$

则$\boldsymbol{C}^2=(a+3)\boldsymbol{E}-\boldsymbol{A}$.

(22)**【解】** (Ⅰ)X 的密度函数为

$$f_X(x)=\begin{cases}1, & 0<x<1,\\ 0, & 其他.\end{cases}$$

(Ⅱ)由 $Y=2-X$ 得 $Z=\dfrac{2-X}{X}$,

$F_Z(z)=P\{Z\leqslant z\}=P\left\{\dfrac{2}{X}-1\leqslant z\right\}$,

当 $z<1$ 时,$F_Z(z)=0$;

当 $z\geqslant 1$ 时,$F_Z(z)=P\left\{X\geqslant \dfrac{2}{z+1}\right\}=\int_{\frac{2}{z+1}}^{1}1\mathrm{d}x=1-\dfrac{2}{z+1}=\dfrac{z-1}{z+1}$,

即
$$F_Z(z)=\begin{cases}0, & z<1,\\ \dfrac{z-1}{z+1}, & z\geqslant 1,\end{cases}$$

故 Z 的密度函数为

$$f_Z(z)=\begin{cases}0, & z\leqslant 1,\\ \dfrac{2}{(z+1)^2}, & z>1.\end{cases}$$

(Ⅲ)$E\left(\dfrac{X}{Y}\right)=E\left(\dfrac{X}{2-X}\right)=\int_0^1\dfrac{x}{2-x}\mathrm{d}x=2\ln 2-1.$